Digital Fabrication and Hidden Concepts in Mathematics Education: The Case of Algebraic Inequalities

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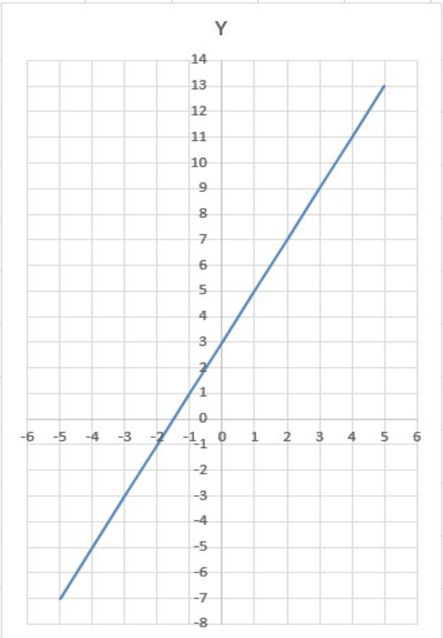
Introduction

When viewed as a content area, mathematics has a split personality.

To use an example from language, there are parts of mathematics that function very much like a noun (the concepts of mathematics), while others function more like a verb (procedures, which many think of as "actually 'doing' math"). Such ideas form the basis for later, more formalized *procedures*. The role technology can play in visualizing these ideas for learners should not be overlooked. This graph of f(x)=2X+3, for example, could be viewed as either depending upon context.

X	Y
-5	-7
-4	-5
-3	-3
-2	-1
-1	1
0	3
1	5
2	7
3	9
4	11
5	13

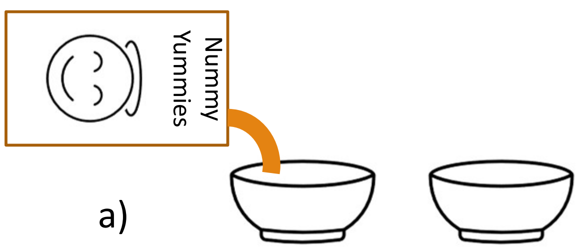


Graph of Y = 2X + 3.

Content and Process

Things get a little complicated when the mathematics described has both noun and verb-like features (i.e., requiring understanding of both *content* and *process* components). For example, the number “2” can be used as a noun describing a position in a sequence or how many of something one might have. In this case, we are clearly using the noun-like features. In a different context, however, “2” can have other meanings such as: a) how many times a process is done – as in the case of filling “2” bowls with cereal; b) a base used by computers to represent other numbers (this is also called Binary); or c) the power to which a quantity is raised as shown in $X^2 + 3$.

a)

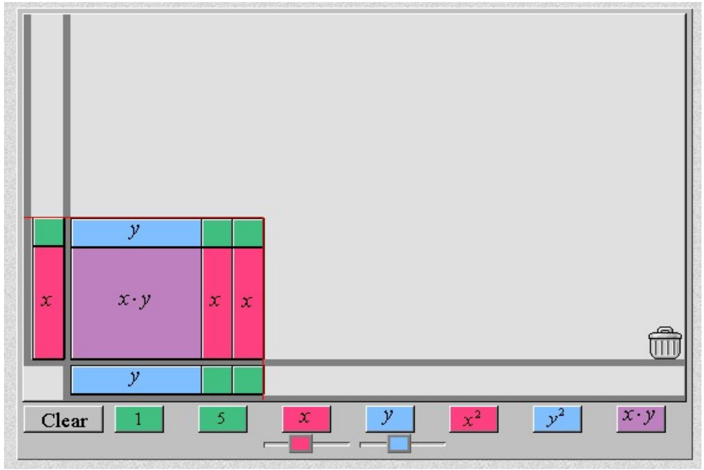


b) $45_{10} = 101101_2$

c) $X^2 + 3$

A Multiplication Example

This can be shown very clearly when considering multiplication strategies. Multiplication is used to compute area, and area can be used to illustrate multiplication – so both the concept and procedure can be illustrated at once*.



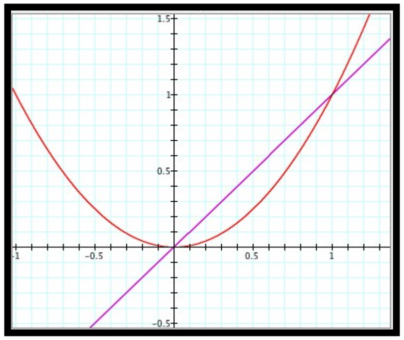
Here we see a rectangle being formed from placing representative tiles along two dimensions: X+1 in the vertical direction, and Y+2 in the horizontal direction. The resulting algebraic product is shown by the area itself. To fill this rectangle the student needs to use an XY piece, two X pieces, one Y piece, and two single squares. When this is written out in standard form it shows that (X+1)(Y+2) = XY + 2X + Y +2.

In order to get to this point, however, students need to be able to utilize both the conceptual and procedural aspects of the representation created through interaction with this application.

Digital Fabrication

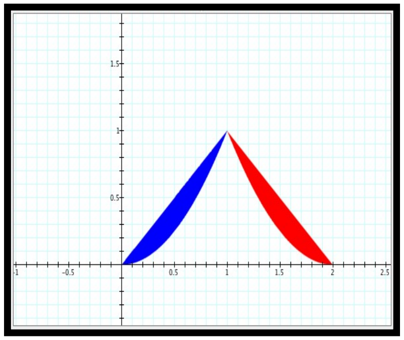
Consider the question of constructing the graphs of the functions $y=x$ and $y=x^2$ in a single drawing.

1



This construction shown in 1, leads to the question of constructing of just the parabolic segment and its reflection on the line $x=1$ as shown in 2.

2



Needed Skills

What skills are necessary to create these two diagrams?

In order to construct the parabolic segment, one must describe the points inside it in the form of inequalities. First, an *x*-coordinate of any point (*x*, *y*) that belongs to the parabolic segment satisfies the inequalities $0<x<1$, where $x=0$ and $x=1$ are the points of intersection of the graphs $y=x$ and $y=x^2$.

Secondly, the y coordinate must satisfy the inequalities $f(x)<y<g(x)$ where $f(x)=x^2$ and $g(x)=x$.

These properties of the points that belong to the parabolic segment can be expressed in the form of simultaneous inequalities

$$x-y>0, \quad y-x^2>0$$

In addition, the reflection of the parabolic segment in the line $x=1$ can be expressed through another set of inequalities by substituting $2-x$ for x

$$(2-x)-y>0, \quad y-(2-x)^2>0$$

Digital Fabrication of $\mathcal{E}$-thick Borders of the Parabolic Segment and its Reflection

Likewise, the set of points that belong to the border of the parabolic segment can be described through inequalities. First, the graph of the upper border (a part of the line $y=x$) can be described as a set of points (*x*, *y*) for which the values of the coordinates *x* and *y* are $\mathcal{E}$-close to each other; that is $|y-x| < \mathcal{E}$

Secondly, the graph of the lower border (a part of the parabola $y=x^2$) can be described as a set of points (*x*, *y*) for which the values of *y* are $\mathcal{E}$-close to the values of $x^{2.}$ Finally, once again, the inequalities $0<x<1$ and $0<2-x<1$ characterize the points that belong to the border.

The Calculator View

In the context of the *Graphing Calculator* these properties of the points that belong to the border of the parabolic segment can be expressed in the form of the union of simultaneous inequalities

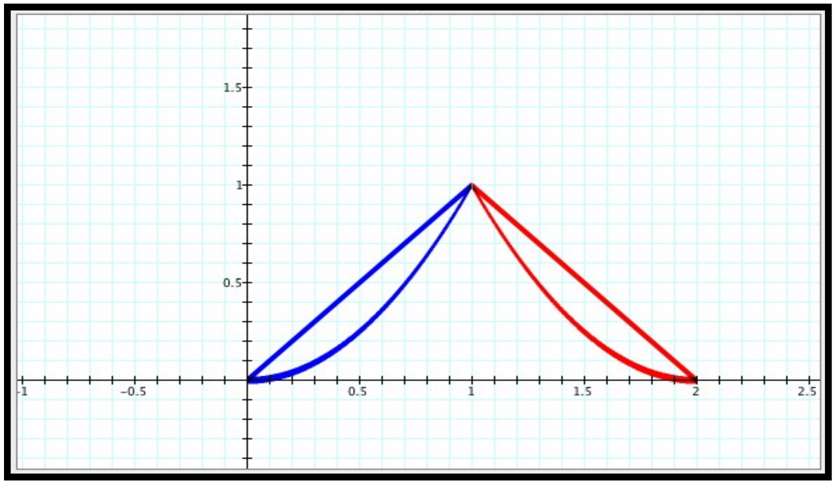
$$\left[\begin{array}{l} |y-x|<\varepsilon, x>0, x<1; \\ |y-x^2|<\varepsilon, x>0, x<1. \end{array}\right.$$

Adding another union of simultaneous inequalities

$$\left[\begin{array}{l} |y-(2-x)|<\varepsilon, 2-x>0, 2-x<1; \\ |y-(2-x)^2|<\varepsilon, 2-x>0, 2-x<1 \end{array}\right.$$

yields the right-hand side of the digital fabrication shown on the next slide.

Software Embodying Mathematical Ideas



Note: in these figures ε=0.02

Using technology to enable students to construct graphs of areas in the plane and their borders by using two-variable inequalities illustrates “the way in which software can embody a mathematical definition” (Conference Board of the Mathematical Sciences, 2001, p. 132).

References

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