Improper Integral: one of endpoints is ∞ instead of constant
- instead take limit as b goes to ∞

$$\int_2^\infty x^{-2}\,dx = \lim_{b\to\infty}\left[-x^{-1}\right]_2^b = \lim_{b\to\infty}\left(\underbrace{-b^{-1}}_{\text{approaches } 0}\right)-\left(-\tfrac{1}{2}\right) = -\left(-\tfrac{1}{2}\right)=\tfrac{1}{2}$$

$$\int_2^\infty e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_2^b = -e^{-b}+e^{-2} = e^{-2}$$

$$\int_4^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln x\right]_4^b = \lim_{b\to\infty}(\ln b)-(\ln 4) \to \infty \text{ (diverges)}$$

Example where the integrand has an infinity discontinuity:

$$\int_0^1 \frac{dx}{\sqrt[4]{x}} = \int_0^1 x^{-1/4}\,dx = \lim_{b\to 0^+}\left[\tfrac{4}{3}x^{3/4}\right]_b^1 = \tfrac{4}{3}(1)^{3/4} - \lim_{b\to 0^+}\tfrac{4}{3}b^{1/4} = \tfrac{4}{3}$$

☆ the function doesn't exist @ x=0, so create b & let b appr. 0

$$\int_0^\pi \frac{x\cos x - \sin x}{x^2}\,dx$$

	u	dv
+	$x\cos x-\sin x$	x^{-2}
	$-x\sin x$	$-x^{-1}$

$$= \lim_{b\to 0^+}\left[\frac{\sin x - x\cos x}{x}\right]_b^\pi - \int_0^\pi \frac{x\sin x}{x}\,dx$$

$$\lim_{b\to 0^+}\left[\frac{\sin x}{x}\right]_b^\pi - \cancel{\lim_{b\to 0^+}\left[\cos x\right]_b^\pi} + \cancel{\lim_{b\to 0^+}\left[\cos x\right]_b^\pi}$$

$$\frac{\sin\pi}{\pi} - \lim_{b\to 0^+}\frac{\sin x}{x}$$

$$\frac{\sin\pi}{\pi} - \cos\pi$$

$$0-1=-1$$

Example where limit indeterminate

$$\int_1^\infty xe^{-x}\,dx$$

	u	dv
+	x	e^{-x}
-	1	$-e^{-x}$
+	0	e^{-x}

$$-xe^{-x}-e^{-x}+0$$

$$\lim_{b\to\infty}\left[-xe^{-x}\right]_1^b - \lim_{b\to\infty}\left[e^{-x}\right]_1^b + \int_1^\infty 0\,dx$$

$$= \left[\lim_{b\to\infty}\left(-\frac{b}{e^b}+\frac{1}{e^1}\right)\right] + \left[\cancel{-\lim_{b\to\infty}}\left(\frac{1}{e^b}+\frac{1}{e}\right)\right]$$

$$= \underbrace{-\lim_{b\to\infty}\frac{b}{e^b}}_{\text{indeterminate so L'Hopitals}} + \frac{2}{e} = \cancel{-\lim_{b\to\infty}\frac{1}{e^b}} + \frac{2}{e} = \frac{2}{e}$$

Sequences, series & partial sums

- series = sum of a sequence
- partial sum $S_4 = u_1+u_2+u_3+u_4$

Example: $u_n = 2^{-n}$ for $n\geq 1$

- $S_n = \frac{2n-1}{2^n}$

Example: expression for the nth partial sum of $\sum_{k=1}^{\infty} -3^{-2k}$

$$\underbrace{S_1 = \frac{-1}{9} \quad S_2 = \frac{-1}{9}+\frac{-1}{81} = \frac{-10}{81}}_{\text{geometric series}}$$

$$S_n = \frac{\frac{-1}{9}\left(1-\left(\frac{1}{9}\right)^n\right)}{1-\frac{1}{9}} \longleftarrow a=-\tfrac{1}{9} \quad r=\tfrac{1}{9}$$

$$S_n = \frac{9^{-n}-1}{8}$$

if we already have the partial sums:

$u_1 = S_1$
$u_2 = S_2 - S_1$
$u_3 = S_3 - S_2$

$u_k = S_k - S_{k-1}$

Example: $S_n = 3n^2-1$

$u_1 = 2$

$u_k = 3k^2-1$

$= (3k^2-1)-(3(k-1)^2-1)$

$= \cancel{3k^2}\cancel{-1} - \cancel{3k^2}+6k-3\cancel{+1}$

$u_k = 6k-3 \quad k\geq 2$ ← first term

☆ Example: $S_n = \ln n$

$u_1 = \ln 1 = 0$

$u_k = \ln k$

$= \ln k - \ln(k-1)$

$= \ln\left(\frac{k}{k-1}\right), \ k\geq 0$

- series converges if add up ∞ number of terms in series & get finite sum $\left(3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}\ldots=6\right)$
- series diverges when sum keeps growing w/o bound

1) Geometric: $\sum_{n=1}^{\infty} ar^n$ or $\sum_{n=0}^{\infty} ar^n$ ⟶ SUM OF A GEOMETRIC SERIES: $\frac{a}{1-r}$

$\boxed{\text{- series converges if } -1<r<1 \text{ and diverges for } |r|\geq 1}$

Example: $1+1.1+1.21\ldots$

$\sum_{n=0}^{\infty}(1.1)^n$ → geometric series w/ r=1.1 → DIVERGES

Example: $0.2+0.04+0.008$

$\sum_{n=1}^{\infty}(0.2)^n$: CONVERGES → $\frac{0.2}{1-0.2} = \frac{1}{4} = \text{sum}$

2) P-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

$\boxed{\text{- series converges if } p>1 \text{ \& diverges for } p\leq 1}$

Example: $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}\ldots$

$\sum_{n=1}^{\infty} \frac{1}{n^3} \rightarrow p=3 \rightarrow$ CONVERGES

Example: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots$

$\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow p=1 \rightarrow$ DIVERGES

just show if it converges to finite #, not what # is

1) Geometric

ar^n $\quad -1<r\leq 1$ conv.

$|r|\geq 1$ div.

2) p series

$\frac{1}{n^p}$ $\quad p>1$ conv.

$p\leq 1$ div

3) nth term for divergence (only ∞ divergence)

if $\lim_{n\to\infty} a_n \neq 0$ or if limit DNE $\rightarrow \sum_{n=1}^{\infty} a_n =$ DIVERGENT

- if individual terms don't approach 0, sum can't

Example: show that $\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots$ is divergent

$\lim_{n\to\infty} a_n = 1 \neq 0 \rightarrow$ series is divergent

☆ inconclusive if $\lim_{n\to\infty} a_n = 0$

4) Direct Comparison

- given 2 positive series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that every $a_n \leq b_n$

- if $\sum_{n=1}^{\infty} b_n$ converges to a limit S, then $\sum_{n=1}^{\infty} a_n$ CONVERGES to a limit T where $T\leq S$

- if $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges

☆ have to pick other that is greater

Ex: show that $\sum_{n=1}^{\infty} \frac{1}{5^n+3}$

- compare to $\frac{1}{5^n}$

- $\frac{1}{5^n+3} \leq \frac{1}{5^n}$

larger denominator

- we know that $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges b/c geometric w/ $r<1$, so $\sum_{n=1}^{\infty} \frac{1}{5^n+3}$ CONVERGES TOO

Ex: show that $\sum_{n=1}^{\infty} \frac{2}{2n-1}$ diverges

- p test $=1$, DIVERGES, so $\sum_{n=1}^{\infty} \frac{2}{2n-1}$ DIVERGES

5) Limit Comparison:

- given two positive series $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ where $\lim_{n\to\infty} \frac{a_n}{b_n} = L>0$, both either C or D

Ex: $\sum_{n=1}^{\infty} \frac{1}{7^n-2}$ $\quad b_n = \frac{1}{7^n}$

$\lim_{n\to\infty} \frac{7^n}{7^n-2} = 1>0$

Both $\frac{1}{7^n-2}$ & $\frac{1}{7^n}$ CONVERGE

Ex: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n-3}}$ $\quad b = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} = \frac{\sqrt{n}}{\sqrt{4n-3}} = \frac{1}{2} > 0$

BOTH DIVERGE

$\frac{\infty}{4\infty-3}$

6) Ratio Test

- given a series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then series is convergent

>1, divergent

$=1$, ratio test inconclusive

☆ use when a_n has factorials or powers

Ex: show that $\sum_{n=1}^{\infty} \frac{2^n}{n!} = c$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\cancel{2^{n+1}}}{(n+1)!} \times \frac{n!}{\cancel{2^n}} = \frac{2\cancel{n!}}{(n+1)\cancel{n!}} = \frac{2}{n+1} = 0 \rightarrow$ convergent

Ex: $\sum_{n=1}^{\infty} \frac{n!}{n^3}$

$\lim_{n\to\infty} \frac{\cancel{(n+1)!}}{(n+1)^2} \cdot \frac{n^3}{\cancel{n!}}$

$\frac{n^3}{(n+1)^2} = \infty =$ DIVERGES

- Some series don't hv formula to calculate infinite sum S
- approx. ∞ sum by adding partial sum of first n terms S_n
  - partial sum off by amnt (error): $S - S_n$
  - maximize error → range of possibilities for infinite sum

7) Integral Test

- given continuous positive decreasing function $f(x)$, $x \geq 1$ & given the integral $\int_1^\infty f(x)\,dx$
  - a) if $\int f(x)\,dx$ converges then $\sum_{n=1}^\infty f(n)$ converges
  - b) if $\int_1^\infty f(x)\,dx$ diverges then $\sum_{n=1}^\infty f(n)$ diverges

Ex: Show that $\sum_{n=1}^\infty \frac{1}{n^2}\cos\left(\frac{1}{n}\right)$ converges

$$\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$$

$$\lim_{x\to\infty}\int_1^b \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$$ ← qualifies 4 integral test b/c cont., pos., & dec.

$$\lim_{b\to\infty}\left[-\sin\left(\frac{1}{x}\right)\right]_1^b$$

$u = x^2$
$\frac{du}{dx} = 2x$
$dx = \frac{du}{2x}$

$$-\sin\left(\frac{1}{b}\right) + \sin\left(\frac{1}{1}\right)$$

$-\sin 0 + \sin 1 = \sin 1$ → integral converges so series $\sum_{n=1}^\infty \frac{1}{n^2}\cos\left(\frac{1}{n}\right)$ converges

Ex: Harmonic series:

$\sum_{n=1}^\infty \frac{1}{n}$ diverges

$$\lim_{n\to\infty}\int_1^b \frac{1}{x}$$

$$\left[\ln x\right]_1^b$$

$\ln b - \ln 1$ → ∞ → integral diverges

Ex: $\sum_{n=1}^\infty \frac{1}{1+n^2}$ estimate sum by adding the first 3 terms

$$\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} = 0.8$$

- to maximize error: since stopped @ 3rd term → $\int_3^\infty \frac{1}{1+x^2}dx$
- SO: sum of series $\sum_{n=1}^\infty \frac{1}{1+n^2}$ btwn 0.8 & 1.122

$$\lim_{x\to\infty}\int_3^b [\arctan(x)]_3^b$$

$\arctan\infty - \arctan 3$

$\frac{\pi}{2} - \arctan 3 \approx 0.322$ ← error

8) Alternating Series

- if for an alternating series $\sum_{n=1}^\infty a_n$
  - a) $|a_{n+1}| < |a_n|$ for sufficiently large n
  - b) $\lim_{n\to\infty}|a_n| = 0$

  THEN SERIES IS CONVERGENT

Ex: show that $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$ converges

$|a_n| = \frac{1}{n}$ $\quad |a_{n+1}| = \frac{1}{n+1}$

$|a_{n+1}| < |a_n|$ for all positive integers n

so $|a_{n+1}| < |a_n|$ for all positive integers n

$\lim_{n\to\infty}\frac{1}{n} = 0$ → converges

ABSOLUTE VS. CONDITIONAL: a series $\sum_{n=1}^\infty a_n$ is absolutely convergent if series $\sum_{n=1}^\infty |a_n|$ is convergent

- Ex: $\sum_{n=1}^\infty \frac{(-1)^n}{n^2}$ is absolutely convergent bc the series $\sum_{n=1}^\infty \frac{1}{n^2}$ is convergent (p-series)

If a series $\sum_{n=1}^\infty a_n$ is convergent but the serie $\sum_{n=1}^\infty |a_n|$ is divergent, then we say the series is conditionally convergent

- Ex: $\sum_{n=1}^\infty \frac{(-1)^n}{n}$ is conditionally convergent b/c series $\sum_{n=1}^\infty \frac{1}{n}$ is not (harmonic), but series $\sum_{n=1}^\infty \frac{(-1)^n}{n}$ is (alternating) if $S=\sum_{n=1}^{\infty} a_n$ is the sum of an alternating series that satisfies $|a_{n+1}|<|a_n|$ for all n & $\lim_{n\to\infty}|a_n|=0$

↳ THEN: the error in taking the first n terms as an approx. to S (the truncation error) is less than the abs. value of the (n+1)th term

$|S-S_n|<|a_{n+1}|$

Ex: how many terms of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ are necessary to find approx. accurate to within 0.001

- alternating → converges absolutely

so $|S-S_n|<|a_{n+1}|=\frac{1}{(n+1)^2}$

thus require $\frac{1}{(n+1)^2}<0.001$

$n>30.62$

Therefore we need to take 31 terms

Ex: find range of values for the sum $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$ by using sum of the 1st three terms as approx.

1) check 4 convergence

$|a_{n+1}|<|a_n|$

$\frac{1}{(n+1)!}<\frac{1}{n!}$ YES

2) $\lim_{n\to\infty}|a_n|=0$?

↳ $\lim_{n\to\infty}\left|\frac{1}{n!}\right|=0$ YES

3) sum of first 3 term:

$\frac{-1}{1!}+\frac{1}{2!}+\frac{-1}{3!}=-\frac{2}{3}$

$-1+\frac{3}{6}-\frac{1}{6}$

4) error is 4th term

$a_4=\frac{(-1)^4}{4!}=\frac{1}{24}$

so sum S is: $-\frac{2}{3}<S<-\frac{2}{3}+\frac{1}{24}$