Graph Coloring Is Hard on Average for Polynomial Calculus and Nullstellensatz

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Joint with Jonas Conneryd, Susanna F. de Rezende, Jakob Nordström, Kilian Risse

    

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Given an n-vertex graph G, is it k-colorable?

Karp's 21 problems, intensively studied. NP-complete when $k \geq 3$.

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Combinatorial reasoning

McDiarmid'84, Beame-Culberson-Mitchell-Moore'05 **Resolution**

Algebraic reasoning

Bayer'82, De Loera'95, De Loera-Lee-Malkin-Margulies'08 ... **Polynomial Calculus**

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Random Graph—Non-k-Colorablility?

Random d-regular graph $G_{n,d}$

Erdös-Rényi-Gilbert $G\left(n, \frac{d}{n}\right)$

Are There Short Proofs of Non-k-Colorability?

For Resolution

$\exp(\Omega_d(n))$ on $G(n, \frac{d}{n})$ Beame-Culberson-Mitchell-Moore'05

For Polynomial Calculus, Nullstellensatz

$\exp(\Omega_d(n))$ on special graph Lauria-Nordström'17, Atserias-Ochreimak'19

$\Omega(g/\chi)$ degree, g is girth, χ is chromatic number Romero-Tunçel'21

$\Omega(n)$ degree on random graphs: open DLLMM'08, LN'17, Lauria'18, ...

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Our algorithm has good practical performance and numerical stability. …our experiments demonstrate that often very low degrees suffice for systems of polynomials coming from graphs.

—De Loera-Lee-Malkin-Margulies'08,
Hilbert's Nullstellensatz and an Algorithm for Proving Combinatorial Infeasibility

Our Result

With high probability, for $G \sim G_{n,d}$ or $G\left(n, \frac{d}{n}\right)$, polynomial calculus requires degree $\Omega_d(n)$ to refute that G is 3-colorable.

Corollary

$\exp(\Omega_d(n))$ size lower bounds for Polynomial Calculus and Nullstellensatz.

Techniques

Extend [Romero-Tunçel'21] to random graphs.

Polynomial ring over field $\mathbb{F}$.

The k-Coloring Axioms on G

Vars: $x_{v,i}$ $(v \in V(G), i \in [k])$ $(x_{v,i}$ is 1 $\leftrightarrow$ v gets color $i)$

$x_{v,i}(x_{v,i} - 1) = 0$ (Boolean)

$\sum_{i \in [k]} x_{v,i} = 1$

$x_{v,i} x_{v,j} = 0 \ (i \neq j)$ (v gets exactly one color)

$x_{u,i} x_{v,i} = 0$ if $\{u, v\} \in E(G)$ (no monochromatic edge)

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Fourier encoding [Bayer'82]
$X_v \in \{1, \zeta, \dots, \zeta^{k-1}\}$
Degree: equivalent

Polynomial Calculus (PC) Clegg-Edmonds-Impagliazzo'96

Axioms $p_1(x_1, \dots, x_n) = 0, \dots, p_m(x_1, \dots, x_n) = 0$

Each step:

$$\frac{p \quad q}{\alpha \cdot p + \beta \cdot q} \ (a, b \in \mathbb{F}) \qquad \frac{p}{x_i \cdot p}$$

Proof/refutation: derive 1.

Complexity Measure

Degree = max deg among all monomials

Size = #(monomials) counted over all lines

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Degree-Size Relation Impagliazzo-Pudlák-Sgall'99

Degree $\Omega(n)$ implies size $\exp(\Omega(n))$

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To Show Deg-D Lower Bounds

Find a linear map R so that:

- $R(\text{axiom}) = 0$
- $\frac{R(p)=0 \quad R(q)=0}{R(\alpha \cdot p + \beta \cdot q)=0} \quad \frac{R(p)=0}{R(x_i \cdot p)=0}$ if $\deg(p) < D$
- $R(1) \neq 0$.

Algebraic Setting

Reduction Operator

"$\succ$" : admissible total ordering on monomials.
Leading monomial of a polynomial (LM)

W a set of polynomials.

Say m is **reducible** by W if: $m = LM(p)$ for some $p \in W$.

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When W is a linear space

$$\mathbb{F}[x_1, \ldots, x_n] = W \oplus \text{span}_{\mathbb{F}}\{m\text{: irred}\}$$

Reduction operator, R_W

Projection to span of irreducibles

- $\text{Ker}(R_W) = W$
- Decrease monomials.

In application: W is an ideal (linear and $p \in W \Rightarrow xp \in W$)

Degree Lower Bound: Local-Global Principle

Deg-D PC—locally powerful, globally not (we believe).

S: a small subset of axioms *"Local set"*

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- Satisfiable
- We have local ideal I_S and local reduction R_S.
- Deg $\leq D$ part of I_S: local conclusions

Let's collect all local sets $\{S_1, S_2, \ldots\}$.

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Meaning: express every line in a deg-D PC proof as

$$p = p_1 + \cdots + p_t,$$

each p_i in some I_S and $\max_{1 \leq i \leq t}(LM(p_i)) = LM(p)$.

Call p "completely reducible" by collection $\{S_1, S_2, \ldots\}$.

If so, we're done. (Each line: LM is reducible by some I_S. 1 is not.)

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Answer: yes… if we don't encounter **Bad Cancellation**.

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Key Question

Do local reductions reduce every line of a deg-D proof?

$$\frac{p + q}{m + \text{smaller terms}}$$

BAD:

p, q: completely reducible

m irreducible by any local I_S.

Answer: yes… if we don't encounter **Bad Cancellation**.

No Bad if and only if A simple case of Buchberger's criterion

$(\star)$: For all i, j and $p_i \in I_{S_i}, p_j \in I_{S_j}$, deg $\leq D$,

$p_i + p_j$ is completely reducible by $\{S_1, S_2, \dots\}$.

E.g. suffices to have $p_i + p_j \in I_{S_k}$ for some k.

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A Sufficient Condition For Degree Lower Bounds

Find $\{S_1, S_2, \ldots\}$ so that

1. Covers all axioms;
2. Each is satisfiable;
3. Satisfy $(\star)$.

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Closed Sets

Buss-Grigoriev-Impagliazzo-Pitassi'99 (implicit), Alekhnovich-Razborov'03, Mikša-Nordström'15...

Pseudo reduction / R-operator Razborov'98

Closed Set for Coloring cf. [Romero-Tunçel'21]

Monomial order ~ Vertex order
Axiom set ~ Vertex set S

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Collection of "closed sets" $\{S_i\}$
—use a stronger requirement than $(\star)$

For all monom m with $\mathrm{Vert}(m) \subseteq S_i$:
m is reducible by $I_T \Rightarrow m$ is reducible by I_{S_i}
for any $|T| \leq 2\max_k |S_k|$.

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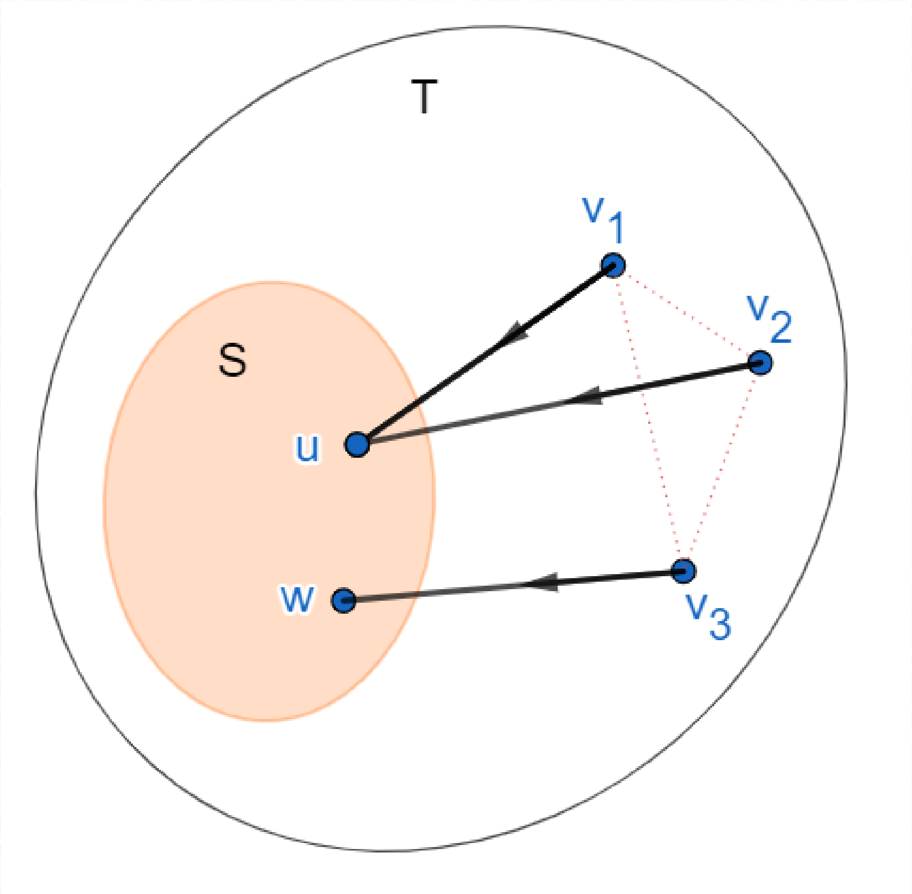
Graph-theoretic condition

1. Boundary is tree-like
   - $\{v_1, v_2, \dots\}$ is independent set
   - v_i has unique neighbor in S
2. $v_i \succ$ its neighbor in S

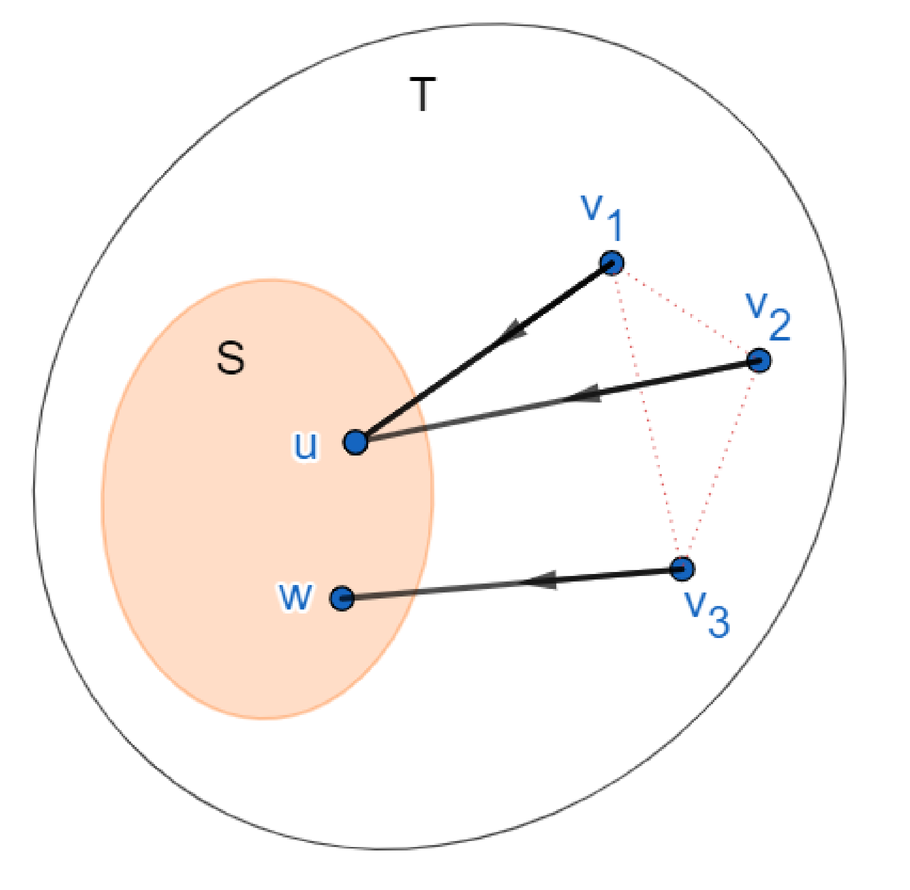


$\{v_1, v_2, \dots\}$: neighbors of S in $T \backslash S$

Closed Set for Coloring cf. [Romero-Tunçel'21]

I.e. S is closed iff:

- S is downward-closed;
  (If ∃directed path from S to v, then $v \in S$.)
- No 2-, 3-hops with respect to S in G.



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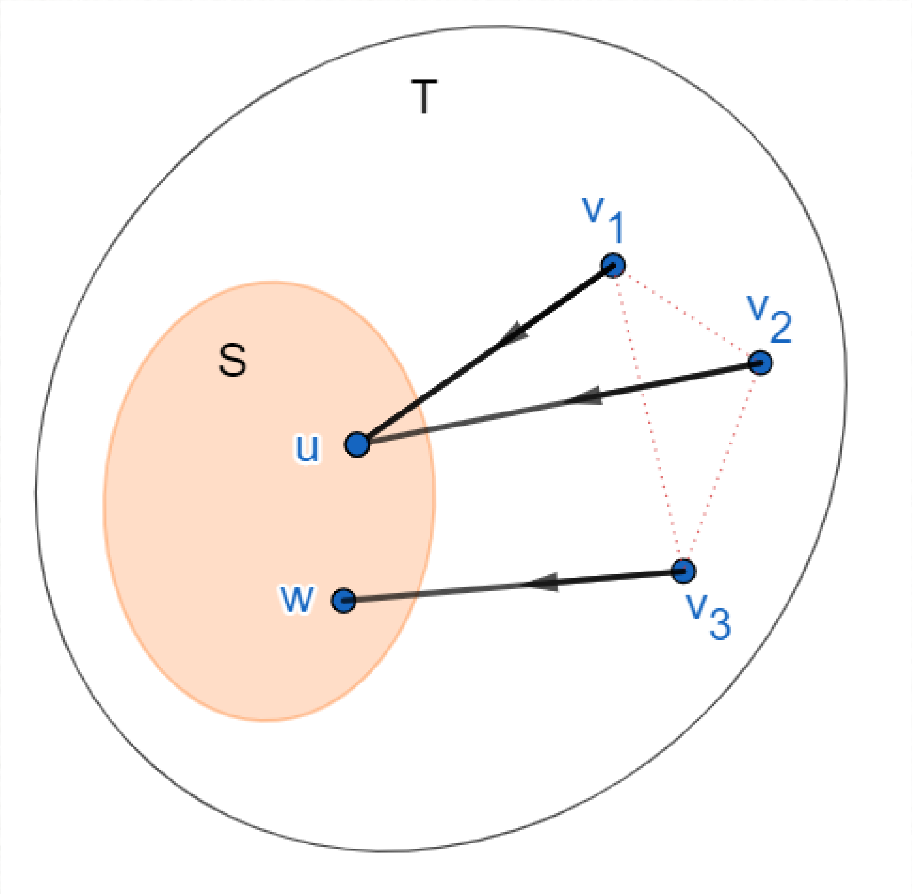

Lemma 1 [Local Reduction]

If $\mathrm{Vert}(m) \subseteq$ closed S, $|T| \leq Cn$, then:

m reducible by $I_T \;\Rightarrow\; m$ reducible by I_S.

Remark. Exlude more shapes for 3-coloring.

(2,3,4,5- and degenerate 5,6-hops)

Closed Set containing given set

$\mathrm{Cl}(S)$

- Take downward-closure;
- Once see a short hop, include it;
- Repeat.

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Collection of closed sets

$\{\mathrm{Cl}(S)\colon |S| \leq \alpha n\}$, α small constant.

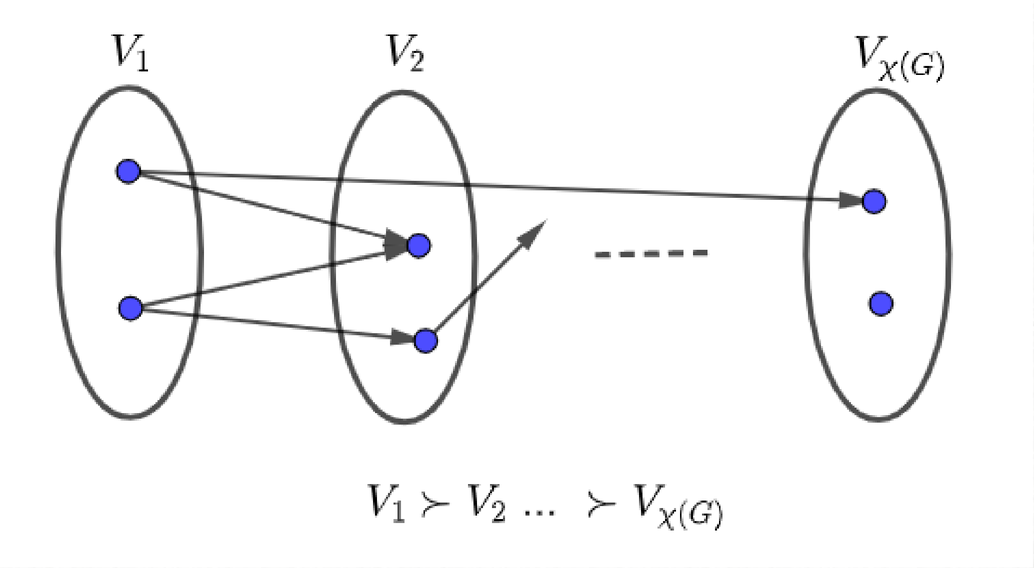
- Covers all axioms
- Satisfies ($\star$) (previous lemma)
- $\mathrm{Cl}(S)$ is small ($\Rightarrow$ satisfiable).

Closure Is Small

Vertex Ordering [RT'21]

Induced by $\chi(G)$ colors.

Directed path has length $\leq \chi$.



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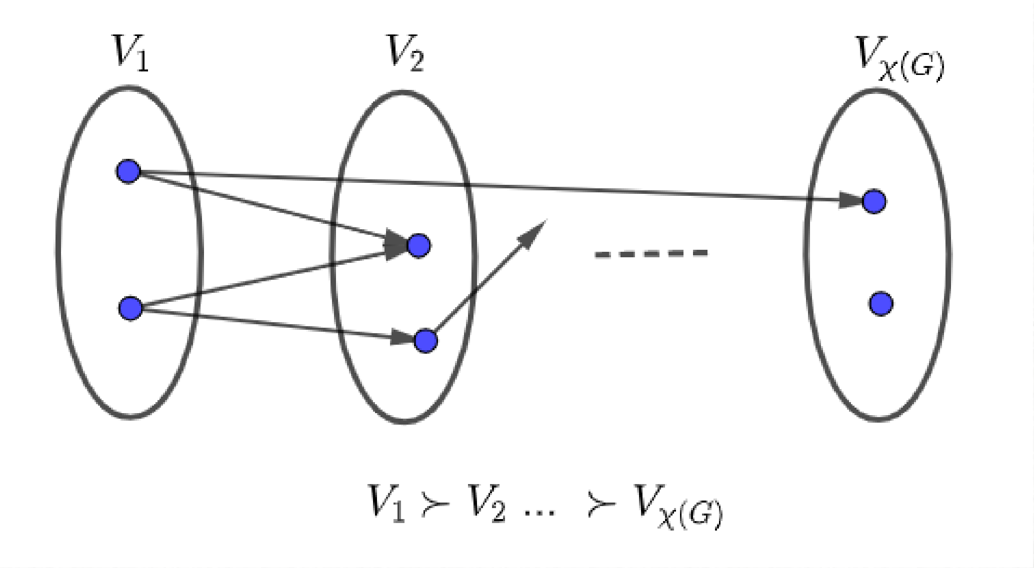


Lemma 2 [Closure Size]

Suppose $\deg(G) \leq d$ and G is locally-sparse. Then:

$$|S| \leq cn \Rightarrow |\mathrm{Cl}(S)| \leq 20 d^{\chi(G)+2} cn.$$

Remark. $G\left(n, \frac{d}{n}\right)$ has large degree vertices. Need other pseudo-random properties.

Lemma 1 [Local Reduction]

If $\text{Vert}(m) \subseteq$ closed S, $|T| \leq Cn$, then:

m reducible by $I_T \Rightarrow m$ reducible by I_S.

Proof. (4-coloring)

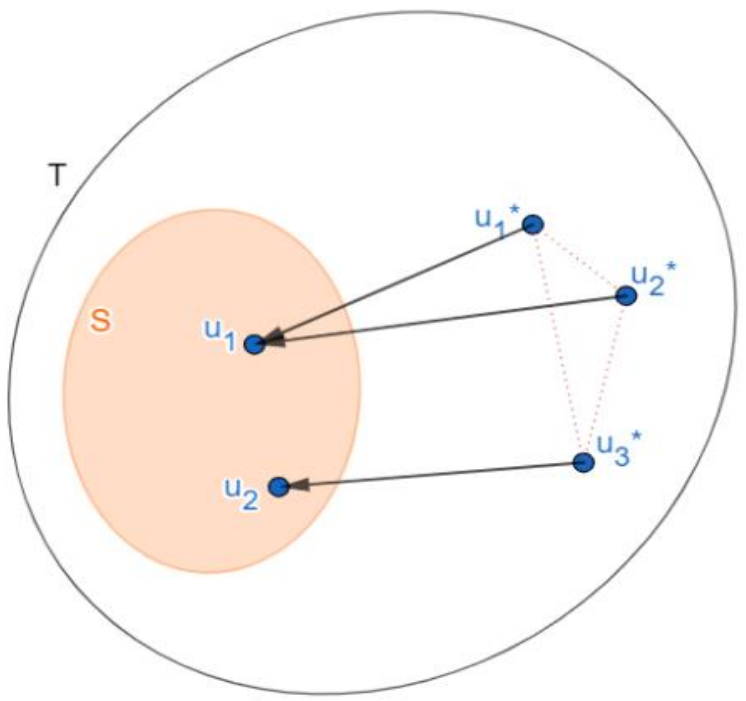
$$m + (\text{lower terms}) = \sum_{S} p_i f_i + \sum_{S,N(S)} q_i g_i + \sum_{\text{others}} r_i h_i$$

1. We can 3-color $T \backslash S$.

- Peeling Lemma

  $\forall A \ |E[A]| < 2|A| \Rightarrow$ graph is 3-colorable.

- Random graph is sparse $\forall |A| < cn \Rightarrow |E([A])| < (1+\epsilon)|A|$ [e.g. Razborov'17]



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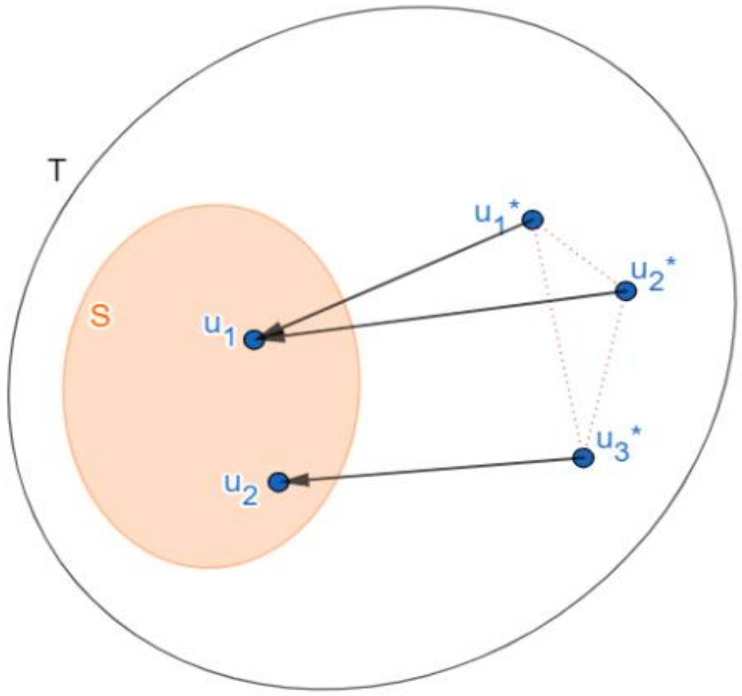

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4. Kill axioms talking about u_1^* & (u_1, u_1^*) by deg-1 substitution.

   $u_1^*(3) \leftarrow u_1(4),\;\; u_1^*(4) \leftarrow \sum_{i \neq 4} u_1(i)$
5. Do the same for $u_2^*, u_3^*, \ldots$

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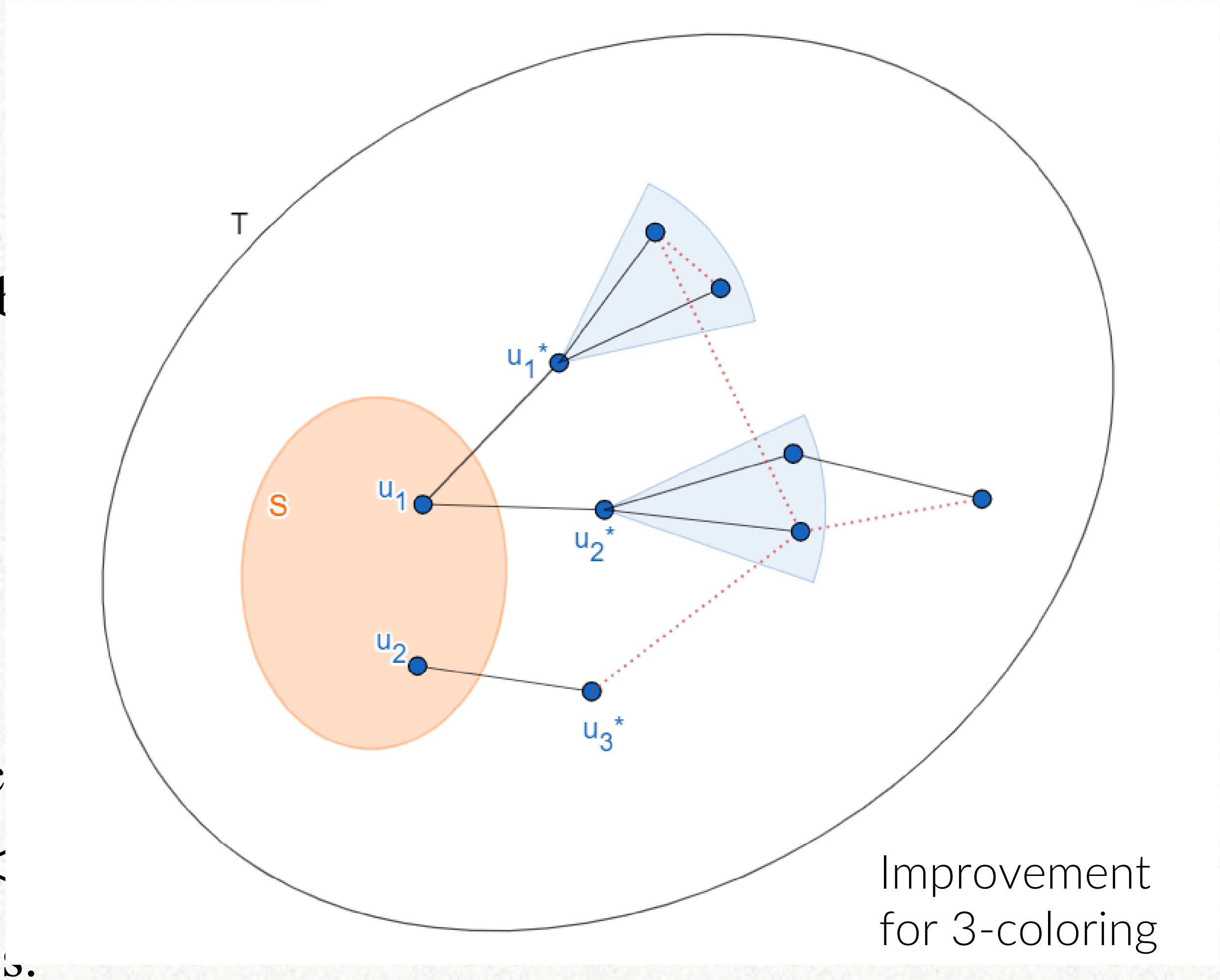

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$(D \coloneqq \frac{c}{20\chi} n)$ $\quad |S| \leq D \;\Rightarrow\; |\mathrm{Cl}(S)| \leq 20 d^{\chi+2} D.$

Proof. Recall $\mathrm{Cl}(S)$ is constructed in rounds.

Claim. There are $\leq 4D$ many rounds.

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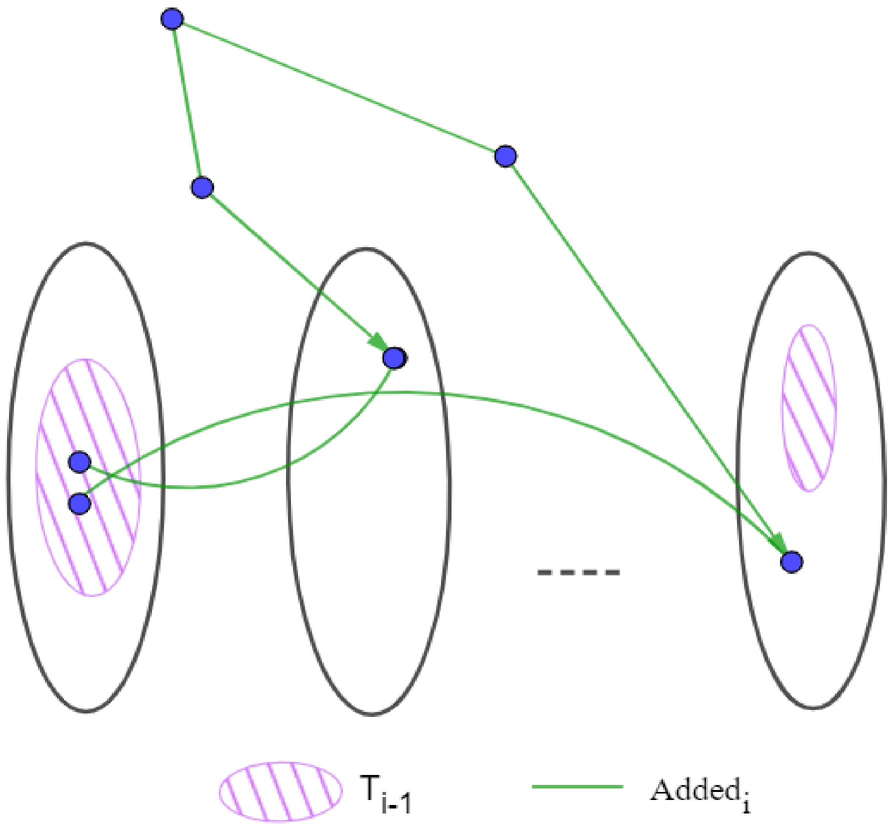

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Reason: inspect edge-density of a set T.

Initially $T_0 \coloneqq S$.

Round i: add new hop P & two **decreasing** paths from T_{i-1} to P.

$$\frac{|\text{added } E|}{|\text{added } V|} \geq \frac{1+|\text{added } V|}{|\text{added } V|} \geq 1 + \frac{1}{2\chi+6} > 1 + 2\epsilon.$$

After $i > 4D$ rounds: edge-density$(T_i) > 1 + \epsilon$. Contradiction.

Cl(S) is downward-closure of T_i, so size $\leq \chi d^{\chi-1}|T_i| \leq 20 d^{\chi+2} D.$ $\square$

Open Problems

1. Closure applied to other (graph-based, perhaps) problems?

2. Sum-of-Squares (SoS) and Sherali-Adams, for $d^{\frac{1}{2}+\epsilon}$-coloring?
   [Kothari-Manohar'21]: $G\left(n, \frac{1}{2}\right)$

Side Remark. [Krivelevich-Vu'02, Coja-Oghalan'03]: ∃deg-2 SoS refutation for $\sqrt{d}$-coloring. With our results ⇒separation

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