Mathematical Representations in the Teaching and Learning of Geometry in the United States

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Presentation Overview

Significant research has been conducted related to geometry curriculum and use of mathematical representations. However, literature that synthesizes across both is scarce. This presentation will share results from a literature search regarding the role of mathematical representations in the teaching and learning of geometry in the United States.

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Introduction

- In the 2001 National Council of Teachers of Mathematics (NCTM) Yearbook, Albert Cuoco challenged the mathematics education community to think beyond ideas of content and pedagogy, to how our students learn (NCTM, 2001).
- One way NCTM has addressed this call is through the introduction of the eight mathematics teaching practices (NCTM, 2014). The aforementioned practices provide students the opportunity to access mathematics through multiple entry points while leveraging multiple mathematical representations (i.e. visual, symbolic, verbal, contextual, and physical) (Lesh et al., 1987).
- The ways in which these connections among mathematical representations can be leveraged - specifically in geometry instruction - will be discussed in this presentation.
- Teaching geometry is crucial in facilitating student opportunities to make connections with the real world (Usiskin, 1980) in addition to experiencing geometry in an integrated and active manner capitalizing on the wonder, joy and beauty of examining the world (NCTM, 2020a).

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Rationale

- Prior research has studied the use of mathematical representations has been researched, as well as about geometry curriculum. There is little literature that synthesizes both.
- Representations cannot fully describe a mathematical construct, and each has different advantages, therefore it becomes crucial that we expose students to using multiple mathematical representations. This allows students to appropriately choose the representation(s) that best works for the given context (Duval, 2002) and for themselves as learners.
- This presentation provides a synthesis on the teaching and learning of geometry at the PK-12 level and the role that mathematical representations can play in enriching the geometry experience for our students.

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Research Questions

Three research questions guided our work:

RQ1: Which frameworks have guided the teaching and learning of geometry in the U.S.?

RQ2: What does the teaching and learning of geometry in the U.S. look like?

RQ3: What is the role of mathematical representations in the teaching and learning of geometry?

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RQ 1: Which frameworks have guided the teaching and learning of geometry in the U.S.?

- While the focus of this presentation is to discuss the role of mathematical representations in the teaching and learning of geometry, it is necessary to first consider the frameworks that have influenced geometry instruction.
- For the purpose of this presentation we define frameworks broadly as contributions that are theoretical frameworks, conceptual frameworks, conceptual models, theories, or similar.
- These frameworks are foundational in understanding how geometry teaching and learning has evolved over time.

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RQ1: Which frameworks have guided the teaching and learning of geometry in the U.S.?

Frameworks

Table 1

Frameworks for Teaching and Learning Geometry

Name	Description
van Hiele (1986)	Children move through five levels of thought in geometry - visual, analytic, abstract, deductive, and rigor.
Theory of Figural Concepts (Fischbein, 1993)	Geometric figures are mental entities which simultaneously possess conceptual and figural properties.
Abstraction (Battista and Clements, 1996)	The process by which the mind registers objects, actions, and ideas in consciousness and memory, and further describes two forms – spatial structuring, and mental models.
Cognitive Processes (Duval, 1998)	Geometrical reasoning involves three kinds of cognitive processes which fulfill specific epistemological functions: visualization processes, construction processes, reasoning processes.

As cited in Desai et al. (2022)

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RQ1: Which frameworks have guided the teaching and learning of geometry in the U.S.?

Frameworks (continued)

Table 1 (cont.)

Frameworks for Teaching and Learning Geometry

Name	Description
Geometric habits of mind (Driscoll, 2007)	Teachers need to develop an understanding of geometric thinking and their own geometric habit of mind including: Reasoning with relationships, generalizing geometric ideas, investigating invariants, balancing exploration and reflection.
Concept learning and the objects of geometric analysis (Battista, 2009)	Students need to analyze objects (physical objects, concepts, and concepts definition) and mental entities to understand and reason about mathematics.
Diagrams and representations (Battista, 2009)	Both diagrams, and physical objects play a major role in geometry.
Spaces for geometric work (SGW) (Goméz-Chacón & Kuzniak, 2015)	Describes the work that people (students, teachers, mathematicians, etc.) perform when they solve geometric tasks.

As cited in Desai et al. (2022)

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RQ 2: What does the teaching and learning of geometry in the U.S. look like?

- Mathematics standards are not isolated concepts – they are connected to each other both within and across grade levels.
- It is crucial for educators to understand these connections so they can link to students' prior knowledge while building a strong foundation for the connections that are still to come (Achieve the Core, n.d.).
- We specifically focus on both the traditional and current approaches to teaching and learning of geometry.

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RQ2: What does the teaching and learning of geometry in the U.S. look like?

Traditional Approaches to Teaching and Learning Geometry

- Geometry is one of the oldest branches of mathematics, and its origins can be traced back to a wide range of cultures and civilizations. Yet, the aims and goals of modern geometry instruction are widely debated (Jones, 2000; The Chicago School Mathematics Project, 1971).
- At least in North America, in over the past hundred years, high school geometry comprised of students using Euclid's *Elements* (Sinclair, 2008). In the 1960s, geometry was then explicitly introduced as a topic in primary schools, and focused primarily on the study of two-dimensional geometry to prepare students for Euclidean geometry (ICME, 1998).

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RQ2: What does the teaching and learning of geometry in the U.S. look like?

Current Approaches to Teaching and Learning Geometry

- More recent studies claim similar purposes for learning geometry and further extend the purpose of elementary school geometry to focus on spatial reasoning (Clements & Battista, 1992; Battista, 2007), and secondary geometry instruction to focus on dynamic geometry software (Hollebrands, 2003) and connections between geometry to algebraic and symbolic manipulations (Knuth, 2000).
- Geometry serves as an essential foundation for space and shape, and also draws on elements of other mathematical ideas such as spatial visualization, measurement and algebra (OECD, 2018).

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Overview of Current *K-8* Geometry Standards in the U.S. (CCSSO & NGA, 2010)

RQ2: What does the teaching and learning of geometry in the U.S. look like?

Table 2a

Common Core State Standards - Geometry Standards (K-8)

Grade	Main Ideas
K	- Identify and describe shapes - Analyze, compare, create, and compose shapes
1	- Reason with shapes and their attributes
2	- Reason with shapes and their attributes
3	- Reason with shapes and their attributes
4	- Draw and identify lines and angles, and classify shapes by properties of their lines and angles
5	- Graph points on the coordinate plane to solve real-world and mathematical problems - Classify two-dimensional figures into categories based on their properties
6	- Solve real-world and mathematical problems involving area, surface area, and volume
7	- Draw, construct, and describe geometrical figures and describe the relationships between them - Solve real-life and mathematical problems involving measure, area, surface area, and volume
8	- Understand congruence and similarity using physical models, transparencies, or geometry software - Understand and apply the Pythagorean theorem - Solve real-world and mathematical problems involving volume or cylinders, cones, and spheres

Adapted from CCSSO and NGA (2010) (as cited in Desai et al. (2022)

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Overview of Current *High School* Geometry Standards in the U.S. (CCSSO & NGA, 2010)

Table 2b

Common Core State Standards - Geometry Standards (High School)

Grade	Categories within Geometry	Main Ideas
High School	Congruence	- Experiment with transformations in the plane - Understand congruence in terms of rigid motions - Prove geometric theorems - Make geometric constructions
	Similarity, Right Triangles, and Trigonometry	- Understand similarity in terms of similarity transformations - Prove theorems involving similarity - Define trigonometric ratios and solve problems involving right triangles - Apply trigonometry to general triangles
	Circles	- Understand and apply theorems about circles - Find arc length and areas of sectors of circles
	Expressing Geometric Properties with Equations	- Translate between the geometric description and the equation for a conic section - Use coordinates to prove simple geometric theorems algebraically
	Geometric Measurement and Dimension	- Explain volume formulas and use them to solve problems - Visualize relationships between two-dimensional and three-dimensional objects
	Modeling with Geometry	- Apply geometric concepts in modeling situations

Adapted from CCSSO and NGA (2010) (as cited in Desai et al. (2022)

RQ2: What does the teaching and learning of geometry in the U.S. look like?

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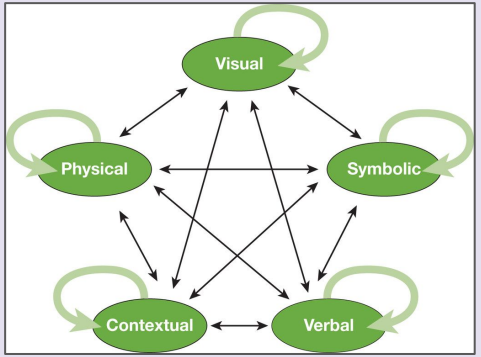
RQ 3: What is the role of mathematical representations in the teaching and learning of geometry?

- Representations as well as connections among representations are essential in PK-12 mathematics.
- While research supports the usefulness of representations and the rich mathematical perspectives that representations provide, transferring between and within representations can be challenging for both teachers and students alike. To understand the role of mathematical representations in geometry, the progression of the main ideas in our current geometry curriculum will be explored.
- At the early childhood and elementary level "Geometry instruction, typically, does not move beyond shape names or definitions, only engaging in low-level thinking" (NCTM, 2020a, p. 119). This is similar at the secondary level where algebraic and symbolic representations are greatly overemphasized (Knuth, 2000).



Huinker (2015)

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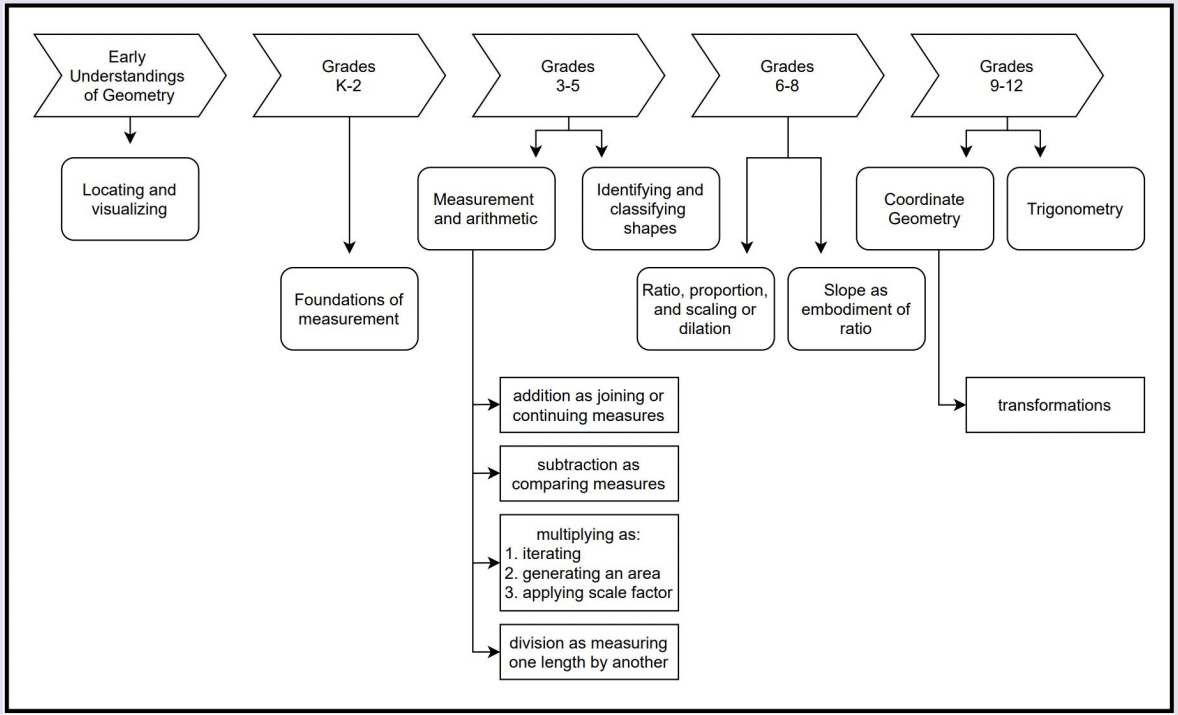
RQ3: What is the role of mathematical representations in the teaching and learning of geometry?

Progression of Main Ideas in Geometry



Note. based on work of Dougherty et al., (2014); Goldenberg et al., (2014); Sinclair et al., (2012a), (2012b) (as cited in Desai et al. (2022))

RQ3: What is the role of mathematical representations in the teaching and learning of geometry?

Early Childhood Geometry Experiences

- Without yet formalizing it, young children are able to understand the distance between themselves and their toys, change location and orientation, and can grasp edges and crawl and run around shapes.
- Initial exposures to geometric representations engage students in informal reasoning which support and build a foundation for informal and formal reasoning in K-12 mathematics and serve as a core in relating other subject areas to mathematics (Clements & Sarama, 2011).

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RQ3: What is the role of mathematical representations in the teaching and learning of geometry?

Elementary School Geometry Experiences

- Students in grades K-2 start to spend time exploring geometry within the context of their own environments and then learn to start engaging in formal activities by identifying and describing the shapes they see and touch (Dixon et al., 2016a).
- In grades 3-5, students build a foundation of geometric ideas such as dividing shapes into equal pieces which connects to ideas even in high school such as to trigonometric ratios such as sine, cosine, and tangent (Dixon et al., 2016b).
- By initially forming connections between the visual, physical, and contextual representations, students are then able to develop formal language to describe the shapes. This progression of geometric understanding that students develop at the K-5 level is important to students' overall mathematical learning.

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RQ3: What is the role of mathematical representations in the teaching and learning of geometry?

Middle School Geometry Experiences

- As students transition from elementary to middle school, visual and physical representations should not fade away, but rather need to be developed alongside symbolic representations (Tripathi, 2008).
- Goal of grade 6-8 geometry is to support students in developing a way to talk about properties of shapes, which is consistent with van Hiele's level 3 (Smith et al., 2017).
- A focus of middle school geometry instruction is developing formulas such as those for surface area and volume, these algebraic manipulations naturally lend themselves to connecting the concrete three-dimensional representation. It is important for students to cultivate this conceptual understanding so they can leverage these connections between and within representations and move beyond memorization and rote application (NCTM, 2020b).

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RQ3: What is the role of mathematical representations in the teaching and learning of geometry?

High School Geometry Experiences

- The four primary focuses of high school geometry include measurement; transformations; geometric arguments, reasoning, and proof; and solving applied problems and modeling in geometry (NCTM, 2018).
- Much of secondary mathematics focuses on formal and rigorous mathematical reasoning, and oftentimes there is a greater emphasis on algebraic or symbolic manipulation and logical deductions (Battista, 2017). While this emphasis is indeed necessary, it is equally important for students to be given experiences with other forms of representations (both static and dynamic) to build their initial conceptions of the topic.
- Teachers need to make intentional efforts to connect the symbolic and algebraic to the physical and visual representations that are often brought forward through integrating geometric connections.
- Teachers can use two- and three-dimensional manipulatives to emphasize connections between algebraic instances—such as multiplying polynomials—with the geometric representations related to the area accounted for through the product of algebraic expressions. Geometric and algebraic understandings and representations reinforce each other, and for students to gain a rich perspective, it is necessary to expose students to both (Safi & Desai, 2017).

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Concluding Remarks

- By providing students access to opportunities to explore multiple mathematical representations, they are no longer limited by the strengths and weaknesses of one particular representation (Elia et al., 2007), and they are able to deepen their mathematical understanding while engaging in meaningful mathematical discourse (Lesh et al., 1987; NCTM, 2014).
- Giving students such opportunities to engage in tasks that allow the use of multiple mathematical representations empowers teachers to create more equitable tasks as they afford a wider range of access to mathematical ideas (Boston et al., 2017).
- "Children enter this world as emergent mathematicians, naturally curious, and trying to make sense of their mathematical environment" (NCTM, 2020a, p. 17). For our students to continue to see themselves as capable learners and doers of mathematics and experience the wonder, joy, and beauty of doing mathematics, it is important that PK-12 instruction provides them opportunities to see connections between mathematics and their daily lives (NCTM 2018, 2020a, 2020b).

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