Limit: value function is approaching as the independent variable approaches specific value

Average ROC: $m = \frac{f(a)-f(b)}{a-b}$

• secant lines can estimate limit

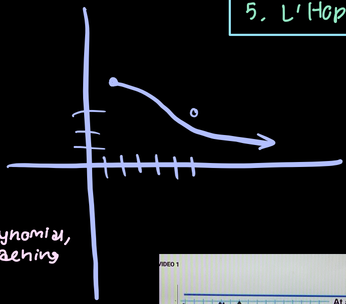
to solve limit:
1. plug in x
2. factor
3. conjugate
4. common denominator
5. L'Hopital

@ certain x value:

$f(6) = 2$

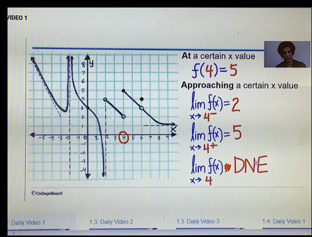
$\lim_{x \to 6^-} f(x) = 3$

$\lim_{x \to 6^+} f(x) = 3$



• to find lim of simple polynomial, plug # that variable approaching & see answer

• $\lim_{x \to x} = \infty$ technically DNE



x	f(x)
2.75	3.4
3.5	4.2

0.8

$0.8 \cdot \frac{1}{3} = 0.2677 + 3.4 = 3.667$

$\lim_{x \to 3} f(x) = 3.667$ ← round to thousandths

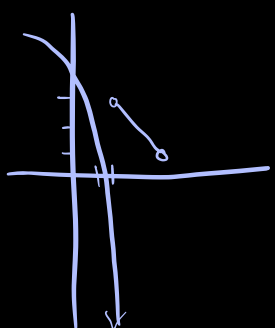
Alg rules of limits:

1. $\lim_{x \to a} k f(x) = k \lim_{x \to a} f(x)$ → Ex: $\lim_{x \to 5} 3x^2 = 3 \lim_{x \to 5} x^2 = 75$

2. if $\lim_{x \to a} f(x) = L_1$ & $\lim_{x \to a} g(x) = L_2$ then $\lim_{x \to a} f(x) + g(x) = L_1 + L_2$

Ex: $\lim_{x \to 5} [x^2 + x^3] = \lim_{x \to 5} x^2 + \lim_{x \to 5} x^3 = 150$

3. $\lim_{x \to 5} [(x^2+1)\sqrt{x-1}] = \lim_{x \to 5} (x^2+1) \lim_{x \to 5} \sqrt{x-1} = 52$



f(2): undefined

$\lim_{x \to 2^-} f(x) = -\infty$

$\lim_{x \to 2^+} f(x) = 3$

$\lim_{x \to 2} f(x) = \text{DNE}$

$\frac{K}{x^2}$ = f(x) always (+) so lim exists

$\frac{K}{x}$ = f(x) dep. on sign of x, one-sided lims

$f(x) = \begin{cases} x^2, & x \neq 2 \\ 5, & x = 2 \end{cases}$

jump discont.

oscillating at certain x = DNE

LIMIT DNE:

1. one-sided limits (left ≠ right)
2. unvalued behavior
3. oscillation btwn 2 fixed values

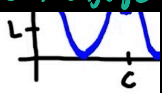
$f(x) = \frac{20}{(x-4)(x+4)}$ v asymp: @ $x = 4, -4$

$\lim_{x \to 4^-} = -\infty$ $\lim_{x \to -4^-} = \infty$

$\lim_{x \to 4^+} = \infty$ $\lim_{x \to -4^+} = -\infty$

indeterminate = $\frac{0}{0}$ ⟶ factor or L'Hospital, common denominator, conjugate

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$



1.	Scalar multiple:	$\lim_{x \to c}[b\, f(x)]$	$= bL$
2.	Sum or difference:	$\lim_{x \to c}[f(x) \pm g(x)]$	$= L \pm K$
3.	Product:	$\lim_{x \to c}[f(x) \cdot g(x)]$	$= L \cdot K$
4.	Quotient:	$\lim_{x \to c}\left[\frac{f(x)}{g(x)}\right]$	$= \frac{L}{K}$, as long as $K \neq 0$
5.	Power:	$\lim_{x \to c}[f(x)]^n$	$= L^n$

$\lim_{x \to 5} \frac{x^2-8x+15}{x^2-3x-10} = \frac{0}{0}$

indeterminate

$\lim_{x \to 5} \frac{(x-3)\cancel{(x-5)}}{\cancel{(x-5)}(x+2)}$

$\lim_{x \to 5} \frac{x-3}{x+2} = \frac{5-3}{5+2} = \boxed{\frac{2}{7}}$

FACTORING METHOD

$\lim_{x \to 5} \frac{\sqrt{x+4}-3}{x-5} \cdot \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}$

$\lim_{x \to 5} \frac{\cancel{(x-5)}}{\cancel{(x-5)}(\sqrt{x+4}+3)}$

$\lim_{x \to 5} \frac{1}{\sqrt{x+4}+3} = \frac{1}{\sqrt{9}+3}$

Conjugates:

$\lim_{h \to 0} \frac{\sqrt{3x+h} - \sqrt{3x}}{h} \cdot \frac{\sqrt{3x+h} + \sqrt{3x}}{\sqrt{3x+h} + \sqrt{3x}}$

$\lim_{h \to 0} \frac{(3x+h) - 3x}{\sqrt{3x+h} + \sqrt{3x}}$

$\lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{3x+h} + \sqrt{3x})}$

$\lim_{h \to 0} \frac{1}{\sqrt{3x+h} + \sqrt{3x}} = \frac{1}{\sqrt{3x} + \sqrt{3x}} = \boxed{\frac{1}{2\sqrt{3x}}}$

$\lim_{x \to 0} f(x) = \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

$\lim_{x \to 0^-} \left(\frac{x}{-x}\right) = \lim_{x \to 0^-} (-1) = -1$

$\lim_{x \to 0^+} (1) = 1$

* $\lim_{x \to -1^+} \frac{2x+3}{x+1} = \infty$

denominator approaches 0 & num approaches 1 thus, unbounded @ $x \to -1^+$ & v asymp @ $x = -1$

TRIG RULES: $\cos^2 x + \sin^2 x = 1$; $\frac{\sin x}{x} = 1$

$\frac{1-\cos x}{x} = 0$ $\frac{\sin x - 2}{x-2} = 1$ $\frac{\sin 2x}{2x} = 1$

$\sec^2\theta - \tan^2\theta = 1$

PIECEWISE LIMITS:

$f(x) = \begin{cases} \frac{|x-6|}{x^2-36}, & x \neq 6 \\ 0, & x = 6 \end{cases}$

$\lim_{x \to 6^-} |x-6| = \lim_{x \to 6^-} (6-x)$

$\lim_{x \to 6^-} \frac{|x-6|}{(x+6)(x-6)}$ ⟵ set =

*if piece wise & asking for $\lim_{x \to y}$ where y is on one side of piecewise, only plug into that function!

• right & left hand limit agreement

↳ $\lim_{x \to 6} f(x) = x$

SQUEEZE THEOREM:

$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x) \le \lim_{x \to a} h(x)$

$\lim_{x \to a} f(x) = L = \lim_{x \to a} (hx)$ have same limit

↳ $\lim_{x \to a} g(x) = L$

$\cos x \le \frac{\sin x}{x} \le 1$

$\cos 0 = 1$

$1 = 1$

thus, $\frac{\sin x}{x} = 1$

$\lim_{x \to 4} h(x) = 10\sqrt{2}$

Ex: $\lim_{x \to 0} x^2 \sin^2 \frac{1}{x}$

$0 \le \sin^2 \frac{1}{x} \le 1$ b/c sin always btwn -1 & 1

mult by x^2:

$0 \le x^2 \sin^2 \frac{1}{x} \le x^2$

$\lim_{x \to 0} 0 = 0$ & $\lim_{x \to 0} x^2 = 0$ so

$\lim_{x \to 0} x^2 \sin^2 \frac{1}{x} = \boxed{0}$

(squeeze thm seen when asked to eval. limit of rational expression w/ sine or cosine)

$\frac{\sin 2x}{x} \cdot \frac{2}{2}$

$\frac{2 \sin 2x}{2x}$

trig limits: (only $\lim_{x \to 0}$)

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\lim_{x \to 0} \frac{x}{\sin x} = 1$

2. $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

3. $\lim_{x \to 0} \frac{\sin ax}{x} = a$

4. $\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$

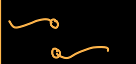
Types of Discontinuities: $\underbrace{\text{infinite, jump}}_{\text{non-removeable}}$, removeable

continuity @ a point:

1. f(c) is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

• polynom. always cont.
trig, rational, piecewise

jump:

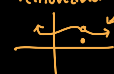


left & right exist but ≠

infinite (essential)

vert. asymp.

removeable:

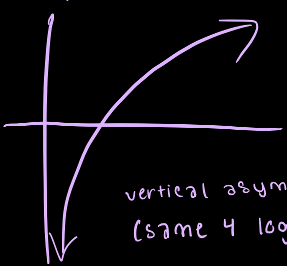


lim exists

∅ or is def. @ point but f(x) value ≠ y-of hole

• continuous "on domain" means that if discontinuity is not in domain, then continuous
• poly, rational, rad, trig, expon., & log funcs continuous on domain
• removeable discont. = ∅ continuous
• f(x) cont. on interval: cont. @ all points on domain
• x value cancel num & den = hole (plug into func. w/o hole to get y)
• x value denominator = v. asymp. (cannot cross b/c und.)
• x value numerator = h. asymptote (end behavior, can cross)
• $\lim_{x \to \infty}$ = h asym

$f(x) = \ln x$



vertical asymp. $x = 0^+$ (same 4 log func)

$f(x) = \begin{cases} x+1, & x < 2 \\ 2x+1, & x \geq 2 \end{cases}$ cont @ x=2?

1. defined
2. left ≠ right
3. ∅

not cont.

Ex: $\ln|x-4|$; x = 4 is v. asymp.

• horizontal asymptotes are trends of end behavior & not restrictions

* e^x dominates x^n

TYPES OF H. ASYMP.:

• degree of d > n = y = 0 $\lim_{x\to\infty} = 0$
• degrees same = $y = \frac{cn}{cd}$ $\lim_{x\to\infty} = \frac{cn}{cd}$
• degree n > d = divide 4 slant, $\lim_{x\to\infty} = \infty$

Intermediate Value Theorem:

1. f is continuous on closed interval [a,b]
2. x is a # btwn f(a) & f(b)
3. there is at least 1 number c
4. such that f(c) = x

EXPONENTIALS:

$\lim_{x\to\infty} \frac{2^x+1}{4+5^x} = \frac{2^\infty+1}{4+5^\infty} = \frac{0}{\infty} = 0$

$\lim_{x\to-\infty} \frac{\frac{1}{2^\infty}+1}{4+\frac{1}{5^\infty}} = \frac{0+1}{4+0} = \frac{1}{4}$

L'Hospital's Rule:

* if limit = $\frac{0}{0}$ or $\frac{\infty}{\infty}$ = indeterminate (USE L'HOSPITAL)

• take deriv. of top & deriv of bottom, keep doing until answer

Ex:

1. $\lim_{x\to0} \frac{\sin x}{x} = \frac{0}{0}$
∴ L'Hospital
$\lim_{x\to0} \frac{\cos x}{1} = 1$

2. $\lim_{x\to0} \frac{2x-\sin x}{x} = \frac{0}{0}$
∴ L'Hospital
$\frac{2-\cos x}{1} = 1$

3. $\lim_{x\to0} \frac{\sqrt{4+x}-2-\frac{x}{4}}{2x^2} = \frac{0}{0}$
∴ L'Hospital
$\frac{\frac{1}{2}(4+x)^{-\frac{1}{2}}-\frac{1}{4}}{4x} = \frac{0}{0}$ ← if u still get $\frac{0}{0}$ DO IT AGAIN!
$\frac{-\frac{1}{4}(4+x)^{-\frac{3}{2}}}{4} = \frac{-\frac{1}{4}\left(\frac{1}{2}\right)^3}{4} = \frac{-\frac{1}{32}}{4} = -\frac{1}{128}$

$\frac{\infty}{\infty}$:

Ex: $\lim_{x\to0^+} x \cot x = 0(\infty)$ ↙ turn into fraction when not alr
∴ L'Hospital
$\frac{x}{\tan x} = \frac{1}{\sec^2 x} = \cos^2 x = 1$

Ex: $\lim_{x\to\infty} xe^{-2x}$
∴ L'Hospital
$\frac{x}{e^{2x}} = \frac{1}{e^{2x}(2)} = 0$

a) $\lim_{x\to0} \frac{\sin 5x}{\sin 4x} \cdot \frac{x}{x}$

$\frac{\sin 5x}{x} \cdot \frac{x}{\sin 4x}$

$5 \cdot \frac{1}{4} = \boxed{\frac{5}{4}}$

b) $\lim_{x\to0} \frac{x^2}{1-\cos^2 x}$

$\lim_{x\to0} \frac{x^2}{\sin^2 x} = 1 \cdot 1 = \boxed{1}$

c) $\lim_{x\to0} \frac{4x}{\tan x}$ $\lim_{x\to0} \frac{4x\cos x}{\sin x}$

$\lim_{x\to0} \frac{4x}{\sin x} \cdot \cos x$

$4 \cdot 1 = \boxed{4}$

c) $\lim_{h\to0} \frac{(5+h)^2-25}{h}$

$\lim_{h\to0} \frac{h(10+h)}{h}$

$= 10$

$\lim_{x\to\infty} \frac{\cos 2x}{x^2} = \boxed{0}$

↳ cos 2x oscillates btwn -1 & 1 so ∅ value @ x = ∞

squeeze thm:

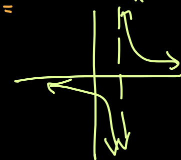
$\frac{-1}{x^2} \leq \frac{\cos 2x}{x^2} \leq \frac{1}{x^2}$ $0 < \frac{\cos 2x}{x^2} < 0$

infinites:

$\lim_{x\to\infty} \frac{1}{x} = 0$ (+) & (−) zero are =

$\lim_{x\to-\infty} = \frac{1}{x} = \frac{1}{-\infty} = 0$

$f(x) = \frac{3}{x-2}$



$\lim_{x\to2^-} = -\infty$

$\lim_{x\to2^+} = \infty$

if k & n are constants w n > 0 then $\lim_{x\to\infty} \frac{k}{x^n} = 0$

when exp. is polynomial/poly, div. each term of num. & den. by highest power of x

Ex: $\lim_{x\to\infty} \frac{3x+5}{7x-2} = \frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{\frac{3x}{x}+\frac{5}{x}}{\frac{7x}{x}-\frac{2}{x}}$

$\lim_{x\to\infty} \frac{3+\frac{5}{x}}{7-\frac{2}{x}} = \frac{3}{7}$

Ex: $\lim_{x\to\infty} \frac{8x^2-4x+1}{16x^2+7x-2} = \frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{\frac{8x^2}{x^2}-\frac{4x}{x^2}+\frac{1}{x^2}}{\frac{16}{x^2}+\frac{7x}{1x^2}-\frac{2}{x^2}}$

$\lim_{x\to\infty} \frac{8-\frac{4}{x}+\frac{1}{x^2}}{16-\frac{1}{x}-\frac{2}{x^2}} = \frac{1}{2}$

Steps to solve limit:

1. plug in → indeterminate form
2. factor
3. $\frac{x}{0}$? check left-hand limits to see if agree, ∅ = DNE
   Ex: $\lim_{x\to2} \frac{8}{x-2} = \frac{8}{0}$
   $\lim_{x\to2^-} = -\infty$ $\lim_{x\to2^+} = \infty$ [ONE SIDED = DNE]
4. div. by highest power of x (only for x → ±∞)

IVT: if f(x) continuous on interval [a, b] & f(c) is btwen f(a) & f(b), there exists value c, a ≤ c ≤ b, such that f(x) = f(c)

Ex:

$h(x) = x^3 - 2x^2 - 4x - 5$, [1,5] , h(x) = 0?

h(1) = -10

h(5) = 50

Yes, ∴ IVT states that b/c h(x) continuous on closed int. [1,5] & h(x) = 0 is between h(1) = -10 & h(5) = 50, there must be 1 ≤ x ≤ 5 such that h(x) = 0.

special case:

if f(x) continuous on interval [a, b] & f(a) & f(b) hv opposite signs, there exists value c, a ≤ c ≤ b such that f(c) = 0

IVT guarantees value c such that f(c) = 0 for x^3+3x-7 on (1,2)

$(1)^3+3(1)-7 = -3$ & $(2)^3+3(2)-7 = 7$, pos & neg ✓

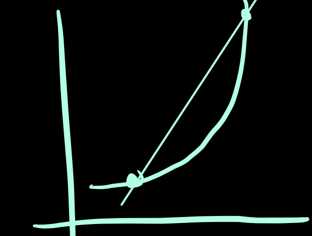
UNIT 2:

secant line = avg ROC

average ROC: rate of change over time:

$\frac{f(x+h)-f(x)}{h}$ ← diff. quotient – use for average & instan. ROC

if changes but start & finish are same avg ROC = 0

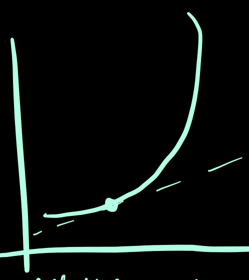


$E(t) = -11t^2 + 900t$

instantaneous ROC @ t=7

$\lim_{h\to0} \frac{E(7+h)-E(7)}{(7+h)-7}$ or

$\lim_{h\to0} \frac{(-11t^2+900t)-(-11(7)^2+900(7))}{t-7}$



tangent line = instantaneous

tangent line
derivative of f(x) @ x : instantaneous ROC

$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$

instantaneous ROC: rate of change @ instant

$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

instantaneous ROC:

$\lim_{x\to0} \frac{\Delta t}{\Delta t}$

Definition of Derivativ:

$\lim_{h\to0} \frac{f(x+h)-f(x)}{h}$ = slope btwn 2 points as x=0 app.

Ex: x^2 @ (2,4)

$\lim_{h\to0} \frac{(2+h)^2-4}{h}$ $\lim_{h\to0} \frac{4+4h+h^2-4}{h} = \frac{h(4+h)}{h} = f'(2) = 4$

Ex: x^2 @ (x_1, x_2)

$\lim_{h\to0} \frac{(x_1+h)^2 - x_1^2}{h} = \frac{x_1^2+2xh+h^2-x^2}{h} = \frac{\cancel{h}(2x+h)}{\cancel{h}} = f'(x) = 2x$

1st Deriv: $f'(x)$ or $\frac{dy}{dx}$
2nd Deriv: $f''(x)$ or $\frac{d^2y}{dx^2}$

DIFFERENTIABLE = DERIVATIVE
(left-hand = right hand deriv)

Estimating derivatives:

→ horizontal: deriv = 0

x	-3	-1	0
y	9	1	0

if not very detailed & not enough info

0.49	44.1584
0.5	44
0.51	43.8384

$\frac{44-44.1584}{0.01} = -15.84$

$\frac{43.8384-44}{0.01} = -16.16$

= 16

$\frac{1-9}{-1--3} = -4$

WAYS TO FIND DERIV.:
1. definition of deriv
2. power rule
3. quotient rule
4. chain rule
5. product rule
6. implicit diff.
7. inverse derivatives

* CONTINUITY:
- if cont. @ x=2, f(2) exists
- if differentiable @ x=2, f(2) exists
! both have to apply for f(2) to exist !

tangent eq: deriv to find slope, find y, point slope

- Differentiability: must be continuity
Ex: f(x) = |x|
- continuous but left & right side differentiability & match

not differentiable (no deriv.): sharp turn w/ multiple possible tangent lines, tangent line = vertical,

$f'(x) = \begin{cases} 3ax^2-4x+1; & x\le2 \\ -4x^3+8ax-5; & x>2 \end{cases}$ > find val. of a that makes f(x) differentiable

$3ax^2-4x+1 = -4x^3+8ax-5$
$x = 2$ $a = \frac{13}{2}$

Power Rule: if $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$
Ex: $x^5 \to 5x^4$
$x^0 \to 0$
$x^1 \to 1$

when power = 1, func linear & deriv = constant
power = 0 function is constant & deriv = 0

trig derivatives:
1. sin x = cos x
2. cos x = −sin x
3. tan x = sec^2x
4. sec x = sec x tan x
5. csc x = −csc x cot x
6. cot x = $-csc^2 x$

$\lim_{h\to0} \frac{\sin(x+h)-\sin x}{h}$
g(x) = cos x
derivative

Constant Rule: f(x) = K, f'(x) = 0 (constant)
Constant Multiple Rule: $\frac{dy}{dx}(kf(x)) = K \frac{dy}{dx}(f(x))$, deriv = constant × deriv of function
- Ex: $5x^{100} = 500x^{99}$

Adding & Subtracting:

$y = \frac{K}{x} \to \frac{dy}{dx} = -\frac{K}{x^2}$

$y = K\sqrt{x} \to \frac{dy}{dr} = \frac{K}{2\sqrt{x}}$

$\frac{3x^2}{x^3}$ $\frac{1}{x}$

$\log_{10} = \log$

more derivatives:
1. $\ln x \to \frac{1}{x}$ or $\ln u \to \frac{u'}{u}$
2. $e^u \to (e^u)(u')$
3. $\log_a u \to \frac{u'}{(\ln a)u}$
4. $a^u \to (a^u)(\ln a)(u')$
5.

Ex: $\ln x^3 \to 3\ln x$

$\log_8(x^2+x) \to \frac{2x+1}{(\ln 8)(x^2+x)}$

$8^{4x^5} \to (8^{4x^5})(\ln 8)(20x^4)$

NEW: $f(x) = x^x$ $f'(x) = ?$
$f(x) = e^{x\ln x}$ $f'(x) = x^x(\ln x + 1)$
$(e^{x\ln x})(x\cdot\frac{1}{x} + 1\cdot\ln x)$
$x^x(1+\ln x)$

$P''(t)(\frac{2\pi}{3})$
$P'''(t) = -2\cos t$
$-2\cos\frac{2\pi}{3} = \boxed{1}$

Ex: $\log_8 \sqrt{\frac{x^3}{1+x^2}}$
$f(x) = \frac{1}{2}[3\log_8 x - \log_8(1+x^2)]$
$f(x) = \frac{1}{2}\left(\frac{3}{\ln 8(x)}\right) - \left(\frac{2x}{\ln 8(1+x^2)}\right)$
$f'(x) = \frac{1}{2\ln 8}\left(\frac{3}{x} - \frac{2x}{1+x^2}\right)$

Ex: $f(x) = 5^{\sqrt{x}}$
$f'(x) = (5^{\sqrt{x}})(\ln 5)(\frac{1}{2}x^{-\frac{1}{2}})$

Ex: $f(x) = \frac{e^{x^3}}{5^{\cos x}}$
$f'(x) = \frac{(5^{\cos x})(e^{x^3})(3x^2) - (e^{x^3})(\ln 5)(5^{\cos x})(-\sin x)}{(5^{\cos x})^2}$
$f'(x) = e^{x^3} \frac{3x^2 + \ln 5 \sin x}{5^{\cos x}}$

Product rule:
$f(x) = uv$ $f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$
don't hv to simplify

Quotient Rule:
$f(x) = \frac{u}{v}$ $f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Chain + Product:
$\frac{dy}{dx} = \frac{1}{2}(5x^3+x)^{-\frac{1}{2}}(15x^2+1)$
$\frac{d^2y}{dx^2} = \frac{1}{2}[-\frac{1}{2}(5x^3+x)^{-\frac{3}{2}}(15x^2+1)(15x^2+1) + (5x^3+x)^{-\frac{1}{2}}(30x)]$

Chain rule:
f(x): f(g(x)) f'(x) = f'(g(x))·g'(x)

Ex: $(5x^3+3x)^5$ $\frac{dy}{dx} = 5(5x^3+3x)^4(15x^2+3)$
Ex: $(x^3-4x)^{\frac{1}{2}}$ $\frac{dy}{dx} = \frac{1}{2}(x^3-4x)^{-\frac{1}{2}}(3x^2-4)$

$\int_1^2 x\,dx \to (2x)(2) = 4x$ (derivative of integral)
* $\int_1^1 \sim = 0$
* tangent to x-axis
- f(x) = 0 AND f'(x) = 0

New Q: $y = 8v^2 - 6v$ & $v = 5x^3$ $\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx}$
1. multiply derivs,
2. plug in for

IMPLICIT DIFFERENTIATION: (when x & y terms mixed) → $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$
$y = x^2 + x \to \frac{dy}{dx} = 2x + 1$ $\frac{dx}{dx} = 1$
$\frac{dy}{dx} = 2x\frac{dx}{dx} + 1\frac{dx}{dx}$

- every time deriv. at term w/ x, mult. by $\frac{dx}{dx}$
y changes w/ respect to y:
$\frac{dy}{dy} = 2x\left(\frac{dx}{dy}\right) + 1\left(\frac{dx}{dy}\right)$
- deriv in terms of y AND x
FIND DERIV OF $3x^2 - 4y^2 + y = 9$ @ (2,1)
const. become 0

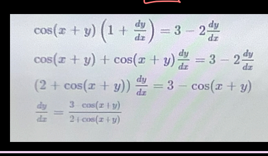
$6x - 8y\frac{dy}{dx} + \frac{dy}{dx} = 0$
$\frac{dy}{dx} = \frac{-6x}{-8y+1}$
$\frac{dy}{dx} = \frac{-12}{-7} = \frac{12}{7}$

Ex: $y^2 + y = 3x^5 - 7x$
$2y\frac{dy}{dx} + \frac{dy}{dx} = 15x^4\frac{dx}{dx} - 7\frac{dx}{dx}$
$2y\frac{dy}{dx} + \frac{dy}{dx} = 15x^4 - 7$
$\frac{dy}{dx} = \frac{15x^4-7}{2y+1}$

Ex 2:
$y^3 - 4y^2 = x^5 + 3x^4$
$3y^2\frac{dy}{dx} - 8y\frac{dy}{dx} = 5x^4 + 12x^3$
$\frac{dy}{dx} = \frac{5x^4+12x^3}{3y^2-8y}$

Ex 3:
$\sin y^2 \cos x^2 = \cos y^2 \sin x^2$
$2y\cos y^2 \frac{dy}{dx} + 2x\sin x^2 = -2y\sin y^2\frac{dy}{dx} + 2x\cos x^2$
$2y\cos y^2\frac{dy}{dx} + 2y\sin y^2\frac{dy}{dx} = 2x\cos x^2 - 2x\sin x^2$
$\frac{dy}{dx} = \frac{2x\cos x^2 - 2x\sin x^2}{2y\cos y^2 + 2y\sin y^2}$

FIND DERIV OF $\frac{2x-5y^2}{4y^3-x^2} = -x$ @ (1,1)
$2x - 5y^2 = -4xy^3 + x^3$
$2 - 10y\frac{dy}{dx} = -4(y^3 + (x)(3y^2\frac{dy}{dx})) + 3x^2$
$2 - 10y\frac{dy}{dx} = -4y^3 - 12xy^2\frac{dy}{dx} + 3x^2$
$2 + 4y^3 - 3x^2 = 10y\frac{dy}{dx} - 12xy^2\frac{dy}{dx}$
$\frac{2+4y^3-3x^2}{10y-12xy^2} = \frac{dy}{dx}$ @ (1,1)
$\frac{2+4-3}{10-12} = \frac{3}{-2} = \boxed{-\frac{3}{2}}$


$\cos(x+y)\left(1+\frac{dy}{dx}\right) = 3 - 2\frac{dy}{dx}$
$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = 3 - 2\frac{dy}{dx}$
$(2+\cos(x+y))\frac{dy}{dx} = 3 - \cos(x+y)$
$\frac{dy}{dx} = \frac{3-\cos(x+y)}{2+\cos(x+y)}$

when g(y) used: deriv = $g'(y)\frac{dy}{dx}$

Differentiating Inverse Functions:
$\frac{d}{dx}f^{-1}(x) = \frac{1}{\frac{d}{dy}f(y)}$ $g'(2) = \frac{1}{f'(-1)}$
$f(x) = x^2 \to$ deriv. of $f^{-1}(x)$ @ x = 9
- find reciprocal of deriv. @ x=a
- f(3) = 9
- x = 3 for original
$f'(x) = 2x \to f'(3) = 6$ $(g^{-1})'(3)$
- reciprocal: $\boxed{\frac{1}{6}}$ $\frac{1}{g'(g^{-1}(x))}$

to find invers:
$y = x^2$
$x = y^2$
$\sqrt{x} = y$

$\frac{1}{2}x^{-\frac{1}{2}}$ plug in inv. x
$\frac{1}{2\sqrt{x}}$

INVERSE DERIVATIVES: find a, find deriv, plug in for x, reciprocal

Ex 2: derivative of inv. of $y = x^3 - 1$ when y = 7
1. $7 = x^3 - 1$, x = 2
2. $3x^2 \to 3(2)^2 = 12$
3. $\frac{1}{12}$

Ex 3: derivative of inv. of $y = x^2 + 4$, y = 29
1. $29 = x^2 + 4$, x = 5
2. $2x \to 10$
3. $\frac{1}{10}$

Ex 4: deriv. of inv. of $y = 2x - x^3$, y = 1
1. $x(2 - x^2) = 1$, x = 1
2. $2 - 3x^2 = 2 - 3 = -1$
3. −1

Differentiating Inv. Trig:
$\frac{d}{dx}(\sin^{-1}u) = \frac{u'}{\sqrt{1-u^2}}$ $-1<u<1$
$\frac{d}{dx}(\tan^{-1}u) = \frac{u'}{1+u^2}$
$\frac{d}{dx}(\sec^{-1}u) = \frac{u'}{|u|\sqrt{u^2-1}}$ $|u|>1$
$\frac{d}{dx}(\cos^{-1}u) = \frac{-u'}{\sqrt{1-v^2}}$ $-1<u<1$
$\frac{d}{dx}(\cot^{-1}u) = \frac{-u'}{1+u^2}$
$\frac{d}{dx}(\csc^{-1}u) = \frac{-u'}{|u|\sqrt{u^2-1}}$ $|u|>1$
* same, but negative

practice!
$y = \tan^3(1-x^2)$
$y\ 3\tan^2(1-x^2)\sec^2(1-x^2)(-2x)$
$y = \sin^{-1}(6x) + \tan^{-1}(6x)$
$\frac{6}{\sqrt{1-36x^2}} + \frac{6}{1+36x^2}$
$3x^5 - 4x^2y^3 + 2y^2 = 1$

Ex: $\frac{d}{dx} = \sin^{-1}x^2 = \frac{2x}{\sqrt{1-x^4}}$
$\frac{d}{dx} = \tan^{-1}5y = \frac{5}{1+25x^2}$
$\frac{d}{dx} = \sec^{-1}(x^2-x) = \frac{2x-1}{|x^2-x|\sqrt{(x^2-x)^2-1}}$

2nd Derivatives using Implicit Differentiation:

$\frac{d^2y}{dx}$ if $y^2+2y=4x^2+2x$

1. Imp. diff to solve for $\frac{dy}{dx}$
2. take deriv again
3. sub in $\frac{dy}{dx}$

$2y\frac{dy}{dx}+2\frac{dy}{dx}=8x+2$

$\frac{dy}{dx}=\frac{8x+2}{2y+2}=\frac{4x+1}{y+1}$

$\frac{(y+1)(4)-(\frac{dy}{dx})(4x+1)}{(y+1)^2}=\frac{(y+1)(4)-\left(\frac{4x+1}{y+1}\right)(4x+1)}{(y+1)^2}=\frac{4(y+1)^2-(4x+1)^2}{(y+1)^3}$

Examples:

Find deriv of each variable w/ respect to t of $x^2+y^2=z^2$

$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt}$

Find $\frac{d^2y}{dx^2}$ if $y^2=x^2-2x$

$2y\frac{dy}{dx}=2x-2$

$\frac{dy}{dx}=\frac{x-1}{y}$

$\frac{(y)-(x-1)(\frac{dy}{dx})}{y^2}$

$\frac{y^2-(x-1)^2}{y^3}$

Position, Velocity, Acceleration:

position $\xrightarrow{\text{deriv.}}$ velocity $\xrightarrow{\text{deriv.}}$ acceleration

velocity → abs value (no direc) → speed

- velocity (−) → particle going left
- velo (+) → particle going right
- $v(t)$ & $a(t)$ same sign = speeding up
- $v(t)$ & $a(t)$ opposite sign = slowing down
- $v(t)=0$ & $a(t)\neq 0$ → momentarily stopped & changing direc
- distance = abs. value of diff. in positions between direc changes

Ex: position: $x(t)=t^3-11t^2+24t$

$v(5)$: $3t^2-22t+24=75-108+24=-11$

$a(5)$: $6t-22=30-22=8$

Ex 2: $x(t)=t^3-12t^2+36t+18$ $t>0$, find where changes direc.

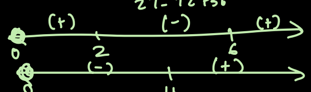
$v(t)=3t^2-24t+36$

$t^2-8t+12$

$(x-2)(x-6)$

$x=2,6$

$\boxed{t=2,6}$



$a(t)=6t-24$

6 $(2)-24=-12$

$6(6)-24=12$

27 − 72 + 36

slowing down: $(0,2)\cup(4,6)$

Ex 3: How far travel btwn 0 – 4 sec if $x(t)=t^4-8t^2$ → $v(t)=4t^3-16t$

$4t(t^2-4)$

$t=0,2,-2$

$|x(4)-x(2)|+|x(2)-x(0)|$

$144+16=160$

$\boxed{\text{If asking for total, always remember initial amnt!}}$

Ex 4: $x(t)=t^3-6t^2+1$ $t>0$, find distance from $t=2$ to $t=5$

$v(t)=3t^2-12t$

$3t(t-4)$

$t=0,4$

$|x(5)-x(4)|+|x(4)-x(2)|=$

$|(-24-(-31))|+|(-31-(-15))|=23$

- displacement $\int_0^{}v(t)\,dt$
- distance $\int_0^{}|v(t)|\,dt$
- position: $\int_0^{}v(t)\,dt+x_0$

1. changes direc
   - $v(t)=0$ & changes signs & $a(t)\neq 0$
2. slowing down
   - $v(t)$ & $a(t)$ diff sign on number line w/ crit. pnts
3. distance
   - points where changes direc ($v(t)$ & 0 & changes sign) in abs value

Equation of tangent line:

$y=5x^2$ @ $(3,45)$

$\frac{dy}{dx}=10x$

$y-45=30(x-3)$

$\left.\frac{dy}{dx}\right|_{x=3}=10(3)=30$

$y=30x-45$

notation 4 plugging in point!

Ex: $y=x^3+x^2$ @ $(3,36)$

$\frac{dy}{dx}=3x^2+2x$

$\left.\frac{dy}{dx}\right|_{x=3}3(3)^2+2(3)=27+6=33$

$y-36=33(x-3)$

$y=33x-63$

when only given x:

$y=\frac{2x+5}{x^2-3}$ @ $x=1$

$y(1)$: $\frac{2(1)+5}{(1)^2-3}=\frac{2+5}{1-3}=\frac{7}{-2}=-\frac{7}{2}$

$(1,-\frac{7}{2})$

$\frac{dy}{dx}=\frac{(x^2-3)(2)-(2x+5)(2x)}{(x^2-3)^2}$

$\left.\frac{dy}{dx}\right|_{x=1}\frac{(1-3)(2)-(2+5)(2)}{(1-3)^2}=$

$\frac{(-2)(2)-(7)(2)}{-2^2}=\frac{-4-14}{4}=-\frac{18}{4}=-\frac{9}{2}$

$y+\frac{7}{2}=-\frac{9}{2}(x-1)$

$y=-\frac{9}{2}x+\frac{9}{2}$

$y=-\frac{9}{2}x+1$

* Asked to find eq. of normal line (perpendicular = negative reciprocal of slope)

$y=x^5-x^4+1$ @ $x=2$

$y(2)=2^5-2^4+1=32-16+1=17$

$(2,17)$

$\frac{dy}{dx}=5x^4-4x^3$

$\left.\frac{dy}{dx}\right|_{x=2}5(2)^4-4(2)^3=48$

$y-17=-\frac{1}{48}(x-2)$

$y_1=ax^2+bx+c$ passes thru $(2,4)$ & is tangent to $y_2=x+1$ @ $(0,1)$ find $a,b,$ & c

$4a+2b+c=4$

$(0,1)\to 0+0+c=1$

$c=1$

$2ax+b=1$

$2a(0)+b=1$

$4a+2+1=4$

$4a=1$

$a=\frac{1}{4}$

$\boxed{c=1,\ b=1,\ a=\frac{1}{4}}$

find the points on the curve $y=2x^3-3x^2-12x+20$ where tangent parallel to x-axis

$\frac{dy}{dx}=0$

$\frac{dy}{dx}=6x^2-6x-12$

$6(x^2-x-2)$

$(x-2)(x+1)$

$x=-1,2$

$2(-1)^3-3(1)^2-12(-1)+20$

$-2-3+12+20$

$-5+32=27$

$2(8)-3(4)-12(2)+20$

$16-12-24+20$

8

$(-1,9)$

$(2,0)$

- if $f(x)$ is cont. on $[a,b]$, there is a number c in $[a,b]$ such that $f(c)\geq f(x)$ for all x in $[a,b]$
- if $f(x)$ dec then inc, there exists "c" on interval such that $f'(c)=0$

$15x^4-(8xy^3-4x^2(3y^2\frac{dy}{dx}))+4y\frac{dy}{dx}$

$-12x^2y^2\frac{dy}{dx}+4y\frac{dy}{dx}=8xy^3-15x^4$

$\frac{dy}{dx}=\frac{8xy^3-15x^4}{-12x^2y^2+4y}$

$f(x)=\sqrt{1-\sec^3(\pi x)}$ @ $x=1$

$\frac{1}{2}(1-\sec^3(\pi x))^{-\frac{1}{2}}(-3\sec^2(\pi x))(\sec(\pi x)\tan(\pi x))(\pi)=0$

PARTICULAR SOLUTION:

initial condition $f(3)=-1$

1. $y\,dy=x\,dx$
2. integrate
3. plug in initial 4 C
4. plug in C = final

RELATED RATES:

Steps:

1. draw & write known
2. relate known in deriv. eq.
3. plug in value
4. solve

Ex: volume of water amnt: $v(t)=8t^2-32t+4$

$\frac{dv}{dt}$ = Rate of volume inc $=16t-32$

Ex 2: $B=64t-2^t$

- how many hours until growth stops?

$\frac{dB}{dt}$ = rate of change of growth

$\frac{dB}{dt}=0=64-(2^t)(\ln 2)$

$\boxed{\text{if value like "r" is fixed, not included in derivative!}}$

Ex 3: circular pool of water expanding @ 16π in²/sec, what rate is radius exp. when radius = 4 inches

$\frac{dA}{dt}=16\pi$

$A=\pi r^2$

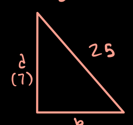
$\frac{dA}{dt}=2\pi r\frac{dr}{dt}$

$16\pi=2\pi r\frac{dr}{dt}$

$\frac{16\pi}{2\pi(4)}=\frac{dr}{dt}$

only plug in AFTER derivative is taken

Ex 4: 25 ft ladder leaning ag. wall & sliding toward floor. foot sliding away from base @ rate of 15 ft/s, how fast is top of ladder sliding when 7 feet from ground



$\frac{db}{dt}=15$

$c=25$

$a=7$

$b=24$

$x^2+y^2=25^2$

$7^2+b^2=25^2$

$2b\frac{db}{dt}=-2a\frac{da}{dt}$

$24(15)=-7(x)$

$\boxed{\frac{da}{dt}=-\frac{360}{7}\text{ ft/s}}$

Ex 5: spherical balloon exp. @ rate of 60π in³/s, how fast is SA of balloon exp. when $r=4$

$V=\frac{4}{3}\pi r^2$

$A=4\pi r^2$

$\frac{dv}{dt}=60\pi$

$r=4$

$V=\frac{4}{3}\pi r^3$

$\frac{dv}{dt}=4\pi r^2\frac{dr}{dt}$

$60\pi=4\pi(4)^2\frac{dr}{dt}$

$\frac{dr}{dt}=\frac{15}{16}$

$\frac{dA}{dt}=8\pi r\frac{dr}{dt}=8\pi(4)\left(\frac{15}{16}\right)=\boxed{30\pi\text{ in}^2\text{/s}}$

to find expression for ___ in terms of c:

1. $\frac{dh}{dt}=$
2. separation of variables
3. initial condition to find "c"
4. simplify

Ex 6: conical tank standing on vertex filled w/ water @ rate of 18π ft³/m height = 30 ft, radius = 15 ft, how fast is water rising when 12 ft deep

$V=\frac{1}{3}\pi\frac{h^3}{4}$

$\frac{dv}{dt}=18\pi$

$h=30$

$r=15$

$\frac{dv}{dt}=\frac{1}{4}\pi h^2\frac{dh}{dt}$

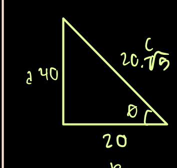
$18\pi=\frac{1}{4}\pi(12)^2\left(\frac{dh}{dt}\right)$

$\boxed{\frac{dh}{dt}=\frac{1}{2}\text{ ft/min}}$

* ratio of height of right, circ. cone to rad. is constant, height = 30, $r=15$

$h=2r$

Ex 7: rocket rising vert. @ rate of 5400 mi/hr, observer standing 20 mi from launch, how fast in rads/sec is θ of elev. btwn ground & observer's line of sight when rocket @ elevation of 40 miles



$c=20\sqrt{5}$

$a=40$

$b=20$

$\frac{da}{dt}=5400$

$\tan\theta=\frac{a}{20}$

$\sec^2\theta\frac{d\theta}{dt}=\frac{1}{20}\frac{da}{dt}$

convert

$\sec^2\theta\frac{d\theta}{dt}=\frac{1}{20}\left(\frac{3}{2}\right)$

$\theta=1.107$

$5\frac{d\theta}{dt}=\frac{3}{40}$

$\frac{d\theta}{dt}=\frac{3}{200}$ rads/s

* rate of change of given ROC?:

$\frac{d^2h}{dt^2}=\ldots$ & plug in $\frac{dh}{dt}$

Differentials:

formula: $f(x+\Delta x)=f(x)+f'(x)\Delta x$

linear approximation

Ex: approx. $\sqrt{9.01}$

$f(x)=\sqrt{x}$ $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$

$x=9$ $f'(x)=\frac{1}{2\sqrt{x}}$

$\Delta=0.01$

$\sqrt{9.01}=\sqrt{9}+\frac{1}{2\sqrt{9}}(0.01)=\boxed{3.00166}$

Ex 2: $\sqrt{9.5}$

$f(x)=\sqrt{x}$ $f'(x)=\frac{1}{2\sqrt{x}}$

$x=9$

$\Delta=0.5$

$\sqrt{9.5}=\sqrt{9}+\frac{1}{2\sqrt{9}}(0.5)$

radius of circle ↑ from 3 to 3.04. find change in area

$A=\pi r^2$ $dr=0.04$ (the change)

$\frac{dA}{dt}=2\pi r\frac{dr}{dt}$

$\frac{dA}{dt}=2\pi(3)(0.04)$

$=\boxed{0.754}$

ANOTHER APPROXIM. FORM.: $dy=f'(x)\,dx$

Ex: $(3.98)^4$

$f(x)=x^4$ $f'(x)=4x^3$

$x=4$

$\Delta x=-0.02$

$(3.98)^4=4^4+(4(4)^3)(-0.02)$

$=250.88$

$f'(x)=2x+1$, $f(1)=4$

approx $f(1.2)$ w/ $x=1$

$y-4=3(1.2-1)$

MEAN VALUE THEOREM:

ie $f(x)$ cont. on $[a,b]$ & differentiable on (a,b), at least one # c such that $a \le c \le b$.

$f'(c) = \frac{f(b)-f(a)}{b-a}$

* in order for MVT, must be cont. @ interval & endpoints

• some pnt. on interval when slope of tangent = slope of secant connecting endpoints

Ex 1: $f(x)=x^2$ on $[1,3]$ (a = 1, b = 3)

$f'(c) = \frac{9-1}{2} = 4$

$f'(x) = 2x$

$f'(c) = 4 = 2x$

$\boxed{c=2}$

Ex 2: $f(x)=x^3-12x$ $[-2,2]$

$f'(c) = \frac{(8-24)-(-8+24)}{4} = \frac{-16-16}{4} = -8$

$-8 = 3x^2-12$

$4 = \sqrt{\frac{4}{3}}$

± important!

$\pm\frac{2}{\sqrt{3}}$

Ex 3: $f(x)=\frac{1}{x}$ $[-2,2]$ x^{-1} $-x^{-2}$

$f'(c) = \frac{\frac{1}{2}+\frac{1}{2}}{4} = \frac{1}{4}$

* take deriv & set = 0

$\frac{1}{4} = -\frac{1}{x^2}$ $\boxed{\text{no c value b/c f(x) not cont.}}$

Rolle's Theorem: special case of MVT

$f(x)$ is cont. on $[a,b]$ & differentiable on (a,b) & if $f(a)=f(b)=0$, at least one value c btwn $a \le c \le b$ such that $f'(c)=0$

* horizontal tangent

Ex: $f(x)=\frac{x^2}{2}-6x$ on $[0,12]$

$f(0)=\frac{0}{2}-6(0)=0$ & $f(12)=\frac{144}{2}-6(12)=0$

$f'(c) = \frac{0-0}{12} = 0$

$f'(x) = x-6$ $c=6$

Ex 2: $f'(c) = \frac{(8+1)-(2)}{1} = 7$

$3x^2=7$

$x^2=\frac{7}{3}$

$\boxed{c=\sqrt{\frac{7}{3}}}$

Ex 3: $f(x)=x^4-x$ on $[0,1]$

$f'(c) = \frac{0-0}{1} = 0$

$4x^3-1=0$

$4x^3=1$

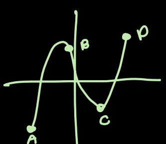
$x^3=\frac{1}{4}$

Extreme Value Theorem:

• if $f(x)$ cont. on interval $[a,b]$ then EVT guarantees f has one max & min value on $[a,b]$

• abs. max/min: no value of f that is higher or lower

• when cont. func has only one crit. point on interval, CP is absolute extrema



abs. max: D
abs. min: A
loc. max: B
loc. min: C

crit. pnts: where extrema could exist

if $f'(x)$ (+) = $f(x)$ increasing
if $f'(x)$ (−) = $f(x)$ decreasing
if $f'(x)=0$ or und = $f(x)$ horiz or vert.

Ex: x^2-8x+6 inc? dec?
$f'(x) = 2x-8$
$x=4$
(−) 4 (+)
inc: $(4,\infty)$
dec: $(-\infty,4)$

inc → dec = max
dec → inc = min

Ex: $2x^3-9x^2-60x+7$
$6x^2-18x-60$ $6(9)+54-60$
$6(x^2-3x-10)$
$(x-5)(x+2)$
$x=5,-2$
(+) −2 (−) 5 (+)
rel max: $(-2,75)$
rel min: $(5,-268)$

$f(x)=2\sin(\frac{\pi}{4}x)$ on $[1,11]$ → value of x can't exceed "

$f'(x) = 2\cos(\frac{\pi}{4}x)(\frac{\pi}{4})$

$f'(x) = \frac{\pi}{2}\cos(\frac{\pi}{4}x)$

$\frac{\pi}{4} \cdot x = \frac{3\pi}{2}$

$\cos\frac{\pi}{4}x = 0$

$\cos\frac{\pi}{2}$ & $\cos\frac{3\pi}{2}$ & $\cos\frac{5\pi}{2}$

$x = 2, 6, 10$

* concave down graph tangents underestimate

1 (+) 2 (−) 6 (+) 10 (−) 11

rel max: $(2,2)$ & $(10,2)$
rel min: $(6,-2)$

* unless domain is restricted, there is no abs. max b/c ends go to ∞

$f(x)=2\sin(\frac{\pi}{4}x)$

x	y
1	1
2	2
6	−2
10	2
11	1

make sure to plug CPs AND endpoints

$2\sin\frac{11\pi}{4}$

abs max: $(2,2)$ $(10,2)$
abs min: $(6,-2)$

$f''(x)>0$ = concave up
$f''(x)<0$ = concave down
$f''(x)=0$ or undef. = point of inflection

Ex: $f(x)=-2x^4+4x^3+24x^2+20x+4$
$f'(x)=-8x^3+12x^2+48x+20$
$f''(x)=-24x^2+24x+48$
$-24(x^2-x-2)$
$(x-2)(x+1)$ $x=-1,2$

Ex 2: $f(x)=e^{-x^2}$

USING TRIG: approx. $\sin 46°$ (Radians!) $\boxed{y=4.6}$

$f(x)=\sin x$ $f'(x)=\cos x$

$x=\frac{\pi}{4}$

$\Delta x = \frac{\pi}{180}$

$\sin\frac{23\pi}{90} = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}(\frac{\pi}{180})$

$= 0.7194$

AGAIN W/ RELATED RATES!

radius of sphere = 4 cm w/ error of ±0.01 cm. approx. error in SA:

$SA = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$

$\frac{dS}{dt} = 8\pi(4)(\pm 0.01)$

2nd Derivative Test:

$f(x) = 4x^3-5x^2-8x-24$

$f'(x) = 12x^2-10x-8$

$f''(x) = 24x-10$

(−) $\frac{5}{12}$ (+)

$f'(x)$ ↝ $x=\frac{4}{3},-\frac{1}{2}$

$(3x-4)(4x+2)$

$f''(x)$ ↝ $24x=10$

$x=\frac{10}{24}=\frac{5}{12}$

$f''(\frac{4}{3}) = 24(\frac{4}{3})-10 = 22$ = conc up

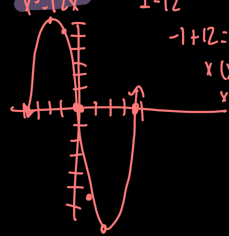
$f''(-\frac{1}{2}) = 24(-\frac{1}{2})-10 = -22$ = conc down

1. find crit. points
2. plug into $f''(x)$ to find concavity
3. use concavity to find extrema

rel max: $(-\frac{1}{2},-21.75)$
rel min: $(\frac{4}{3},-34.074)$
POI = $(\frac{5}{12},-27.912)$
inc: $(-\infty,-\frac{1}{2})\cup(\frac{4}{3},\infty)$
dec: $(-\frac{1}{2},\frac{4}{3})$
CU: $(\frac{5}{12},\infty)$
CD: $(-\infty,\frac{5}{12})$

Sketching:

x^3-12x



$1-12$

$-1+12=11$

$x(x^2-12)$

$x=0,\pm\sqrt{12}$

$x^2=12$

$3x^2=12$

$x^2=4$

$x=\pm 2$

$f'(x)=3x^2-12$ $f''(x)=6x$

x int: $(0,0)$, $(\sqrt{12},0)$, $(-\sqrt{12},0)$

y int: $(0,0)$

crit pnt: $(2,-16)$ $(-2,16)$

POI: $(0,0)$

Finding A Cusp:

$f(x)=2-x^{\frac{2}{3}}$

$2-x^{\frac{2}{3}}=0$

$-x^{\frac{2}{3}}=-2$

$x^{\frac{2}{3}}=2$

$x=2^{\frac{3}{2}}$

$x=\sqrt{8}$

$x \pm 2\sqrt{2}$

$f'(x)=-\frac{2}{3}x^{-\frac{1}{3}}$

$x=0$

cusp:

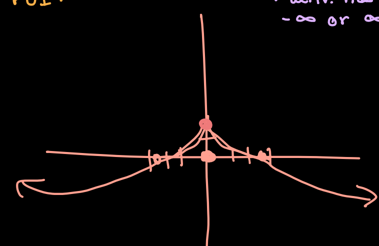
$\lim_{x\to 0^+} -\frac{2}{3}x^{-\frac{1}{3}} = -\infty$

$\lim_{x\to 0^-} -\frac{2}{3}x^{-\frac{1}{3}} = \infty$

$f''(x)=\frac{2}{9}x^{-\frac{4}{3}}$

x int: $(2\sqrt{2},0)$ & $(-2\sqrt{2},0)$
y int: $(0,2)$
crit: $(0,2)$ cusp
POI:

1. f(c) doesn't exist
2. deriv. has a limit of −∞ or ∞



ASYMPTOTES:

$y=\frac{3x}{x+2}$

$\frac{(x+2)(3)-(3x)}{(x+2)^2}$

$3x+6-3x$

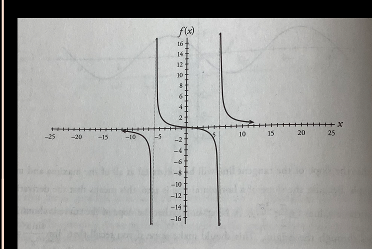
$6(x+2)^{-2}$

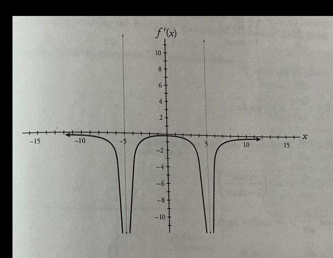
$(x+2)^{-2}=0$

v asymp: $x=-2$
x int: $(0,0)$ only set num = 0
y int: $(0,0)$
crit: 0
POI: 0

* if x in den. ∅ crit. pnt w/ constant in num.

DERIVATIVE GRAPHS:





$f'(x) = (e^{-x^2})(-2x)$

$f''(x) = (e^{-x^2})(-2x)(-2x) + (e^{-x^2})(-2)$

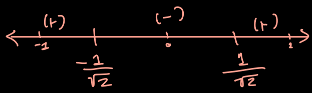
$(e^{-x^2})(-2x)(-2x) = -(e^{-x^2})(-2)$

$(-2x)^2 = 2$

$4x^2 = 2$

$x^2 = \frac{1}{2}$

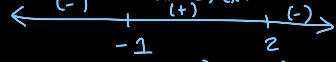
$x = \pm\frac{1}{\sqrt{2}}$



CD: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

CU: $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

POI: $\left(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$ & $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$



CU: $(-1, 2)$ CD: $(-\infty, -1) \cup (2, \infty)$

OPTIMIZATION:

- max or min of $f(x)$ when $f'(x) = 0$ or DNE
- relative max or min: horz. tangent but not highest or lowest value of (x)
- abs. max/min: highest/lowest value w/ $f(x)$ w/ endpoints
  - relative max/min can also be absolute
- steps: find deriv, set $=0$, find crit. values, test values w/ 2nd deriv test
  - relative max if $f''(c) < 0$ & rel min if $f''(c) > 0$, if $f''(c) = 0 \rightarrow$ POI
- EVT (Extreme Value Thm): if $f(x)$ cont. on $[a,b]$ then on (a,b) there exists max/min value of $f(c)$

EX: $y = ax^2$ if $a > 0$

$2ax = 0$

$x = 0 \rightarrow$ crit. point @ $(0,0)$

$y'' = 2a \rightarrow$ second deriv is positive so $(0,0)$ is rel min.

EX2: $8x^2 - 176x + 1800$

$16x - 176 = 0$

$16x = 176$ $x = 11$

$x = 11$ is rel min

$f''(x) = 16 \rightarrow$ concave up

- when domain of $f(x)$ is restricted, hv to test endpoints of interval b/c highest & lowest may be @ end p.
- for AP exam: endpoints separate from crit. values

Ex 3: $h(t) = 1600 + 196t - 4.9t^2$, find max height & t

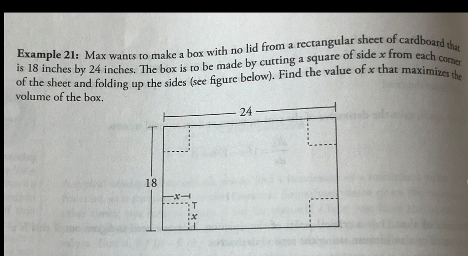
$h'(t) = 196 - 9.8t = 0$

$-9.8t = -196 \rightarrow t = 20$

$h''(t) = -9.8 \rightarrow$ max

$t: 20$

$h(20) = 3{,}560$ m @ $t = 20$ s



Example 21: Max wants to make a box with no lid from a rectangular sheet of cardboard that is 18 inches by 24 inches. The box is to be made by cutting a square of side x from each corner of the sheet and folding up the sides (see figure below). Find the value of x that maximizes the volume of the box.

width · height · depth = V

$(24-2x)(18-2x)(x)$

$V = 4x^3 - 84x^2 + 432x$

$V' = 12x^2 - 168x + 432 = 0$

$x = 7 \pm \sqrt{13} = 3.4, 10.6$

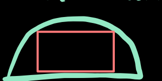
$V'' = 24x - 168 = 0$

$24x = 168$

$24(3.4) - 168 = (-)$ (max

$\boxed{\text{volume of box maximized @ } x = 3.4}$

Ex: rectangle inside semicircle w/ radius 4, one side along diameter, largest area of rectangle?



eq. of circle w/ radius 4: $x^2 + y^2 = 16$ (r^2)

$y^2 = 16 - x^2$

$y = \sqrt{16 - x^2}$

rectangle h: $\sqrt{16-x^2}$ b: $2x$

area $= 2x(16-x^2)^{\frac{1}{2}}$

a': $2x \cdot \frac{1}{2}(16-x^2)^{-\frac{1}{2}}(-2x) + 2(16-x^2)^{\frac{1}{2}}$

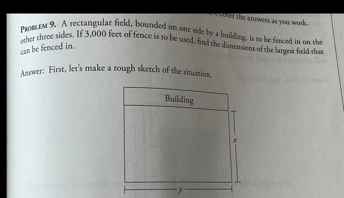
$\frac{2x^2}{\sqrt{16-x^2}} = 2\sqrt{16-x^2}$

$x = \pm\sqrt{8}$

x	y
-4	0
$\sqrt{8}$	16
4	0

no such thing as neg. area

max area of 16 when $x = \sqrt{8}$



... cover the answers as you work.

PROBLEM 9. A rectangular field, bounded on one side by a building, is to be fenced in on the other three sides. If 3,000 feet of fence is to be used, find the dimensions of the largest field that can be fenced in.

Answer: First, let's make a rough sketch of the situation.

Area: $x \cdot y$

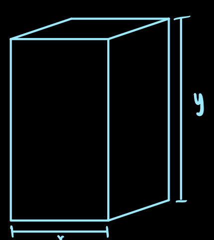
Perimeter: $2x + y = 3{,}000$

$y = 3000 - 2x$

$x \cdot (3000 - 2x)$

$A = 3000x - 2x^2$

↳ process



$V = 256$ in^3

find dimensions w/ min SA

$SA = x^2 + 4xy$

$V: x^2y = 256$

$y = \frac{256}{x^2}$

$SA = x^2 \cdot 4x\left(\frac{256}{x^2}\right)$

$SA = x^2 + \frac{1024}{x}$

↳ process

Find point on curve $y = \sqrt{x}$ that is min distance from point $(4,0)$

$L = y = \sqrt{x}$

$D^2 = (4-x)^2 + (y-0)^2$

$x^2 - 8x + 16 + y^2$

$y = \sqrt{x}$

$x^2 - 8x + 16 + x$

$L = x^2 - 7x + 16$

↳ process

$L = D^2$ b/c mins @ same place

FUNDAMENTAL THM OF CALCULUS:

- if $f(x)$ cont. on $[a,b]$, deriv of func $F(x) = \int_a^x f(t)\,dt$ is $f(x)$

Crit. Points of implicit functions:

Ex: $3x^2 + 5y^2 - 2x + 2y = x$

$6x + 10y\frac{dy}{dx} - 2 + 2\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{3-6x}{10y+2}$

$3 - 6x = 0$

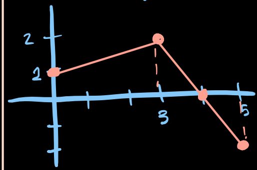
$x = \frac{1}{2}$ (find y by plugging into og)

$10y + 2 = 0$

$y = -\frac{1}{5}$

Accumulation:

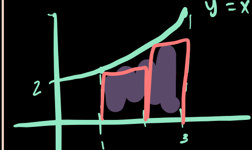
- definite integral: area of region under function



$\frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}u^2 = $ area

accum. of change = total distance

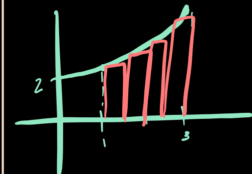
$y = x^2 + 2$ find area from $x = 1$ to $x = 3$



use rectangles: width of each rec is 1 so plug 1 into $y = x^2 + 2$

1st: $1 \cdot 3$ 2nd: $1 \cdot 6$

area under curve $= 9$



LHS: $\frac{1}{2}\left(3 + \frac{17}{4} + 6 + \frac{33}{4}\right) = \frac{43}{4}$

RHS: $\frac{1}{2}\left(\frac{17}{4} + 6 + \frac{33}{4} + 11\right) = \frac{59}{4}$

$\frac{43}{4} + \frac{59}{4} \cdot \frac{1}{2} = \boxed{\frac{51}{4}u^2}$

FORMULAS:

LHS $= \left(\frac{b-a}{n}\right)[y_0 + y_1 \ldots]$

RHS: $\left(\frac{b-a}{n}\right)[y_1 + y_2 \ldots]$

number of rectangles: n

(a,b) = interval

midpoint:

$\frac{1}{2}\left(\frac{57}{16} + \frac{81}{16} + \frac{113}{16} + \frac{153}{16}\right) = 12.625$

MPS: $\left(\frac{b-a}{n}\right)[y_{\frac{1}{2}} + y_{\frac{3}{2}} \ldots]$

USE INTEGRAL!

$\int_1^3 (x^2 + 2)\,dx$

Table Riemann Sum:

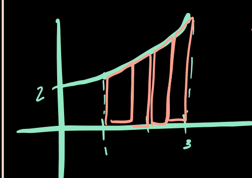
x	2	4	6	8	10	12
f(x)	10	13	15	14	9	3

RHS: $2(13 + 15 + 14 + 9 + 3) = 108$

x	0	2	5	11	19	22	23
f(x)	4	6	16	18	22	29	50

$2(4) + 3(6) + 6(16) + 8(18) + 3(22) + 1(29) = 361$

TRAPEZOID RULE:



area of trapezoid $= \frac{1}{2}(b_1 + b_2)h$

solve: find area of each trapezoid & add up

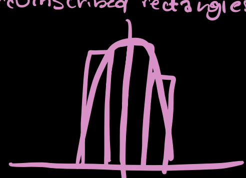
Ex: area under $y = x^3$ from $(2,3)$ using 4 trapezoids

height of each trapezoid: $\frac{1}{4}$

area: $\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(2^3 + 2\left(\frac{9}{4}\right)^3 + 2\left(\frac{5}{2}\right)^3 + 2\left(\frac{11}{4}\right)^3 + 3^3\right) = 1{,}045/64$

inscribed rectangles

circumscribed rectangles:



SUMMATION NOTATION:

$\frac{b-a}{n} = x$ (width of each rectangle)

$y_1 = f(a + \Delta x)$

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(a + i\Delta x)$

Ex: what integral repres. by

$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(6 + \frac{4}{n}i\right)^2 - \left(6 + \frac{4}{n}i\right)$

Ex: find area under curve $y = x^3$ from $(2,3)$ using ∞ # of right-hand rectangles

$\int_2^3 (x^3)\,dx$

$\Delta x = \frac{3-2}{n} = \frac{1}{n}$

↓ plug into formula

$\int_2^3 x^3\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(2 + \frac{1}{n}i\right)^3$

Ex: find $\frac{d}{dx}\int_2^x \cos(t)\,dt$

$f(t) = \cos x - 0$

Ex: $\frac{d}{dx}\int_2^x (1-t^3)\,dt = 1-x^3$

Ex: $\frac{d}{dx}\int_1^x \frac{1}{1-\sqrt[3]{t}}\,dt = \frac{1}{1-\sqrt[3]{x}}$

• if upper limit is func. of x, mult. answer by deriv. of term

$\frac{d}{dx}\int_2^{x^2}(1-t^3)\,dx = (1-x^6)(2x)$ ← this is deriv.

$\int_2^{x^2}(1-t^3)\,dx = \left(t-\frac{t^4}{4}\right)(2x)$ ← find ad, then plug in

$F(x) = \int_a^x f(t)\,dt$ is accumulation function b/c value of integral increases as $x \uparrow$

Ex: $\int_0^{\frac{\pi}{6}} \sin t\,dt = -\cos\frac{\pi}{6} + \cos 0 = -\frac{\sqrt{3}}{2} + 1 = 0.134$

$f(x) = \int_0^x h(x)$

$f'(x) = h(x)$

Ex: $\int_0^1 t^2\,dt : \frac{t^3}{3} : \frac{(1)^3}{3} = \frac{1}{3}$

PROPERTIES OF DEFINITE INTEGRALS:

$\int_a^b f(x)\,dx = M$ & $\int_b^c f(x)\,dx = N$

following properties:

1: $k\int_a^b f(x)\,dx = kM = kM$

2: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = M+N$

3: $\int_b^a f(x)\,dx = -M$

4: $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Ex: $\int_2^{20} f(x)\,dx = 45$, $\int_2^5 f(x)\,dx = 12$, $\int_5^{14} f(x)\,dx = 20$

$\int_{14}^{20} f(x)\,dx = 45 - 12 - 20 = \boxed{13}$

$\boxed{\text{FTC: } \int_a^b f(x)\,dx = F(b) - F(a),\ F(x) \text{ is anti-deriv of } f(x)}$

Ex: $\int_1^3 (x^2+2)\,dx$

$\frac{x^3}{3} + 2x = \left(\frac{27}{3}+6\right) - \left(\frac{1}{3}+2\right)$

$15 - \frac{1}{3} = \frac{45}{3} - \frac{7}{3} = \boxed{\frac{38}{3}}$

Anti-derivatives:

- anti-deriv of f(x) written w/ capital (F(x))
- anti-deriv of any func, add +C since deriv of constant = 0
- add dx or dy!

Power Rule: $f(x) = x^n$ then $\int f(x)\,dx = \frac{x^{n+1}}{n+1} + C$

- Ex: $\int x^3\,dx = \frac{x^4}{4} + C$
- Ex: $\int x^{-3}\,dx = \frac{x^{-2}}{-2} + C$

More Rules:

$\int k f(x)\,dx = k\int f(x)\,dx$

$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

$\int k\,dx = kx + C$

Ex: $\int (3x^2+2x)\,dy = x^3 + x^2 + C$

Integrals of Trig Functions:

$\int \sin 5x\,dx = \frac{-\cos 5x}{5} + C$

$\int \sec^2 3x\,dx : \frac{\tan 3x}{3} + C$

RULE: $\int \ast\_\,ax = \frac{\_\,ax}{a}$

Adding & Subtracting:

$\int x^3 + x^2 - x = \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + C$

Ex: Find equation of y where $\frac{dy}{dx} = 3x+5$ & $y=6$ when $x=0$

$\int 3x+5\,dx \quad \frac{3x^2}{2} + 5x + C = y$

* solve 4 c by plugging in x & y

$c = 6$

$\boxed{y = \frac{3x^2}{2} + 5x + 6}$

Ex: find f(x) if $f'(x) = \sin x - \cos x$ & $(\pi, 3)$

$\int(\sin x - \cos x)\,dx = -\cos x - \sin x + C = y$

$3 = -\cos\pi - \sin\pi + C$

$3 = 1 - 0 + C$

$2 = C$

$-\cos x - \sin x + 2 = f(x)$

Ex: eval $\int 4\sin x - 3\cos x\,dx$

$-4\cos x - 3\sin x + C$

U-Sub: $\int u^n\,du = \frac{u^{n+1}}{n+1} + C$

if smth can't cancel w/ du rewrite in terms of u using u=

$\int (x-4)^{10}$ $\quad u = x-4$

$\int u^{10}\,du$ $\quad \frac{du}{dx} = 1$ $\quad dx = \frac{du}{1}$

$\frac{u^{11}}{11} + C$

$\frac{(x-4)^{11}}{11} + C$

$\int 2x\sqrt{x^2-5}\,dx$ $\quad u = x^2-5$ $\quad \frac{du}{dx} = 2x$ $\quad dx = \frac{du}{2x}$

$\int \sqrt{u}\,du$

$\frac{2}{3}u^{\frac{3}{2}} + C$

$\frac{2}{3}(x^2-5) + C$

$\int 3\sin(3x-1)\,dx$ $\quad u = 3x-1$ $\quad \frac{du}{dx} = 3$ $\quad dx = \frac{du}{3}$

$\int \sin u\,dx$

$-\cos(3x-1) + C$

$\int (5x+7)^{20}\,dx$ $\quad u = 5x+7$ $\quad \frac{du}{dx} = 5$ $\quad dx = \frac{du}{5}$

$\frac{1}{5}\int u^{20}\,du$

$\frac{1}{5}\ \frac{u^{21}}{21} = \boxed{\frac{(5x+7)^{21}}{105} + C}$

Substitution w/ Inverse Trig Functions:

derivatives of inverse trig: anti-derivs:

$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ $\quad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ $\quad \int \frac{dx}{1+x^2} = \tan^{-1}x + C$

$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{\ }$ $\quad \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + C$

Ex: Eval $\int \frac{8\,dx}{1+4x^2} = 8\int \frac{1}{1+(2x)^2}\,dx$ $\quad u = 2x$ $\quad \frac{du}{dx} = 2$ $\quad dx = \frac{du}{2}$

$\frac{1}{2}(8\tan^{-1}(2x) + C)$

$\boxed{4\tan^{-1}(2x) + C}$

#2: Eval: $\int e^x \frac{1}{1+(e^x)^2}\,dx$ $\quad u = e^x$

Ex 3: Eval: $\int \frac{1}{x^2+6x+10}\,dx$

$b - a = 4$ & $a = 6$ so $b = 10$

$\boxed{\int_6^{10} (x^2 - x)\,dx}$

DIFFERENTIAL EQ:

Ex: given $f'(x) = 6t^2 - 2t + 7$, verify that $y = 2t^3 - t^2 + 7t + 6$ is a solution

find deriv. & plug in for deriv.

$6t^2 - 2t + 7$

$f(x) = 2t^3 - t^2 + 7t + C$ ← general solution

$f(x) = 2t^3 - t^2 + 7t + 5$ ← particular solu.

Ex2: verify $y = e^{-4x}$ is solution to $\frac{dy}{dx} + 4y = 0$

1. find deriv of solution
2. plug in for $\frac{dy}{dx}$
3. plug in y

$y' = e^{-4x}(-4)$ $\quad \frac{dy}{dx}$ $\quad y$

$\underbrace{\frac{dy}{dx} + 4y}_{0} = \underbrace{-4e^{-4x} + 4e^{-4x}}_{0} = 0$

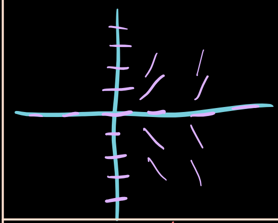
$y' = e^{-4x}(-4)$

$\frac{dy}{dx} + 4y = -4e^{-4x} + 4e^{-4x} =$

SLOPE FIELDS:

given $\frac{dy}{dx} = xy$, slope field

• slope fields resemble their functions



General Solutions w/ Separation of Variables

$\frac{dy}{dx} = \frac{5x^2}{6y^2}$

$\int 6y^2\,dy = \int 5x^2\,dx$

$2y^3 = \frac{5x^3}{3} + C$

$y^3 = \frac{5x^3}{6} + C$

$y = \left(\frac{5x^3}{6} + \frac{C}{2}\right)^{\frac{1}{3}}$

Ex2: $\frac{dy}{dx} = 6\cos 2x$

$dy = \int 6\cos 2x$

$= 3\sin 2x +$

* ANYTIME $\frac{dy}{dx}$ & y are on opp. sides of equation

Ex 3: $\frac{dy}{dx} = xy^2$

$\int y^{-2}\,dy = \int x\,dx$

$-\frac{1}{y} = \frac{x^2}{2} + C$

$-y = \frac{2}{x^2+C}$

$y = -\frac{2}{x^2+C}$

Particular solutions using Initial Conditions & Sep. of variables

$\frac{dy}{dx} = \frac{4x}{y}$ & $(0,5)$, find eq. 4 y in terms of x

$\int y\,dy = \int 4x\,dx$

$\frac{y^2}{2} = 2x^2 + C$

$\frac{y^2}{2} = 2x^2 + \frac{25}{2}$

$$\boxed{\begin{aligned} \ln y &= x^3 + C \\ y &= e^{x^3+C} \\ y &= Ce^{x^3} \end{aligned}}$$

Ex: $a(t) = -32$ ft/sec^2

$v(t) = 64$ ft/sec, $x(0) = 32$

a) $v(t)$: $-32t + 64 = v$

$\int -32$

$-32t + C = v$

$-32(0) + C = 64$

$C = 64$

b) $x(t) = -16t^2 + 64t + 32$

$\int -32t + 64$

c) max height = 96 ft

$-32t = -64$

$t = 2$

$-64 + 128 + 32 = 96$

Ex: $\frac{dy}{dx} = \frac{y^2}{x}$ $\quad \left(1, \frac{1}{3}\right)$

$\int y^{-2} = \int x^{-1}\,dx$

$-\frac{1}{y} = \ln|x| + C$

$\boxed{y = -\frac{1}{\ln x - 3}}$

$\frac{1}{3} = -\frac{1}{\ln 1 + C}$

Example 14: The rate of growth in the number of bacteria in a petri dish is proportional to the number in the dish at any time. Initially, there are 100 bacteria. Two hours later, there are 160 bacteria. Find an equation for the number of bacteria, B, in the dish at time t, where t is in hours.

$\frac{dB}{dx} = kB$

$\ln B = kt + C$

$B = Ce^{kt}$

$B = 100e^{kt}$

$160 = 100e^{k2}$

$\boxed{B = 100e^{\frac{1}{2}\ln\frac{8}{5}t}}$

1. $\frac{dB}{dt} = kB$
2. differentiate
3. set C = initial
4. solve for K w/ (x, y)
5. new eq

$\boxed{\tan^{-1}(e^x)+C}$

$\frac{du}{dx} = e^x$ $dx = \frac{du}{e^x}$

$x^2+6x+9+1$ $(x+3)^2+1$

$\frac{1}{(u^2)+1} = \boxed{\tan^{-1}(x+3)+C}$

du $dx = \frac{du}{1}$

* inverse trig if denominator has quadratic that is not easily factored

MORE RULES!

$\int \frac{du}{u} = \ln|u|+C$

Ex: $\int \frac{2x}{x^2+1}\,dx$ $u = x^2+1$ $\frac{du}{dx} = 2x$ $dx = \frac{du}{2x}$

$\int \frac{1}{u}\,dx$

$\boxed{\ln(x^2+1)+C}$

PULL OUT COEFFICENT!

Derivatives or Differentiation Formulas	Antiderivatives or Integration Formulas
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

Ex: $\int \tan x\,dx$

$\int \frac{\sin x}{\cos x}\,dx$ $u = \cos x$ $dx = \frac{du}{-\sin x}$

$\int -\frac{1}{u}\,dx$

$-\ln(\cos x)+C$

Ex: $\int \sec x\,dx \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$

$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$ $u = \sec x + \tan x$ $dx = \frac{du}{\sec x \tan x + \sec^2 x}$

$\int \frac{1}{u}\,dx$

$\boxed{\ln(\sec x + \tan x)+C}$

$\int e^u\,du = e^u + C$

Ex: $\int e^{7x} = \frac{1}{7}e^{7x} + C$ $u = 7x$ $dx = \frac{du}{7}$

$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

Ex: $\int x e^{3x^2+1}\,dx$ $u = 3x^2+1$ $\frac{du}{dx} = 6x$ $dx = \frac{du}{6x}$

$\int x e^u \frac{1}{6x}\,du$

$\frac{1}{6}e^{3x^2+1}+C$

$\int a^u\,du = \frac{1}{\ln a}a^u + C$

Ex: $\int 5^x\,dx \rightarrow \frac{1}{\ln 5}5^x + C$

IF U-SUB, YOU CAN ONLY U SUB ONE! JUST CANCEL OUT THE OTHER

#1: $\int \frac{3x^2}{x^3-1}\,dx$ $u = x^3-1$ $\frac{du}{dx} = 3x^2$ $dx = \frac{du}{3x^2}$

$\ln|x^3-1|+C$

abs. value

#2: $\int 2^{3x}\,dx$ $u = 3x$ $dx = \frac{du}{3}$

$\frac{1}{\ln 2}2^{3x}+C$

$\frac{2^{3x}}{3\ln 2}+C$

Integrating w/ Long Division & Completing the Square

#1: $\int \frac{4x^2-14x+9}{x-3} \xrightarrow{LD} \int 4x-2+\frac{3}{x-3} \rightarrow 2x^2-2x+3\ln|x-3|+C$

#2: $4\int \frac{1}{x^2-4x+4-4+5} = 4\int \frac{1}{(x-2)^2+1}$ $u = x-2$

$\boxed{4\tan^{-1}(x-2)+C}$

#3: $8\int \frac{1}{\sqrt{-8+6x-x^2}}\,dx$

$8\int \frac{1}{\sqrt{-x^2+6x-8}} = 8\int \frac{1}{-\sqrt{x^2-6x+9-9+8}} = 8\int \frac{1}{\sqrt{1-(x-3)^2}} = 8\sin^{-1}(x-3)+C$

Ways to find anti-d:
1: FTC
2: trig
3: u-sub
4: inv. trig
5: $\ln|u|$
6: long div.
7: comp. the square

Ex: $\int (x-3)\sqrt{4x^2-24x+7}\,dx$ $u = 4x^2-24x+7$ $\frac{du}{dx} = 8x-24$ $dx = \frac{du}{8x-24}$

$\int (x-3)\sqrt{u}\;\frac{1}{8}\left(\frac{1}{x-3}\right)dx$

$\frac{1}{8}\int \sqrt{u} = \frac{1}{12}u^{\frac{3}{2}} = \frac{1}{2}(4x^2-24x+7)^{\frac{3}{2}}+C$

Ex: $\int (\tan^5\theta \sec^4\theta)\,d\theta$ $u = \tan\theta$ $du = \sec^2\theta$

$\int \tan^5\theta \sec^2\theta \sec^2\theta$

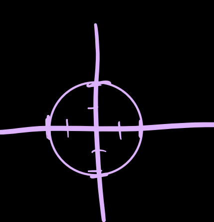
$\int \tan^5\theta(1+\tan^2\theta)\sec^2\theta$

$\int u^5 + (1+u^2) \Rightarrow \int u^5 + u^7 = \frac{u^6}{6} + \frac{u^8}{8} + C$

$\boxed{\frac{\tan^6\theta}{6} + \frac{\tan^8\theta}{8} + C}$

VOLUME W/ CROSS-SECTIONS:

$x^2+y^2=4$, crosssec = squares perp to x-axis $= V = \int a^2$



$\int_{-2}^{2} \left(2\sqrt{4-x^2}\right)^2 dx = \frac{128}{3}$

Steps:
1. find side of cross-sec
2. plug side into eq.
4 area of cross-sec
3. integrate w/ bounds

Cross Sec Area:
square: s^2
rectangle: ks^2
equilateral tri: $\frac{\sqrt{3}}{4}s^2$
semicircle: $\frac{\pi}{8}s^2$
hyp of isosceles right tri: $s^2/4$
isos right tri: $s^2/2$

$y^2 = 4-x^2$

$y = \sqrt{4-x^2}$

Ex 2: $y = 20-x^2$, rectangles, $h = 2b$

$B = 20-x^2 \cdot h = 40-2x^2 = 800-80x^2+2x^4$

$\int_0^{\sqrt{20}} 2(20-x^2)^2\,dx$

$\frac{8}{5} = e^{2k}$

$\ln\frac{8}{5} = 2k$

$k = \frac{1}{2}\ln\frac{8}{5}$

Example 15: A capacitor is fully charged at 1,000 millifarads. After 5 milliseconds, it only has 10 millifarads left. If the amount of charge in the capacitor, C, is proportional to the amount at time t, where t is measured in milliseconds, find (a) an equation for the charge in terms of time and (b) the amount of charge at 8 milliseconds.

$\frac{dy}{dt} = ky$

$y = Ce^{kt}$

$y = 1{,}000e^{kt}$

$10 = 1{,}000e^{5t}$

$\ln 0.01 = 5t$

$\frac{\ln 0.01}{5} = t$

a) $C = 1000e^{\frac{1}{5}\ln 100\,t}$

b) $C = 1000e^{\frac{1}{5}\ln 800}$

if eq. in form of
* $\frac{dy}{dt} = ky \rightarrow y = Ce^{kt}$
* $y(0) = A \rightarrow y = Ae^{kt}$
* if given two (x,y) use both to solve y & k & set equal to find k

Average Value of Function:
- MVT applies
- area under $f(x)$ on $[a,b]$ = to value of function @ some value $c \cdot$ length of interval
- finding area of rec w/ base of interval and height $f(c)$

$\boxed{f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx}$

Ex: avg value of $f(x)=x^2$ on $[2,4]$

$\frac{1}{2}\int_2^4 x^2\,dx$

$\frac{1}{2}\left[\frac{x^3}{3}\right]_2^4 = \frac{1}{2}\left(\frac{64}{3}-\frac{8}{3}\right) = \boxed{\frac{28}{3}}$

Ex 2: avg val. of $f(x)=\sin x$ on $(0,\pi]$

$\frac{1}{\pi}\int_0^\pi \sin x\,dx$

$\frac{1}{\pi}((-\cos\pi)-(-\cos 0))$

$= \frac{1}{\pi}(1+1) = \boxed{\frac{2}{\pi}\ v^2}$

POSITION, V, & A W/ INTEGRALS

$\int a(t)\,dt = v(t)+v_0$

$\int v(t)\,dt = s(t)+s_0$

Ex: $s_0 = 120$ $a(t) = -10$ $v_0 = -8$

$v(t) = -10x-8$

$s(t) = -5x^2-8x+120$

Ex 2: $a(t) = 2t-8$ $v(0) = 4$

$t^2-8t+4 = v(t)$

$x(t) = \frac{t^3}{3} - 4t^2 + 4t$

ACCUMULATION INTEGRALS:

$\frac{dv}{dt} = -t^2+14t$ initially 50, rate, how much after 12 hrs?

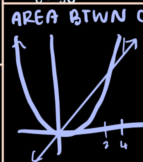
$50+\int_0^{12} -t^2+14t = 50+\left[-\frac{t^3}{3}+7t^2\right]_0^{12} = -\frac{12^3}{3}+7(144) = 432+50$

$= \boxed{482}$

rate added: $48t$
rate leaking: $\frac{dw}{dt} = e^{0.03t}$
initially: 120 gal
$t = 30$

$120+48t-\int_0^{30} e^{0.03t}\,dt = \boxed{1{,}511.347}$

AREA BTWN CURVES:



$\int_2^4 (x^2-x)\,dx$ ← top - bottom b/c perp to x-axis

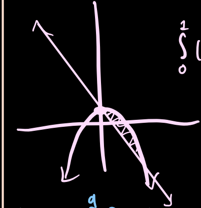
x Rule: $\int_a^b [f(x)-g(x)]\,dx$

* integrate area to find volume

#2: $y = 1-x^2$ & $y = 1-x$

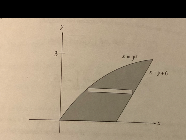
$1-x^2 = 1-x$
x^2-x
$x(x-1)$ $x = 0, 1$

$\int_0^1 (1-x^2)-(1-x)\,dx$



Steps:
1. find intersection pnts by setting =
2. set up integral w/ bounds & > - <
3. solve
* if t&b/l&r switch, 2 integrals needed

y rule $\int_c^d [f(y)-g(y)]\,dy$



When you slice up the area horizontally, the right end of each section is the curve $x = y + 6$, and the left end of each section is always the curve $x = y^2$. Now set up the integral.

$$\int_0^3 (y+6-y^2)\,dy$$

Evaluating this gives us the area.

$$\int_0^3 (y+6-y^2)\,dy = \left(\frac{y^2}{2}+6y-\frac{y^3}{3}\right)\Big|_0^3 = \frac{27}{2}$$

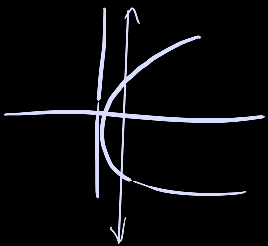
Ex: $y = \sqrt{x+3}$ & $y = \sqrt{3-x}$ $(-3,3)$

slice x:

0 3

DISC METHOD:

$y=\sqrt{x}$, $y=0$, $x=0$, $x=1$



$\pi\int_0^1 (\sqrt{x})^2\,dx = \frac{\pi}{2}$

$$\boxed{\pi\int_a^b (\text{radius})^2\,dx}$$

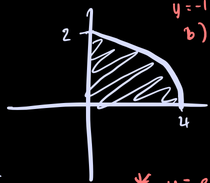
Ex: $y=x$, $x=0$, $x=1$

$\pi\int_0^1 x^2\,dx = \frac{\pi}{3}$

Ex 2:

$x=4-y^2$, 1st quad,

$\pi\int_0^2 (4-y^2)^2\,dy = \frac{256\pi}{15}$



what if axis of rot. was $y=-2$?

$\pi\int_0^2 (4-y^2+2)^2\,dy = \frac{256\pi}{15}$

* radius inc by 2 so add 2

* $y=\cos x$ $x=\frac{\pi}{2}$

x axis

a) $\pi\int_0^{\frac{\pi}{2}} (\cos x)^2\,dx$

$y=-1$

b) $\pi\int_0^{\frac{\pi}{2}} (\cos x+1)^2\,dx$

* $y=e^x$, $y=\ln 3$

$\ln y = x$

y axis

a) $\pi\int_0^{\ln 3} (\ln y)^2\,dy$

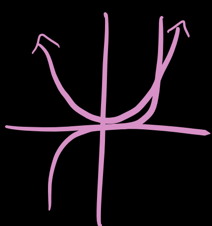
$x=-2$

b) $\pi\int_0^{\ln 3} (\ln y+2)^2\,dy$

WASHER METHOD:

$y=x^3$ $y=x^2$ $(2,4)$

$\pi\int_2^4 (x^3-x^2)^2\,dx$



outer radius: x^3

inner radius: x^2

$\pi\int (R^2-r^2)$

$\pi\int_a^b [f(x)^2-g(x)^2]\,dx$

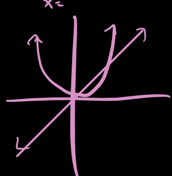
Ex: $y=x$, $y=x^2$, $(0,1)$

x: $\pi\int_0^1 (x-x^2)^2\,dx$

y: $\pi\int_0^1 (\sqrt{y}-y)^2\,dy$

$x=x^2$ $x(x-1)$

$x=$



$\sqrt{y}=y$

$y=y^2$

y^2-y

Ex: $y=x^2$ $y=4x$, rev arnd $y=-2$

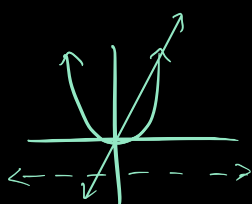
$\pi\int_0^4 (4x+2)^2-(x^2+2)^2$

$x^2=4x$

x^2-4x

$x(x-4)$

$x=0,4$



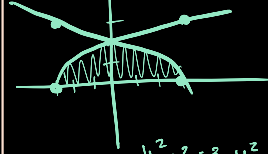
* if radius ↑ , add to radii

* if radius ↓ , subtract from radii

* have to integ. for each

Steps:

1. draw pic
2. slice vert or horz
3. intersec pnts
4. integral w/ bounds & t-b or r-l



$\int_{-3} \sqrt{x+3}\,dx + \int_0 \sqrt{3-x}\,dx = 4\sqrt{3}\,u^2$

Slicey:

$\int_0^{\sqrt{3}} (3-y^2)-(y^2-3)\,dy$ ← right - left

$=4\sqrt{3}$

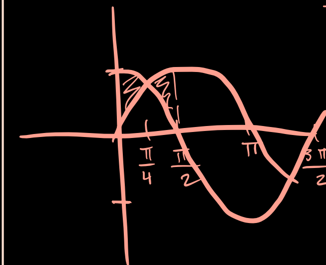
$y^2-3=3-y^2$

$y^2-3=x$ $2y^2-6$

$3-y^2=x$ $2(y^2-3)$

$x=0,\sqrt{3}$

Ex: $y=\sin x$, $y=\cos x$ $(0,\frac{\pi}{2})$



$\int_0^{\frac{\pi}{4}} \cos x-\sin x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x-\cos x\,dx = 2\sqrt{2}-2$