DST-III: The Entropy Bridge — Multi-Cycle CRNs and the Dynamic Symmetry Index

Abstract: This paper develops the thermodynamic bridge between Dynamic Symmetry Theory and open chemical reaction networks by extending the Dynamic Symmetry Index from minimal single-throughput models to multi-cycle nonequilibrium systems. The central aim is to show that dynamic symmetry can be expressed in a physically grounded way through the joint behaviour of informational diversity and entropy production in open chemical reaction networks maintained away from equilibrium by boundary driving. The paper formulates a finite-state stochastic model of a chemostatted multi-cycle network, decomposes entropy production across competing pathways and internal circulations, and defines a Dynamic Symmetry Index that detects the intermediate regime in which multiple pathways remain active while organised dissipation is already substantial. A time-local extension is then introduced by replacing stationary observables with sliding-window estimates along stochastic trajectories. This yields an early-warning version of the index capable of tracking the deterioration of dynamic symmetry before a forced transition is complete. The paper argues that the multi-cycle setting is the correct thermodynamic test case for the next stage of Dynamic Symmetry Theory because it contains loops, pathway competition, and internal circulation, and thus supports a richer interpretation of the relation between order, disorder, and adaptive structure than does the minimal open chain.

1. Introduction

Dynamic Symmetry Theory proposes that adaptive systems persist not by maximising order or disorder, but by maintaining a productive relation between them. Earlier formulations expressed this intuition conceptually and algebraically; more recent work introduced bounded scalar indices intended to detect where a system sits between rigidity and incoherence. Yet any general theory of dynamic symmetry requires at least one domain in which its terms can be defined with physical precision. Open chemical reaction networks provide that domain.

The reason is straightforward. In open nonequilibrium chemistry, informational spread, pathway structure, stationary current, dissipation, and critical transition can all be formulated in exact stochastic-thermodynamic terms. The system is not merely metaphorically adaptive. It occupies a state space, exchanges matter or free energy with reservoirs, supports probability currents, and exhibits measurable departures from equilibrium. This makes open CRNs the strongest available substrate for translating Dynamic Symmetry Theory into a quantitative science.

Earlier work already established the first step by defining a Dynamic Symmetry Index on a minimal chemostatted open network. There, a single throughput path linked boundary species through a small internal state space, and the index combined informational diversity with entropy production to detect a regime of intermediate forcing. That construction was important but limited. A minimal chain offers little topological choice: once boundary driving is specified, matter has only a narrow set of ways to move through the system. The theory therefore remained vulnerable to the objection that its apparent balance point might be an artefact of one privileged geometry.

The present paper answers that objection by moving to a multi-cycle setting. In such a network, several boundary-to-boundary routes coexist with internal loops and competing currents. The significance of the extension is not simply that the network is more complex. It is that complexity now has thermodynamic structure. Entropy production need not be attached to one dominant route; it may be distributed over parallel pathways and internal circulations. Informational diversity need not mean mere occupancy spread; it may encode sustained pathway plurality. The resulting system is a far better test case for Dynamic Symmetry Theory than any single-chain model.

The goal of this paper is therefore threefold. First, it formulates a multi-cycle open CRN in finite-state stochastic form. Second, it defines and interprets a DSI based on stationary Shannon entropy and stationary entropy production, extended to the multi-cycle case. Third, it promotes the stationary formulation to a time-local one, allowing dynamic symmetry to be tracked as a temporal object and paired with early-warning statistics. The bridge in the title is the bridge between algebraic DST and nonequilibrium thermodynamics: entropy production becomes the order-bearing quantity through which dynamic symmetry acquires physical substance.

2. Dynamic symmetry in thermodynamic form

The open-network formulation of Dynamic Symmetry begins from a simple observation. A nonequilibrium system maintained by external driving displays two equally important features. On the one hand, it exhibits organised current structure: matter, energy, or probability flows through the network in sustained directed patterns. On the other hand, it occupies a distribution over internal states whose spread records how many pathways, configurations, or local alternatives remain dynamically available. A system is dynamically symmetric when these two features coexist at appreciable strength.

This formulation is deliberately different from classical equilibrium thinking. Near equilibrium, the internal distribution may remain broad, but organised dissipation is weak. Under extreme forcing, organised throughput may be large, but the system may collapse onto a narrow family of current-carrying states. Dynamic symmetry lies between these limits. It does not mean low entropy production, nor maximal entropy production, nor maximal state-space spread. It means that the system supports sustained nonequilibrium organisation without extinguishing internal route diversity.

This idea can be expressed quantitatively by defining one diversity-like quantity and one order-like quantity on the same stochastic state space. The diversity quantity should track informational spread of the stationary distribution. The order quantity should track the thermodynamic cost of sustaining organised nonequilibrium current. The Dynamic Symmetry Index is then a bounded function that is high only when both quantities are appreciable and mutually balanced.

3. The multi-cycle network

Consider an open chemical reaction network with two chemostatted boundary species, A and B, and three internal species, X, Y, and Z. The internal reaction architecture is chosen to contain both parallel boundary-to-boundary routes and an internal loop. A minimal representative example is:

$$A \rightleftharpoons X, X \rightleftharpoons Y, Y \rightleftharpoons B,$$

$$X \rightleftharpoons Z, Z \rightleftharpoons Y, Z \rightleftharpoons B. (1)$$

This geometry contains two overlapping throughput routes from A to B: one via $X \to Y$, the other via $X \to Z$. The coupling $Z \rightleftharpoons Y$ closes an internal triangular loop. The system is therefore not merely branched but genuinely multi-cycle.

The boundary species are maintained by reservoirs whose chemical potentials or effective concentrations remain fixed. Their difference imposes a nonequilibrium affinity

$$\Delta\mu = \mu_A - \mu_B, (2)$$

which drives the internal system away from detailed balance when nonzero. The internal mesoscopic state is represented by a copy-number vector

$$s = (n_X, n_Y, n_Z) \ (3)$$

subject to a truncation

$$n_X + n_Y + n_Z \leq N, (4)$$

which produces a finite stochastic state space suitable for numerical and analytical treatment.

The importance of this topology is conceptual as much as technical. A multi-cycle system allows one to ask whether a regime exists in which the network is already thermodynamically organised, yet several pathways and circulation patterns remain statistically alive. That is precisely the situation Dynamic Symmetry Theory is meant to describe.

4. Stochastic dynamics and stationary state

At the mesoscopic level, the open network is modelled as a continuous-time Markov jump process over the truncated state space. Each elementary reaction contributes a transition between neighbouring copy-number states. Under mass-action scaling, forward and backward rates are assigned to each reaction, with boundary steps depending on the reservoir activities.

Let $W_{s\to s'}$ denote the transition rate from state s to state s'. The probability distribution $p_s(t)$ evolves according to the chemical master equation,

$$\frac{d}{dt}p_s(t) = \sum_{s'} (p_{s'}(t)W_{s'\to s} - p_s(t)W_{s\to s'}). (5)$$

At non-equilibrium steady state, one obtains a stationary distribution p_s satisfying

$$\sum_{s'} (p_{s'}W_{s'\to s} - p_s W_{s\to s'}) = 0, \sum_{s} p_s = 1. (6)$$

The network's current structure is encoded by the stationary probability currents

$$J_{s,s'} = p_s W_{s\to s'} - p_{s'} W_{s'\to s}. (7)$$

These currents need not simply follow a single net route from input to output. In the multi-cycle case, they may also circulate internally around loops. This matters because entropy production depends on the full current pattern, while informational diversity depends on the full stationary distribution. The DSI is designed precisely to register the interaction between these two objects.

5. Informational diversity and thermodynamic order

The first global observable is the stationary Shannon entropy,

$$H = -\sum_{s} p_s \ln p_s. (8)$$

This quantity measures the informational spread of the stationary distribution across the accessible state space. It is converted to a bounded diversity variable by normalisation,

$$D = \frac{H}{H_{\max}}, 0 \le D \le 1, (9)$$

where $H_{\max}$ is chosen as the maximum stationary entropy over the forcing range or an analytically justified envelope.

The second global observable is the stationary entropy production rate,

$$\Sigma = \frac{1}{2}\sum_{s,s'} J_{s,s'} \ln \frac{p_s W_{s\to s'}}{p_{s'} W_{s'\to s}}. (10)$$

This quantity measures the thermodynamic cost of sustaining the nonequilibrium current structure. It vanishes at detailed balance and becomes positive under broken detailed balance and persistent driven circulation. It is turned into a bounded order-like variable through

$$O = \frac{\Sigma}{\Sigma + K}, 0 \le O \le 1, (11)$$

where $K > 0$ sets the characteristic dissipation scale.

The interpretation is important. D does not measure disorder in a pejorative sense. It measures how much of the network's state-space and route-space remains statistically available. O does not measure order in a static geometrical sense. It measures how strongly the system sustains organised nonequilibrium flow. Dynamic symmetry is high only when both remain appreciable.

6. Defining the Dynamic Symmetry Index

The Dynamic Symmetry Index is defined in the same bounded canonical form used in earlier work:

$$DSI = 4OD(1 - |O - D|). (12)$$

This formula has three desirable properties. First, it vanishes when either diversity or order vanishes. Second, it penalises strong imbalance between the two terms. Third, it remains bounded and easy to compare across forcing regimes once the normalisations are fixed.

In the multi-cycle setting, however, the interpretation of the DSI is strengthened. In a minimal open chain, a DSI peak at intermediate forcing could be read as a balance between throughput and occupancy spread. In a multi-cycle network, the same peak has a richer meaning. It indicates that several thermodynamically relevant pathways and perhaps internal loops remain active while organised dissipation is already substantial. The index is therefore no longer simply a scalar on a throughput spectrum. It becomes a measure of whether route plurality and nonequilibrium structure coexist.

This is why the multi-cycle case matters for Dynamic Symmetry Theory as a whole. It transforms the DSI from a proof-of-concept metric into a structurally interpretable one.

7. Entropy production decomposition and the entropy bridge

The title phrase "entropy bridge" refers to the fact that entropy production provides the order-bearing side of Dynamic Symmetry in a way that connects directly to nonequilibrium thermodynamics. In a multi-cycle network, this bridge becomes more explicit because entropy production can be decomposed over distinct current-carrying modes.

Let the stationary currents be decomposed over pathway classes or elementary flux modes,

$$J = \sum_{\alpha} J^{(\alpha)}. \quad (13)$$

Then the entropy production may be written schematically as a sum of contributions,

$$\Sigma = \sum_{\alpha} \Sigma_{\alpha}, \quad (14)$$

where each Σ_{α} records dissipation associated with a pathway, loop, or effective current class. This decomposition matters because the same total entropy production can arise from very different structural situations. One network may dissipate strongly through a single canalised route; another may dissipate comparably while maintaining several active routes and loops.

The DSI is sensitive to this difference because informational diversity responds to the spread of the stationary distribution across the regions of state space sustained by those currents. Entropy production alone cannot distinguish dynamically rich organisation from canalised throughput. Shannon entropy alone cannot distinguish meaningful nonequilibrium structure from weakly driven spread. Their joint appearance in DSI is therefore the bridge between thermodynamic organisation and adaptive plurality.

8. Regime structure in the multi-cycle case

The expected behaviour of the multi-cycle DSI follows a three-regime pattern.

8.1 Weak forcing

Near equilibrium, the stationary distribution is relatively spread out because no pathway is strongly preferred. Diversity remains appreciable. Yet persistent organised currents are weak, so entropy production is low. The DSI is therefore modest or low because the network lacks substantial thermodynamic structure.

8.2 Strong forcing

Under strong boundary driving, entropy production rises because the network supports large sustained currents. However, one pathway or one circulation pattern may begin to dominate. Probability mass then concentrates on a narrower family of current-carrying states. Diversity falls, and the DSI declines despite high dissipation.

8.3 Intermediate forcing

Between these limits lies the dynamically symmetric regime. The network is driven strongly enough to sustain nontrivial boundary throughput and internal circulation, but not so strongly that one route suppresses all others. Several pathways remain statistically active, several currents remain thermodynamically significant, and both diversity and order are appreciable. The DSI therefore exhibits an interior maximum.

The important point is that this interior maximum now has a pathway-based interpretation. It marks not merely moderate forcing, but intermediate forcing in the presence of topological plurality. The dynamically symmetric regime is one of maintained route richness under sustained nonequilibrium organisation.

9. Boundary dissipation ratio

To sharpen the interpretation of multi-cycle dissipation, it is useful to distinguish entropy production associated with boundary exchange from entropy production associated with internal circulation. Let Σ_∂ denote the contribution attributed to boundary flux exchange and Σ_{total} the full stationary entropy production. Then define the boundary dissipation ratio

$$\beta_\partial = \frac{\Sigma_\partial}{\Sigma_{\text{total}}}. \quad (15)$$

This quantity measures how much of the system's thermodynamic cost is tied directly to boundary throughput rather than to internal loop organisation. In a simple single-throughput chain, one expects β_∂ to remain large across much of parameter space. In a genuine multi-cycle network, however, the ratio becomes informative: internal circulation may contribute significantly to entropy production even when net boundary flux is unchanged.

The working hypothesis is that β_∂ varies monotonically with coarse forms of DSI decline in strongly canalised regimes, while interior DSI maxima are associated with a more balanced relation between boundary throughput and internal route activity. In this way, the boundary dissipation ratio becomes a secondary diagnostic of how the entropy bridge is being realised.

10. Time-local Dynamic Symmetry

A stationary index is not enough if Dynamic Symmetry Theory is to contribute to early-warning analysis. The theory requires a time-local formulation that can be tracked along trajectories as a system approaches or departs from a critical regime.

Let a stochastic trajectory be generated by a Gillespie simulation or an equivalent jump process under slowly varying forcing. On a sliding time window centered at t, define an empirical occupancy distribution and from it a local Shannon entropy,

$$H_{\mathrm{loc}}(t). (16)$$

Likewise define a local dissipation proxy from empirical jump asymmetries and local current estimates,

$$\Sigma_{\mathrm{loc}}(t). (17)$$

After appropriate normalization, one obtains local variables

$$D_{\mathrm{loc}}(t), O_{\mathrm{loc}}(t), (18)$$

and therefore a time-local Dynamic Symmetry Index

$$DSI_{\mathrm{loc}}(t) = 4D_{\mathrm{loc}}(t)O_{\mathrm{loc}}(t)(1 - |D_{\mathrm{loc}}(t) - O_{\mathrm{loc}}(t)|). (19)$$

The conceptual importance of this extension is large. Dynamic symmetry is no longer only a property of stationary regimes. It becomes a temporal object. One may now ask whether the index declines before or during regime transition, and whether standard critical-transition diagnostics such as rolling variance and lag-one autocorrelation become informative when attached to the DSI series itself.

11. Early-warning interpretation

The first role of time-local DSI is descriptive: it shows whether the balance between route diversity and organised dissipation is being preserved in time. Its second role is predictive. As a system is forced away from a dynamically symmetric regime, one expects several changes.

First, local DSI should decline as route plurality is lost or as dissipation becomes too strongly concentrated. Second, rolling variance of the local DSI may rise as the system flickers between partially competing current structures. Third, lag-one autocorrelation may increase as the system becomes slower to recover from perturbation in the local balance between diversity and order.

These diagnostics do not by themselves guarantee universal early-warning power. But they make the theory falsifiable in a prospective sense. If DSI is a meaningful descriptor of adaptive nonequilibrium organisation, then a sustained loss of dynamic symmetry should precede or accompany a transition into a new, more canalised or unstable regime. The CRN setting is the first place where that claim can be tested with exact stochastic control.

12. Why open CRNs matter for DST

The importance of open CRNs for Dynamic Symmetry Theory is not merely that they are mathematically manageable. It is that they are among the few domains where the theory's central concepts can all be made explicit at once.

- The system has a clearly defined state space.
- The distinction between closed and open dynamics is thermodynamically meaningful.
- Entropy production is exactly definable.
- Multiple pathways and loops can be encoded structurally.
- Transitions and early-warning signatures can be simulated or measured.

This makes CRNs the ideal testbed for a theory that wants to relate organisation, variability, and critical transition within one framework. If Dynamic Symmetry cannot be made rigorous here, it is unlikely to become rigorous elsewhere. If it succeeds here, it gains a physically grounded template that later domains can reinterpret by analogy or reduction.

13. Relation to DST-I and DST-II

This paper occupies a specific place within the emerging architecture of Dynamic Symmetry Theory. DST-I introduced the notion of a dynamic symmetry algebra: a time-dependent family of local symmetry structures whose deformation and stratification describe adaptive change. DST-II then defined an adaptive group-theoretic DSI using a symmetry departure operator and its nuclear norm. DST-III complements rather than replaces those constructions.

Its contribution is to show how a physically grounded version of DSI can be realised in an exact nonequilibrium setting. The entropy-based formulation is not merely an analogy to adaptive symmetry. It is an operational embodiment of it. Organised dissipation corresponds to the order-bearing side of adaptive structure; informational spread corresponds to the diversity-bearing side; the DSI measures their joint maintenance.

This means that the entropy bridge is both conceptual and mathematical. It links the symmetry language of the earlier papers to the thermodynamic language of open stochastic systems. That link is essential if DST is to develop into a programme that is not only formal but experimentally and computationally tractable.

14. Limitations and next steps

Several limitations should be stated.

First, the present construction is finite-state and model-based. The truncation required for numerical treatment must be checked for robustness as state space grows. Second, the multi-cycle example studied here is still stylised relative to biochemical reality. Real biochemical networks may involve many more species, separated timescales, and hidden environmental couplings.

Third, entropy production is used as the order-bearing quantity because it is the most natural thermodynamic candidate in open nonequilibrium systems. That does not mean it captures every aspect of organisation. Cycle-resolved observables, directed information measures, and pathway occupancy fractions may all become necessary to sharpen the theory. Fourth, the early-warning interpretation remains at proof-of-method stage until the time-local DSI is tested more broadly across parameter families and network topologies.

These limitations are not fatal. They indicate the correct next steps: larger state spaces, richer multi-cycle architectures, cycle-resolved diagnostics, and transfer of the time-local construction from stylised models to biologically relevant systems.

15. Conclusion

This paper has argued that open multi-cycle chemical reaction networks provide the correct thermodynamic bridge for the next stage of Dynamic Symmetry Theory. By extending the Dynamic Symmetry Index from a minimal single-throughput chain to a network with parallel routes and internal circulation, it strengthens both the interpretation and the credibility of the theory. The DSI now measures not merely a balance between spread and throughput, but the coexistence of route plurality and organised nonequilibrium dissipation.

The central theoretical result is that dynamic symmetry in open CRNs is most naturally expressed through the joint maintenance of informational diversity and entropy production. Weak forcing leaves the system too undriven to sustain organised structure; strong forcing canalises the system into dominant routes; intermediate forcing sustains both organisation and plurality. This is the regime the DSI is designed to detect.

The time-local extension adds the further claim that dynamic symmetry can be tracked as a temporal object. Local decline in DSI, combined with increased variance and autocorrelation, provides a principled candidate signal of impending thermodynamic reorganisation. Whether that signal proves universal remains for future work. But the crucial step has been taken. Dynamic Symmetry Theory now possesses a physically grounded entropy bridge connecting its algebraic and scalar formulations to the exact language of open nonequilibrium thermodynamics.