The Language of Symmetry in Music

Robert Quinney
Associate Professor, Music, University of Oxford

Abstract
Robert Quinney considers symmetry in the Western tradition of music. He first examines mirror images in a rondeau by Guillaume de Machaut, then he discusses symmetries in the circle of fifths and harmony. Quinney highlights examples of the interplay between consonance and dissonance throughout.

'Notes pass quickly away; numbers, however, though stained by the corporeal touch of pitches and motions, remain.' [1]

How might we define musical symmetry? We could begin with some examples of order in music – the internal organisation of pieces of music, their 'form' or structure – and see whether or not these match the definition of symmetry: *transformations that leave the object unchanged*. We could fetch some scores from the library and look for evidence.

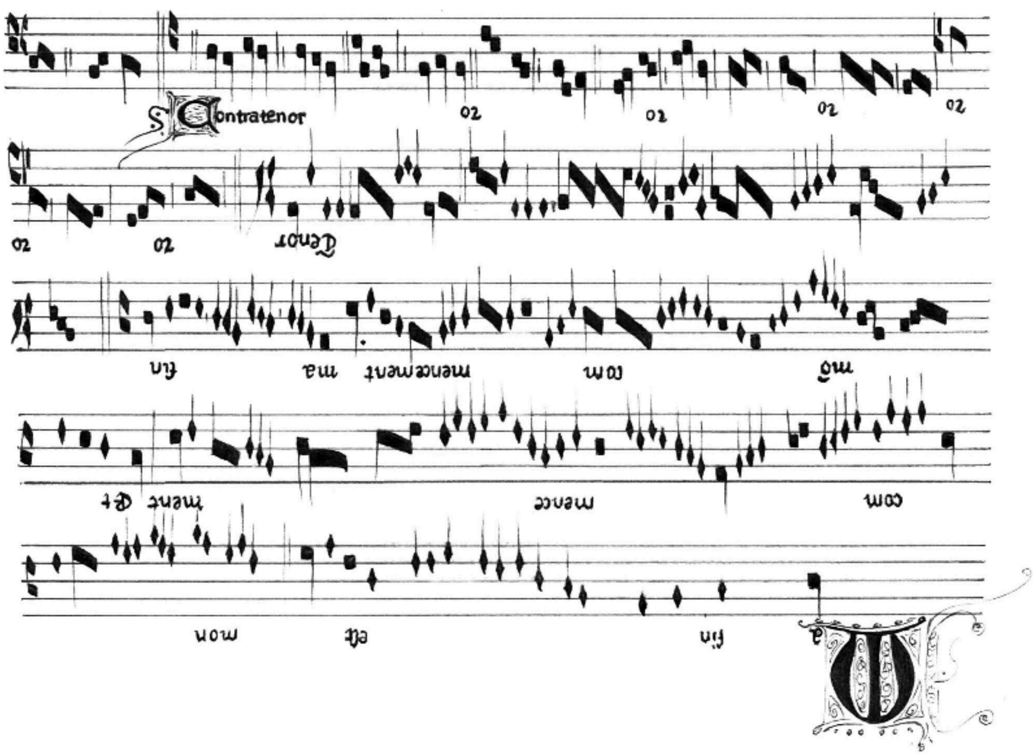
Music, however, cannot be reduced to a text. It is also, in fact it is primarily, an act. [2]

So simply *looking* at music will not help us: we need to proceed with ears open as well as eyes, asking what the music's effect on us might be as listeners.

One more caveat: the examples here are all situated within the Western art music tradition, what is popularly known as 'classical music'. The music under discussion here has all come down to us via documents, the existence and survival of which depended upon their association with high social status and wealth. While it is probable that symmetry can exist in all music, here I focus on the music of a specific tradition—one which, due to globalisation, continues to be a pervasive influence in millions of lives. [3]

Symmetry is most often thought of by mathematical dunces like me in terms of reflection. This is only rarely how symmetry manifests in music, however. Why? Most music contains some sort of *narrative*: not songs only, but ‘abstract’ music too. Music unfolds in time, but not as a succession of unrelated moments. It isn’t simply one thing after another; rather, in music one thing usually *leads* to another. Lift the lid of most musical works, and what is revealed is an ordered collection of mutually reliant constituent parts, whose relationship to one another is defined by their *successive* nature (or, sometimes, a rhetorically unsuccessive gesture like a sudden pause or a surprising chord). The principal medium for this successiveness is *harmony*, the means by which melodies are simultaneously combined with one another—everything from a solo voice over a bass line to the densest orchestral score. And because it is virtually impossible to narrate backwards, the uses of mirror (reflection) symmetry in music are limited. We shall look further at narrative in music, and the ways in which symmetry helps to generate it. But not before we encounter two pieces that do employ mirror symmetry, both for the delight of an audience of *cognoscenti* and to demonstrate the composers’ ingenuity.

Machaut, *Ma fin est mon commencement.* Paris, Bibliothèque Nationale, MS Fonds Français 9221, f.136



The clue is often in the title: never more so than in this *rondeau* by Guillaume de Machaut (d. 1377), *Ma fin est mon commencement.* Those performing it in the days before modern editions (see Fig. 1) would have needed to take the text to heart: in particular the line 'Mes tiers chans trois fois seulement se retrogade et einsi fin' ('my third voice reverses itself three times only and thus ends'). For a start, there are only two notated voices, and one voice has only half the notes of the other. The solution is for one singer to read from the shorter part, twice: once forwards, then backwards. This part is a reflection of itself, one 'side' of the reflection sung after the other. Meanwhile, the second and third singers share the written-out music, but they read at 180º to each other, one ending where the other began. One reads right-way-up, the other upside down: in other words, these two parts are a simultaneous reflection of each other, designed to sound simultaneously.

Nearly two hundred years after Machaut's death, a joint publication by Thomas Tallis and William Byrd included as its final number another tour-de-force of musical symmetry. It takes a short text, 'Miserere nostri, Domine' (Have mercy on us, Lord) and spins a seven-voice web: two canons arrayed across six voices, and one 'free' part. The upper two voices, called *Superius* and *Superius secundus*, have a straightforward canon of the Frère Jacques sort: they sing the same melody at a temporal distance. We can hear the relationship clearly, since the two voices are not far apart temporally, and sing at the same pitch: had theirs been a canon at the fifth (with the second voice singing the melody five notes higher), our brains would probably have given up on it.

Above: Tallis/Byrd, *Miserere nostri* (*Cantiones Sacrae*, 1575), Discantus part. Note (a) the attribution to Byrd, unique to this partbook, and (b) the symbols [X] showing where each of the three parts that sings a rhythmically augmented transformation of these notes comes to an end. All three parts derived from this one sang from their own partbook, with a realisation of this canonic *dux* provided (and attributed to Tallis).

The other canon is another matter. Below the two Superius voices sits Discantus, singing the *dux* (leader) from which three other voices are canonically derived (Fig. 2). Incidentally, in the Discantus partbook this piece is attributed to Byrd, whereas the remaining sources all name Tallis as the composer: evidence, perhaps, of a joint effort whereby one contributed the basic material and the other 'realised' it. This four-part canon is doubly obscure to the listener. It is arrayed across two pairs of voices: *Discantus–Contra Tenor*, and *Bassus–Bassus secundus*. The lower pair sings an inversion of the *Discantus* melody: where the *dux* falls, the melody in the *Bassus* pair rises by the same degree: a mirror image of the original. Unlike the *Superius* canon, the four lower voices all begin at the same time, but move at different, proportionally related speeds. Like shadows cast at intervals by the setting sun, their dimensions are different, but all are proportionally related to their source. *Contra Tenor* sings the *dux* in double augmentation—each note is four times the length of the original. Meanwhile, *Bassus secundus* sings the *dux* at half the original speed, and *Bassus* [*primus*] in triple augmentation, at eight times the original duration. All this is well beyond our cognitive ability: indeed, can we experience this canon at all, or is it just a conceit to amuse the knowing reader?

In the *Superius* canon we can hear the music travelling forward. The fact that the second voice literally follows the first, their phrases overlapping, lends a gently propulsive quality to the arrangement. By contrast, the complex duration relationships of the *Discantus* canon (which we could represent as 8:2:1:4, working from the highest to the lowest voice) give their music a quality we might characterise not as two parallel straight lines, but as four concentric circles. This sense of circularity is a common characteristic of complex canons. Time seems to be standing still—or, at least, moving rather more slowly than usual. This music is not, as it were, normal. Why?

For Machaut, Tallis, Byrd, and for musicians of their times and some considerable time later, composing was not an exercise of the creative ego, but a process of *inventing*—from the Latin *invenire*, to 'find out'. Any individual piece of music was subject to ancient immutable laws, the numerical expression of which was attributed to Pythagoras. *Pythagorean* has come to refer to ratios by which musical sound is ordered: that is, the relationships between different notes. Famously, a taut string stopped halfway along its length, and thus vibrating at a ratio of 2:1 to its unstopped self, produces a note one *octave* above that produced when it is full length. Outward from this simple physical reality extends a whole system of proportional relationships, adopted by theorists from Boethius (d. c.524) onward to explain musical phenomena of all kinds. Music, to this worldview, is a fundamental constituent of the created universe: it is sounding evidence of the perfection of that creation. Music was literally everywhere. Following Plato, Boethius proposed three types of music: of the universe, of the body, and finally the music actually produced by human activity.

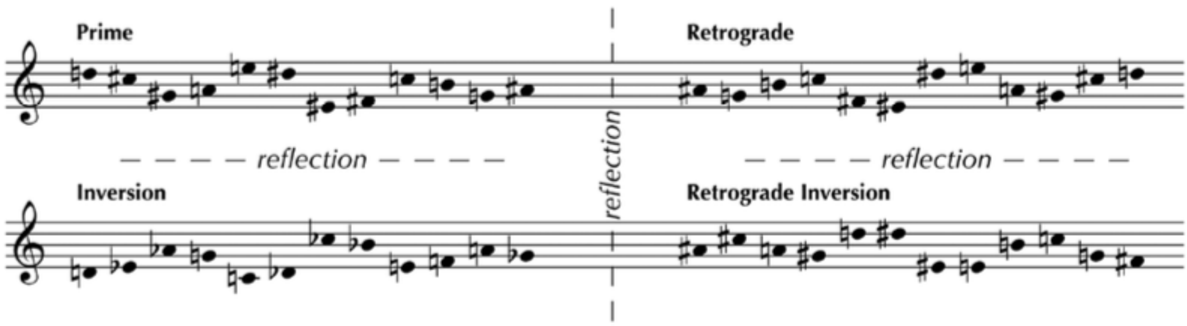
In order to make their intricate canonic constructions work according to the conventions of their time and place, the composers we have so far considered had to favour *consonance* and mostly eliminate *dissonance*. The relationships between two sounding pitches – their relative frequencies, what we call the *intervals* between notes – were codified according to ideas of perfection and imperfection. We are dealing here with *Harmony*. Its roots are in the Pythagorean numbers, and specifically the *harmonic series*: the pitches that resonate, in unvarying order, above the 'fundamental' pitch of any note struck, plucked, blown or sung. The two are interconnected: the first harmonic is the octave (2:1), and the interval of an octave (or two octaves, or three, and so on) is considered *perfect*; likewise the interval of a fifth, whose ratio to the fundamental is 3:2, and which appears third in the harmonic series (in fact as a 'twelfth' i.e. an octave plus a fifth above the fundamental). Less perfect but still *consonant* were the third and sixth, and these two turn out to be helpful in the composition of music that has more than two parts: imagine two notes a third apart, and move the upper note down an octave, it becomes a sixth, and *vice versa*. In other words, these intervals are *invertible*. All other intervals were *dissonant*: they were literally unharmonious. But without them music was just one consonance after another, unvarying and bland.

How was dissonance released into the musical stream without polluting it? There were two methods. First, dissonances could pass, unnoticed, between adjacent consonances. Second, and more importantly for our purposes, dissonances could be hit head-on, *accented*, before *resolving* into consonance. A ‘passing note’ is – no surprise – a form of passing dissonance, while an *appoggiatura* and a ‘suspension’ fall into the category of accented dissonance.

Ma fin est mon commencement and *Miserere nostri* work by avoiding accented dissonance entirely. There are passing dissonances, gently brushing the consonant surface, but nothing that must be *resolved*. Accented dissonance is prominent: we notice that it doesn’t fit. It poses a problem that requires resolution, and convention dictates that the resolution happen in a fixed temporal and melodic direction: it must resolve forward in time and downward in pitch. To do this in reverse would create chaos: dissonance springing from nowhere, and left hanging (like the end of sentence spoken as if it were a question?). Just as only certain dispositions of letters will create a palindrome, only a limited number of rhythms and pitches can be deployed in a complex canonic array. For the most part, it might seem, symmetry was ruled out of this conception of music.

It is certainly true that symmetry came into its own when the system described above, which had endured since before the days of Tallis and Byrd, collapsed around the turn of the twentieth century. As the dust settled, composers looked for an alternative set of conventions – a new framework for their music – and symmetrical transformations acquired a new and prominent role.

In the early 1920s Arnold Schoenberg sought a language for music that would replace the old system with something equally rigorous and ‘universal’. The twelve notes of the ‘chromatic scale’ would now have equal status, and order imposed by using an unvarying ‘row’ or ‘series’ of all twelve notes, in any order, without repetition. [4] Crucially, the row would be subject to transformations, including reflection (either inversion or retrograde motion, or both) and translation (in musical terms, transposition so the row begins on a note different to the ‘prime’ form: see Fig. 3). Here was a new musical language, spoken most eloquently, at first, by Schoenberg and his pupils Anton Webern and Alban Berg: the triumvirate often referred to as the ‘Second Viennese School’. It says something about human creativity that these these composers, all using the same system, each produced such astonishingly individual music.



Above: An example of a ‘tone row’ and its basic permutations. The sounding pitch may be changed (e.g. all notes of the Prime form moved up three semitones, the shorthand for which is P3), but the intervals between the notes must remain the same (e.g. in P3 the first note would be F natural, the second E, the third B, etc.). This ‘row’ and its permutations provide all the melodic material for a piece of music.

Of the three, Webern was arguably the most concerned with the internal coherence of his music. In a revealing doodle, he inscribed one of his compositional sketches with this square palindrome:

S	A	T	O	R
A	R	E	P	O
T	E	N	E	T
O	P	E	R	A
R	O	T	A	S

The ‘sator square’ could be a composition by Webern, if we replaced the letters with notes (except he had twelve notes at his disposal, not eight letters). His music speaks of a belief, not uncommon in the mid-twentieth century, in order as an end in itself; specifically, of music purged of decadent appeals to emotion. After his death in 1945, his influence spread across Europe and the United States, and an early standard-bearer was Pierre Boulez, who in wrote an article entitled ‘Schoenberg est mort’ (indeed he was, but this was no admiring obituary). Boulez’s teacher Olivier Messiaen had introduced serialism to the manipulation of rhythm as well as pitch. Now Boulez included dynamics (relative loudness), and ‘attacks’ (e.g. how sharply a piano key was struck and released) in an ultra-constructivist system known as ‘total serialism’. Series of pitches, rhythms, dynamics and attacks were allotted numbers and fed into matrices that more-or-less instructed the composer how the piece should turn out. In other words, this was composition by algorithm—a conscious repudiation of ‘romantic’ ideas of artistic autonomy. Boulez later reminisced that he had wanted ‘to strip music of its accumulated dirt and give it the structure it had lacked since the Renaissance’. [5]

Had music come full circle? Hardly, for Boulez and his colleagues’ interest in the music of ‘the Renaissance’ resided in the ways in which it was controlled by ‘structure’. In reality, the conventions of harmony, of consonance and dissonance, were far broader and more permissive than the straightjacket of total serialism. The two strictly canonic pieces we have considered were quite unlike the vast majority of music produced by Machaut, Byrd and Tallis. Symmetry is there even in less outwardly symmetrical music—indeed, we might even hold it responsible for some of the ‘dirt’ decried by Boulez. And so to Johann Sebastian Bach (whose initials were arranged in near-symmetry for his seal – see below).

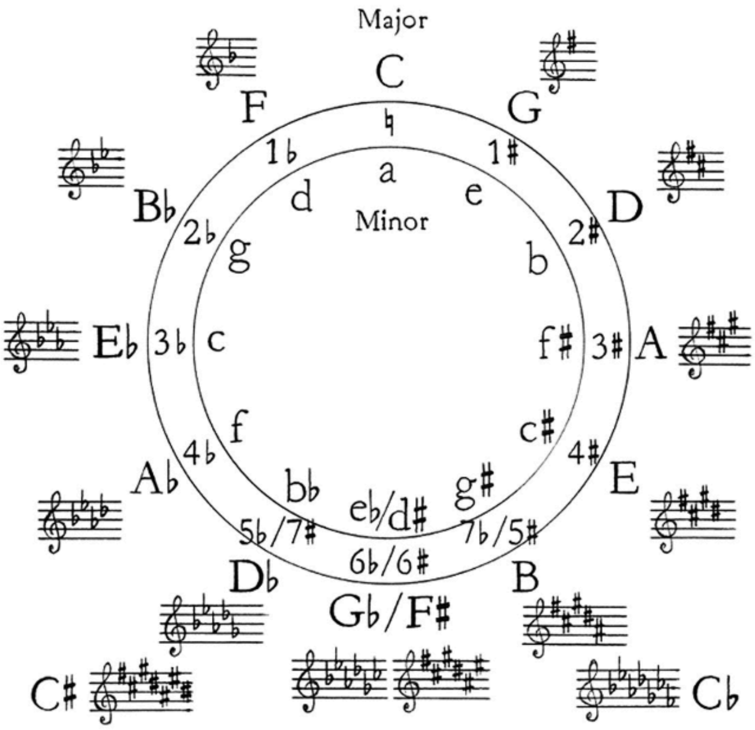
J. S. Bach’s seal, designed in 1722. To the left and right: the initials J, S, B extracted from the seal, demonstrating their near-perfect reflection symmetry.

But first, more Pythagoras. The system of *tonality* employed by Bach derived from the same numerical relationships explored above. Each tonal area – what we would call a *key* – had as its boundary an octave (2:1). We call the first note (or *degree*) of the scale the *tonic*. The fifth degree of the scale – 3:2 to the tonic – is called the *dominant*. In ‘tonal’ music the tonic and dominant exist in creative tension. Tonal music begins in the tonic key, and ends in it too, with the tonic placed reassuringly at the bottom of the final chord.

This system does not give the appearance of symmetry. If the octave (between, say, a note C and the C above) were divided exactly in half, the mid-point would not be a fifth above the lower note. Imagine moving inwards from an octave played on a piano keyboard. The mid-point is F sharp, or G flat. In Pythagorean terms, however, those notes are not one and the same: it is only by dividing the octave into twelve equal semitones that such an 'enharmonic' equivalence is made possible, and if we divide things that way, the ratio between the tonic and dominant (C and G) is nota perfect 3:2. There is not enough space here to plot the history of tuning systems, but we need to unlock this apparent problem that the dominant is not produced by a symmetrical, 50/50 division of the scale. The key to this lock is the idea of *inversion*—or, more properly, *invertibility*.

In the first of his two-part *Inventions*, Bach shows us how we might 'find out' a modest piece of music (Fig. 5). First, an assortment of notes is played by the right hand: the *subject*. It is immediately answered by the left hand, which plays the same subject an octave lower; meanwhile, the right hand continues with a complementary set of notes above. Note that the subject begins on C, the tonic, and rises to G, the dominant: by the time the left hand reaches this latter point at the start of bar 2, the right hand has risen to D, which is a fifth above G and therefore it is dominant. The listener senses that something has changed: we're no longer where we started. Sure enough, we now hear the subject again, in the same exchange between the hands, but starting this time on G. Very soon after beginning, the music has reached the opposite pole of the tonal planet, the dominant. All that has happened, as far as we can see in the score, is that everything has simply moved up five notes—but we have in fact travelled, in not much more than an instant, far from home. Then, just as quickly, we are back: the penultimate right hand note of bar 2, unlike its counterpart in bar 1, reverses the direction of travel, leading to a *perfect cadence*—an unequivocal statement of tonal movement, here from dominant to tonic. The next time we hear the subject, it is firmly in the dominant key, whose arrival has been announced with a perfect cadence from bar 6 to bar 7. In order to see the symmetry here, we need conceive of the various keys as a *circle* rather than a straight line.

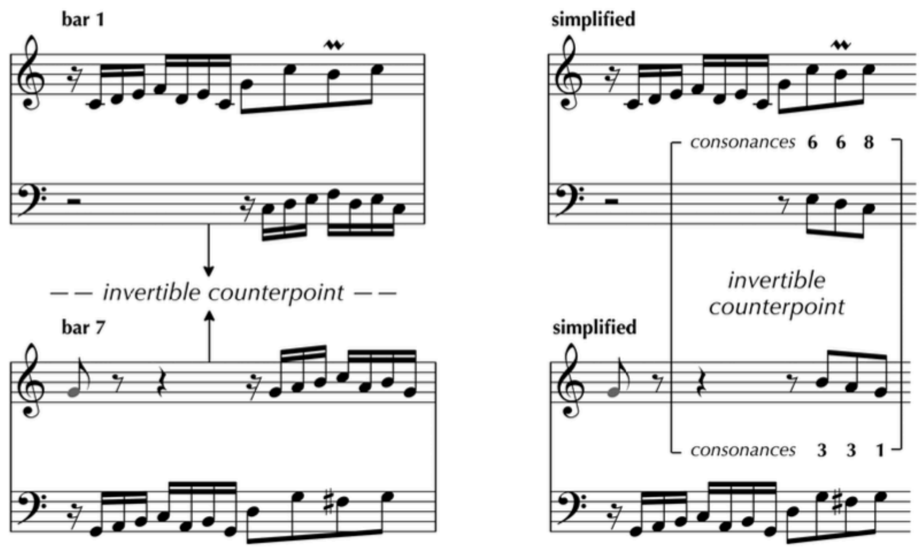
Indeed we should, because if we move from one key to another by perfect fifth we will eventually arrive where we began. Thus the subject, and the piece based upon it, has been transformed by rotation.



Above: Circle/cycle of fifths

Nor is rotational symmetry is confined to tonality. In bar 7, the subject appears in the left hand first, then the right hand. The point here is not simply that the order has been reversed between the hands, but that the subject now appears *above* its accompaniment, or the *countersubject*, not below as before. They have been *inverted*, not in the sense of melodic inversion as we saw in *Miserere nostri*, but an inversion of order. Bach's music, perhaps more than any other music we have, relies on and exploits this invertibility. Remember how the interval of a third above a note becomes the sixth below if inverted? Both intervals are consonant, and 'invertible counterpoint' depends upon such flexible intervals: thirds, sixths and octaves (see Fig. 7a). Fifths are tricky, because they become dissonant fourths when inverted, but Bach knew how to incorporate fourths and other dissonant intervals into his invertible counterpoint so that the ear does not recognise them as dissonant.

Bach, *Inventio*. BWV 772.



Above: Bach, *Inventio* showing invertible counterpoint

Above: Bach, *Inventio* showing reflection symmetry

A further symmetry exists here, one which combines melodic and tonal movement and depends upon both reflection and rotation. The music that Bach uses to move from tonic to dominant in bars 3 to 7, and which then takes us off on an extended journey around keys, is an inversion of the subject's opening: its first seven notes now fall then rise, instead of the other way around (Fig. 7b). It has been transformed into a linear pattern of intervals, a free-wheeling 'sequence' of notes that repeat at regular intervals. Sequences such as this fulfil a purpose entirely different that of the subject. Instead of defining a tonal area, they blur the boundaries. They thus enable free movement from one key to another, or provide a brief excursion away from an otherwise settled key or simply a sort of holding pattern. They need not have anything to do with the subject in terms of pitches and rhythms. Here, however, those basic constituents are exactly the same, except that here they have been inverted—or, in symmetrical terms, transformed by reflection. Furthermore, because sequences exist to move between keys – to make free but transitory associations until called to order by a cadence – and do so around the circle of fifths, they are constantly subject to rotational transformations. [6]

All this from the first few bars of a piece in only two parts, which takes about a minute to play. And there is, of course, far more to say. We might, for example, note that Bach's fascination with invertible counterpoint persisted to the very end of his life, with the posthumously published collection *Die Kunst der Fuge*, an encyclopaedia of the different ways a single subject could be subjected to fugal and canonic treatment. Among the fugues are two 'mirror fugues', in which symmetrical reflection governs the music: each is really two fugues, but they are vertical mirror-images, *rectus* and *inversus*, of each other. Another rotational symmetry is incorporated into the 'Goldberg' Variations, published in 1741: here every third variation is a canon, but the interval of the canon expands successively upward, eventually crossing over itself at the *Canon at the octave* like a traveller crossing the International Date Line.

Small this Invention might be, but it tells us things that are of fundamental importance to our understanding of Bach's music. Invention and invertible counterpoint, both reliant on symmetrical transformations, are the foundation for all his often speculative music adventures. After Bach's death, the tonal ('diatonic') system, with tonic and dominant as its poles, continued to command the world of art music with its overarching mirror symmetry. In the diverse works of Haydn, Mozart, Beethoven, Schumann (Clara and Robert), Mendelssohn (Fanny and Felix), and so many others, the tonal space defined by the tonic and dominant remained reassuringly closed, and the symmetrical underpinning of diatonic harmony continued to operate on a great variety of music.

As we have already seen, even when conventions and styles have changed radically, symmetry has not vanished from music. How could it? We would be wrong to think that 'total serialism' was the high-watermark of musical symmetry. Indeed, at the very moment when that might have seemed the case, a group of musicians who were more interested in jazz and rock than high art and the academy were developing new symmetries in their own ‘minimalist’ compositions.

In Steve Reich’s *Piano Phase* (1967), a repeated pattern is played on two pianos, one of which – imperceptibly at first – begins to fall out of step with the other, eventually becoming a quasi-independent voice in its own right, then moving gradually back into phase. Over the course of the piece the same thing happens several times: as the pianos diverge their rhythmic interplay becomes frenetic, then calms as the second ‘divergent’ piano rotates to a point where it is playing different notes to the first, but in rhythmic unison—two notes sounding at the same time. The pattern of pitches never changes, but depending on the combination of notes being played, we intuit different rhythmic groupings; we hear accents emerge, then fade. There is a kind of counterpoint here, between the predictable and the unexpected—or rather the *unexpectable*, because our cognition of the patterns does not extend to predicting their effect.

Piano Phase and Bach’s Invention both set their ‘subjects’ on a path and moving them forward through time. Symmetry plays an essential part in generating and governing that movement. Indeed, musical symmetries – whether they engender a sense of circularity or of propulsion, stillness or activity – are perhaps even responsible for manipulating our experience of time itself. [7]

References

1. *Scolica enchiriadis (ca. 850 ce), trans. Lawrence Rosenwald, in Pierro Weiss and Richard Taruskin, Music in the Western World: A History in Documents (Belmont, CA: Thomson/Schirmer, 2007), 34.*
2. The distinction, first made by Richard Taruskin in a record review of 1987, later became the title of his collection of essays, *Text and Act: Essays on Music and Performance* (New York: Oxford UP, 1995)
3. For one example of non-Western musical symmetry, see Dave Benson, *Music: A Mathematical Offering* (Cambridge: Cambridge University Press, 2006), 322-324.
4. Josef Hauer (1883-1959) had, in fact, experimented with a 'twelve-tone' technique a few years before Schoenberg, who saw some of Hauer's keyboard music in 1916. Other composers had also 'discovered' the technique. See Taruskin, *Oxford History of Western Music, vol. 4: Music in the Early Twentieth Century* (Oxford: Oxford University Press, 2010), 680-686.
5. Joan Peyser, *To Boulez and Beyond* (revised edition) (Lanham, MD: Scarecrow Press, 2008), 147. The quotation first appears in the same author's *Pierre Boulez: composer, conductor, enigma* (1977), which reported conversations between the author and Boulez during his tenure as Music Director of the New York Philharmonic Orchestra.
6. For a full analysis of BWV 772, see Dreyfus, Laurence, *Bach and the Patterns of Invention* (Cambridge, Mass.; London: Harvard University Press, 1996), 10-26.
7. This essay was reproduced from *The Language of Symmetry* (Eds. Rattigan, Noble & Hatta), 2023