Ch 11. Pricing American Options by Monte Carlo Simulation

- This chapter introduces the methods to price American options with the Monte Carlo simulation. The introduced methods include Tilley (1993), Barraquand and Martineau (1995), Raymar and Zwecher (1997), Broadie and Glasserman (1997), and Longstaff and Schwartz (2001). Among these models, the most important method is the least-squares method proposed by Longstaff and Schwartz (2001).

I. Strengths and Weaknesses of Monte Carlo Simulation

- The advantage of the Monte Carlo simulation method is to deal with path dependent options. The superiority of the Monte Carlo simulation method is that it can simulate the underlying asset price path by path, calculate the payoff associated with the information for each simulated path, e.g., $S_{\max}$ or S_{ave}, and utilize the average discounted payoff to approximate the expected discounted payoff, which is the value of path-dependent options.

  ⊙ However, the advantage of the Monte Carlo simulation method causes the difficulty to apply it to pricing American options. This is because it is difficult to derive the holding value (or the continuation value) at any time point t based on one single subsequent path.

  ⊙ Someone may try to apply the multiple-tier Monte Carlo simulation to estimating the holding value and thus price American options, but this method is infeasible for a large number of early exercise time points, n.

  ⊙ Note that in the tree-based model, the holding value for each node is determined through $e^{-r\Delta t}(p_u C_u + p_d C_d)$, where C_u and C_d are the option values corresponding to the upper and lower branches and they already take the possible early exercise in the future into account.

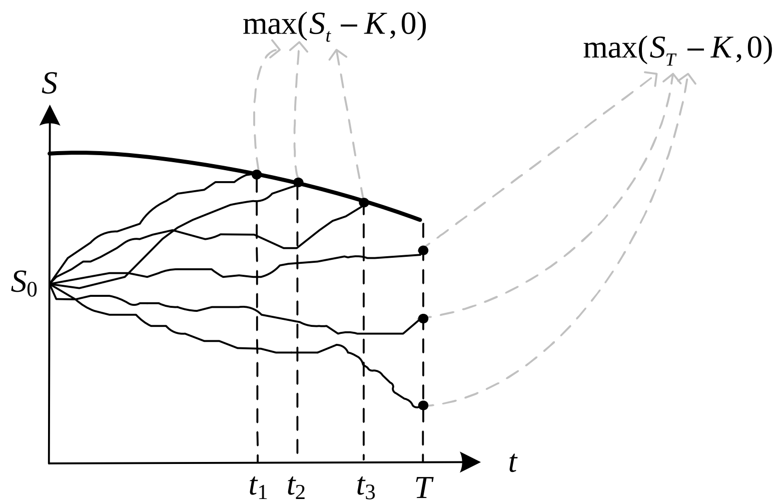
II. The Pioneer

- Tilley (1993): the first one trying to price American options by proposing a bundling algorithm based on simulation.

  ⊙ The main idea is to devise a method based on the Monte Carlo simulation to decide the early exercise boundary.

  ⊙ Once the early exercise boundary is determined, an American option can be viewed as a knocked-and-exercised option. As long as the simulated path touches the early exercise boundary, the payoff is, taking calls for example, $\max(S_t - K, 0)$. If the simulated path never touches the early exercise boundary before the maturity date, the payoff for that simulated path will be $\max(S_T - K, 0)$. The option value can be expressed as $\frac{1}{n}\sum_i e^{-r\tau}\max(S_{i\tau} - K, 0)$, where τ represents the optimal exercise time for each path.

  Figure 11-1



  ⊙ Five steps to decide the early exercise boundary and thus price American options:

  (i) Simulate N stock paths, $S(j,t)$, where t is the time point and j is the index for stock paths. In addition, each path is partitioned into n subperiods and thus $\Delta t = T/n$.

  (ii) Decide the payoff of each stock path on the maturity date, T, i.e., $V(j,T) = \max(S(j,T)$ $-K, 0)$.

(iii) For $t = (n-1)\Delta t$ to 0,

(1) Sort N paths with respect to the values of $S(j,t)$ from the minimum to the maximum.

(2) Classify all paths into Q groups and q is the index for groups with the values to be $1,\ldots,Q$. So, there are $\frac{N}{Q} = M$ paths in each group.

(3) For each **group**, estimate a holding value $H(q,t) = e^{-r\Delta t}\frac{1}{M}\sum_{\substack{\text{path } j \\ \in \text{ group } q}} V(j,t+\Delta t)$.

(4) For each **path** j at the time point t, the exercise value $= E(j,t) = \max(S(j,t) - K, 0)$, and define

$$x(j,t)=\begin{cases} 1 & \text{if} \quad E(j,t) \geq H(q,t) \\ 0 & \text{if} \quad E(j,t) < H(q,t) \end{cases}.$$

(5) Decide the "sharp" boundary (the early exercise boundary), as illustrated in Figure 11-2 on the next page.

(6) Define $y(j,t)=\begin{cases} 1 & \text{for} \quad j \geq j^*(t) \\ 0 & \text{for} \quad j < j^*(t) \end{cases}$.

That is, for paths below or equal to $j^*(t)$, it is optimal to be early exercised. Hence the value of $V(j,t)$ can be decided as follows. For $y(j,t) = 1$, $V(j,t) = \max(S(j,t) - K, 0)$, and for $y(j,t) = 0$, $V(j,t) = e^{-r\Delta t} \cdot V(j,t+\Delta t)$. The value of $V(j,t)$ is prepared for the backward induction process at the previous time point $t - \Delta t$.

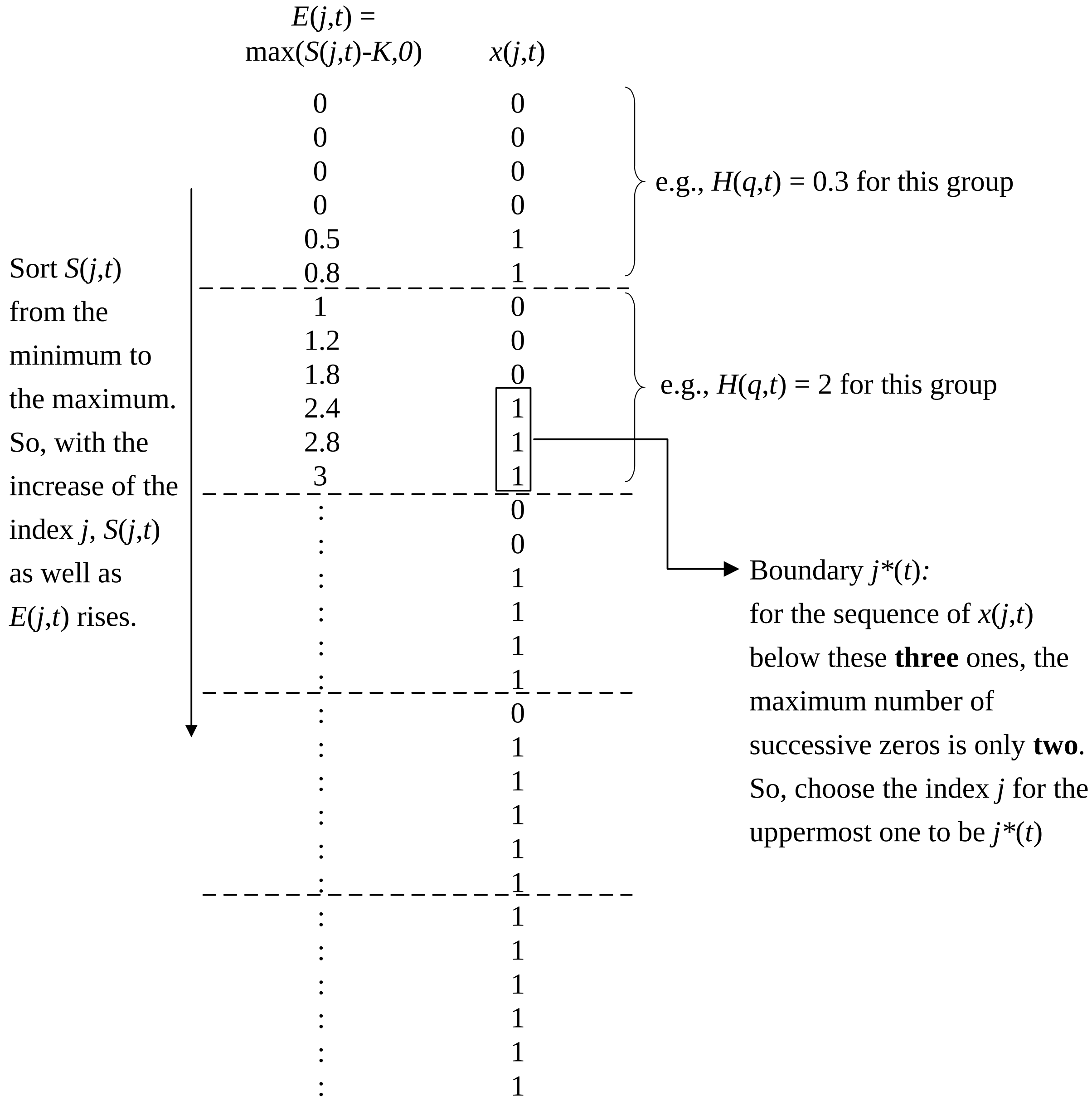
(iv) For each path, find the first time point such that $y(j,t) = 1$, which is the early exercise point for that path. In addition, along that path, define the variable $I(j,t)$ to be 1 at that time point and 0 for other time points, i.e.,

$$I(j,t)=\begin{cases} 1 & \text{if } y(j,t) = 1 \text{ and } y(j,s) = 0 \text{ for all } s < t \\ 0 & \text{o/w} \end{cases}.$$

If there is no $y(j,t) = 1$ along the stock path, set $I(j,T)$ to be 1 on the maturity date.

(v) Since $I(j,t)$ indicates the early exercise time point for each path, an American option can be priced as $C = \frac{1}{N}\sum_{\text{path } j}\sum_{\text{all } t} e^{-rt} \cdot I(j,t) \cdot \max(S(j,t) - K, 0)$.

Figure 11-2


E(j,t) =
max(S(j,t)-K,0)
x(j,t)
0 0
0 0
0 0
0 0
0.5 1
0.8 1
1 0
1.2 0
1.8 0
2.4 1
2.8 1
3 1
: 0
: 0
: 1
: 1
: 1
: 1
: 0
: 1
: 1
: 1
: 1
: 1
: 1
: 1
: 1
: 1
: 1
: 1
e.g., H(q,t) = 0.3 for this group
e.g., H(q,t) = 2 for this group
Sort S(j,t) from the minimum to the maximum. So, with the increase of the index j, S(j,t) as well as E(j,t) rises.
Boundary j*(t): for the sequence of x(j,t) below these three ones, the maximum number of successive zeros is only two. So, choose the index j for the uppermost one to be j*(t)

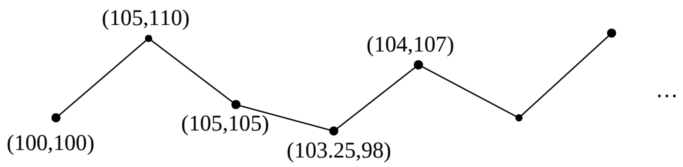
III. Stratified State Methods

- Barraquand and Martineau (1995)

  They proposed the method of Stratified State Aggregate along the Payoff (SSAP) to price American rainbow option, $C_\mathcal{T} = \max(\max(S_{1\mathcal{T}}, S_{2\mathcal{T}}, \ldots, S_{n\mathcal{T}}) - K, 0)$, in which the stock price paths are sorted according to a state variable (rather than the stock price) to determine payoff. Here the method of SSAP is illustrated for pricing American arithmetic average option, $C_\mathcal{T} = \max(S_{\text{ave},\mathcal{T}} - K, 0)$.

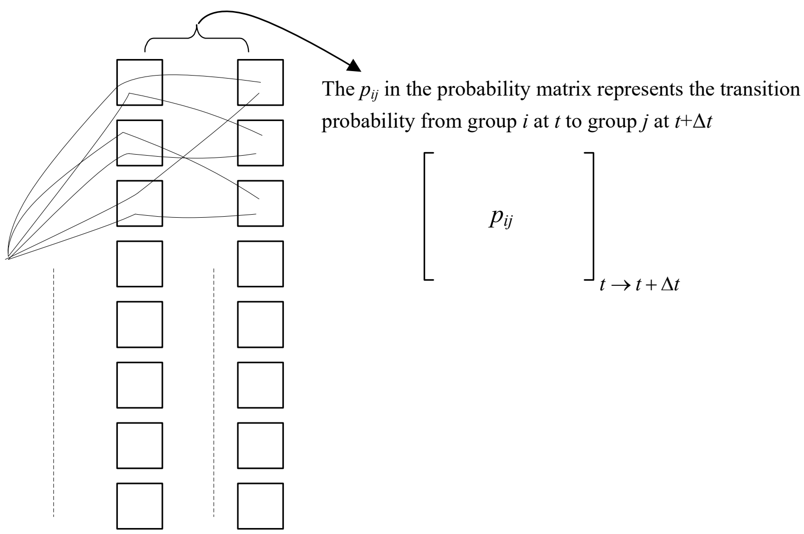
  (i) Along each stock price path, it is necessary to record the pair of ($S_{\text{ave},t}$, S_t) at each time point, where $S_{\text{ave},t}$ (also known as the prefix average price) is the realized arithmetic average price until t. For example,

  Figure 11-3



  (ii) For each time point, sort stock price paths according to the prefix average price and next classify them into 8 groups. In addition, calculate the transition probability P_{ij} by counting the paths from i-th group at time t to j-th group at time $t + \Delta t$.

  Figure 11-4



  (iii) For $T = n\Delta t$, the payoff for each path is $\max(S_{\text{ave},T} - K, 0)$. In addition, calculate the average of the payoffs of all paths for each group, and the result is treated as the option value for this group. (In this algorithm, **the optimal exercise boundary is determined group by group**.)

(iv) For $t = (n-1)\Delta t$ to 0,

For each group i, calculate

(1) the exercise value for this group, which is the average of the exercise values of all paths in this group.

(2) the holding value for this group, which is $e^{-r\Delta t} \sum_{j=1}^{8} p_{ij}$(option value of the group j at $t + \Delta t$).

If (1) $\geq$ (2), the option value for this group = (1) (exercise)

(1) $<$ (2), the option value for this group = (2) (hold)

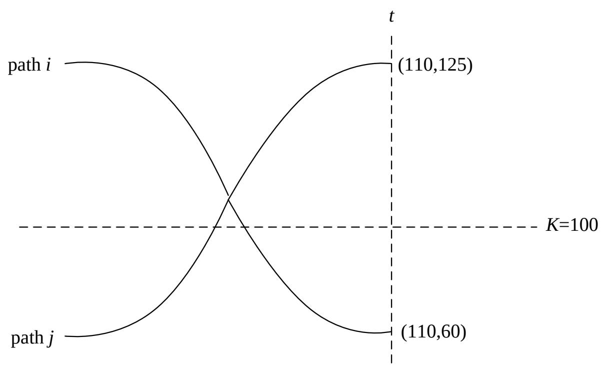
(v) Continue the process until $t = 0$, since there is only one node, the option value for this node will be the value for the American arithmetic average option.

- Problems for the model in Barraquand and Martineau (1995).

Since it classifies paths into groups according to the prefix average price, the approximation for the **exercise value of each group is acceptable** because the exercise value highly depends on the prefix average price. However, the approximation for the **holding value is not accurate**, so it is possible to make wrong early exercise decisions.

⊙ For example, paths i and j are with the same prefix arithmetic price but with different stock price at time point t.

Figure 11-5



According to the BM's model, paths i and j will be classified into the same group and they will be early exercised or held together. However, the option holder, in fact, will make different early exercise decisions for these two paths.

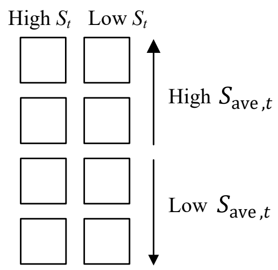
For path i: the arithmetic average price will become lower, so it is optimal to early exercise at time point t to earn $110 - 100 = 10$.

For path j: the arithmetic average price will become higher, so it could earn more profit to postpone the exercise.

- Raymar and Zwecher (1997) fix the problems in the BM's model.

  The main idea of RZ's model is to introduce more factors to classify paths into different groups. For arithmetic average options, it is possible to employ the stock price as an additional factor to classify paths. More specifically, paths are sorted and classified into 4 groups according to the arithmetic average price and then each group are further divided into 2 groups according to the stock price.

  Figure 11-6



  ⊙ It is possible to increase the number of groups, but more groups mean less paths in each group given the same number of simulated stock paths, which in turn influence the accuracy of the approximations for the holding and exercise values for each group. In Raymar and Zwecher (1997), in order to show the superiority of this classification, they maintain the same number of simulated stock paths and the number of groups is fixed to be 8.

- Raymar and Zwecher (1997) also propose a three-phase simulation framework to avoid the **foresight bias problem**. The foresight bias appears if the exercise criteria is calculated by the same simulated samples as those to determine the exercise values. During the backward induction process, whenever the option holder makes early-exercise decisions, he already knows some information of following stock price paths, that is helpful to make better decision than those could be made in the real world. See page 11-9 for more explanations of the foresight bias problem.

  (i) First phase: simulate N paths and decide critical prices to separate groups.

  (ii) Second phase: simulate N paths and classify paths into groups according to the critical prices derived in (i), and then

  (1) Decide the transition probabilities

  (2) Perform the backward induction process in the BM's model

  (3) Record the information of whether early exercise or not for each group

  (iii) Third phase: simulate N paths

  (1) As long as the path i reaches the group for early exercise, exercise immediately and derive the payoff for this path. If the path is not early exercised until the maturity, derive the payoff for this path on the maturity date.

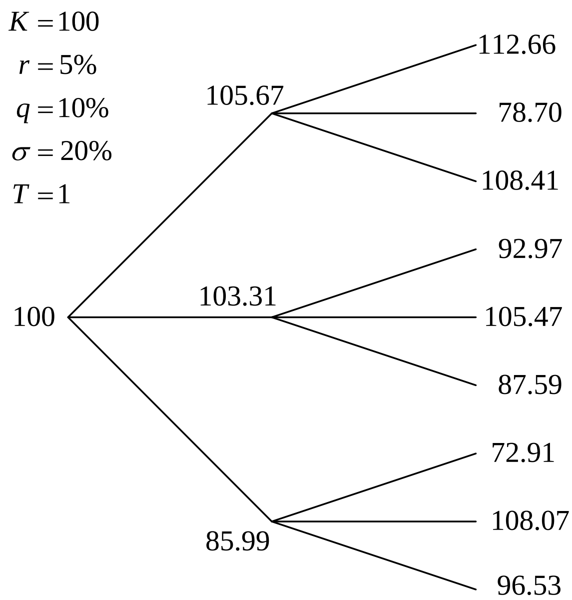
(2) Discount payoffs for all paths and the take the average of these discounted payoffs to generate the value for the American option.

IV. Simulated Tree Method

- Broadie and Glasserman (1997) propose the method of the simulated tree to price American options, which can derive the upper and lower bounds for American options.

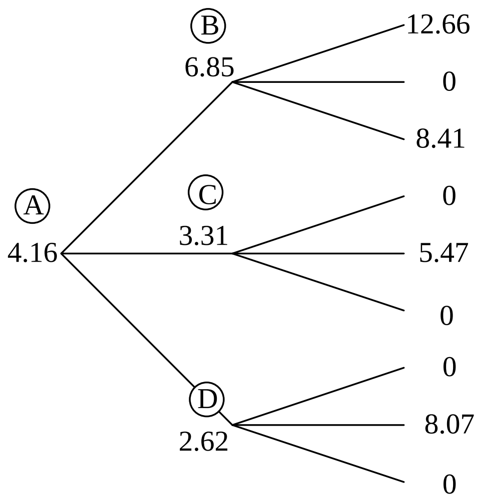
  ⊙ The simulated tree for stock price is as follows, which is in essence a non-recombined trinomial tree with simulated stock prices.

Figure 11-7



⊙ Derive the upper bound (similar to the standard backward induction process in tree-based models)

Figure 11-8



B: $\begin{cases} \text{holding value } = \frac{1}{3}(12.66+0+8.41)\times e^{-0.05\times 0.5} = 6.85 \\ \text{exercise value} = 5.67 \end{cases}$

C: $\begin{cases} \text{holding value } = \frac{1}{3}(0+5.47+0)\times e^{-0.05\times 0.5} = 1.78 \\ \text{exercise value} = 3.31 \end{cases}$

D: $\begin{cases} \text{holding value } = \frac{1}{3}(0+8.07+0)\times e^{-0.05\times 0.5} = 2.62 \\ \text{exercise value} = 0 \end{cases}$

A: $\begin{cases} \text{holding value } = \frac{1}{3}(6.85+3.31+2.62)\times e^{-0.05\times 0.5} = 4.16 \\ \text{exercise value} = 0 \end{cases}$

∗ Since the same set of simulated sample is employed to decide exercise values and whether it is optimal to early exercise, the **foresight bias** emerges, which will **overestimate** the value of American options. The reason is analyzed as follows.

- If the realized average return at the next time point is overestimated, i.e., larger than r, the holding value today is overestimated as well. The option holder will postpone the exercise to obtain higher profit, that increases the option value.
- If the realized average return at the next time point is underestimated, i.e., smaller than r, the holding value today is underestimated as well. The option holder will early exercise the option today. This behavior also increases the option value.

∗ In a word, the foresight bias problem appears when employing the above Monte Carlo simulation methods for pricing American options, which results overestimated American option values and thus forms upper bounds for American option values.

∗ For tree models, because the appropriate values of p, u, and d are derived such that the growth and volatility of the stock price equal r and σ exactly, there is no foresight bias for tree models.

$\odot$ Derive the lower bound (to separate the random samples for calculating holding values and making the early exercise decision)

$*$ The main idea of the algorithm is as follows. For the branch j ($j = 1, 2, 3$), use the other two branches to decide the expected holding value, and compare it with the exercise value of the examined node. If it is optimal to hold the option, the option value corresponding to the branch j is the present value of the period-end option value of the branch j. Otherwise, the option value corresponding to the branch j is the exercise value of the examined node.

For node B: exercise value is $\max(105.67 - 100, 0) = 5.67$

	holding value	option value corresponding to branch j
$j = 1$	$\frac{1}{2}(0 + 8.41)e^{-0.05\times0.5} = 4.10$	5.67
$j = 2$	$\frac{1}{2}(12.66 + 8.41)e^{-0.05\times0.5} = 10.27$	$0 \cdot e^{-0.05\times0.5}$
$j = 3$	$\frac{1}{2}(12.66 + 0)e^{-0.05\times0.5} = 6.17$	$8.41e^{-0.05\times0.5}$

$\Rightarrow$ Option value for node B is 4.62 ($= \frac{(5.67+0+8.41e^{-0.05\times0.5})}{3}$).

For node C: exercise value is $\max(103.31 - 100, 0) = 3.31$

	holding value	option value corresponding to branch j
$j = 1$	$\frac{1}{2}(5.47 + 0)e^{-0.05\times0.5} = 2.67$	3.31
$j = 2$	$\frac{1}{2}(0 + 0)e^{-0.05\times0.5} = 0$	3.31
$j = 3$	$\frac{1}{2}(0 + 5.47)e^{-0.05\times0.5} = 2.67$	3.31

$\Rightarrow$ Option value for node C is 3.31 ($= \frac{3.31+3.31+3.31}{3}$).

For node D: exercise value is $\max(85.99 - 100, 0) = 0$

	holding value	option corresponding to branch j
$j = 1$	$\frac{1}{2}(8.07 + 0)e^{-0.05\times0.5} = 3.94$	$0 \cdot e^{-0.05\times0.5}$
$j = 2$	$\frac{1}{2}(0 + 0)e^{-0.05\times0.5} = 0$	$8.07e^{-0.05\times0.5}$
$j = 3$	$\frac{1}{2}(0 + 8.07)e^{-0.05\times0.5} = 3.94$	$0 \cdot e^{-0.05\times0.5}$

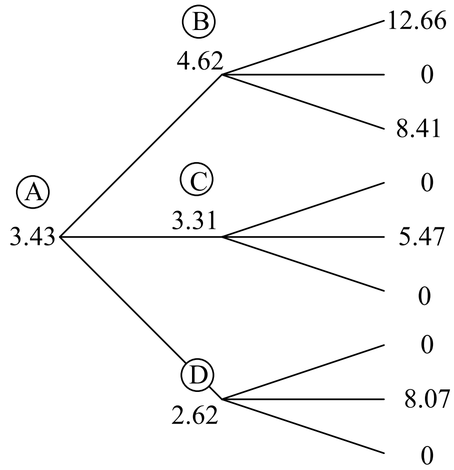
$\Rightarrow$ Option value for node D is 2.62 ($= \frac{0+8.07e^{-0.05*0.5}+0}{3}$).

For node A: exercise value is $\max(100 - 100, 0) = 0$

	holding value	option value corresponding to branch j
$j = 1$	$\frac{1}{2}(3.31 + 2.62)e^{-0.05\times0.5} = 2.89$	$4.62e^{-0.05\times0.5}$
$j = 2$	$\frac{1}{2}(4.62 + 2.62)e^{-0.05\times0.5} = 3.53$	$3.31e^{-0.05\times0.5}$
$j = 3$	$\frac{1}{2}(4.62 + 3.31)e^{-0.05\times0.5} = 3.87$	$2.62e^{-0.05\times0.5}$

$\Rightarrow$ Option value for node A is 3.43 ($= \frac{4.62e^{-0.05*0.5}+3.31e^{-0.05*0.5}+2.62e^{-0.05*0.5}}{3}$).

Figure 11-9



∗ For the above pricing approach, it does not commit the foresight bias.

∗ However, the early exercise decisions generated by the above pricing approach are neither optimal. The suboptimal early exercise decisions result in a undervalued pricing result than the true American option value. Moreover, this lower bound should be higher than the corresponding European option value. This is because among all early exercise opportunities, a higher exercise value in general leads to a higher probability of early exercise. Accumulation of these advantages should generate a option value higher than that of the corresponding European option, which does not have any early exercise opportunity.

V. Least-Squares Approach

- Longstaff and Schwartz (2001), "Valuing American Option by Simulation: A Simple Least-Squares Approach," *Review of Financial Studies* 14, pp. 113–147.

  ⊙ A numerical example for pricing American puts is employed to illustrate how this approach works. Suppose $S_0 = 1$, $K = 1.1$, $T = 3$yrs, $r = 0.06$, $q = 0$, and $\Delta t = 1$yr. The simulated 8 stock prices are listed in the following table.

Stock Price	$t = 0$	$t = 1$	$t = 2$	$t = 3$
path 1	1	1.09	1.08	1.34
path 2	1	1.16	1.26	1.54
path 3	1	1.22	1.07	1.03
path 4	1	0.93	0.97	0.92
path 5	1	1.11	1.56	1.52
path 6	1	0.76	0.77	0.90
path 7	1	0.92	0.84	1.01
path 8	1	0.88	1.22	1.34

  ⊙ Step 1: Determine the payoff for each path at maturity ($t = 3$).

	payoff ($t = 3$)
path 1	0
path 2	0
path 3	0.07
path 4	0.18
path 5	0
path 6	0.20
path 7	0.09
path 8	0

⊙ Step 2: For $t = 2$, the path-wise holding values are calculated as the present values of the option values at the next time point, i.e., at $t = 3$. The path-wise holding and exercise values at $t = 2$ are listed as follows. Note that due to the foresight bias, the path-wise holding value cannot be used to determine the early exercise decision.

	exercise value (EV) ($t = 2$)	path-wise holding value (HV) ($t = 2$)	payoff ($t = 3$)
path 1	0.02	0	0
path 2	0	0	0
path 3	0.03	$0.0659 = 0.07 \cdot e^{-0.06}$	0.07
path 4	0.13	$0.1695 = 0.18 \cdot e^{-0.06}$	0.18
path 5	0	0	0
path 6	0.33	$0.1884 = 0.2 \cdot e^{-0.06}$	0.20
path 7	0.26	$0.0848 = 0.09 \cdot e^{-0.06}$	0.09
path 8	0	0	0

$*$ For in-the-money paths at $t = 2$, i.e., paths 1, 3, 4, 6, and 7, decide whether or not it is optimal to early exercise for these paths. The main idea to achieve this goal is to employ a regression equation, e.g., $HV = a + bS + cS^2 + \upsilon$, where S is the stock price at $t = 2$ for paths 1, 3, 4, 6, and 7, and υ is the white noise and $\upsilon \sim ND(0, \eta^2)$, to estimate the expected holding value (conditional on S), $E[HV] = \hat{a} + \hat{b}S + \hat{c}S^2$.

	for $HV = a + bS + cS^2 + \upsilon$						$E[HV]$ EV	
path 1	0	1.08	1.08^2				$0.0369 > 0.02$	
path 3	0.0659	1.07	1.07^2		$\hat{a} = -1.070$		$0.0461 > 0.03$	
path 4	0.1695	0.97	0.97^2	$\Rightarrow$	$\hat{b} = 2.983$	$\Rightarrow$	$0.1176 < 0.13$	$\checkmark$
path 6	0.1884	0.77	0.77^2		$\hat{c} = -1.813$		$0.1520 < 0.33$	$\checkmark$
path 7	0.0848	0.84	0.84^2				$0.1565 < 0.26$	$\checkmark$

$*$ For paths 4, 6, and 7, since $EV > E[HV]$, the option values for these paths at $t = 2$ are the corresponding exercise values. In addition, set the option value for these paths at the subsequent time point, i.e., $t = 3$ at the current step, to be zero. For other paths, the option values are the corresponding path-wise holding values, HV. (Setting the option value to be zero is not necessary, but this step can enhance the understanding of the optimal exercise time point for each path.)

⊙ Step 3: For $t = 1$, the path-wise holding values are calculated as the present value of the option values at the sebsequent time point, i.e., at $t = 2$. The path-wise holding and exercise values at $t = 1$ are listed in the following table.

	EV $(t=1)$	path-wise HV $(t=1)$	payoff $(t=2)$	payoff $(t=3)$
path 1	0.01	0	0	0
path 2	0	0	0	0
path 3	0	$0.06208 = 0.0659 \cdot e^{-0.06}$	0.0659	0.07
path 4	0.17	$0.1224 = 0.13 \cdot e^{-0.06}$	0.13	0
path 5	0	0	0	0
path 6	0.34	$0.3108 = 0.33 \cdot e^{-0.06}$	0.33	0
path 7	0.18	$0.2449 = 0.26 \cdot e^{-0.06}$	0.26	0
path 8	0.22	0	0	0

∗ For in-the-money paths at $t = 1$, i.e. paths 1, 4, 6, 7, and 8, perform the regression analysis to estimate the expected holding value, $E[HV]$.

	for $HV = a + bS + cS^2 + \upsilon$						$E[HV]$	EV	
path 1	0	1.09	1.09^2				$0.0139 >$	0.01	
path 4	0.1224	0.93	0.93^2		$\hat{a} = 2.038$		$0.1092 <$	0.17	√
path 6	0.3108	0.76	0.76^2	⇒	$\hat{b} = -3.335$	⇒	$0.2866 <$	0.34	√
path 7	0.2449	0.92	0.92^2		$\hat{c} = 1.356$		$0.1175 <$	0.18	√
path 8	0	0.88	0.88^2				$0.1533 <$	0.22	√

∗ For paths 4, 6, 7, and 8, since $EV > E[HV]$, the option values for these paths at $t = 1$ are the corresponding exercise values. In addition, set the option value for these path at the subsequent time point, i.e., at $t = 2$, to be zero. For other paths, the option values are the corresponding path-wise holding values, HV. See the following table.

	$t=1$	$t=2$	$t=3$
path 1	0	0	0
path 2	0	0	0
path 3	0.06208	0.0659	0.07
path 4	0.17	0	0
path 5	0	0	0
path 6	0.34	0	0
path 7	0.18	0	0
path 8	0.22	0	0

⊙ Step 4: The American put option value at $t = 0$:

$\frac{1}{8}(0 + 0 + 0.06208e^{-0.06} + 0.17e^{-0.06} + 0 + 0.34e^{-0.06} + 0.18e^{-0.06} + 0.22e^{-0.06})$

$= 0.1144 > \underline{1.1 - 1.0}$.

(The early exercise value today is 0.1, which is smaller than the option value today, 0.1144, so it is not optimal to early exercise today.)

⊙ Moreover, Longstaff and Schwartz (2001) also apply the (weighted) Laguerre polynomials as the basis function.

$$
\begin{aligned}
B_0(S) &= \exp(-S/2),\\
B_1(S) &= \exp(-S/2)(1 - S),\\
B_2(S) &= \exp(-S/2)(1 - 2S + S^2/2),\\
&\vdots\\
B_j(S) &= \exp(-S/2)\frac{e^S}{j!}\frac{d^j}{dS^j}(S^j e^{-S}).
\end{aligned}
$$

Since B_i and B_j are orthogonal functions over $[0, \infty)$, the independency feature of orthogonal functions can satisfy the need for the regression analysis that the explanatory variables should be independent.

⊙ The regression equation may not be $HV = a + bS + cS^2 + v$. There is no common way to decide the regression equation and the explanatory variables in it. Longstaff and Schwartz (2001) suggest to employ any relevant variables into consideration in the regression equation. Furthermore, the degree of the polynomial functions for variables is also uncertain. For instance, in this case, it is possible to consider the regression equation to be $HV = a + bS + cS^2 + dS^3 + v$.

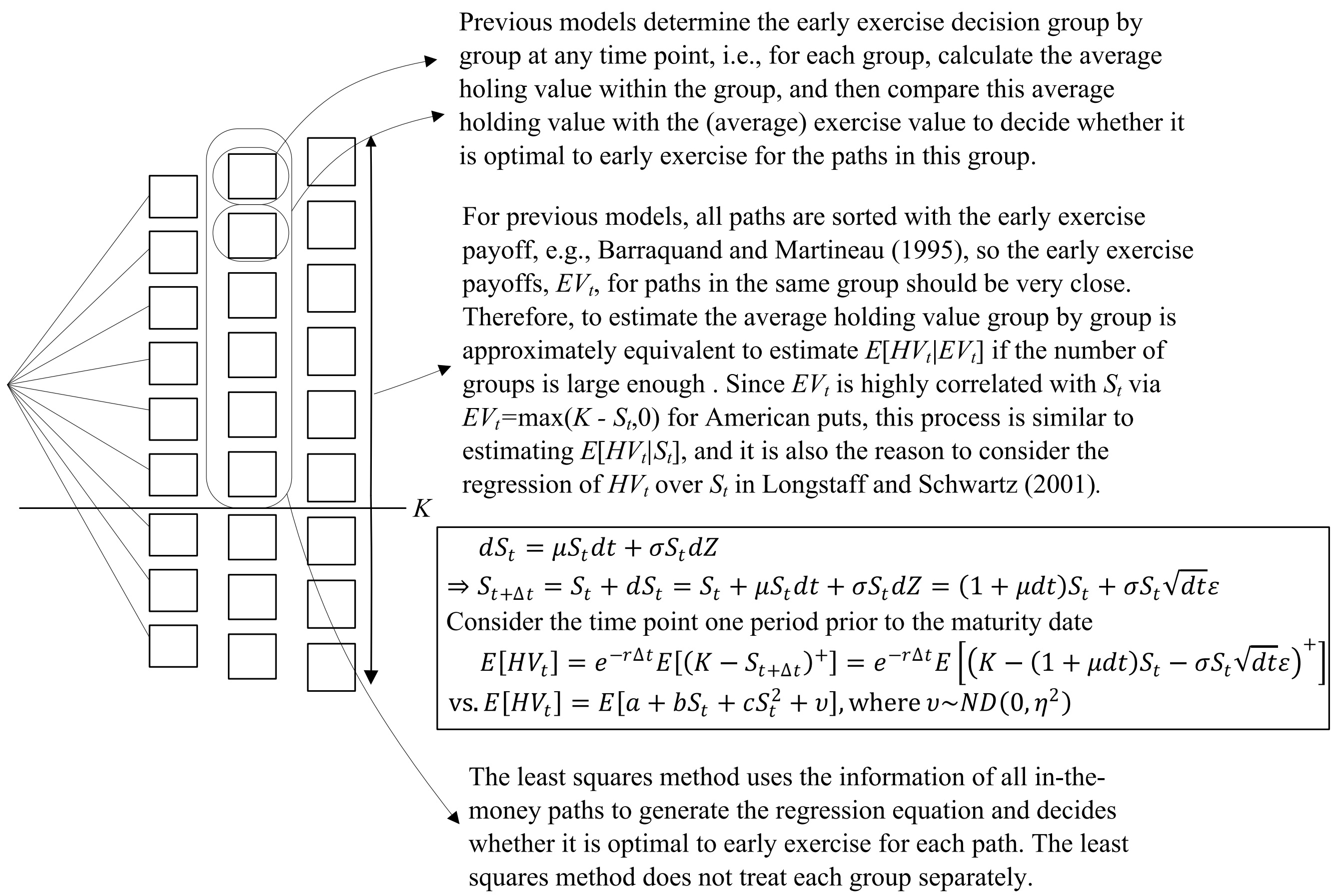
⊙ In Longstaff and Schwartz (2001), they examine the performance of the least-squares approach for many different options, including the plain vanilla put, the average option, the rainbow option, etc. The accuracy of this least-squares approach is examined by comparing with the pricing results based on lattice models.

⊙ Since there is no general rule to decide the relevant variables and the degree of the polynomial functions, it is never known whether the authors adopt the trial and error method to decide the relevant variables and the degree of the polynomial functions to make the pricing results of the least-squares approach be close to those from lattice models.

⊙ The aim to develop the least-squares approach should be as follows: for newly designed exotic American options, since there is no lattice models or it is difficult to develop a lattice model to price that options, the best way is to resort to the least-squares approach. However, without correct benchmark answers from lattice models, it is impossible to make sure that whether we take sufficient relevant variables or degree for polynomial function into consideration, which is the most serious problem for applying the least-squares approach.

- Comparisons with previous simulation-based models.

Figure 11-10



- Is it appropriate to consider only in-the-money paths? For pricing convertible bonds, since the holder can convert the bonds they own into stock shares even in marginal out-of-the-money cases (the stock price is slightly lower than the conversion price). That is, for both in-the-money and out-of-the-money paths, we need to know the holding value to decide whether it is optimal to convert.

⊙ The first solution is to distinguish in-the-money and out-of-the-money paths and conduct the regression for each of them separately. The second solution is to conduct only one regression for all paths. According to the results of my experiments, the second solution provides more accurate and better convergent convertible bond values.

- Relevant variables in the regression of the LSM for pricing different options:

  ⊙ When pricing average options, it is necessary to consider both S_t and the prefix average price. Otherwise, you will make a mistake like Barraquand and Martineau (1995) did. The solution proposed by Raymar and Zwecher (1997) to fix this problem is equivalent to include the prefix arithmetic average price into the regression equation.

  ⊙ To improve accuracy for pricing average options, it is suggested to include ($S_t \times$ prefix average price) as an additional explanatory variable in the regression of the LSM.

  ⊙ For arithmetic average reset options, it is necessary to consider S_t, the current strike price K_t, the prefix arithmetic average price after the previous reset date A_t, and the products of any two of these three variables in the regression of the LSM.

  ⊙ For lookback options, it is necessary to consider S_t, the updated $S_{\max}$, and the product of them in the regression of the LSM.

  ⊙ For Parisian options, in addition to S_t, maybe the sojourning time over the barrier should be considered, because this information will influence the possibility of knocking in or out, and thus affect the likelihood of early exercise. In addition, the product of these two variables is also suggested to be included as an additional explanatory variable in the regression of the LSM.

- Many papers improve the LSM by using more advanced regression methods, such as ridge regression (Tompaidis and Yang, 2014), least absolute shrinkage and selection operator (LASSO) (Tompaidis and Yang, 2014; Chen et al., 2019), weighted least squares regression (Fabozzi et al., 2017; Ibáñez and Velasco, 2018), and non-parametric kernel regression (Belomestny, 2011; Ludkovski, 2018).