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Additional Mathematics

Val Hanrahan

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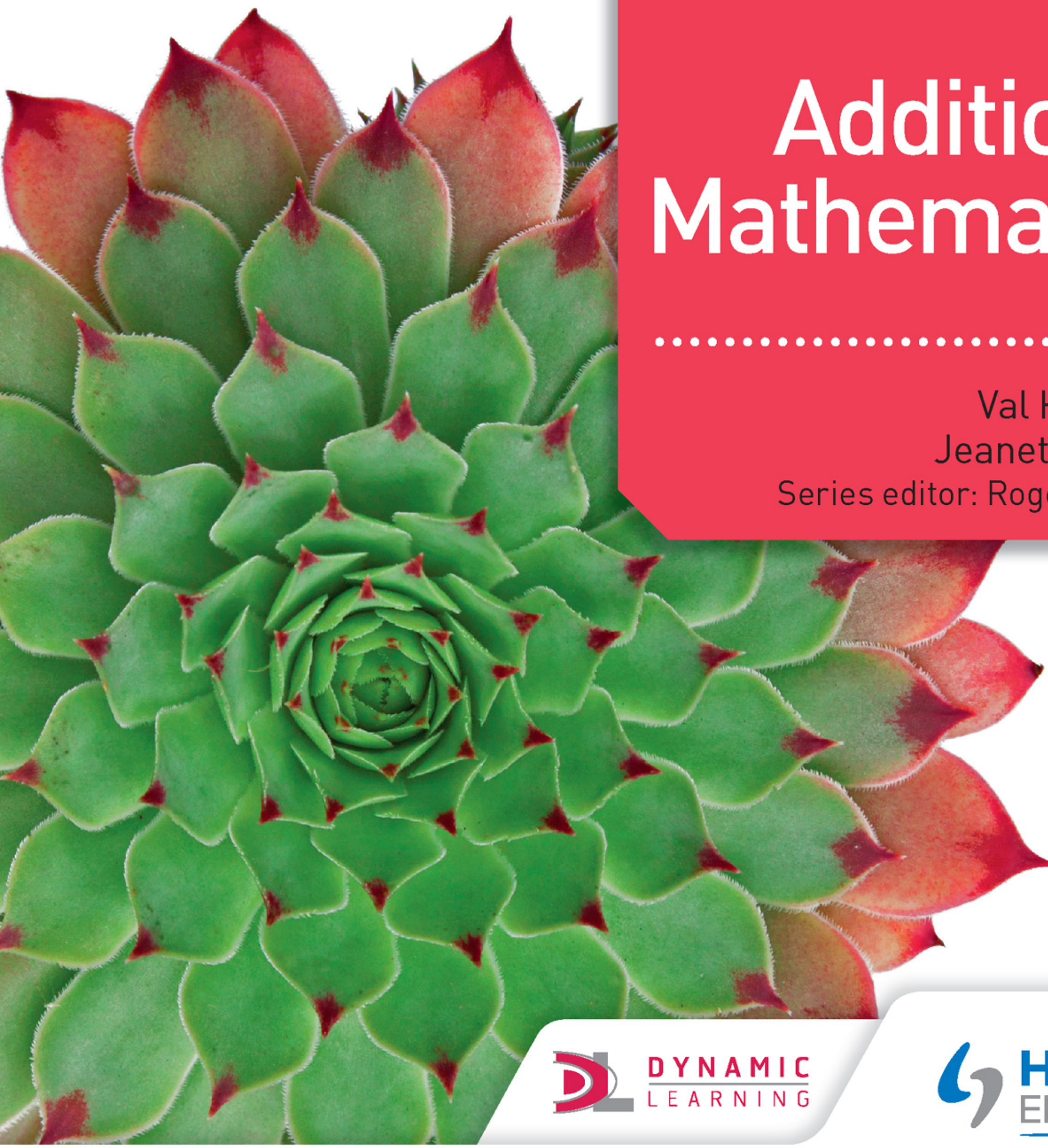

DYNAMIC LEARNING

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Contents

Introduction

This book has been written for all students of Cambridge IGCSE® and O Level Additional Mathematics syllabuses (0606/4037). It carefully and precisely follows the syllabus from Cambridge Assessment International Education. It provides the detail and guidance that are needed to support you throughout the course and help you to prepare for your examinations.

Prior knowledge

Throughout this book, it is assumed that readers are competent and fluent in the basic algebra that is covered in Cambridge IGCSE® / O Level Mathematics:

- working with expressions and formulae, simplifying and collecting like terms
- substituting numbers into algebraic expressions
- linear and quadratic factorisation and the use of brackets
- solving simple, simultaneous and quadratic equations
- working with inequalities
- changing the subject of a formula
- plotting and sketching graphs.

Chapter 1 Functions

As well as basic algebra, this chapter assumes a knowledge of graphs and the basic trigonometrical functions. This chapter introduces readers to the idea of a function, and the associated vocabulary; this pervades the rest of the book.

Chapter 2 Quadratic functions

In this chapter, readers are expected to be familiar with the basic algebra associated with quadratic functions, including drawing their graphs. Quadratic functions are used in many places in this book.

Chapter 3 Equations, inequalities and graphs

This chapter assumes that readers are competent in basic algebra. It builds on the work on the modulus function in Chapter 1. Chapter 2 included solving quadratic equations and this is now extended to solving cubic equations.

Chapter 4 Indices and surds

In this chapter readers are expected to be familiar with numbers, including squares, square roots and cubes. They are also expected to be familiar with techniques from basic algebra, especially working with brackets and collecting like terms.

Chapter 5 Factors of polynomials

This chapter builds on and starts to generalise ideas developed in the context of quadratic and cubic functions in Chapters 2 and 3. It requires competence in basic algebra and familiarity with curve sketching.

Chapter 6 Simultaneous equations

In this chapter readers solve simultaneous equations in two unknowns, including cases where one of the equations is quadratic and the other linear. It builds on work in Chapters 2 and 3, which in turn requires fluency in basic algebra.

Chapter 7 Logarithmic and exponential functions

This chapter develops new ideas based on the work on indices covered in Chapter 4.

Chapter 8 Straight line graphs

In this chapter, readers are required to use techniques relating to the graphs of straight lines that they covered in Cambridge IGCSE® / O Level, including drawing a straight line given its equation. They are expected to know that $y = mx + c$ is the equation of a straight line with gradient m that crosses the y-axis at $(0, c)$ and to be able to find the gradient of a given line. They should also be able to find the equation of a straight line given two points on it, or one point and its gradient.

Chapter 9 Circular measure

In this chapter, it is assumed that readers are familiar with the standard formulae for the area and circumference of a circle.

Chapter 10 Trigonometry

Readers are expected to understand the three basic trigonometrical functions, sin, cos and tan, and how they are used to find unknown sides and angles in right-angled triangles. They are expected to be able to work in radians, which are met in Chapter 8, as well as in degrees. This chapter involves drawing and transforming the graphs of the various trigonometrical functions, so it builds on ideas from Chapters 1, 2 and 3.

Chapter 11 Permutations and combinations

The work in this chapter is essentially new and no prior knowledge is required.

Chapter 12 Series

This chapter develops ideas about sequences and series. Readers who have prior knowledge of this topic will find it helpful but it is not essential. Much more important is fluency in basic algebra. Knowledge of the work in Chapter 4 on indices is expected.

Chapter 13 Vectors in two dimensions

Much of the work in this chapter will be new to readers. However, they are expected to be competent in basic trigonometry and also in translations, both from Cambridge IGCSE® / O Level and from Chapter 10.

Chapter 14 Differentiation

This is the first of three chapters on calculus. Readers are expected to be fluent with basic algebra from Cambridge IGCSE® / O Level and from the earlier chapters in this book. They are also expected to be familiar with work on straight line graphs from Chapter 8.

Chapter 15 Integration

This is the second chapter on calculus. It follows on from Chapter 14 and readers are expected to understand the ideas and techniques developed there.

Chapter 16 Kinematics

This chapter shows how the ideas in the two previous chapters on calculus can be applied to motion. So knowledge of both Chapters 14 and 15 is assumed. Readers are also expected to be familiar with Chapter 8 on straight line graphs.

How to use this book

To make your study of Additional Mathematics as rewarding and successful as possible, this Cambridge endorsed textbook offers the following important features:

Organisation

The content is generally in the same order as the syllabus, although the material within each chapter is presented in a natural teaching order to aid both teaching and learning. Where possible, the chapter titles and chapter section headings match those of the Cambridge IGCSE® and O Level Additional Mathematics syllabuses; however, the long, final section on calculus is split into three chapters: Differentiation, Integration and Kinematics, so that it is easily manageable for students.

Approach

Each chapter is broken down into several sections, with each section covering a single topic. Topics are introduced through clear **explanations**, with **key terms** picked out in bold type.

The modulus function

The **modulus** of a number is its positive value even when the number itself is negative.

The modulus is denoted by a vertical line on each side of the number and is sometimes called the **magnitude** of the quantity.

Worked examples

The worked examples cover important techniques and question styles. They are designed to reinforce the explanations, and give you step-by-step help for solving problems.

Worked example

Show that the two lines $y = \frac{1}{2}x - 4$ and $x - 2y - 6 = 0$ are parallel.

Solution

Start by rearranging the second equation into the form $y = mx + c$.

$$\begin{aligned} x - 2y - 6 = 0 &\Rightarrow x - 6 = 2y \\ &\Rightarrow 2y = x - 6 \\ &\Rightarrow y = \frac{1}{2}x - 3 \end{aligned}$$

Both lines have a gradient of $\frac{1}{2}$ so are parallel.

Exercises

These appear throughout the text, and allow you to apply what you have learned. There are plenty of routine questions covering important examination techniques.

Exercise 1.2

1. Given that $f(x) = 3x + 2$, $g(x) = x^2$ and $h(x) = 2x$, find:
   a $fg(2)$ **b** $fg(x)$ **c** $gh(x)$ **d** $fgh(x)$
2. Given that $f(x) = \sqrt{2x+1}$ and $g(x) = 4 - x$, find:
   a $fg(-4)$ **b** $gf(12)$ **c** $fg(x)$ **d** $gf(x)$
3. Given that $f(x) = x + 4$, $g(x) = 2x^2$ and $h(x) = \frac{1}{2x+1}$, find:
   a $f^2(x)$ **b** $g^2(x)$ **c** $h^2(x)$ **d** $hgf(x)$
4. For each function, find the inverse and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes. Use the same scale on both axes.
   a $f(x) = 3x - 1$ **b** $f(x) = x^3, x > 0$
5. Solve the following equations:
   a $|x - 3| = 4$ **b** $|2x + 1| = 7$ **c** $|3x - 2| = 5$ **d** $|x + 2| = 2$
6. Sketch the graph of each function:
   a $y = x + 2$ **b** $y = |x + 2|$ **c** $y = |x + 2| + 3$

Commentaries

The commentaries provide additional explanations and encourage full understanding of mathematical principles.

Worked example

Given that $f(x) = x^3 - x^2 - 4x + 12$

a Show that $(x - 3)$ is a factor of $f(x)$.

b Solve the equation $f(x) = 0$.

Solution

To show that $(x - 3)$ is a factor, you need to show that $f(3) = 0$.

a $f(3) = 3^3 - 3^2 - 9(3) + 9$

$= 27 - 9 - 27 + 9 = 0$

This shows that $(x - 3)$ is a factor of $f(x)$.

b Once you have found one linear factor of a cubic expression, the remaining factor is quadratic. With practice you will be able to do this step by inspection.

Check by multiplying out that you agree with this answer.

$x^3 - x^2 - 10x + 12 = (x - 3)(x^2 + 2x - 4)$

How do you know it can't be factorised?

$x^2 + 2x - 4$ cannot be factorised so use the quadratic formula for the next step.

$$ax^2 + bx + c = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this example, $a = 1$, $b = 2$, $c = -4 \rightarrow x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-4)}}{2(-1)}$

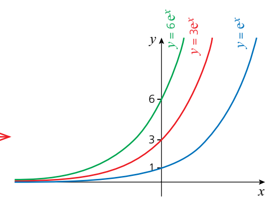
$$\rightarrow x = \frac{-2 \pm \sqrt{20}}{-2}$$

$$\rightarrow x = 1 \pm \sqrt{5}$$

The solution to the equation is therefore $x = 3$ or $x = 1 \pm \sqrt{5}$.

The family of curves $y = ke^x$, where k is a positive integer, is different set of transformations of the curve $y = e^x$. These represent stretches of the curve $y = e^x$ in the y-direction.

Notice that the curve $y = ke^x$ crosses the y-axis at $(0, k)$.



Similarly, for a fixed value of n, graphs of the family $y = ke^{nx}$ are represented by stretches of the graph $y = e^{nx}$ by scale factor k in the y-direction.

One additional transformation gives graphs of the form $y = ke^{nx} + a$.

Discussion point

Why is $\sqrt{4}$ not a surd?

Discussion point

This is the Singapore Flyer. It has a radius of 75 metres. It takes about 30 minutes to complete one rotation, travelling at a constant speed. How fast do the capsules travel?

Discussion points

These are points you should discuss in class with your teacher or fellow students, to encourage deeper exploration and mathematical communication.

Learning outcomes

Each chapter ends with a summary of the learning outcomes and a list of key points to confirm what you should have learned and understood.

Learning outcomes

Now you should be able to:

★ interpret the equation of a straight line graph in the form $y = mx + c$
★ solve questions involving midpoint and length of a line
★ know and use the condition for two lines to be parallel or perpendicular, including finding the equation of perpendicular bisectors
★ transform given relationships, including $y = ax^n$ and $y = Ab^x$, to straight line form and hence determine unknown constants by calculating the gradient or intercept of the transformed graph.

Key points

✔ An equation of the form $y = mx + c$ represents a straight line that has gradient m and intersects the y-axis at $(0, c)$.
✔ The midpoint of the line joining the points (x_1, y_1) and (x_2, y_2) is given by:

midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

✔ The length of the line joining the points (x_1, y_1) and (x_2, y_2) is given by:

length $= \sqrt{\left((x_2 - x_1)^2 + (y_2 - y_1)^2\right)}$.

✔ Two lines are parallel if they have the same gradient.
✔ Two lines are perpendicular if they intersect at an angle of 90°.
✔ When the gradients of two parallel lines are given by m_1 and m_2, $m_1 m_2 = -1$.
✔ Logarithms can be used to describe the relationship between two variables in the following cases:

i $y = ax^n$ Taking logs, $y = ax^n$ is equivalent to $\log y = \log a + n \log x$. Plotting $\log y$ against $\log x$ gives a straight line of gradient n that intersects the vertical axis at the point $(0, \log a)$.

ii $y = Ab^x$ Taking logs, $= Ab^x$ is equivalent to $\log y = \log A + x \log b$. Plotting $\log y$ against x gives a straight line of gradient $\log b$ that intersects the vertical axis at the point $(0, \log A)$.

Additional support

The Workbook provides additional practice for students. These write-in workbooks are designed to be used throughout the course.

Assessment

For both Cambridge IGCSE® and O Level Additional Mathematics you will take two examination papers; Paper 1 and Paper 2:

- 2 hours each
- 50% each
- scientific calculators are required.

Command words

Command word	What it means
Calculate	Work out from given facts, figures or information, generally using a calculator
Describe	State the points of a topic / give characteristics and main features
Determine	Establish with certainty
Explain	Set out purposes or reasons / make the relationships between things evident / provide why and / or how and support with relevant evidence
Give	Produce an answer from a given source or recall / memory
Plot	Mark point(s) on a graph
Show (that)	Provide structured evidence that leads to a given result
Sketch	Make a simple freehand drawing showing the key features
State	Express in clear terms
Verify	Confirm a given statement / result is true
Work out	Calculate from given facts, figures or information with or without the use of a calculator
Write	Give an answer in a specific form
Write down	Give an answer without significant working

From the authors

We very much hope you enjoy this book. It introduces you to some of the exciting ideas of mathematics. These will broaden your understanding of the subject and prove really helpful when you go on to further study. They include topics such as identities, vectors and particularly calculus; all of these are covered in the later chapters of the book. In order to handle such topics confidently, you will need to be fluent in algebra and numerical work and to be able to communicate the mathematics you are doing. The early chapters are designed to build on your previous experience in a way that develops these essential skills and at the same time expands the techniques you are able to use.

Val Hanrahan

Jeanette Powell

Roger Porkess

1 Functions

If A equals success, then the formula is A equals X plus Y plus Z, with X being work, Y play, and Z keeping your mouth shut.

Albert Einstein (1879 – 1955)

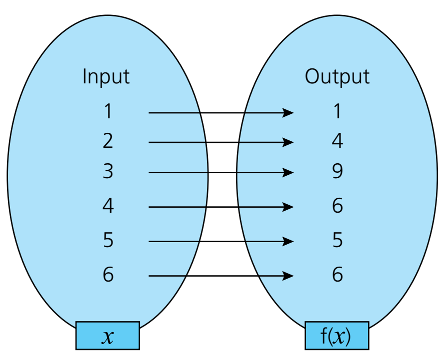
Discussion point

Look at the display on this fuel pump. One of the quantities is measured and one is calculated from it. Which is which?

You can think of the display as a mapping. Some of the values are shown below.

Amount of fuel (litres)	$\rightarrow$	Cost (\$)
1	$\rightarrow$	2.40
2	$\rightarrow$	4.80
3	$\rightarrow$	7.20
4	$\rightarrow$	9.60
5	$\rightarrow$	12.00
10	$\rightarrow$	24.00
50	$\rightarrow$	120.00
100	$\rightarrow$	240.00

If x is an element of the first set, then $f(x)$ denotes the associated element from the second set. For example, this mapping diagram shows integers mapped onto the final digit of their squares.



Discussion point

Which digits will never appear in the output set of the previous example?

A **function** is a rule that associates each element of one set (the **input**) with only one element of a second set (the **output**). It is possible for more than one input to have the same output, as shown above.

You can use a **flow chart** (or **number machine**) to express a function.

This flow chart shows a function, f, with two operations. The first operation is $\times 2$ and the second operation is $+3$.

Input $\longrightarrow$ $\boxed{\times 2}$ $\longrightarrow$ $\boxed{+3}$ $\longrightarrow$ Output

You can write the equation of a line in the form $y = 2x + 3$ using **function notation**.

Read this as 'f of x equals two x plus three'.

$$f(x) = 2x + 3$$

or

$$f: x \mapsto 2x + 3$$

Read this as 'f maps x onto two x plus three'.

Using this notation, you can write, for example:

$$f(4) = 2 \times 4 + 3 = 11$$

or

$$f:(-5) \mapsto 2 \times (-5) + 3 = -7$$

The domain and range

The **domain** of a function f(x) is the set of all possible inputs. This is the set of values of x that the function operates on. In the first mapping diagram of the next worked example, the domain is the first five positive odd numbers. If no domain is given, it is assumed to be all real values of x. This is often denoted by the letter $\mathbb{R}$.

Real numbers are all of the rational and irrational numbers.

The **range** of the function f(x) is all the possible output values, i.e. the corresponding values of f(x). It is sometimes called the **image set** and is controlled by the domain.

In certain functions one or more values must be excluded from the domain, as shown in the following example.

Worked example

For the function $f(x) = \frac{1}{2x+1}$:

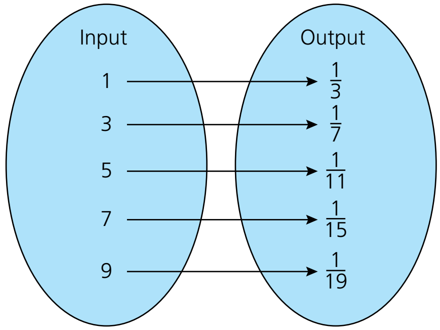
a Draw a mapping diagram showing the outputs for the set of inputs odd numbers from 1 to 9 inclusive.

b Draw a mapping diagram showing the outputs for the set of inputs even numbers from 2 to 10 inclusive.

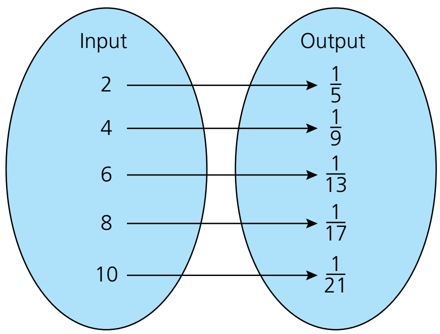
c Which number cannot be an input for this function?

Solution

a



b



c A fraction cannot have a denominator of 0, so $2x + 1 \neq 0$

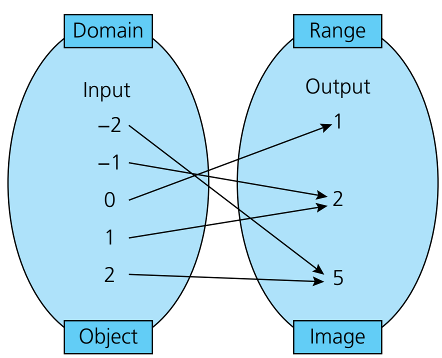
$\Rightarrow x = -\frac{1}{2}$ must be excluded.

Mappings

A mapping is the process of going from an object to its image.

For example, this mapping diagram shows the function $f(x) = x^2 + 1$ when the domain is the set of integers $-2 \leqslant x \leqslant 2$.

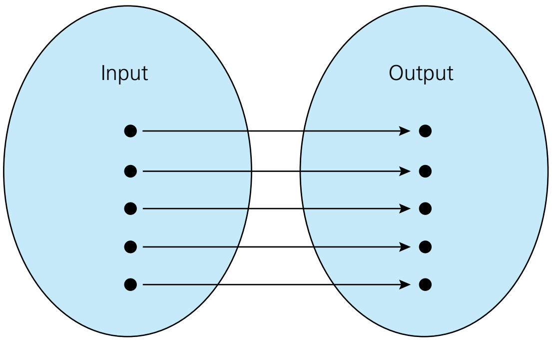
A mapping diagram is one way to illustrate a function.

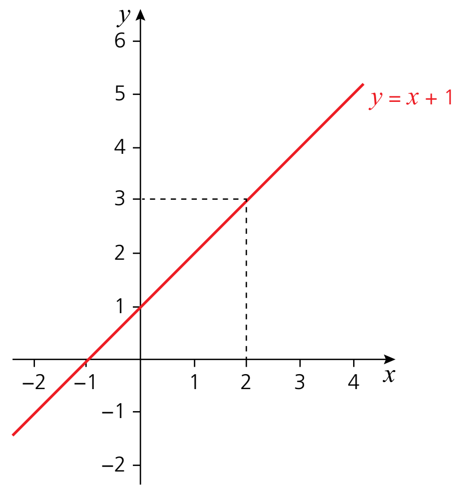


There are four different types of mappings.

One-one

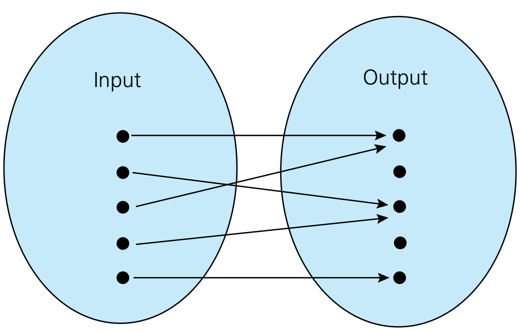
Every object has a unique image and every image comes from only one object.

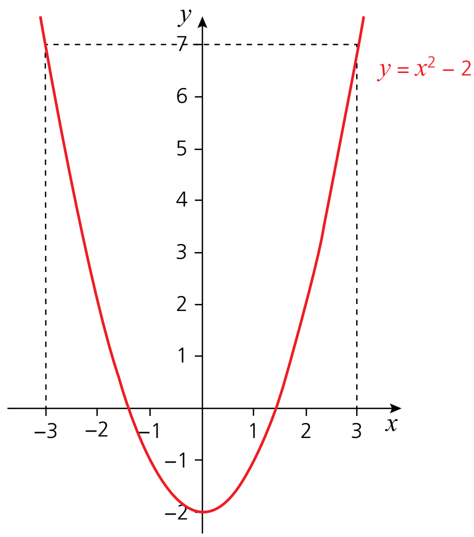




Many-one

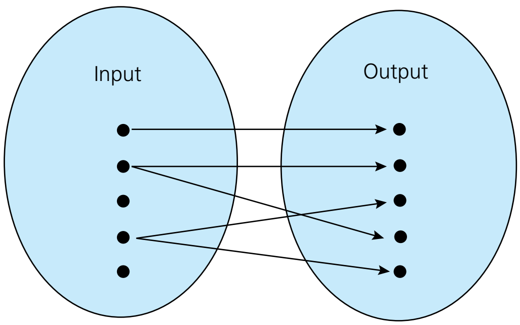
Every object has a unique image but at least one image corresponds to more than one object.

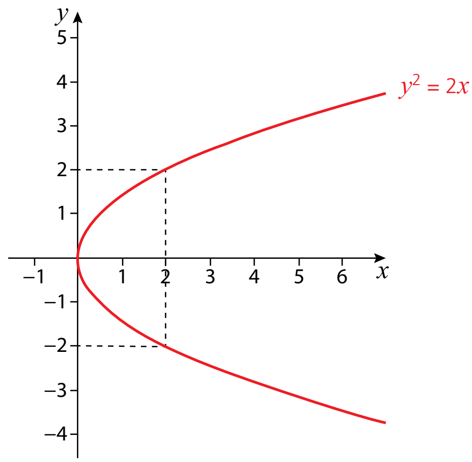




One-many

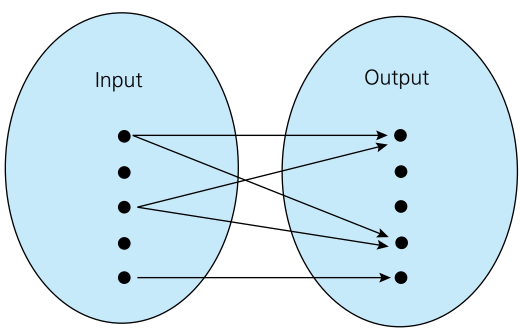
There is at least one object that has more than one image but every image comes from only one object.

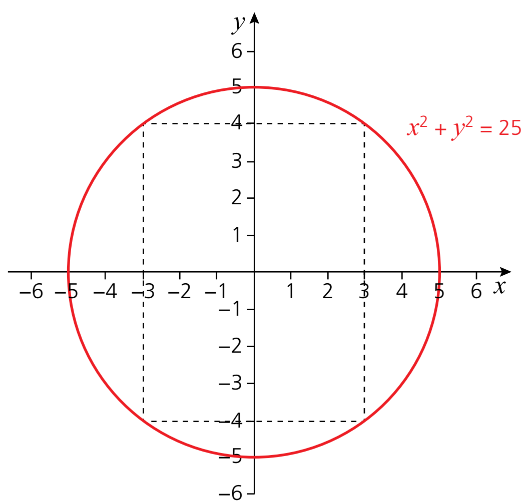




Many-many

There is at least one object that has more than one image and at least one image that corresponds to more than one object.





Types of function

A function is a mapping that is either one-one or many-one.

For a one-one function, the graph of y against x doesn't 'double back' on itself.

Below are some examples of one-one functions.

- All straight lines that are not parallel to either axis.
- Functions of the form $y = x^{2n+1}$ for integer values of n.
- Functions of the form $y = a^x$ for $a > 0$.
- $y = \cos x$ for $0° \leqslant x \leqslant 180°$.

These are examples of many-one functions:

- all quadratic curves,
- cubic equations with two turning points.

Worked example

Sketch each function and state whether it is one-one or many-one.

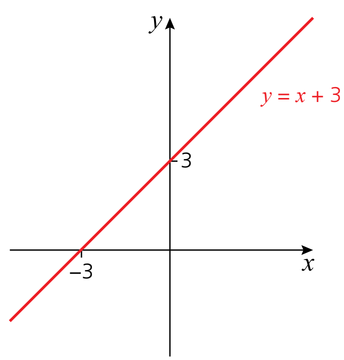
a $y = x + 3$ **b** $y = x^2 - 1$

Solution

a $y = x + 3$ is a straight line.

When $x = 0$, $y = 3$, so the point $(0, 3)$ is on the line.

When $y = 0$, $x = -3$, so the point $(-3, 0)$ is on the line.



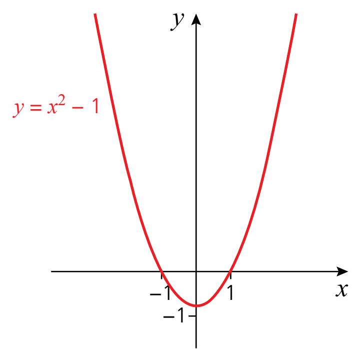
$y = x + 3$ is a one-one function.

b $y = x^2$ is a ∪-shaped curve through the origin.

$y = x^2 - 1$ is the same shape, but has been moved down one unit so crosses the y-axis at $(0, -1)$.

$y = x^2 - 1$ factorises to $y = (x + 1)(x - 1)$

$\Rightarrow$ When $y = 0$, $x = 1$ or $x = -1$.



$y = x^2 - 1$ is a many-one function since, for example, $y = 0$ corresponds to both $x = 1$ and $x = -1$.

Inverse function

The inverse function reverses the effect of the function. For example, if the function says 'double', the inverse says 'halve'; if the function says 'add 2', the inverse says 'subtract 2'. All one-one functions have an inverse; many-one functions do not.

Worked example

a Use a flow chart to find the inverse of the function $f(x) = \frac{3x+2}{2}$.

b Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes. Use the same scale on both axes.

c What do you notice?

Solution

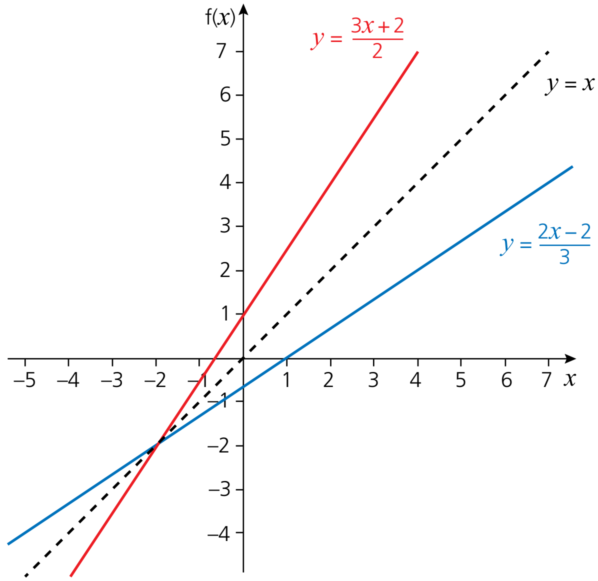
a For $f(x) = \frac{3x+2}{2}$:

$$\begin{array}{ccccccccc}
\text{Input} & \longrightarrow & \boxed{\times 3} & \longrightarrow & \boxed{+2} & \longrightarrow & \boxed{\div 2} & \longrightarrow & \text{Output} \\
x & \longrightarrow & 3x & \longrightarrow & 3x + 2 & \longrightarrow & \frac{3x+2}{2} & \longrightarrow & f(x)
\end{array}$$

Reversing these operations gives the inverse function.

$$\begin{array}{ccccccccc}
\text{Output} & \longleftarrow & \boxed{\div 3} & \longleftarrow & \boxed{-2} & \longleftarrow & \boxed{\times 2} & \longleftarrow & \text{Input} \\
f^{-1}(x) & \longleftarrow & \frac{2x-2}{3} & \longleftarrow & 2x - 2 & \longleftarrow & 2x & \longleftarrow & x
\end{array}$$

b



Reflecting in the line $y = x$ has the effect of switching the x- and y-coordinates.

c The graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ are reflections of each other in the line $y = x$.

An alternative method is to interchange the coordinates, since this gives a reflection in the line $y = x$, and then use an algebraic method to find the inverse as shown in the next example.

Worked example

a Find $\mathrm{g}^{-1}(x)$ when $\mathrm{g}(x) = \frac{x}{3} + 4$.

b Sketch $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$ on the same axes. Use the same scale on both axes.

Solution

a Let $y = \frac{x}{3} + 4$.

Interchange x and y. $\qquad x = \frac{y}{3} + 4$

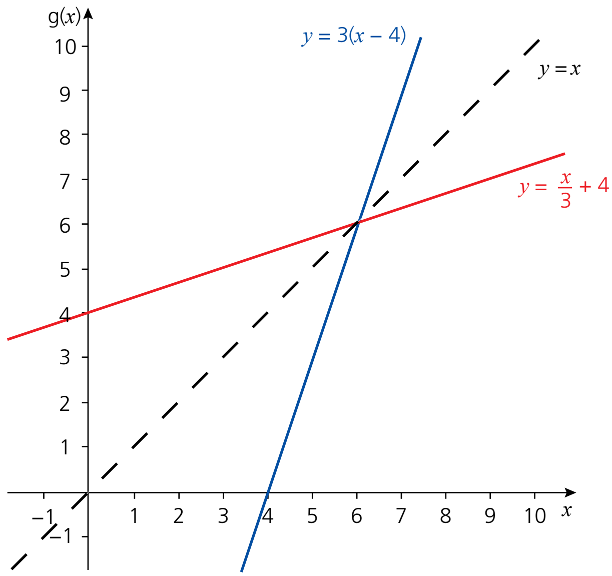
Rearrange to make y the subject. $\qquad x - 4 = \frac{y}{3}$

$$\Rightarrow y = 3(x - 4)$$

The inverse function is given by $\mathrm{g}^{-1}(x) = 3(x - 4)$.

Rearranging and interchanging x and y can be done in either order.

b



Worked example

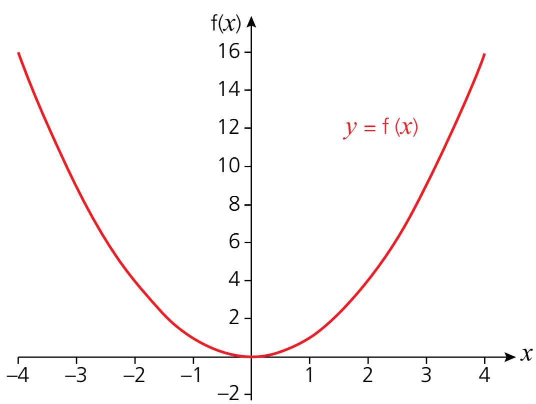
a Sketch the graph of the function $f(x) = x^2$ for $-4 \leqslant x \leqslant 4$.

b Explain, using an example, why $f(x)$ does not have an inverse with $-4 \leqslant x \leqslant 4$ as its domain.

c Suggest a suitable domain for $f(x)$ so that an inverse can be found.

Solution

a



b The function does not have an inverse with $-4 \leqslant x \leqslant 4$ as its domain because, for example, $f(2)$ and $f(-2)$ both equal 4. This means that if the function were reversed, there would be no unique value for 4 to return to. In other words, $f(x) = x^2$ is not a one-one function for $-4 \leqslant x \leqslant 4$.

c Any domain in which the function is one-one, for example, $0 \leqslant x \leqslant 4$.

Exercise 1.1

1 For the function $f(x) = 3x + 4$, find:

a $f(3)$ **b** $f(-2)$ **c** $f(0)$ **d** $f\left(\frac{1}{2}\right)$

2 For the function $g(x) = (x + 2)^2$, find:

a $g(4)$ **b** $g(-6)$ **c** $g(0)$ **d** $g\left(\frac{1}{2}\right)$

3 For the function $h: x \to 3x^2 + 1$, find:

a $h(2)$ **b** $h(-3)$ **c** $h(0)$ **d** $h\left(\frac{1}{3}\right)$

4 For the function $f: x \to \frac{2x+6}{3}$, find:

a $f(3)$ **b** $f(-6)$ **c** $f(0)$ **d** $f\left(\frac{1}{4}\right)$

$\sqrt{2x+1}$ is the notation for the positive square root of $2x+1$

5 For the function $f(x) = \sqrt{2x+1}$:

a Draw a mapping diagram to show the outputs when the set of inputs is the odd numbers from 1 to 9 inclusive.

b Draw a mapping diagram to show the outputs when the set of inputs is the even numbers from 2 to 10 inclusive.

c Which number must be excluded as an input?

6 Find the range of each function:

a $f(x) = 3x - 2$; domain $\{1, 2, 3, 4, 5\}$

b $g(x) = \frac{x-4}{2}$; domain $\{-2, -1, 0, 1, 2\}$

c $h(x) = 2x^2$; domain $x \in \mathbb{R}$

d $f: x \to x^2 + 6$; domain $x \in \mathbb{R}$

7 Which value(s) must be excluded from the domain of these functions?

a $f(x) = \frac{1}{x}$ **b** $f(x) = \sqrt{x-1}$

c $f(x) = \frac{3}{2x-3}$ **d** $f(x) = \sqrt{2-x^2}$

8 Find the inverse of each function:

a $f(x) = 7x - 2$ **b** $g(x) = \frac{3x+4}{2}$

c $h(x) = (x-1)^2$ for $x \geqslant 1$ **d** $f(x) = x^2 + 4$ for $x \geqslant 0$

9 a Find the inverse of the function $f(x) = 3x - 4$.

b Sketch $f(x)$, $f^{-1}(x)$ and the line $y = x$ on the same axes. Use the same scale on both axes.

Plot: Start with a table of values. Sketch: Show the main features of the curve.

10 a Plot the graph of the function $f(x) = 4 - x^2$ for values of x such that $0 \leqslant x \leqslant 3$. Use the same scale on both axes.

b Find the values of $f^{-1}(-5)$, $f^{-1}(0)$, $f^{-1}(3)$ and $f^{-1}(4)$.

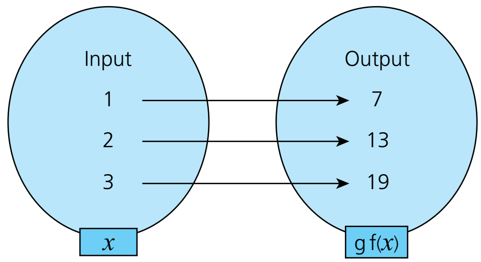
c Sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same axes. Use the range -6 to $+6$ for both axes.

Composition of functions

When two functions are used one after the other, the single equivalent function is called the **composite function**.

For example, if $f(x) = 3x + 2$ and $g(x) = 2x - 3$, then the composite function $gf(x)$ is obtained by applying f first and then applying g to the result.

Input	$\longrightarrow$	$\boxed{f(x)}$	$\longrightarrow$	$\boxed{g\,f(x)}$	$\longrightarrow$	Output
x	$\longrightarrow$	$3x+2$	$\longrightarrow$	$2(3x+2)-3$	$\longrightarrow$	$6x+1$



If $f(x)$ and $g(x)$ are two functions such that the range of f is the domain of g, then $gf(x) = g(f(x))$ means that you apply f first and then apply g to the result.

$f^2(x)$ is the same as $f(f(x))$ and means that you apply the same function twice.

The order in which these operations are applied is important, as shown below.

Worked example

Given that $f(x) = 2x$, $g(x) = x^2$ and $h(x) = \frac{1}{x}$, find:

a $fg(x)$ **b** $gf(x)$ **c** $h^2(x)$

d $fgh(x)$ **e** $hgf(x)$

Solution

a $fg(x) = f(x^2)$
$= 2x^2$

b $gf(x) = g(2x)$
$= (2x)^2$
$= 4x^2$

c $h^2(x) = h[h(x)]$
$= h\left(\frac{1}{x}\right)$
$= 1 \div \frac{1}{x}$
$= x$

d $fgh(x) = fg\left(\frac{1}{x}\right)$
$= f\left[\left(\frac{1}{x}\right)^2\right]$
$= f\left(\frac{1}{x^2}\right)$
$= \frac{2}{x^2}$

e $hgf(x) = hg(2x)$
$= h((2x)^2)$
$= h(4x^2)$
$= \frac{1}{4x^2}$

Worked example

a Find $f^{-1}(x)$ when $f(x) = \frac{2x-1}{4}$

b Find $f[f^{-1}(x)]$.

c Find $f^{-1}[f(x)]$.

d What do you notice?

Solution

a Write $f(x)$ as $y = \frac{2x-1}{4}$

Interchange x and y. $x = \frac{2y-1}{4}$

$$\Rightarrow 4x = 2y - 1$$

$$\Rightarrow 2y = 4x + 1$$

$$\Rightarrow y = \frac{4x+1}{2}$$

$$\Rightarrow f^{-1}(x) = \frac{4x+1}{2}$$

b $f[f^{-1}(x)] = f\left[\frac{4x+1}{2}\right]$

$$= \frac{2\left(\frac{4x+1}{2}\right) - 1}{4}$$

$$= \frac{(4x+1)-1}{4}$$

$$= \frac{4x}{4}$$

$$= x$$

c $f^{-1}[f(x)] = f^{-1}\left(\frac{2x-1}{4}\right)$

$$= \frac{4\left(\frac{2x-1}{4}\right) + 1}{2}$$

$$= \frac{(2x-1)+1}{2}$$

$$= \frac{2x}{2}$$

$$= x$$

This result is true for all functions that have an inverse.

d Questions **a** and **b** show that $f[f^{-1}(x)] = f^{-1}[f(x)] = x$.

The examples above show that applying a function and its inverse in either order leaves the original quantity unchanged, which is what the notation $f(f^{-1})$ or $f^{-1}(f)$ implies.

Worked example

Using the functions $f(x) = \sin x$ and $g(x) = x^2$, express the following as functions of x:

a $fg(x)$ **b** $gf(x)$ **c** $f^2(x)$

Solution

a $\text{fg}(x) = \text{f}[\text{g}(x)]$
$= \sin(x^2)$

b $\text{gf}(x) = \text{g}[\text{f}(x)]$
$= (\sin x)^2$

c $\text{f}^2(x) = \text{f}[\text{f}(x)]$
$= \sin(\sin x)$

Notice that $\sin(x^2)$ is not the same as $(\sin x)^2$ or $\sin(\sin x)$.

The modulus function

The **modulus** of a number is its positive value even when the number itself is negative.

The modulus is denoted by a vertical line on each side of the number and is sometimes called the **magnitude** of the quantity.

For example, $|28| = 28$ and $|-28| = 28$

$|x| = x$ when $x \geqslant 0$ and $|x| = -x$ when $x < 0$

Therefore for the graph of the modulus function $y = |\text{f}(x)|$, any part of the corresponding graph of $y = \text{f}(x)$ where $y < 0$, is reflected in the x-axis.

Worked example

For each of the following, sketch $y = \text{f}(x)$ and $y = |\text{f}(x)|$ on separate axes:

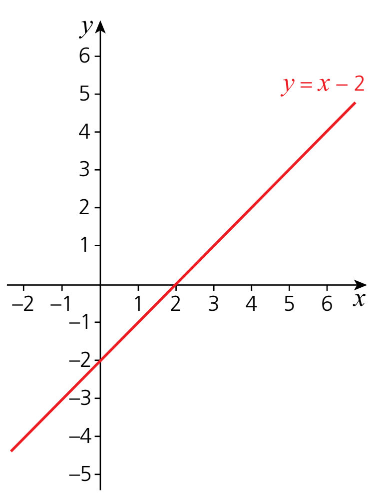
a $y = x - 2;\quad -2 \leqslant x \leqslant 6$

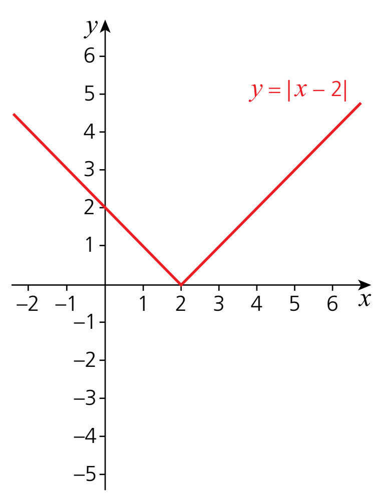
b $y = x^2 - 2;\quad -3 \leqslant x \leqslant 3$

c $y = \cos x;\quad 0° \leqslant x \leqslant 180°$

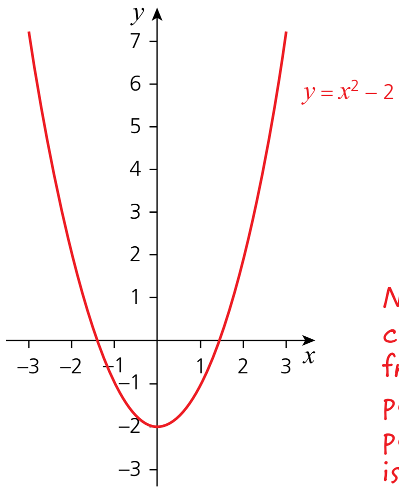
Solution

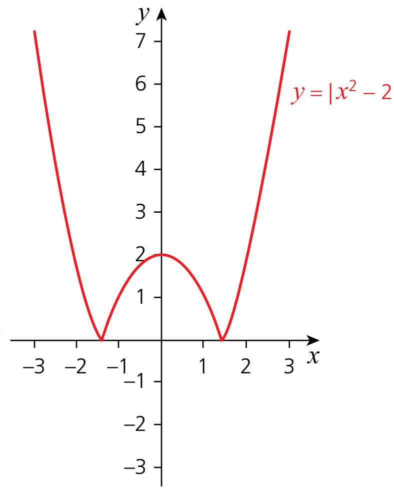
a





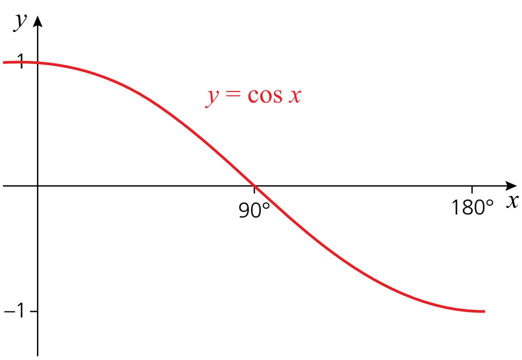
b

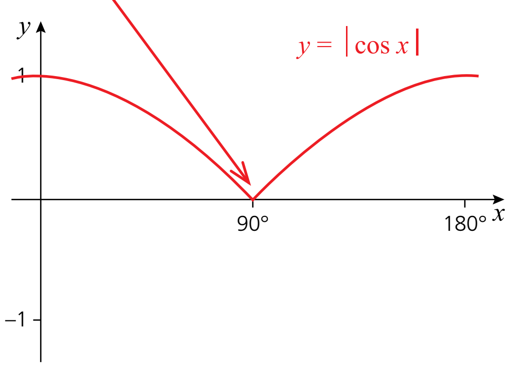




Notice the sharp change of gradient from negative to positive, where part of the graph is reflected. This point is called a 'cusp'.

c





Exercise 1.2

1 Given that $f(x) = 3x + 2$, $g(x) = x^2$ and $h(x) = 2x$, find:

a $fg(2)$ **b** $fg(x)$ **c** $gh(x)$ **d** $fgh(x)$

2 Given that $f(x) = \sqrt{2x+1}$ and $g(x) = 4 - x$, find:

a $fg(-4)$ **b** $gf(12)$ **c** $fg(x)$ **d** $gf(x)$

3 Given that $f(x) = x + 4$, $g(x) = 2x^2$ and $h(x) = \frac{1}{2x+1}$, find:

a $f^2(x)$ **b** $g^2(x)$ **c** $h^2(x)$ **d** $hgf(x)$

4 For each function, find the inverse and sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes. Use the same scale on both axes.

a $f(x) = 3x - 1$ **b** $f(x) = x^3, x > 0$

5 Solve the following equations:

a $|x - 3| = 4$ **b** $|2x + 1| = 7$ **c** $|3x - 2| = 5$ **d** $|x + 2| = 2$

6 Sketch the graph of each function:

a $y = x + 2$ **b** $y = |x + 2|$ **c** $y = |x + 2| + 3$

7 Sketch these graphs for $0° \leqslant x \leqslant 360°$:

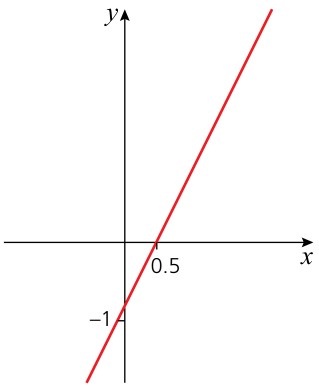
a $y = \cos x$ **b** $y = \cos x + 1$

c $y = |\cos x|$ **d** $y = |\cos x| + 1$

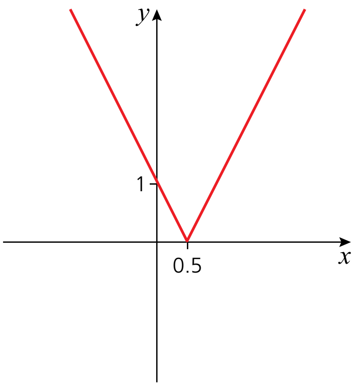
8 Graph 1 represents the line $y = 2x - 1$. Graph 2 is related to Graph 1 and Graph 3 is related to Graph 2.

Write down the equations of Graph 2 and Graph 3.

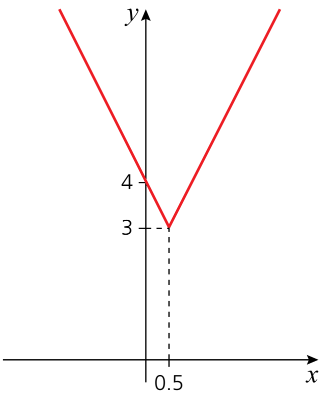
Graph 1



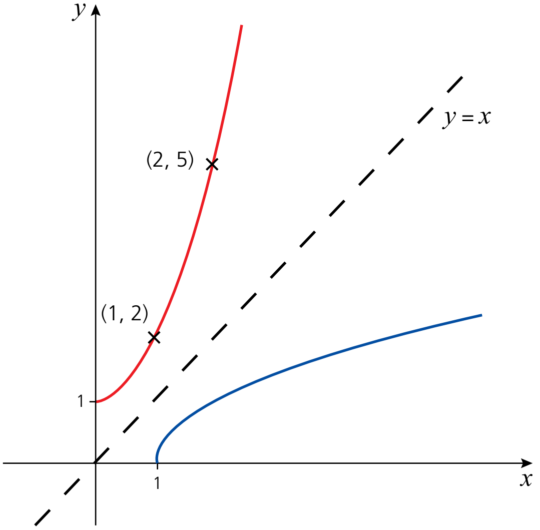
Graph 2



Graph 3



9 The graph shows part of a quadratic curve and its inverse.



a What is the equation of the curve?

b What is the equation of the inverse?

10 a Sketch the graphs of these functions:

i $y = 1 - 2x$ **ii** $y = |1 - 2x|$

iii $y = -|1 - 2x|$ **iv** $y = 3 - |1 - 2x|$

b Use a series of transformations to sketch the graph of $y = |3x + 1| - 2$.

Exercise 1.2 (cont)

11 For each part:

a Sketch both graphs on the same axes.

b Write down the coordinates of their points of intersection.

i $y = |x|$ and $y = 1 - |x|$

ii $y = 2|x|$ and $y = 2 - |x|$

iii $y = 3|x|$ and $y = 3 - |x|$

Past-paper questions

1 The functions f and g are defined by

$$\mathrm{f}(x) = \frac{2x}{x+1} \text{ for } x > 0,$$

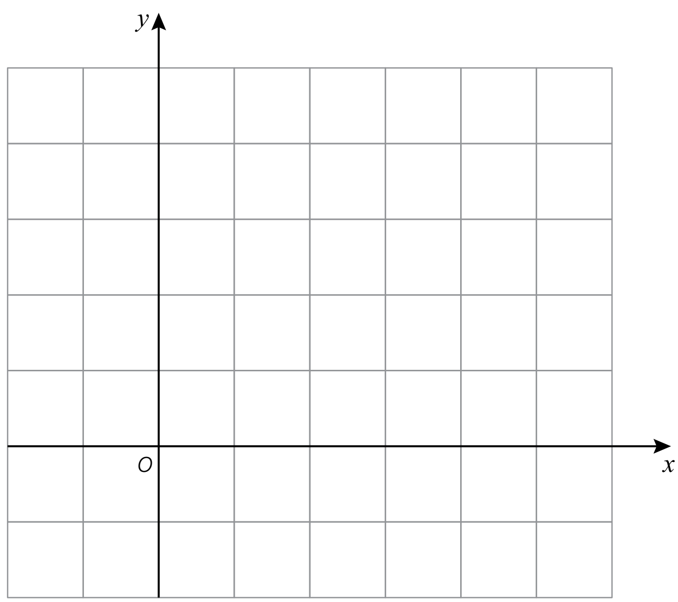
$$\mathrm{g}(x) = \sqrt{x+1} \text{ for } x > -1$$

(i) Find fg(8). [2]

(ii) Find an expression for $\mathrm{f}^2(x)$, giving your answer in the form $\dfrac{ax}{bx+c}$, where a, b and c are integers to be found. [3]

(iii) Find an expression for $\mathrm{g}^{-1}(x)$, stating its domain and range. [4]

(iv) On axes like the ones below, sketch the graphs of $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$, indicating the geometrical relationship between the graphs. [3]



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Paper 21 Q12 June 2014
Cambridge IGCSE Additional Mathematics 0606
Paper 21 Q12 June 2014

2 **(i)** Sketch the graph of $y = |3x - 5|$, for $-2 \leqslant x \leqslant 3$, showing the coordinates of the points where the graph meets the axes. [3]

(ii) On the same diagram, sketch the graph of $y = 8x$. [1]

(iii) Solve the equation $8x = |3x - 5|$. [3]

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Paper 13 Q7 November 2010
Cambridge IGCSE Additional Mathematics 0606
Paper 13 Q7 November 2010

Learning outcomes

Now you should be able to:

- ★ understand the terms function, domain, range (image set), one-one function, inverse function and composite function
- ★ use the notation $f(x) = \sin x$, $f: x \mapsto g(x), x > 0$, $f^{-1}(x)$ and $f^2(x)$ $[= f(f(x))]$
- ★ understand the relationship between $y = f(x)$ and $y = |f(x)|$, where $f(x)$ may be linear, quadratic or trigonometric
- ★ explain in words why a given function is a function or why it does not have an inverse
- ★ find the inverse of a one-one function and form composite functions
- ★ use sketch graphs to show the relationship between a function and its inverse.

Key points

- A **mapping** is a rule for changing one number into another number or numbers.
- A **function**, $f(x)$, is a rule that maps one number onto another single number.
- The **graph of a function** has only one value of y for each value of x. However, two or more values of x may give the same value of y.
- A **flow chart** can be used to show the individual operations within a function in the order in which they are applied.
- The **domain** of a function is the set of **input values**, or **objects**, that the function is operating on.
- The **range** or **image set** of a function is the corresponding set of **output values** or **images**, $f(x)$.
- A **mapping diagram** can be used to illustrate a function. It is best used when the domain contains only a small number of values.
- In a **one-one function** there is a unique value of y for every value of x and a unique value of x for every value of y.
- In a **many-one** function two or more values of x correspond to the same value of y.
- In a **one-many** function one value of x corresponds to two or more values of y.
- In a **many-many** function two or more values of x correspond to the same value of y and two or more values of y correspond to the same value of x.
- The **inverse** of a function reverses the effect of the function. Only one-one functions have inverses.
- The term **composition of functions** is used to describe the application of one function followed by another function(s).The notation $fg(x)$ means that the function g is applied first, then f is applied to the result.
- The **modulus** of a number or a function is always a positive value. $|x| = x$ if $x \geqslant 0$ and $|x| = -x$ if $x < 0$.
- The modulus of a function $y = f(x)$ is denoted by $|f(x)|$ and is illustrated by reflecting any part of the graph where $y < 0$ in the x-axis.

2 Quadratic functions

One really can't argue with a mathematical theorem.

Stephen Hawking (1942–2018)

Early mathematics focused principally on arithmetic and geometry. However, in the sixteenth century a French mathematician, François Viète, started work on 'new algebra'. He was a lawyer by trade and served as a privy councillor to both Henry III and Henry IV of France. His innovative use of letters and parameters in equations was an important step towards modern algebra.

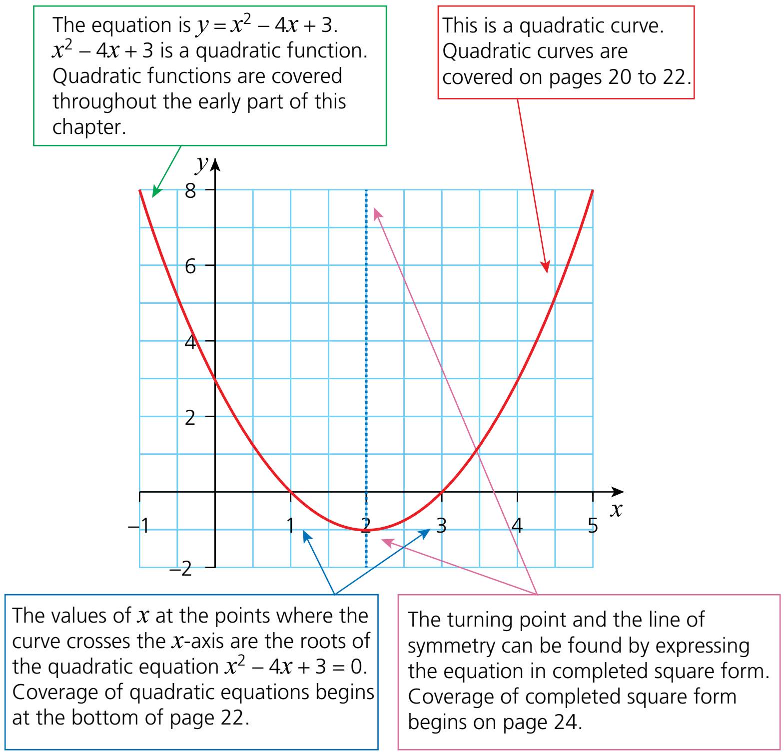
François Viète (1540–1603)

Discussion point

Viète presented methods of solving equations of second, third and fourth degrees and discovered the connection between the positive roots of an equation and the coefficients of different powers of the unknown quantity. Another of Viete's remarkable achievements was to prove that claims that a circle could be squared, an angle trisected and the cube doubled were untrue. He achieved all this, and much more, using only a ruler and compasses, without the use of either tables or a calculator! In order to appreciate the challenges Viète faced, try to solve the quadratic equation $2x^2 - 8x + 5 = 0$ without using a calculator. Give your answers correct to two decimal places.

This chapter is about quadratic functions and covers a number of related themes.

The graph below illustrates these themes:



Maximum and minimum values

A **polynomial** is an expression in which, with the exception of a constant, the terms are positive integer powers of a variable. The highest power is the **order** of the polynomial.

A **quadratic function** or expression is a polynomial of order 2. $x^2 + 3$, a^2 and $2y^2 - 3y + 5$ are all quadratic expressions. Each expression contains only one variable (letter), and the highest power of that variable is 2.

The graph of a quadratic function is either ∪-shaped or ∩-shaped. Think about the expression $x^2 + 3x + 2$. When the value of x is very large and positive, or very large and negative, the x^2 term dominates the expression, resulting in large positive values. Therefore the graph of the function is ∪-shaped.

Similarly, the $-2x^2$ term dominates the expression $5 - 4x - 2x^2$ for both large positive and large negative values of x giving negative values of the expression for both. Therefore the graph of this function is $\cap$-shaped.

Although many of the quadratic equations that you will meet will have three terms, you will also meet quadratic equations with only two, or even one term. These fall into two main categories.

1 Equations with no constant term, for example, $2x^2 - 5x = 0$.
This has x as a common factor so factorises to $x(2x - 5) = 0$

$$\Rightarrow x = 0 \text{ or } 2x - 5 = 0$$
$$\Rightarrow x = 0 \text{ or } x = 2.5$$

2 Equations with no 'middle' term, which come into two categories:

i The sign of the constant term is negative, for example, $a^2 - 9 = 0$ and $2a^2 - 7 = 0$.

$$a^2 - 9 = 0 \text{ factorises to } (a + 3)(a - 3) = 0$$
$$\Rightarrow a = -3 \text{ or } a = 3$$
$$2a^2 - 7 = 0 \Rightarrow a^2 = 3.5$$
$$\Rightarrow a = \pm\sqrt{3.5}$$

ii The sign of the constant term is positive, for example, $p^2 + 4 = 0$.
$p^2 + 4 = 0 \quad \Rightarrow \quad p^2 = -4$, so there is no real-valued solution.

Note

Depending on the calculator you are using, $\sqrt{(-4)}$ may be displayed as 'Math error' or '2i', where i is used to denote $\sqrt{(-1)}$. This is a **complex number** or **imaginary number** which you will meet if you study Further Mathematics at Advanced Level.

The vertical line of symmetry

Graphs of all quadratic functions have a vertical line of symmetry. You can use this to find the maximum or minimum value of the function. If the graph crosses the horizontal axis, then the line of symmetry is halfway between the two points of intersection. The maximum or minimum value lies on this line of symmetry.

Worked example

a Plot the graph of $y = x^2 - 4x - 5$ for values of x from –2 to +6.

b Identify the values of x where the curve intersects the horizontal axis.

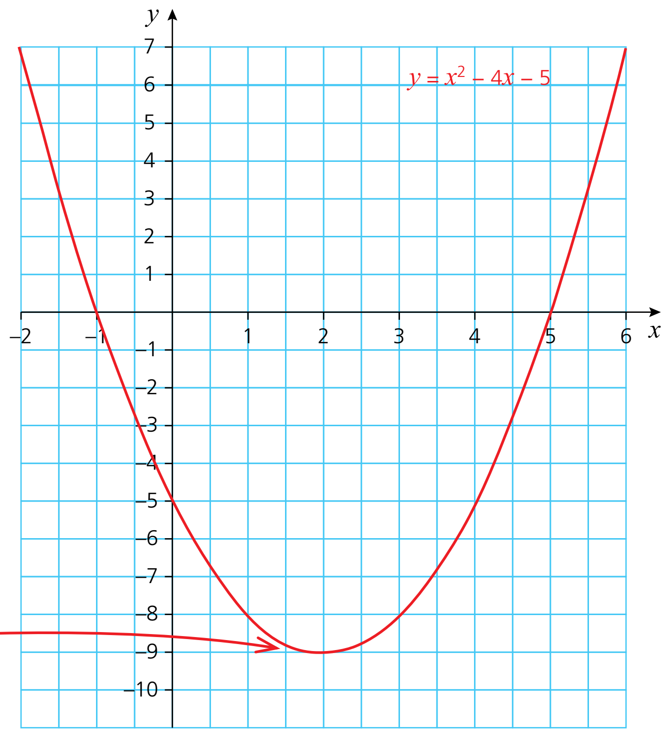
c Hence find the coordinates of the maximum or minimum point.

Solution

First create a table of values for $-2 \leqslant x \leqslant 6$.

a

x	–2	–1	0	1	2	3	4	5	6
y	7	0	–5	–8	–9	–8	–5	0	7



This point is often referred to as the **turning point** of the curve.

This is also shown in the table.

b The graph intersects the horizontal axis when $x = -1$ and when $x = 5$.

The line $x = 2$ passes through the turning point. It is a **vertical line of symmetry** for the curve.

c The graph shows that the curve has a minimum turning point halfway between $x = -1$ and $x = 5$. The table shows that the coordinates of this point are (2, –9).

Factorising

Drawing graphs by hand to find maximum or minimum values can be time-consuming. The following example shows you how to use algebra to find these values.

Worked example

The first step is to factorise the expression. One method of factorising is shown, but if you are confident using a different method then continue to use it.

Find the coordinates of the turning point of the curve $y = x^2 + x - 6$. State whether the turning point is a maximum or a minimum value.

Solution

Find two integers (whole numbers) that multiply together to give the **constant** term, –6.

Possible pairs of numbers with a product of –6 are: 6 and –1, 1 and –6, 3 and –2, 2 and –3.

Identify any of the pairs of numbers that can be added together to give the coefficient of x (1). 3 and –2 are the only pair with a sum of 1, so use this pair to split up the x term.

Both expressions in the brackets must be the same. Notice the sign change due to the negative sign in front of the 2.

$$\begin{aligned} x^2 + x - 6 &= x^2 + 3x - 2x - 6 \\ &= x(x+3) - 2(x+3) \\ &= (x+3)(x-2) \end{aligned}$$

Note

You would get the same result if you used $3x$ and $-2x$ in the opposite order:

$$\begin{aligned} x^2 + x - 6 &= x^2 - 2x + 3x - 6 \\ &= x(x-2) + 3(x-2) \\ &= (x-2)(x+3) \end{aligned}$$

The graph of $y = x^2 + x - 6$ crosses the x-axis when $(x+3)(x-2) = 0$, i.e. when $x = -3$ and when $x = 2$.

The x-coordinate of the turning point is halfway between these two values, so:

$$\begin{aligned} x &= \frac{-3+2}{2} \\ &= -0.5 \end{aligned}$$

Substituting this value into the equation of the curve gives:

$$\begin{aligned} y &= (-0.5)^2 + (-0.5) - 6 \\ &= -6.25 \end{aligned}$$

The equation of the curve has a positive x^2 term so its graph is $\cup$-shaped. Therefore the minimum value is at (–0.5, –6.25).

The method shown above can be adapted for curves with an equation in which the coefficient of x^2 is not +1, for example, $y = 6x^2 - 13x + 6$ or $y = 6 - x - 2x^2$, as shown in the next example.

Worked example

For the curve with equation $y = 6 - x - 2x^2$:

a Will the turning point of the curve be a maximum or a minimum? Give a reason for your answer.

b Write down the coordinates of the turning point.

c State the equation of the line of symmetry.

Solution

a The coefficient of x^2 is negative so the curve will be $\cap$-shaped. This means that the turning point will be a maximum.

Continue to use any alternative methods of factorising that you are confident with.

b First multiply the constant term and the coefficient of x^2, i.e. $6 \times -2 = -12$. Then find two whole numbers that multiply together to give this product.

Possible pairs are: 6 and −2, −6 and 2, −3 and 4, −3 and 4, 1 and −12, −1 and 12.

Identify any of the pairs of numbers that can be added together to give the coefficient of x (-1).

−3 and 4 are the only pair with a sum of −1, so use this pair to split up the x term.

$$\begin{aligned} 6 - x - 2x^2 &= 6 + 3x - 4x - 2x^2 \\ &= 3(2 + x) - 2x(2 + x) \\ &= (2 + x)(3 - 2x) \end{aligned}$$

Both expressions in the brackets must be the same. Notice the sign change is due to the sign in front of the 2.

The graph of $y = 6 - x - 2x^2$ crosses the x-axis when $(2 + x)(3 - 2x) = 0$, i.e. when $x = -2$ and when $x = 1.5$.

The x-coordinate of the turning point is halfway between these two values.

$$\begin{aligned} x &= \frac{-2 + 1.5}{2} \\ &= -0.25 \end{aligned}$$

Substituting this value into the equation of the curve gives:

$$\begin{aligned} y &= 6 - (-0.25) - 2(-0.25)^2 \\ &= 6.125 \end{aligned}$$

So the turning point is $(-0.25, 6.125)$

c The equation of the line of symmetry is $x = -0.25$.

Completing the square

The methods shown in the previous examples will always work for curves that cross the x-axis. For quadratic curves that do not cross the x-axis, you will need to use the method of **completing the square**, shown in the next example.

Another way of writing the quadratic expression $x^2 + 6x + 11$ is $(x + 3)^2 + 2$ and this is called **completed square form**. Written like this the expression consists of a squared term, $(x + 3)^2$, that includes the variable, x, and a constant term +2.

In the next example you see how to convert an ordinary quadratic expression into completed square form.

Worked example

a Write $x^2 - 8x + 18$ in completed square form.

b State whether it is a maximum or minimum.

c Sketch the curve $y = f(x)$.

Solution

a Start by halving the coefficient of x and squaring the result.

$$-8 \div 2 = -4$$
$$(-4)^2 = 16$$

Now use this result to break up the constant term, +18, into two parts:

$$18 = 16 + 2$$

and use this to rewrite the original expression as:

You will always have a perfect square in this expression.

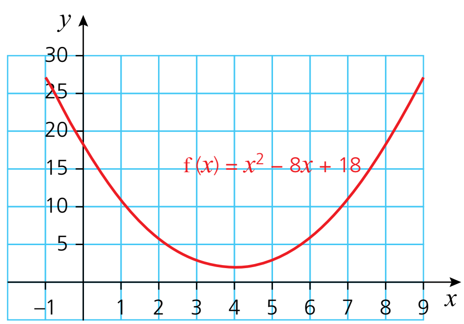
$$f(x) = x^2 - 8x + 16 + 2$$
$$= (x - 4)^2 + 2$$
$$(x - 4)^2 \geqslant 0 \text{ (always)}$$
$$\Rightarrow f(x) \geqslant 2 \text{ for all values of } x$$

In completed square form, $x^2 - 8x + 18 = (x - 4)^2 + 2$

b $f(x) \geqslant 2$ for all values of x so the turning point is a minimum.

c The function is a ∪-shaped curve because the coefficient of x^2 is positive. From the above, the minimum turning point is at (4, 2) so the curve does not cross the x-axis. To sketch the graph, you will also need to know where it crosses the y-axis.

$f(x) = x^2 - 8x + 18$ crosses the y-axis when $x = 0$, i.e. at (0, 18).



Worked example

Use the method of completing the square to work out the coordinates of the turning point of the quadratic function $f(x) = 2x^2 - 8x + 9$.

Solution

$$f(x) = 2x^2 - 8x + 9$$
$$= 2(x^2 - 4x) + 9$$
$$= 2((x - 2)^2 - 4) + 9$$
$$= 2(x - 2)^2 + 1$$

$(x-2)^2 \geqslant 0$ (always), so the minimum value of $f(x)$ is 1.

When $f(x) = 1$, $x = 2$.

Therefore the coordinates of the turning point (minimum value) of the function $f(x) = 2x^2 - 8x + 9$ are $(2, 1)$.

Sometimes you will be asked to sketch the graph of a function $f(x)$ for certain values of x. This set of values of x is called the **domain** of the function. The corresponding set of y-values is called the **range**.

Worked example

The domain of the function $y = 6x^2 + x - 2$ is $-3 \leqslant x \leqslant 3$.

Sketch the graph and find the range of the function.

Solution

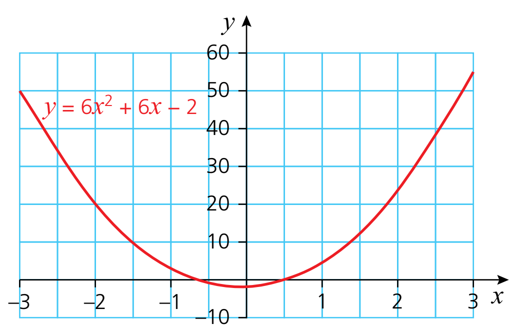
The coefficient of x^2 is positive, so the curve is $\cup$-shaped and the turning point is a minimum.

The curve crosses the x-axis when $6x^2 + x - 2 = 0$.

$$\begin{aligned} 6x^2 + x - 2 &= (3x+2)(2x-1) \\ \Rightarrow (3x+2)(2x-1) &= 0 \\ \Rightarrow \quad (3x+2) &= 0 \text{ or } (2x-1) = 0 \end{aligned}$$

So the graph crosses the x-axis at $(-\frac{2}{3}, 0)$ and $(\frac{1}{2}, 0)$.

The curve crosses the y-axis when $x = 0$, i.e. at $(0, -2)$.



The curve has a vertical line of symmetry passing halfway between the two points where the curve intersects the x-axis. Therefore the equation of this line of symmetry is $x = \dfrac{-\frac{2}{3} + \frac{1}{2}}{2}$ or $x = -\frac{1}{12}$.

When $x = -\frac{1}{12}$, $y = 6\left(-\frac{1}{12}\right)^2 + \left(-\frac{1}{12}\right) - 2 = -2\frac{1}{24}$, the minimum value of the function.

To find the range, work out the values of y for $x = -3$ and $x = +3$.

The larger of these gives the maximum value.

When $x = -3$, $y = 6(-3)^2 + (-3) - 2 = 49$.
When $x = 3$, $y = 6(3)^2 + 3 - 2 = 55$.

The range of the function corresponding to the domain $-3 \leqslant x \leqslant 3$ is therefore $-2\frac{1}{24} \leqslant y \leqslant 55$.

Exercise 2.1

1 Solve each equation by factorising:
- **a** $x^2 + x - 20 = 0$
- **b** $x^2 - 5x + 6 = 0$
- **c** $x^2 - 3x - 28 = 0$
- **d** $x^2 + 13x + 42 = 0$

2 Solve each equation by factorising:
- **a** $2x^2 - 3x + 1 = 0$
- **b** $9x^2 + 3x - 2 = 0$
- **c** $2x^2 - 5x - 7 = 0$
- **d** $3x^2 + 17x + 10 = 0$

3 Solve each equation by factorising:
- **a** $x^2 - 169 = 0$
- **b** $4x^2 - 121 = 0$
- **c** $100 - 64x^2 = 0$
- **d** $12x^2 - 27 = 0$

4 For each of the following curves:
- **i** Factorise the function.
- **ii** Work out the coordinates of the turning point.
- **iii** State whether the turning point is a maximum or minimum.
- **iv** Sketch the graph, labelling the coordinates of the turning point and any points of intersection with the axes.

- **a** $y = x^2 + 7x + 10$
- **b** $f(x) = 16 - 6x - x^2$
- **c** $y = 5 - 9x - 2x^2$
- **d** $f(x) = 2x^2 + 11x + 12$

5 Write each quadratic expressions in the form $(x + a)^2 + b$:
- **a** $x^2 + 4x + 9$
- **b** $x^2 - 10x - 4$
- **c** $x^2 + 5x - 7$
- **d** $x^2 - 9x - 2$

6 Write each quadratic expression in the form $c(x + a)^2 + b$.
- **a** $2x^2 - 12x + 5$
- **b** $3x^2 + 12x + 20$
- **c** $4x^2 - 8x + 5$
- **d** $2x^2 + 9x + 6$

7 Solve the following quadratic equations. Leave your answers in the form $x = p \pm \sqrt{q}$.
- **a** $x^2 + 4x - 9 = 0$
- **b** $x^2 - 7x - 2 = 0$
- **c** $2x^2 + 6x - 9 = 0$
- **d** $3x^2 + 9x - 15 = 0$

8 For each of the following functions:
- **i** Use the method of completing the square to find the coordinates of the turning point of the graph.
- **ii** State whether the turning point is a maximum or a minimum.
- **iii** Sketch the graph.

- **a** $f(x) = x^2 + 6x + 15$
- **b** $y = 8 + 2x - x^2$
- **c** $y = 2x^2 + 2x - 9$
- **d** $f : x \rightarrow x^2 - 8x + 20$

9 Sketch the graph and find the corresponding range for each function and domain.
- **a** $y = x^2 - 7x + 10$ for the domain $1 \leqslant x \leqslant 6$
- **b** $f(x) = 2x^2 - x - 6$ for the domain $-2 \leqslant x \leqslant 2$

Real-world activity

1 Draw a sketch of a bridge modelled on the equation $25y = 100 - x^2$ for $-10 \leqslant x \leqslant 10$. Label the origin O, point A$(-10, 0)$, point B$(10, 0)$ and point C$(0, 4)$.

2 1 unit on your graph represents 1 metre. State the maximum height of the bridge, OC, and the span, AB.

3 Work out the equation of a similar bridge with a maximum height of 5 m and a span of 40 m.

The quadratic formula

The **roots** of a quadratic equation f(x) are those values of x for which $y = 0$ for the curve $y = \mathrm{f}(x)$. In other words, they are the x-coordinates of the points where the curve either crosses or touches the x-axis. There are three possible outcomes.

1. The curve crosses the x-axis at two distinct points. In this case, the corresponding equation is said to have *two real distinct roots.*
2. The curve touches the x-axis, in which case the equation has *two equal (repeating) roots.*
3. The curve lies completely above or completely below the x-axis so it neither crosses nor touches the axis. In this case, the equation has *no real roots.*

The method of completing the square can be generalised to give a formula for solving quadratic equations. The next example uses this method in a particular case on the left-hand side and shows the same steps for the general case on the right-hand side, using algebra to derive the formula for solving quadratic equations.

Worked example

Solve $2x^2 + x - 4 = 0$.

Solution

$$2x^2 + x - 4 = 0$$

$$\Rightarrow \quad x^2 + \frac{1}{2}x - 2 = 0$$

$$\Rightarrow \quad x^2 + \frac{1}{2}x = 2$$

$$\Rightarrow \quad x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)x^2 = 2 + \left(\frac{1}{4}\right)^2$$

$$\Rightarrow \quad \left(x + \frac{1}{4}\right)^2 = \frac{33}{16}$$

$$\Rightarrow \quad \left(x + \frac{1}{4}\right) = \pm\frac{\sqrt{33}}{4}$$

$$\Rightarrow \quad x = -\frac{1}{4} \pm \frac{\sqrt{33}}{4}$$

$$= \frac{-1 \pm \sqrt{33}}{4}$$

Generalisation

$$ax^2 + bx + c = 0$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Rightarrow \quad x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The result $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is known as the **quadratic formula**. You can use it to solve any quadratic equation. One root is found by taking the + sign, and the other by taking the – sign. When the value of $b^2 - 4ac$ is negative, the square root cannot be found and so there is no real solution to that quadratic equation. This occurs when the curve does not cross the x-axis.

In an equation of the form $(px + q)^2 = 0$, where p and q can represent either positive or negative numbers, $px + q = 0$ gives the only solution.

Note

The part $b^2 - 4ac$ is called the **discriminant** because it discriminates between quadratic equations with no roots, quadratic equations with one repeated root and quadratic equations with two real roots.

- If $b^2 - 4ac > 0$ there are 2 real roots.
- If $b^2 - 4ac = 0$ there is 1 repeated root.
- If $b^2 - 4ac < 0$ there are no real roots.

Worked example

a Show that the equation $4x^2 - 12x + 9 = 0$ has a repeated root by:

i factorising **ii** using the discriminant.

b State with reasons how many real roots the following equations have:

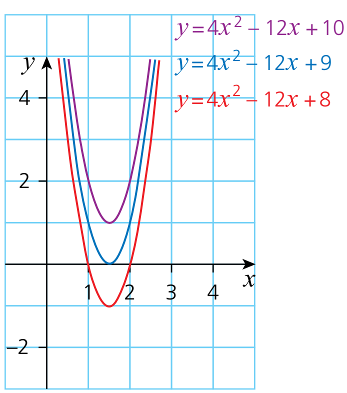
i $4x^2 - 12x + 8 = 0$ **ii** $4x^2 - 12x + 10 = 0$

Solution

a i

$$\begin{aligned} 4x^2 - 12x + 9 &= 0 \\ \Rightarrow (2x - 3)(2x - 3) &= 0 \\ 2x - 3 &= 0 \\ \Rightarrow \qquad x &= 1.5 \end{aligned}$$

ii The equation has a repeated root because the discriminant $b^2 - 4ac = (-12)^2 - 4(4)(9) = 0$.



b i The curve $y = 4x^2 - 12x + 8$ is 1 unit below $y = 4x^2 - 12x + 9$ and crosses the x-axis in 2 points. So the equation has two real roots.

ii The curve $y = 4x^2 - 12x + 10$ is 1 unit above $y = 4x^2 - 12x + 9$ and does not cross the x-axis. So the equation $4x^2 - 12x + 10 = 0$ has no real roots.

In some cases, such as in the previous example, the factorisation is not straightforward. In such cases, evaluating the discriminant is a reliable method to obtain an accurate result.

Worked example

Show that the equation $3x^2 - 2x + 4 = 0$ has no real solution.

Solution

The most straightforward method is to look at the discriminant. If the discriminant is negative, there is no real solution.

Substituting these values into the discriminant

For $3x^2 - 2x + 4 = 0$, $a = 3$, $b = -2$ and $c = 4$.

$$\begin{aligned} b^2 - 4ac &= (-2)^2 - 4(3)(4) \\ &= -44 \end{aligned}$$

Since the discriminant is negative, there is no real solution.

The intersection of a line and a curve

The general equation of a straight line is $y = mx + c$. This has alternate forms, e.g. $ax + by + c = 0$.

The examples so far have considered whether or not a curve intersects, touches, or lies completely above or below the x-axis ($y = 0$). The next example considers the more general case of whether or not a curve intersects, touches or lies completely above or below a particular straight line.

Worked example

a Find the coordinates of the points of intersection of the line $y = 4 - 2x$ and the curve $y = x^2 + x$.

b Sketch the line and the curve on the same axes.

Solution

The y-values of both equations are the same at the point(s) of intersection.

a To find where the curve and the lines intersect, solve $y = x^2 + x$ simultaneously with $y = 4 - 2x$.

$$\begin{aligned} x^2 + x &= 4 - 2x \\ \Rightarrow \quad x^2 + 3x - 4 &= 0 \\ \Rightarrow \quad (x + 4)(x - 1) &= 0 \\ \Rightarrow \quad x = -4 &\text{ or } x = 1 \end{aligned}$$

It is more straightforward to substitute into the linear equation.

To find the y-coordinate, substitute into one of the equations.

When $x = -4$, $y = 4 - 2(-4) = 12$.
When $x = 1$, $y = 4 - 2(1) = 2$.
The line $y = 4 - 2x$ intersects the curve $y = x^2 + x$ at $(-4, 12)$ and $(1, 2)$.

b The curve has a positive coefficient of x^2 so is ∪-shaped.

It crosses the x-axis when $x^2 + x = 0$.

$\Rightarrow \quad x(x + 1) = 0$

$\Rightarrow \quad x = 0$ or $x = -1$

So the curve crosses the x-axis at $x = 0$ and $x = -1$.

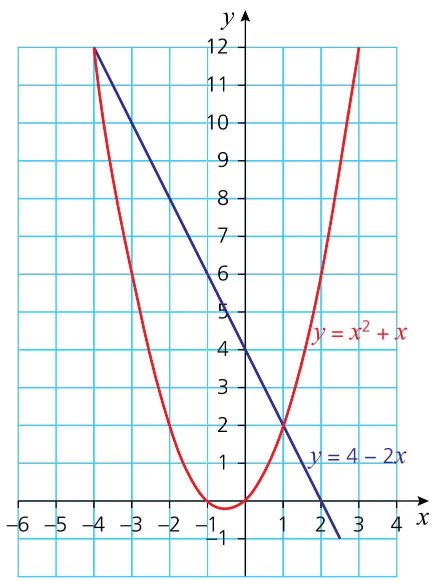
It crosses the y-axis when $x = 0$.

Substituting $x = 0$ into $y = x^2 + x$ gives $y = 0$.

So the curve passes through the origin.

The line $2x + y = 4$ crosses the x-axis when $y = 0$. When $y = 0$, $x = 2$.

The line $2x + y = 4$ crosses the y-axis when $x = 0$. When $x = 0$, $y = 4$.



It is possible for a quadratic curve to touch a general line, either sloping or parallel to the x-axis. You can see this when you solve the equations of the line and the curve simultaneously. If you get a repeated root, it means that they touch at only one point. The line is a tangent to the curve. This is shown in the next example.

Worked example

a Use algebra to show that the line $y = 6x - 19$ touches the curve $y = x^2 - 2x - 3$ and find the coordinates of the point of contact.

b Sketch the line and curve on the same axes.

Solution

a Solving the equations simultaneously

$$x^2 - 2x - 3 = 6x - 19$$
$$\Rightarrow \quad x^2 - 8x + 16 = 0$$
$$\Rightarrow \quad (x - 4)^2 = 0$$
$$\Rightarrow \quad x = 4$$

The repeated root $x = 4$ shows that the line and the curve touch.

Substitute $x = 4$ into either equation to find the value of the y-coordinate.

It is more straightforward to substitute into the line equation.

$$y = 6(4) - 19$$
$$= 5$$

Therefore the point of contact is (4, 5).

b The coefficient of x^2 is positive so the curve is $\cup$-shaped.

Substituting $x = 0$ into $y = x^2 - 2x - 3$ shows that the curve intersects the y-axis at (0, –3).

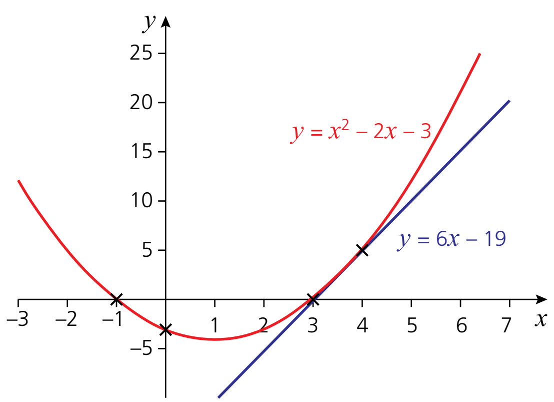
Substituting $y = 0$ into $y = x^2 - 2x - 3$ gives $x^2 - 2x - 3 = 0$.

$\Rightarrow \quad (x - 3)(x + 1) = 0$

$\Rightarrow \quad x = -1$ or $x = 3$

So the curve intersects the x-axis at (–1, 0) and (3, 0).

You need two points to draw a line. It is best to choose points with whole numbers that are not too large, such as (3, –1) and (5, 11).



Discussion point

Why is it not possible for a quadratic curve to touch a line parallel to the y-axis?

There are many situations when a line and a curve do not intersect or touch each other. A straightforward example of this occurs when the graph of a quadratic function is a $\cup$-shaped curve completely above the x-axis, e.g. $y = x^2 + 3$, and the line is the x-axis.

You have seen how solving the equations of a curve and a line simultaneously gives a quadratic equation with *two* roots when the line crosses the curve, and a quadratic equation with a *repeated* root when it touches the curve. If solving the two equations simultaneously results in *no* real roots, i.e. the discriminant is negative, then they do not cross or touch.

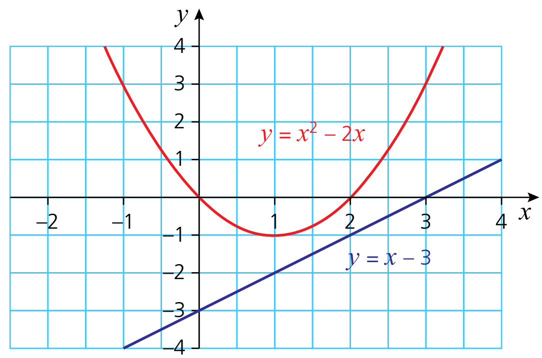
Worked example

a Sketch the graphs of the line $y = x - 3$ and the curve $y = x^2 - 2x$ on the same axes.

b Use algebra to prove that the line and the curve don't meet.

Solution

a



b $x^2 - 2x = x - 3$ ← Solving the two equations simultaneously

$\Rightarrow x^2 - 3x + 3 = 0$

This does not factorise, so solve using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1$, $b = -3$ and $c = 3$

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(3)}}{2(1)}$

$= \frac{3 \pm \sqrt{-3}}{2}$

You do not actually need to write down the solution. Once you see that the value of the discriminant is negative, as in this case where it is −3, you know that the equation has no real roots, so the line and the curve don't meet.

Since there is a negative value under the square root, there is no *real* solution. This implies that the line and the curve do not meet.

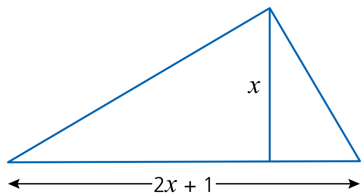
Note

It would have been sufficient to consider only the discriminant $b^2 - 4ac$. Solving a quadratic equation is equivalent to finding the point(s) where the curve crosses the horizontal axis (the roots).

Using quadratic equations to solve problems

Worked example

A triangle has a base of $(2x + 1)$ cm, a height of x cm and an area of 68 cm^2.



a Show that x satisfies the equation $2x^2 + x - 136 = 0$.

b Solve the equation and work out the base length of the triangle.

Solution

a Using the formula for the area of a triangle, area $= \frac{1}{2}$base × height:

$$\text{Area} = \frac{1}{2} \times (2x + 1) \times x$$
$$= \frac{1}{2}(2x^2 + x)$$

The area is 68 cm^2, so:

$$\frac{1}{2}(2x^2 + x) = 68$$
$$\Rightarrow \quad 2x^2 + x = 136$$
$$\Rightarrow \quad 2x^2 + x - 136 = 0$$

Alternatively, you can use a calculator to solve the equation.

b It is not easy to factorise this equation – it is not even obvious that there are factors – so use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 2$, $b = 1$ and $c = -136$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-136)}}{2(2)}$$
$$\Rightarrow x = \frac{-1 \pm \sqrt{1089}}{4}$$
$$\Rightarrow x = \frac{-1 \pm 33}{4}$$
$$\Rightarrow x = 8 \text{ or } x = -8.5$$

Since x is a length, reject the negative solution.

Substitute $x = 8$ into the expression for the base of the triangle, $2x + 1$, and work out the length of the base of the triangle, 17 cm.

Check that this works with the information given in the original question.

$$\frac{1}{2} \times 17\,\text{cm} \times 8\,\text{cm} = 68\,\text{cm}^2$$

Solving quadratic inequalities

The quadratic inequalities in this section all involve quadratic expressions that factorise. This means that you can find a solution either by sketching the appropriate graph or by using line segments to reduce the quadratic inequality to two simultaneous linear inequalities.

The example below shows two valid methods for solving quadratic inequalities. You should use whichever method you prefer. Your choice may depend on how easily you sketch graphs or if you have a graphic calculator that you can use to plot these graphs.

Worked example

Solve these quadratic inequalities.

a $x^2 - 2x - 3 < 0$

b $x^2 - 2x - 3 \geqslant 0$

Solution

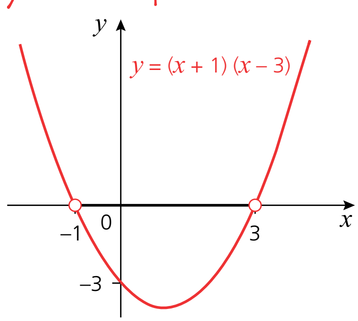
Method 1

$x^2 - 2x - 3 = (x + 1)(x - 3)$

So the graph of $y = x^2 - 2x - 3$ crosses the x-axis when $x = -1$ and $x = 3$.

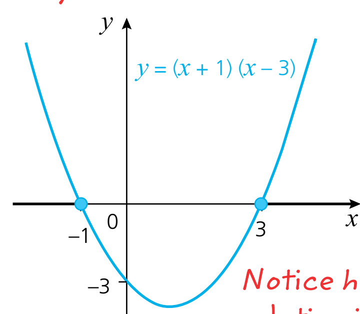
Look at the two graphs below.

Here the end points are not included in the solution, so you draw open circles: ○



The solution is $-1 < x < 3$.

Here the end points are included in the solutions, so you draw solid circles: ●



The solution is $x \leqslant -1$ or $x \geqslant 3$.

Notice how the solution is in two parts when there are two line segments.

a The answer is the values of x for which $y < 0$, i.e. where the curve is below the x-axis.

b The answer is the values of x for which $y \geqslant 0$, i.e. where the curve crosses or is above the x-axis.

Method 2

This method identifies the values of x for which each of the factors is 0 and considers the sign of each factor in the intervals between these critical values.

	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
Sign of $(x + 1)$	–	0	+	+	+
Sign of $(x - 3)$	–	–	–	0	+
Sign of $(x + 1)(x - 3)$	$(-) \times (-) = +$	$(0) \times (-) = 0$	$(+) \times (-) = -$	$(+) \times (0) = 0$	$(+) \times (+) = +$

From the table, the solution to:

a $(x + 1)(x - 3) < 0$ is $-1 < x < 3$

b $(x + 1)(x - 3) \geqslant 0$ is $x \leqslant -1$ or $x \geqslant 3$

If the inequality to be solved contains > or <, then the solution is described using > and <. If the original inequality contains ⩾ or ⩽, then the solution is described using ⩾ and ⩽.

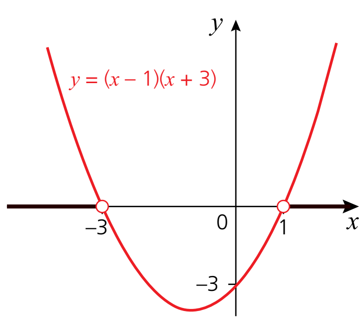
If the quadratic inequality has the variable on both sides, collect the terms involving the variable on one side first in the same way as you would before solving a quadratic equation.

Worked example

Solve $2x + x^2 > 3$.

Solution

$$2x + x^2 > 3 \Rightarrow x^2 + 2x - 3 > 0$$
$$\Rightarrow (x - 1)(x + 3) > 0$$



From the graph, the solution is $x < -3$ or $x > 1$.

Exercise 2.2

1 For each of the following equations, decide if there are two real and different roots, two equal roots or no real roots. Solve the equations with real roots.

a $x^2 + 3x + 2 = 0$ b $t^2 - 9 = 0$ c $x^2 + 16 = 0$

d $2x^2 - 5x = 0$ e $p^2 + 3p - 18 = 0$ f $x^2 + 10x + 25 = 0$

g $15a^2 + 2a - 1 = 0$ h $3r^2 + 8r = 3$

2 Solve the following equations by:

i completing the square **ii** using the quadratic formula.

Give your answers correct to two decimal places.

a $x^2 - 2x - 10 = 0$ **b** $x^2 + x = 0$
c $2x^2 + 2x - 9 = 0$ **d** $2x^2 + x - 8 = 0$

3 Try to solve each of the following equations. Where there is a solution, give your answers correct to two decimal places.

a $4x^2 + 6x - 9 = 0$ **b** $9x^2 + 6x + 4 = 0$
c $(2x + 3)^2 = 7$ **d** $x(2x - 1) = 9$

4 Use the discriminant to decide whether each of the following equations has two equal roots, two distinct roots or no real roots:

a $9x^2 - 12x + 4 = 0$ **b** $6x^2 - 13x + 6 = 0$ **c** $2x^2 + 7x + 9 = 0$
d $2x^2 + 9x + 10 = 0$ **e** $3x^2 - 4x + 5 = 0$ **f** $4x^2 + 28x + 49 = 0$

5 For each pair of equations determine if the line intersects the curve, is a tangent to the curve or does not meet the curve. Give the coordinates of any points where the line and curve touch or intersect.

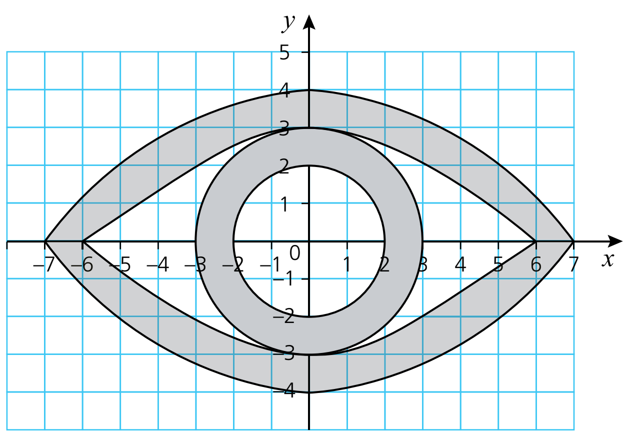
a $y = x^2 + 12x$; $y = 9 + 6x$ **b** $y = 2x^2 + 3x - 4$; $y = 2x - 6$
c $y = 6x^2 - 12x + 6$; $y = x$ **d** $y = x^2 - 8x + 18$; $y = 2x + 3$
e $y = x^2 + x$; $2x + y = 4$ **f** $y = 4x^2 + 9$; $y = 12x$
g $y = 3 - 2x - x^2$; $y = 9 + 2x$ **h** $y = (3 - 2x)^2$; $y = 2 - 3x$

6 Solve the following inequalities and illustrate each solution on a number line:

a $x^2 - 6x + 5 > 0$ **b** $a^2 + 3a - 4 \leqslant 0$ **c** $4 - y^2 > 0$
d $x^2 - 4x + 4 > 0$ **e** $8 - 2a > a^2$ **f** $3y^2 + 2y - 1 > 0$

Real-world activity

Anna would like to design a pendant for her mother and decides that it should resemble an eye. She starts by making the scale drawing, shown below.



The pendant is made up of the shaded area.

The equations of the two circles are $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

The rest of the pendant is formed by quadratic curves.

The scale is 2 units represents 1 cm.

1 Find the equations of the four quadratic curves.
2 Anna decides to make some earrings using a smaller version of the pendant design. She reduces the size by a factor of 2. Find the equations of the four quadratic curves for the earrings.

Past-paper questions

1 (i) Express $2x^2 - x + 6$ in the form $p(x - q)^2 + r$, where p, q and r are constants to be found. [3]

(ii) Hence state the least value of $2x^2 - x + 6$ and the value of x at which this occurs. [2]

Cambridge O Level Additional Mathematics 4037
Paper 21 Q5 June 2014
Cambridge IGCSE Additional Mathematics 0606
Paper 21 Q5 June 2014

2 Find the set of values of k for which the curve $y = 2x^2 + kx + 2k - 6$ lies above the x-axis for all values of x. [4]

Cambridge O Level Additional Mathematics 4037
Paper 12 Q4 June 2013
Cambridge IGCSE Additional Mathematics 0606
Paper 12 Q4 June 2013

3 The line $y = mx + 2$ is a tangent to the curve $y = x^2 + 12x + 18$. Find the possible values of m. [4]

Cambridge O Level Additional Mathematics 4037
Paper 13 Q3 November 2010
Cambridge IGCSE Additional Mathematics 0606
Paper 13 Q3 November 2010

Learning outcomes

Now you should be able to:

- ★ find the maximum or minimum value of the quadratic function f: $x \mapsto ax^2 + bx + c$ by any method
- ★ use the maximum or minimum values of $f(x)$ to sketch the graph or determine the range for a given domain
- ★ know the conditions for $f(x) = 0$ to have two real roots, two equal roots or no real roots and know the related conditions for a given line to intersect a given curve, be a tangent to a given curve or not intersect a given curve
- ★ solve quadratic equations for real roots and find the solution set for quadratic inequalities.

Key points

- ✔ A **quadratic function** has the form $f(x) = ax^2 + bx + c$, where a, b and c can be any number (positive, negative or zero) provided that $a \neq 0$. The set of possible values of x is called the **domain** of the function and the set of y values is called the **range**.
- ✔ To plot the graph of a quadratic function, first calculate the value of y for each value of x in the given range.
- ✔ The graph of a quadratic function is symmetrical about a vertical line. It is $\cup$-shaped if the coefficient of x^2 is positive and $\cap$-shaped if the coefficient of x^2 is negative.
- ✔ To sketch the graph of a quadratic function:
  - look at the coefficient of x^2 to determine the shape
  - substitute $x = 0$ to determine where the curve crosses the vertical axis
  - solve $f(x) = 0$ to determine any values of x where the curve touches or crosses the horizontal axis.
- ✔ If there are no real values for x for which $f(x) = 0$, then the curve will be either completely above or completely below the x-axis.
- ✔ A **quadratic equation** is of the form $ax^2 + bx + c$ with $a \neq 0$.
- ✔ To **factorise** a quadratic equation of the form $x^2 + bx + c = 0$, look for two numbers, p and q, with the sum b and the product c. The factorised form is then $(x - p)(x - q) = 0$. To factorise an equation of the form $ax^2 + bx + c = 0$, look for two numbers with the sum b and the product ac.

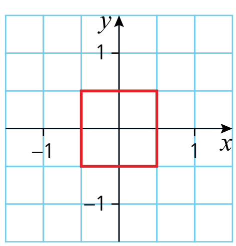
- ✔ The **discriminant** of a quadratic equation ($ax^2 + bx + c = 0$) is $b^2 - 4ac$. If $b^2 - 4ac > 0$, a quadratic equation will have two distinct solutions (or roots). If $b^2 - 4ac = 0$, the two roots are equal so there is one repeating root. If $b^2 - 4ac < 0$, the roots have no real values.
- ✔ An expression of the form $(px + q)^2$ is called a **perfect square**.
- ✔ $x^2 + bx + c$ can be written as $\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$ using the method of **completing the square**. For expressions of the form $ax^2 + bx + c$, first take a out as a factor.
- ✔ The **quadratic formula** for solving an equation of the form $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
- ✔ To find the point(s) where a line and a curve touch or intersect, substitute the expression for y from one equation into the other to give a quadratic equation in x.
- ✔ When solving a **quadratic inequality**, it is advisable to start by sketching the associated quadratic graph.

3 Equations, inequalities and graphs

It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that that we ignore its true merit.

Pierre-Simon, Marquis de Laplace (1749 – 1827)

The picture shows a quadrat. This is a tool used by biologists to select a random sample of ground; once it is in place they will make a record of all the plants and creatures living there. Then they will throw the quadrat so that it lands somewhere else.



This diagram illustrates a 1 metre square quadrat. The centre point is taken to be the origin and the sides to be parallel to the x- and y-axes.

Discussion point

What is the easiest way to describe the region it covers?

Many practical situations involve the use of inequalities.

How economical is a Formula 1 car?

The Monaco Grand Prix, consisting of 78 laps and a total distance of approximately 260 km, is a well-known Formula 1 race. In 2017 it was won by Sebastian Vettel in 1 hour 44 minutes. Restrictions on the amount and use of fuel mean that drivers need to manage the performance of their car very carefully throughout the race.

A restriction in 2017 was that the total amount of fuel used during the race was limited to 105kg which is approximately 140 litres.

Using: f to denote the total amount of fuel used in litres

d to represent the distance travelled in kilometres

E to represent the fuel economy in litres per kilometre $\left(E = \frac{f}{d}\right)$

the restriction can be represented as $E \leqslant \frac{140}{260}$

$$\Rightarrow E \leqslant 0.538$$

This shows that, at worst, the fuel economy of Vettel's Ferrari Formula 1 car is 0.538 litres per kilometre.

Discussion point

How does this compare with an average road car?

Modulus functions and graphs

For any real number x, the modulus of x is denoted by $|x|$ and is defined as:
$|x| = x$ if $x \geqslant 0$
$|x| = -x$ if $x < 0$.

For any real number, the **absolute value**, or **modulus**, is its positive size whether that number is positive or negative. It is denoted by a vertical line on each side of the quantity. For example, $|5| = 5$ and $|-5| = 5$ also. The absolute value of a number will always be positive or zero. It can be thought of as the distance between that point on the x-axis and the origin.

You have already met graphs of the form $y = 3x + 4$ and $y = x^2 - 3x - 4$. However you might not be as familiar with graphs of the form $y = |3x + 4|$ and $y = |x^2 - 3x - 4|$.

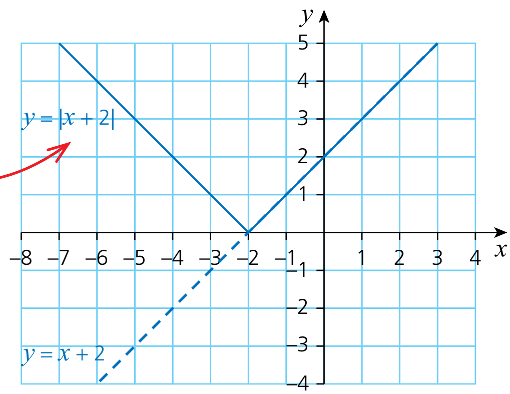
Worked example

Set up a table for the graphs $y = x + 2$ and $y = |x + 2|$ for $-6 \leqslant x \leqslant 2$. Draw both graphs on the same axes.

Solution

y	–6	–5	–4	–3	–2	–1	0	1	2
$x + 2$	–4	–3	–2	–1	0	1	2	3	4
$\lvert x + 2\rvert$	4	3	2	1	0	1	2	3	4

The equation of this part of the graph is $y = -(x + 2)$.



Notice that effect of taking the modulus is a transformation in which the positive part of the original graph (above the x-axis) remains the same and the negative part of the original graph (below the x-axis) is reflected in the x-axis.

Solving modulus equations

Worked example

Solve the equation $|2x + 3| = 5$

a graphically

b algebraically.

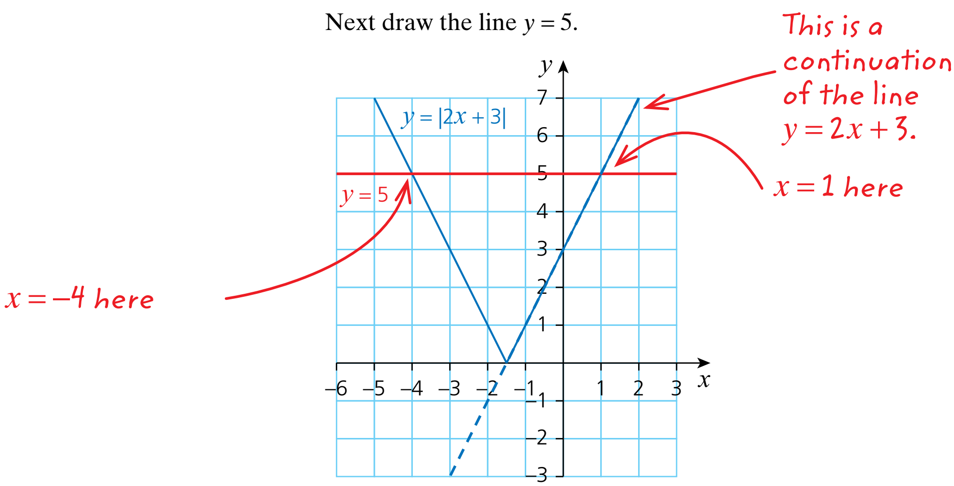
Solution

a First draw the graph of $y = 2x + 3$.

Start by choosing three values of x and calculating the corresponding values of y, for example, $(-2, -1)$, $(0, 3)$ and $(2, 7)$.

Then reflect in the x-axis any part of the graph that is below the x-axis to give the graph of $y = |2x + 3|$.

Next draw the line $y = 5$.



The solution is given by the values of x where the V-shaped graph meets the line $y = 5$

$\Rightarrow x = 1$ or $x = -4$.

b $|2x + 3| = 5 \Rightarrow 2x + 3 = 5$ or $2x + 3 = -5$

$\Rightarrow 2x = 2$ or $2x = -8$

$\Rightarrow x = 1$ or $x = -4$

Discussion point

Notice that in the solution three points are used to draw the straight line when only two are necessary. Why is this good practice?

Either of these methods can be extended to find the points where two V-shaped graphs intersect. However, the graphical method will not always give an accurate solution.

Worked example

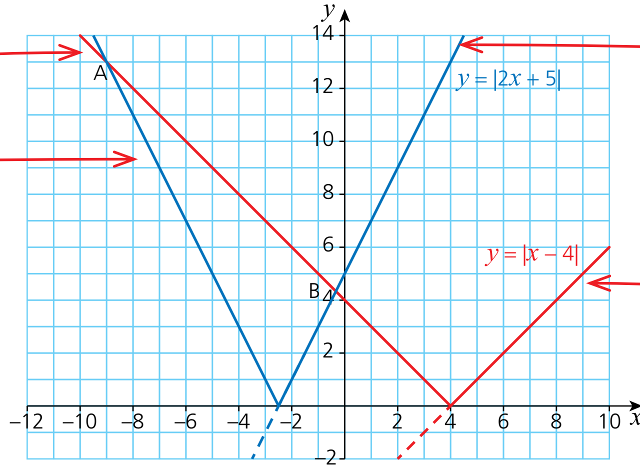
Solve the equation $|2x + 5| = |x - 4|$.

Solution

Start by drawing the graphs of $y = |2x + 5|$ and $y = |x - 4|$ on the same axes.

This is part of the line $y = -(x - 4)$.

This is part of the line $y = -(2x + 5)$.



This is part of the line $y = 2x + 5$.

This is part of the line $y = x - 4$.

This shows a failing of the graphical method. However, the graph is useful in determining the equation of the line required for an algebraic solution.

The graph shows that the point A is (–9, 13), but the coordinates of B are not clear.

The graph shows that both points of intersection occur where the reflected part of the line $y = x - 4$, i.e. the line $y = -(x - 4)$ intersects the graph of $y = |2x + 5|$.

At A, $y = 4 - x$ meets $y = -2x - 5$

$\Rightarrow 4 - x = -2x - 5$

$\Rightarrow 2x - x = -5 - 4$

$\Rightarrow x = -9$

When $x = -9$, $y = 4 - (-9) = 13$, i.e. A is the point (–9, 13).

At B, $y = 4 - x$ meets $y = 2x + 5$

$\Rightarrow 2x + 5 = 4 - x$

$\Rightarrow 3x = -1$

$\Rightarrow x = -\frac{1}{3}$

When $x = -\frac{1}{3}$, $y = 4 - \left(-\frac{1}{3}\right) = 4\frac{1}{3}$, i.e. B is the point $\left(-\frac{1}{3}, 4\frac{1}{3}\right)$.

Exercise 3.1

For questions 1–3, sketch each pair of graphs on the same axes.

1 **a** $y = x$ and $y = |x|$ **b** $y = x - 1$ and $y = |x - 1|$
c $y = x - 2$ and $y = |x - 2|$

2 **a** $y = 2x$ and $y = |2x|$ **b** $y = 2x - 1$ and $y = |2x - 1|$
c $y = 2x - 2$ and $y = |2x - 2|$

3 **a** $y = 2 - x$ and $y = |2 - x|$ **b** $y = 3 - x$ and $y = |3 - x|$
c $y = 4 - x$ and $y = |4 - x|$

Exercise 3.1 (cont)

4 **a** Draw the graph of $y = |x + 1|$.
b Use the graph to solve the equation $|x + 1| = 5$.
c Use algebra to verify your answer to part **b**.

5 **a** Draw the graph of $y = |x - 1|$.
b Use the graph to solve the equation $|x - 1| = 5$.
c Use algebra to verify your answer to part **b**.

6 **a** Draw the graph of $y = |2x + 3|$.
b Use the graph to solve the equation $|2x + 3| = 7$.
c Use algebra to verify your answer to part **b**.

7 **a** Draw the graph of $y = |2x - 3|$.
b Use the graph to solve the equation $|2x - 3| = 7$.
c Use algebra to verify your answer to part **b**.

8 Solve the equation $|x + 1| = |x - 1|$ both graphically and algebraically.

9 Solve the equation $|x + 5| = |x - 5|$ both graphically and algebraically.

10 Solve the equation $|2x + 4| = |2x - 4|$ both graphically and algebraically.

Solving modulus inequalities

When illustrating an inequality in one variable:

- An open circle at the end of a line shows that the end point is excluded.
- A solid circle at the end of a line shows that the end point is included.
- The line is drawn either in colour or as a solid line.

For example, the inequality $-2 < x \leqslant 3$ is shown as:

Worked example

a Solve algebraically the inequality $|x - 3| > 2$.

b Illustrate the solution on a number line.

An open circle is used to show that the value there is not part of the solution.

Solution

a $|x - 3| > 2 \Rightarrow x - 3 > 2$ or $x - 3 < -2$

$\Rightarrow x > 5$ or $x < 1$

The blue lines show the required parts of the number line.

b

Worked example

Write the inequality $-3 \leqslant x \leqslant 9$ in the form $|x - a| \leqslant b$ and show a and b on a number line.

Solution

$|x - a| \leqslant b \quad \Rightarrow \quad -b \leqslant x - a \leqslant b$

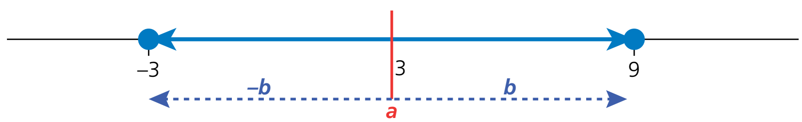
$\quad \Rightarrow \quad a - b \leqslant x \leqslant a + b$

You are finding the values of x **within** $a \pm b$.

Solve $a + b = 9$ and $a - b = -3$ simultaneously.

Adding: $2a = 6$, so $a = 3$

Subtracting: $2b = 12$, so $b = 6$



Substituting in $a - b \leqslant x \leqslant a + b$ gives $-3 \leqslant x \leqslant 9$.
Substituting in $|x - a| \leqslant b$ gives $|x - 3| \leqslant 6$.

Worked example

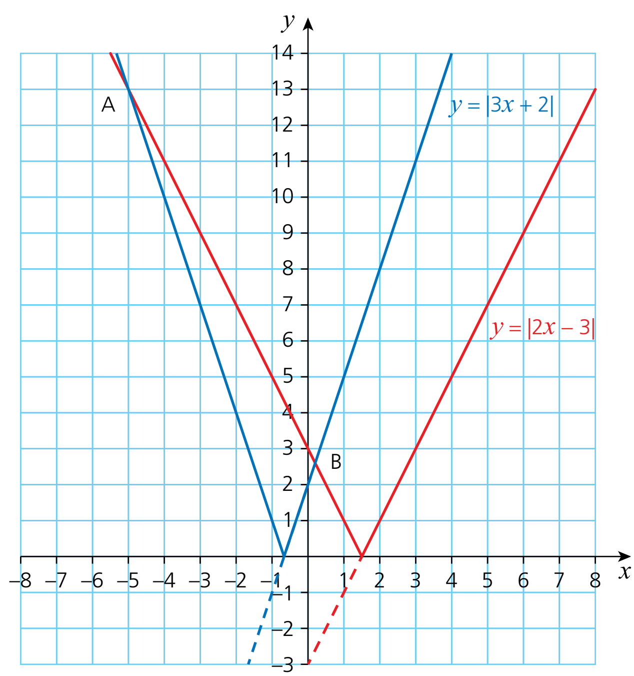
Solve the inequality $|3x + 2| \leqslant |2x - 3|$.

Solution

Draw the graphs of $y = |3x + 2|$ and $y = |2x - 3|$. The inequality is true for values of x where the unbroken blue line is below or crosses the unbroken red line, i.e. between (and including) the points A and B.

Draw the line $y = 3x + 2$ as a straight line through $(0, 2)$ with a gradient of $+3$. Reflect in the x-axis the part of the line that is below this axis.

Draw the line $y = 2x - 3$ as a straight line through $(0, -3)$ with a gradient of $+2$. Reflect in the x-axis the part of the line that is below this axis.



The graph shows that $x = -5$ at A, but the exact value for x at B is not clear. The algebraic solution gives a more precise value.

At A, $-(3x + 2) = -(2x - 3) \Rightarrow 3x + 2 = 2x - 3$

$$\Rightarrow x = -5$$

Substituting in either of the equations gives $y = 13$, so A is the point $(-5, 13)$.

At B, $3x + 2 = -(2x - 3) \Rightarrow 3x + 2 = -2x + 3$

$$\Rightarrow 5x = 1$$

$$\Rightarrow x = 0.2$$

Substituting in either equation gives $y = 2.6$, so B is the point $(0.2, 2.6)$.

The inequality is satisfied for values of x between A and B, i.e. for $-5 \leqslant x \leqslant 0.2$.

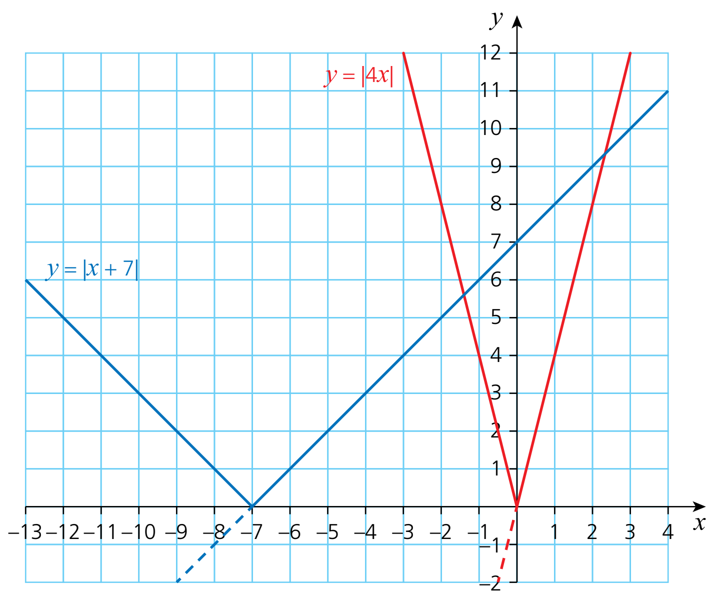
Worked example

Solve the inequality $|x + 7| < |4x|$.

Solution

The question does not stipulate a particular method, so start with a sketch graph.

The sketch graph shows that the points where the two graphs intersect do not have integer coordinates. This means that a graphical method is unlikely to give an accurate solution.



Use algebra to find the points when $|x + 7| = |4x|$, i.e. when $x + 7 = 4x$ and when $x + 7 = -4x$.

Discussion point

Why is it sufficient to consider only these two cases? Why do you not need to consider when $-(x + 7) = 4x$?

$x + 7 = 4x$ when $x = \frac{7}{3}$.

$x + 7 = -4x$ when $x = -\frac{7}{5}$.

Think about a point to the left of $x = -\frac{7}{5}$, such as $x = -2$.

This tells you that part of the solution is to the left of $x = -\frac{7}{5}$, i.e. $x < -\frac{7}{5}$.

When $x = -2$, $|x + 7| < |4x|$ gives $5 < 8$. This is true so the inequality is satisfied.

Next think about a value of x in the interval $\left(-\frac{7}{5}, \frac{7}{3}\right)$, for example, $x = 0$.

When $x = 0$, $|x + 7| < |4x|$ gives $7 < 0$, which is false.

Any other value in this interval will also give a false result.

Finally consider a value greater than $\frac{7}{3}$, for example, $x = 3$.

When $x = 3$, $|x + 7| < |4x|$ gives $10 < 12$. This is true so the inequality is satisfied.

Therefore, the solution is $x < -\frac{7}{5}$ or $x > \frac{7}{3}$.

Inequalities in two dimensions are illustrated by regions. For example, $x > 2$ is shown by the part of the x–y plane to the right of the line $x = 2$ and $x < -1$ by the part to the left of the line $x = -1$.

If you are asked to illustrate the region $x \geqslant 2$, then the line $x = 2$ must be included as well as the region $x > 2$.

Note

- When the boundary line is included, it is drawn as a **solid line**; when it is excluded, it is drawn as a **dotted line**.
- The answer to an inequality of this type is **a region of the x–y plane**, not simply a set of points. It is common practice to specify the region that you want (called the **feasible region**) by **shading out the unwanted region**. This keeps the feasible region clear so that you can see clearly what you are working with.

Worked example

Illustrate the inequality $3y - 2x \geqslant 0$ on a graph.

Solution

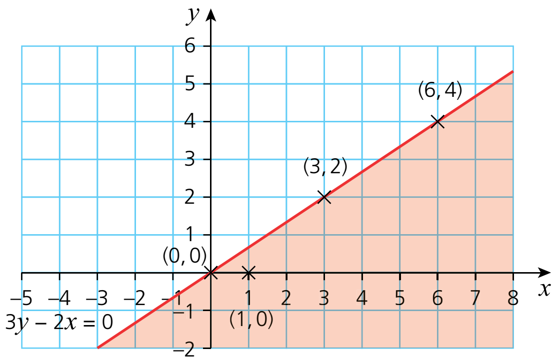
Draw the line $3y - 2x = 0$ as a **solid** line through (0, 0), (3, 2) and (6, 4).

Discussion point

Why are these points more suitable than, for example, $\left(1, \frac{2}{3}\right)$?

Choose a point which is not on the line as a test point, for example, (1, 0).

Using these values, $3y - 2x = -2$. This is clearly not true, so this point is not in the feasible region. Therefore shade out the region containing the point (1, 0).



Exercise 3.2

1 Write each of the following inequalities in the form $|x - a| \leqslant b$:
a $-3 \leqslant x \leqslant 15$ b $-4 \leqslant x \leqslant 16$ c $-5 \leqslant x \leqslant 17$

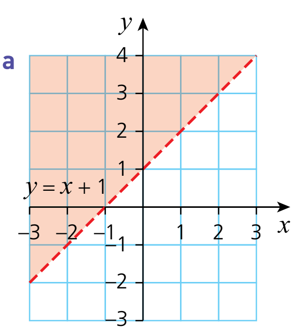
2 Write each of the following expressions in the form $a \leqslant x \leqslant b$:
a $|x - 1| \leqslant 2$ b $|x - 2| \leqslant 3$ c $|x - 3| \leqslant 4$

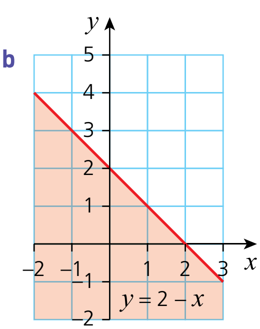
3 Solve the following inequalities and illustrate each solution on a number line:
a $|x - 1| < 4$ b $|x - 1| > 4$ c $|2x + 3| < 5$
d $|2x + 3| > 5$

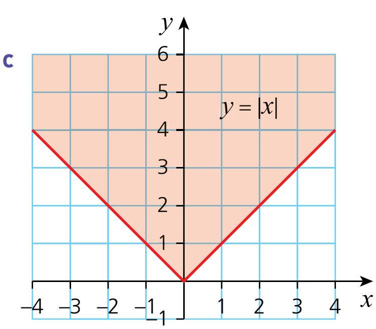
4 Illustrate each of the following inequalities graphically by shading the unwanted region:
a $y - 2x > 0$ b $y - 2x \leqslant 0$ c $2y - 3x > 0$
d $2y - 3x \leqslant 0$

5 Solve the following inequalities:
i graphically ii algebraically.
a $|x - 1| < |x + 1|$ b $|x - 1| > |x + 1|$
c $|2x - 1| \leqslant |2x + 1|$ d $|2x - 1| \geqslant |2x + 1|$

6 Each of the following graphs represents an inequality. Name the inequality.

a


b


c


Using substitution to solve quadratic equations

Sometimes you will meet an equation which includes a square root. Although this is not initially a quadratic equation, you can use a substitution to solve it in this way, as shown in the following example.

Worked example

Use the substitution $x = u^2$ to solve the equation $x - 3\sqrt{x} = -2$.

Solution

Substituting $x = u^2$ in the equation $x - 3\sqrt{x} = -2$ gives $u^2 - 3u = -2$

$\Rightarrow u^2 - 3u + 2 = 0$

Factorising

$\Rightarrow (u - 1)(u - 2) = 0$

$\Rightarrow u = 1$ or $u = 2$

Since $x = u^2$, $x = 1$ or $x = 4$.

Checking these values

When $x = 1$, $1 - 3\sqrt{1} = -2$, so $x = 1$ is a valid solution.

When $x = 4$, $4 - 3\sqrt{4} = -2$, so $x = 4$ is also a valid solution.

It is always advisable to check possible solutions, since in some cases not all values of u will give a valid solution to the equation, as shown in the following example.

Worked example

Solve the equation $x - \sqrt{x} = 6$.

Solution

Substituting $x = u^2$ in the equation $x - \sqrt{x} = 6$ gives $u^2 - u = 6$

$\Rightarrow u^2 - u - 6 = 0$

Factorising

$\Rightarrow (u - 3)(u + 2) = 0$

$\Rightarrow u = 3$ or $u = -2$

Since $x = u^2$, $x = 9$ or $x = 4$.

Checking these values

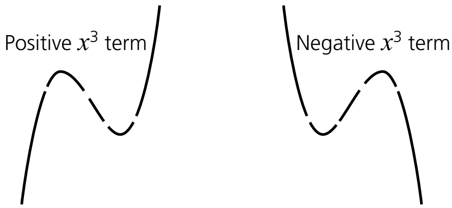
When $x = 9$, $9 - \sqrt{9} = 6$, so $x = 9$ is a possible solution.

When $x = 4$, $4 - \sqrt{4} = 2$, so reject $x = 4$ as a possible solution.

The only solution to this equation is $x = 9$.

Using graphs to solve cubic inequalities

Cubic graphs have distinctive shapes determined by the coefficient of x^3.



The centre part of each of these curves may not have two distinct turning points like those shown above, but may instead 'flatten out' to give a **point of inflection.** When the modulus of a cubic function is required, any part of the curve below the x-axis is reflected in that axis.

Worked example

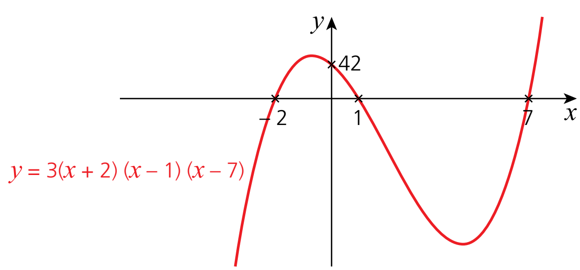
a Sketch the graph of $y = 3(x + 2)(x - 1)(x - 7)$. Identify the points where the curve cuts the axes.

b Sketch the graph of $y = |3(x + 2)(x - 1)(x - 7)|$.

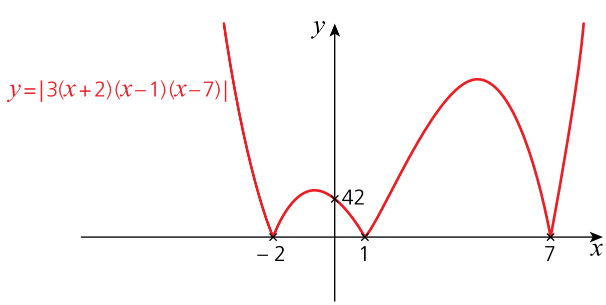
Solution

a The curve crosses the x-axis at -2, 1 and 7. Notice that the distance between consecutive points is 3 and 6 units respectively, so the y-axis is between the points -2 and 1 on the x-axis, but closer to the 1.

The curve crosses the y-axis when $x = 0$, i.e. when $y = 3(2)(-1)(-7) = 42$.



b To obtain a sketch of the modulus curve, reflect any part of the curve which is below the x-axis in the x-axis.



You are asked for a sketch graph, so although it must show the main features, it does not need to be absolutely accurate. You may find it easier to draw the curve first, with the positive x^3 term determining the shape of the curve, and then position the x-axis so that the distance between the first and second intersections is about half that between the second and third, since these are 3 and 6 units respectively.

Worked example

Solve the inequality $3(x + 2)(x - 1)(x - 7) \leqslant -100$ graphically.

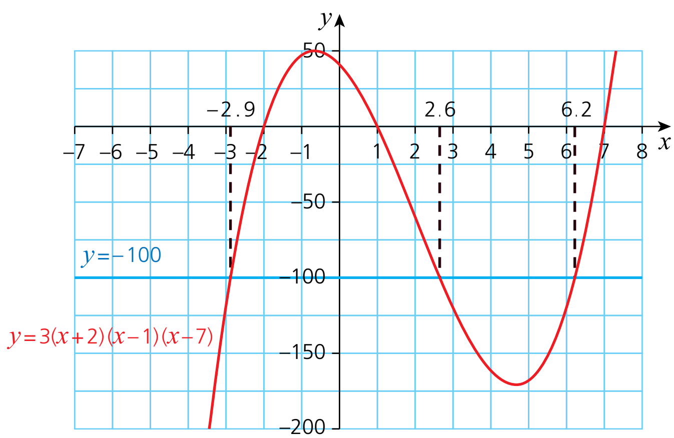
Solution

Because you are solving the inequality graphically, you will need to draw the curve as accurately as possible on graph paper, so start by drawing up a table of values.

$y = 3(x + 2)(x - 1)(x - 7)$

x	−3	−2	−1	0	1	2	3	4	5	6	7	8
$(x + 2)$	−1	0	1	2	3	4	5	6	7	8	9	10
$(x - 1)$	−4	−3	−2	−1	0	1	2	3	4	5	6	7
$(x - 7)$	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
x	−120	0	48	42	0	−60	−120	−162	−168	−120	0	210

The solution is given by the values of x that correspond to the parts of the curve on or below the line $y = -100$.



From the graph, the solution is $x \leqslant -2.9$ or $2.6 \leqslant x \leqslant 6.2$.

Exercise 3.3

1 Where possible, use the substitution $x = u^2$ to solve the following equations:

a $x - 4\sqrt{x} = -4$
b $x + 2\sqrt{x} = 8$
c $x - 2\sqrt{x} = 15$
d $x + 6\sqrt{x} = -5$

2 Sketch the following graphs, indicating the points where they cross the x-axis:

a $y = x(x - 2)(x + 2)$
b $y = |x(x - 2)(x + 2)|$
c $y = 3(2x - 1)(x + 1)(x + 3)$
d $y = |3(2x - 1)(x + 1)(x + 3)|$

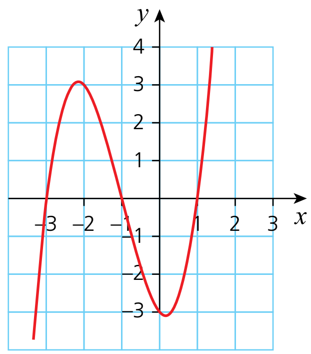
3 Solve the following equations graphically. You will need to use graph paper.

a $x(x + 2)(x - 3) \geqslant 1$
b $x(x + 2)(x - 3) \leqslant -1$
c $(x + 2)(x - 1)(x - 3) > 2$
d $(x + 2)(x - 1)(x - 3) < -2$

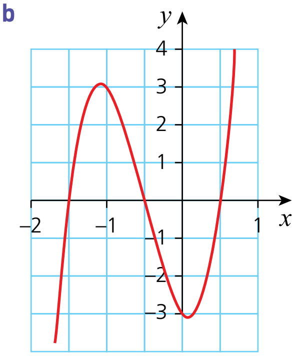
Exercise 3.3 (cont)

4 Identify the following cubic graphs:

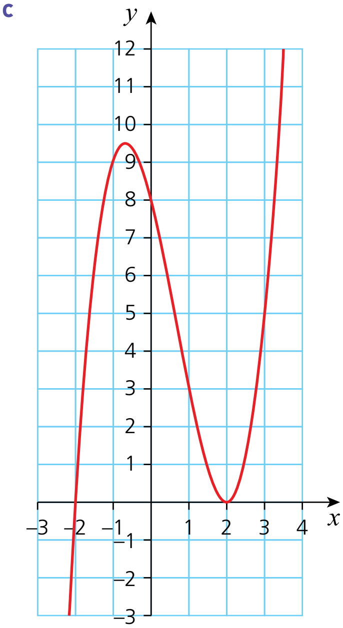
a



b

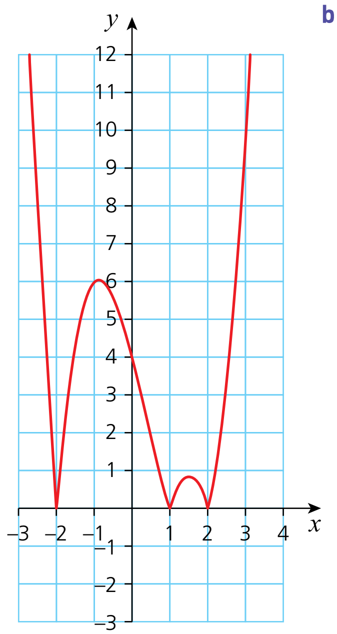


c

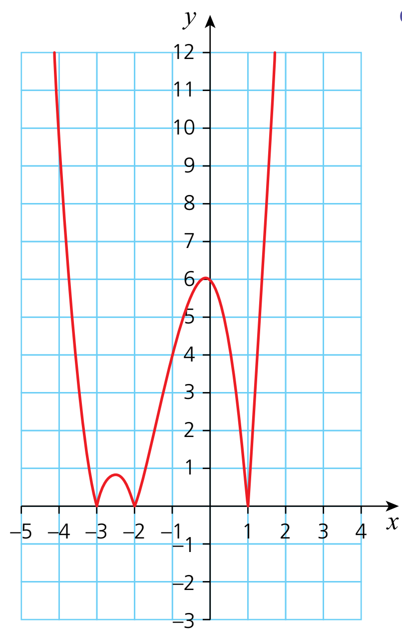


5 Identify these graphs. (They are the moduli of cubic graphs.)

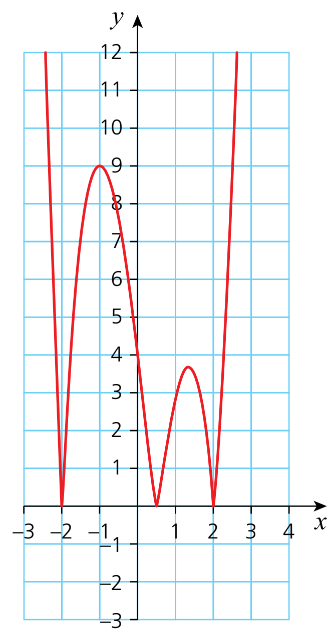
a



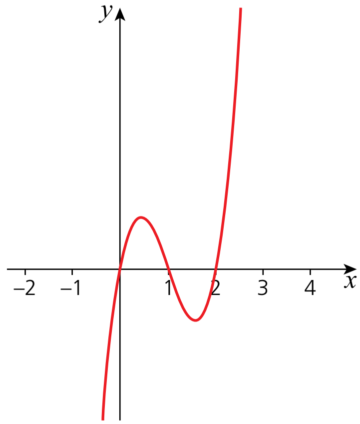
b



c



6 Why is it not possible to identify the following graph without further information?



Past-paper questions

1 **(i)** Sketch the graph of $y = |(2x + 3)(2x - 7)|$. [4]

(ii) How many values of x satisfy the equation $|(2x + 3)(2x - 7)| = 2x$? [2]

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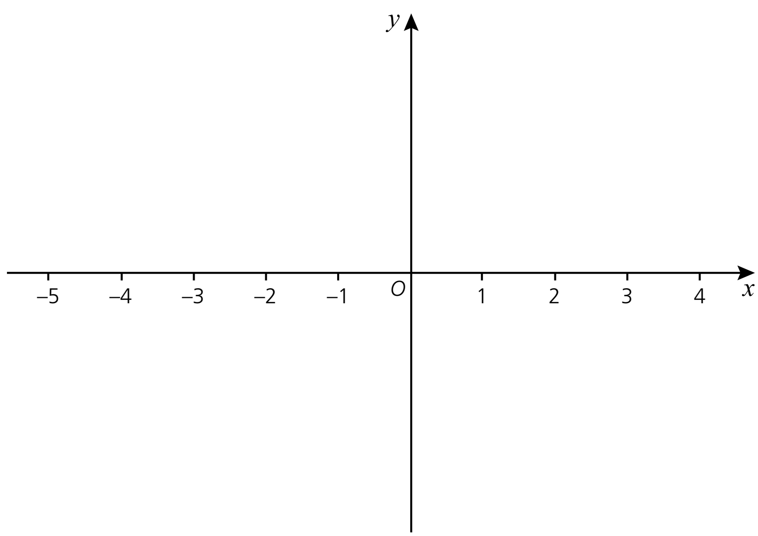
2 **(i)** On a grid like the one below, sketch the graph of $y = |(x - 2)(x + 3)|$ for $-5 \leqslant x \leqslant 4$, and state the coordinates of the points where the curve meets the coordinate axes. [4]



(ii) Find the coordinates of the stationary point on the curve $y = |(x-2)(x+3)|$. [2]

(iii) Given that k is a positive constant, state the set of values of k for which $|(x-2)(x+3)| = k$ has 2 solutions only. [1]

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3 Solve the inequality $9x^2 + 2x - 1 < (x+1)^2$. [3]

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Paper 22 Q2 November 2014

Learning outcomes

Now you should be able to:

- ★ solve graphically or algebraically equations of the type $|ax + b| = c$ $(c \geqslant 0)$ and $|ax + b| = |cx + d|$
- ★ solve graphically or algebraically inequalities of the type $|ax + b| > c$ $(c \geqslant 0)$, $|ax + b| \leqslant c$ $(c > 0)$and $|ax + b| \leqslant (cx + d)$
- ★ use substitution to form and solve a quadratic equation in order to solve a related equation
- ★ sketch the graphs of cubic polynomials and their moduli, when given in factorised form $y = k(x - a)(x - b)(x - c)$
- ★ solve cubic inequalities in the form $k(x - a)(x - b)(x - c) \leqslant d$ graphically.

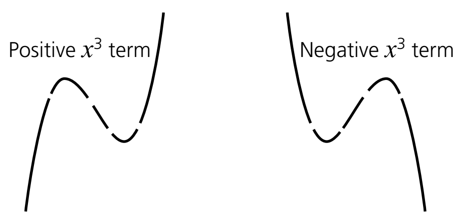
Key points

- ✔ For any real number x, the **modulus** of x is denoted by $|x|$ and is defined as:

  $|x| = x$ if $x \geqslant 0$
  $|x| = -x$ if $x < 0$.

- ✔ A modulus equation of the form $|ax + b| = b$ can be solved either graphically or algebraically.
- ✔ A modulus equation of the form $|ax + b| = |cx + d|$ can be solved graphically by first drawing both graphs on the same axes and then, if necessary, identifying the solution algebraically.

- ✔ A modulus inequality of the form $|x - a| < b$ is equivalent to the inequality $a - b < x < a + b$ and can be illustrated on a number line with an open circle marking the ends of the interval to show that these points are not included. For $|x - a| \leqslant b$, the interval is the same but the end points are marked with solid circles.
- ✔ A modulus inequality of the form $|x - a| > b$ or $|x - a| \geqslant b$ is represented by the parts of the line outside the intervals above.
- ✔ A modulus inequality in two dimensions is identified as a region on a graph, called the **feasible region.** It is common practice to shade out the region not required to keep the feasible region clear.
- ✔ It is sometimes possible to solve an equation involving both x and $\sqrt{x}$ by making a substitution of the form $x = u^2$. You must check all answers in the original equation.
- ✔ The graph of a cubic function has a distinctive shape determined by the coefficient of x^3.



4 Indices and surds

An estate had seven houses;
Each house had seven cats;
Each cat ate seven mice;
Each mouse ate seven grains of wheat.
Wheat grains, mice, cats and houses,
How many were there on the estate?

Ancient Egyptian problem

Discussion point

How can you write down the answer to this problem without doing any calculations?

Indices

The word 'index' (plural indices) has many meanings in real life including a list of names, the index for a book and a price index, but the focus in this chapter is, of course, related to numbers.

Index notation is a shorthand way of writing numbers.

For example, $2 \times 2 \times 2 \times 2 \times 2$ can be written as 2^5.

2 is the base number.
5 is the index or power.

Operations using indices

There are a number of rules that you need to learn when you are working with indices.

Multiplying

$5^4 \times 5^3 = (5 \times 5 \times 5 \times 5) \times (5 \times 5 \times 5)$

$= 5^7$

Rule: To multiply numbers in index form where the base number is the same, add the indices.
$a^m \times a^n = a^{m+n}$

Dividing

$$3^7 \div 3^4 = \frac{3 \times 3 \times 3 \times \cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3}}$$

$= 3^3$

Rule: To divide numbers in index form where the base number is the same, subtract the second index from the first.
$a^m \div a^n = a^{m-n}$

A power raised to a power

Rule: To raise a power to a power, multiply the indices. $(a^m)^n = a^{m \times n}$

$$(2^3)^4 = (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (2 \times 2 \times 2)$$
$$= 2^{12}$$

Index zero

Using the rule for division, $6^3 \div 6^3 = 6^{3-3}$
$= 6^0$

Rule: Any number with an index zero equals one. $a^0 = 1$

However, dividing a number by itself always gives the result 1, so $6^0 = 1$.

Negative indices

$$4^2 \div 4^5 = \frac{4 \times 4}{4 \times 4 \times 4 \times 4 \times 4}$$
$$= \frac{1}{4^3}$$

Using the rule for division, $4^2 \div 4^5 = 4^{2-5}$
$= 4^{-3}$

Rule: $\frac{1}{a^m} = a^{-m}$

Worked example

Write $\left(\frac{3}{2}\right)^{-3}$ as a fraction.

Solution

$$\left(\frac{3}{2}\right)^{-3} = 1 \div \left(\frac{3}{2}\right)^3$$
$$= 1 \times \left(\frac{2}{3}\right)^3$$
$$= \frac{8}{27}$$

Fractional indices

What number multiplied by itself equals 5?

The answer to this is usually written as $\sqrt{5}$, but it can also be written in index form.

Let $5^p \times 5^p = 5$

Using the rule for multiplication, $5^{2p} = 5^1$ so $p = \frac{1}{2}$

This gives $\sqrt{5} = 5^{\frac{1}{2}}$

Similarly, $\sqrt[3]{5} = 5^{\frac{1}{3}}$

Rule: $\sqrt[n]{a} = a^{\frac{1}{n}}$

These rules can be combined further to give other rules.

Replacing n by $\frac{1}{n}$ in the rule $(a^m)^n = a^{mn}$ gives the result

$$(a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$$

This can also be written as $\left(a^{\frac{1}{n}}\right)^m$ or $(\sqrt[n]{a})^m$

Worked example

Calculate $25^{\frac{3}{2}}$

Solution

$$25^{\frac{3}{2}} = \left(25^{\frac{1}{2}}\right)^3$$
$$= \left(\sqrt{25}\right)^3$$
$$= 5^3$$
$$= 125$$

It is usually more straightforward to use the fractional index first since you are then using smaller numbers and are more likely to recognise the values.

Worked example

Simplify the following, leaving your answers in standard form:

a $(5\times10^5)\times(4\times10^2)$ **b** $(8\times10^5)\div(4\times10^2)$

Solution

a $(5\times10^5)\times(4\times10^2) = (5\times4)\times(10^5\times10^2)$
$= 20\times10^7$
$= 2\times10^8$

b $(8\times10^5)\div(4\times10^2) = (8\div4)\times(10^5\div10^2)$
$= 2\times10^3$

Exercise 4.1

Use a calculator to check your results only.

1 Simplify the following, giving your answers in the form x^n:
a $2^3\times2^7$ **b** $5^{-3}\times5^4$ **c** $3^6\div3^3$
d $6^5\div6^{-4}$ **e** $(4^2)^3$ **f** $(5^2)^{-2}$

2 Simplify the following, leaving your answers in standard form:
a $(3\times10^5)\times(2\times10^9)$ **b** $(2\times10^4)\times(3\times10^{-3})$
c $(8\times10^5)\div10^3$ **d** $(9\times10^9)\div(3\times10^{-3})$

3 Rewrite each of the following as a number raised to a positive integer power:
a 3^{-2} **b** 5^{-4} **c** $\left(\frac{2}{3}\right)^{-3}$ **d** $\left(\frac{1}{3}\right)^{-6}$

4 Simplify the following, leaving your answers in standard form:
a $(5\times10^7)\times(3\times10^{-3})$ **b** $(4\times10^{-2})\times(6\times10^4)$
c $(4\times10^7)\div(8\times10^{-2})$ **d** $(3\times10^3)\div(6\times10^6)$

5 Find the value of each of the following, giving your answer as a whole number or fraction:
a $(3^4\times3^{-2})$ **b** $6^{-6}\times6^6$ **c** $5^5\div5^2$ **d** $(2^3)^4$
e $(3^2)^2$ **f** 7^{-2} **g** $\left(\frac{1}{4}\right)^{-3}$ **h** 2^{-5}
i $\left(\frac{3}{4}\right)^{-2}$ **j** $9^{\frac{1}{2}}$ **k** $81^{\frac{1}{4}}$ **l** $16^{\frac{3}{2}}$
m $27^{\frac{2}{3}}$ **n** $256^{-\frac{1}{4}}$ **o** $128^{-\frac{5}{7}}$

6 Rank each set of numbers in order of increasing size:
a $3^5, 4^4, 5^3$ b $2^7, 3^5, 4^4$ c $2^{-5}, 3^{-4}, 4^{-3}$

7 Find the value of x in each of the following:
a $\dfrac{3^3 \times 3^6}{3^7 \div 3^5} = 3^x$ b $\dfrac{(2^2 \times 2^4)^3}{2 \times 2^3} = 2^x$
c $\dfrac{5^4 \times 5^3 \times 5^2}{5^2 \times 5^x} = 5^6$ d $\dfrac{(7^x \times 7^3)^2}{7^4 \div 7^2} = 7^3$
e $\left(\dfrac{1}{2}\right)^x = 8$ f $4^x = \dfrac{1}{64}$
g $2^x = 0.125$ h $4^x = 0.0625$

8 Simplify the following:
a $3a^2 \times 2a^5$ b $6x^4y^2 \times 2xy^{-4}$
c $10b^5 \div 2b^2$ d $12p^{-4}q^{-3} \div 3p^2q^2$
e $(4m)^3$ f $(2s^2t)^6$

9 Find integers x and y such that $2x \times 3y = 6^4$.

Surds

Surds are irrational numbers that cannot be expressed exactly. $\sqrt{2}$, $\sqrt{3}$, $2+\sqrt{3}$ and $\sqrt{5} - \sqrt{3}$ are all examples of surds.

Discussion point

Why is $\sqrt{4}$ not a surd?

Although your calculator will simplify expressions containing square roots of numbers, it is often easier to work with surds in their exact form. You need to know the rules for manipulating surds so you can work with them in an algebraic setting such as $(\sqrt{a} + \sqrt{b})(3\sqrt{a} - 4\sqrt{b})$.

Operations using surds

Simplifying surds

To simplify a surd, start by writing the number under the square root sign as a product of two factors, one of which is the largest possible perfect square.

Worked example

Simplify $\sqrt{18}$.

Solution

Writing $\sqrt{18} = \sqrt{6} \times \sqrt{3}$ does not help since neither 6 nor 3 is a perfect square.

$$\begin{aligned}\sqrt{18} &= \sqrt{9 \times 2}\\ &= \sqrt{9} \times \sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

Worked example

Simplify $\sqrt{\frac{2}{9}}$.

Solution

$$\sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{\sqrt{9}}$$
$$= \frac{\sqrt{2}}{3}$$

Adding and subtracting surds

You can add and subtract surds using the same methods as for other algebraic expressions, keeping the rational numbers and the square roots separate.

For example:

$$\sqrt{3} + \sqrt{3} = 2\sqrt{3}$$
$$3\sqrt{5} + 2\sqrt{5} = 5\sqrt{5}$$
$$\sqrt{48} - \sqrt{12} = 4\sqrt{3} - 2\sqrt{3}$$
$$= 2\sqrt{3}$$

Expanding brackets containing surds

To expand brackets containing surds, use the same methods that you use for other algebraic operations.

Worked example

Simplify $\sqrt{3}\left(\sqrt{3} + 2\right)$.

Solution

$$\sqrt{3}\left(\sqrt{3} + 2\right) = \left(\sqrt{3}\right)^2 + 2\sqrt{3}$$
$$= 3 + 2\sqrt{3}$$

Worked example

Simplify $\sqrt{2}(\sqrt{6} + \sqrt{2})$.

Solution

$$\sqrt{2}\left(\sqrt{6} + \sqrt{2}\right) = \sqrt{12} + \left(\sqrt{2}\right)^2$$
$$= \left(\sqrt{4} \times \sqrt{3}\right) + 2$$
$$= 2\sqrt{3} + 2$$

The same principles apply to expanding a pair of brackets. Use whichever method/s you feel most comfortable with.

Worked example

Simplify $(\sqrt{3}+\sqrt{2})(\sqrt{3}+\sqrt{2})$.

Solution

$$(\sqrt{3}+\sqrt{2})(\sqrt{3}+\sqrt{2}) = (\sqrt{3})^2 + \sqrt{6} + \sqrt{6} + (\sqrt{2})^2$$
$$= 3 + 2\sqrt{6} + 2$$
$$= 5 + 2\sqrt{6}$$

When the brackets have the form of the difference of two squares, the result is a numerical value.

Worked example

Simplify $(3+\sqrt{5})(3-\sqrt{5})$.

Solution

$$(3+\sqrt{5})(3-\sqrt{5}) = 3^2 - (\sqrt{5})^2$$
$$= 9 - 5$$
$$= 4$$

Rationalising the denominator

Multiplying both the top line (numerator) and the bottom line (denominator) of any fraction by the same amount doesn't change the value of the fraction. You can use this to rationalise the denominator of a fraction involving a surd, i.e. eliminate the surd so that the denominator has an integer value.

Worked example

Simplify $\frac{9}{\sqrt{3}}$.

Solution

Multiplying the numerator and denominator by $\sqrt{3}$

For all values of a, $\sqrt{a} \times \sqrt{a} = a$.

$$\frac{9}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{9\sqrt{3}}{3}$$
$$= 3\sqrt{3}$$

Worked example

Simplify $\frac{1}{(\sqrt{5}-1)}$.

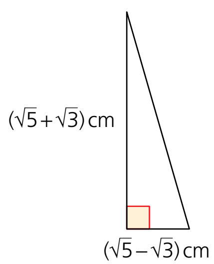
Solution

Using the technique for the difference of two squares, multiply the top and bottom of the fraction by $(\sqrt{5}+1)$.

$$\begin{aligned}\frac{1}{(\sqrt{5}-1)} &= \frac{1}{(\sqrt{5}-1)} \times \frac{(\sqrt{5}+1)}{(\sqrt{5}+1)}\\ &= \frac{\sqrt{5}+1}{5-1}\\ &= \frac{\sqrt{5}+1}{4}\\ &= \frac{1+\sqrt{5}}{4}\end{aligned}$$

Worked example

A right-angled triangle has shorter sides of lengths $(\sqrt{5}+\sqrt{3})$ cm and $(\sqrt{5}-\sqrt{3})$ cm. Work out the length of the hypotenuse.



Solution

Let the length of the hypotenuse be h.

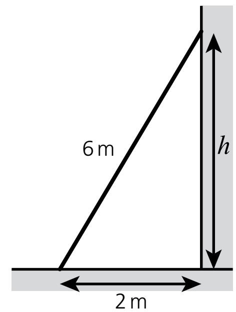
Using Pythagoras' theorem

$$\begin{aligned}h^2 &= \left(\sqrt{5}+\sqrt{3}\right)^2 + \left(\sqrt{5}-\sqrt{3}\right)^2\\ &= \left(5+2\sqrt{15}+3\right)+\left(5-2\sqrt{15}+3\right)\\ &= 8+2\sqrt{15}+8-2\sqrt{15}\\ &= 16\end{aligned}$$

The length of the hypotenuse is 4 cm.

Worked example

A ladder of length 6 m is placed 2 m from a vertical wall at the side of a house. How far up the wall does the ladder reach? Give your answer as a surd in its simplest form.



Solution

Using Pythagoras' theorem

$$6^2 = 2^2 + h^2$$
$$\Rightarrow \quad 36 = 4 + h^2$$
$$\Rightarrow \quad 32 = h^2$$
$$\Rightarrow \quad h = \sqrt{32}$$
$$= 4\sqrt{2}$$

h must be positive.

The ladder reaches $4\sqrt{2}$ metres up the wall.

Exercise 4.2

1 Write each of the following in its simplest form:

a $\sqrt{12}$ **b** $\sqrt{75}$ **c** $\sqrt{300}$

d $3\sqrt{5} + 6\sqrt{5}$ **e** $\sqrt{48} + \sqrt{27}$ **f** $3\sqrt{45} - 2\sqrt{20}$

2 Express each of the following as the square root of a single number:

a $3\sqrt{6}$ **b** $5\sqrt{5}$ **c** $12\sqrt{3}$ **d** $10\sqrt{17}$

3 Simplify the following:

a $\sqrt{\frac{25}{49}}$ **b** $\sqrt{\frac{24}{9}}$ **c** $\sqrt{\frac{12}{15}}$ **d** $\sqrt{\frac{6}{121}}$

4 Simplify the following by collecting like terms:

a $(3+\sqrt{2})+(5+4\sqrt{2})$ **b** $4(\sqrt{3}-1)+4(\sqrt{3}+1)$

5 Expand and simplify:

a $(\sqrt{3}+2)(\sqrt{3}-2)$ **b** $\sqrt{3}(5-\sqrt{3})$

c $(4+\sqrt{2})^2$ **d** $(\sqrt{6}-\sqrt{3})(\sqrt{6}-\sqrt{3})$

6 Rationalise the denominators, giving each answer in its simplest form:

a $\frac{1}{\sqrt{6}}$ **b** $\frac{12}{\sqrt{3}}$ **c** $\frac{\sqrt{6}}{2\sqrt{2}}$

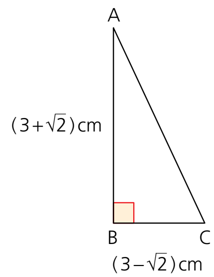
d $\frac{1}{(\sqrt{5}-\sqrt{2})}$ **e** $\frac{3+\sqrt{2}}{4-\sqrt{2}}$ **f** $\frac{3-\sqrt{5}}{5+\sqrt{5}}$

7 Write the following in the form $a + b\sqrt{c}$ where c is an integer and a and b are rational numbers:

a $\frac{1+\sqrt{2}}{3-\sqrt{2}}$ **b** $\frac{3\sqrt{5}}{3+\sqrt{5}}$ **c** $\frac{2\sqrt{6}}{\sqrt{6}-2}$

Exercise 4.2 (cont)

8 Work out the length of AC.



9 A square has sides of length x cm and diagonals of length 12 cm. Use Pythagoras' theorem to find the exact value of x and work out the area of the square.

10 An equilateral triangle has sides of length $\sqrt{3}$ cm.
Work out:
a the height of the triangle
b the area of the triangle in its simplest surd form.

Past-paper questions

1 Without using a calculator, find the positive root of the equation
$(5-2\sqrt{2})x^2-(4+2\sqrt{2})x-2=0$
giving your answer in the form $a+b\sqrt{2}$, where a and b are integers. [6]

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2 (a) Solve the equation $16^{3x-2}=8^{2x}$. [3]

(b) Given that $\dfrac{\sqrt{a^{\frac{4}{3}}b^{-\frac{2}{5}}}}{a^{-\frac{1}{3}}b^{\frac{3}{5}}}=a^pb^q$, find the value of p and of q. [2]

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3 (i) Given that $2^{5x}\times4^y=\frac{1}{8}$, show that $5x+2y=-3$. [3]

(ii) Solve the simultaneous equations $2^{5x}\times4^y=\frac{1}{8}$ and $7^x\times49^{2y}=1$. [4]

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Learning outcomes

Now you should be able to:

★ perform simple operations with indices and with surds, including rationalising the denominator.

Key points

✔ Use these rules for manipulating indices (powers).

- Multiplication: $a^m \times a^n = a^{m+n}$
- Division: $a^m \div a^n = a^{m-n}$
- Power of a power: $(a^m)^n = a^{mn}$
- Power zero: $a^0 = 1$
- Negative indices: $a^{-m} = \frac{1}{a^m}$
- Fractional indices: $a^{\frac{1}{n}} = \sqrt[n]{a}$

✔ Use these rules for simplifying surds (square roots).

- Leave the smallest number possible under the square root sign, e.g. $\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$
- Expand a surd expression in brackets in the same way any other algebraic expression.
- To rationalise the denominator of a surd:
  - (i) If the denominator contains a single term, multiply numerator and denominator by that term, e.g. $\frac{7}{\sqrt{5}} = \frac{7 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{7\sqrt{5}}{5}$
  - (ii) If the denominator contains two terms, multiply numerator and denominator by a similar expression with the opposite sign, e.g. $\frac{7}{6-\sqrt{2}} = \frac{7}{(6-\sqrt{2})} \times \frac{(6+\sqrt{2})}{(6+\sqrt{2})} = \frac{7(6+\sqrt{2})}{6^2-(\sqrt{2})^2} = \frac{7(6+\sqrt{2})}{32}$

5 Factors of polynomials

A **quadratic expression** is any expression of the form $ax^2 + bx + c$, where x is a variable and a, b and c are constants with $a \neq 0$.
An expression of the form $ax^3 + bx^2 + cx + d$ that includes a term in x^3 is called a **cubic expression**.
A **quartic** has a term in x^4 as its highest power, a **quintic** one with x^5 and so on.
All of these are **polynomials** and the highest power of the variable is called the **order** of the polynomial.

There are things of an unknown number which when divided by 3 leave 2, by 5 leave 3, and by 7 leave 2. What is the smallest number?

Sun-Tzi (544 BC - 496 BC)

Discussion point

Sun Tzi posed his problem in the Chinese Han dynasty and it is seen as the forerunner of the Remainder Theorem, which you will meet in this chapter. What is the answer? What is the next possible answer to Sun-Tzi's problem? How do you find further answers?

It is believed that the way that numbers were written during the Han dynasty laid the foundation for the abacus, an early form of hand calculator.

In Chapter 2 you met quadratic expressions like $x^2 - 4x - 12$ and solved quadratic equations such as $x^2 - 4x - 12 = 0$.

Multiplication and division of polynomials

There are a number of methods for multiplying and dividing polynomials. One method is shown in the worked example, but if you already know and prefer an alternative method, continue to use it.

Multiplication

Worked example

Multiply $(x^2 - 5x + 2)$ by $(2x^2 - x + 1)$.

Solution

This is an extension of the method you used to multiply two brackets that each contain two terms. If you are familiar with a different method, then use that.

$$
\begin{aligned}
(x^2 - 5x + 2) \times (2x^2 - x + 1) &= x^2(2x^2 - x + 1) - 5x(2x^2 - x + 1) + 2(2x^2 - x + 1) \\
&= 2x^4 - x^3 + x^2 - 10x^3 + 5x^2 - 5x + 4x^2 - 2x + 2 \\
&= 2x^4 + x^3(-1 - 10) + x^2(1 + 5 + 4) + x(-5 - 2) + 2 \\
&= 2x^4 - 11x^3 + 10x^2 - 7x + 2
\end{aligned}
$$

Division

Worked example

Divide $(x^3 - x^2 - 2x + 8)$ by $(x + 2)$.

Solution

Let $(x^3 - x^2 - 2x + 8) = (x + 2)(ax^2 + bx + c)$

This bracket must be a quadratic expression.

Multiplying each term in the second bracket by x and then by 2

$$= x(ax^2 + bx + c) + 2(ax^2 + bx + c)$$

$$= ax^3 + bx^2 + cx + 2ax^2 + 2bx + 2c$$

Collecting like terms

$$= ax^3 + (b + 2a)x^2 + (c + 2b)x + 2c$$

Comparing coefficients:

$a = 1$

Since $a = 1$

$b + 2a = -1 \Rightarrow b = -3$

Since $b = -3$

$c + 2b = -2 \Rightarrow c = 4$

Checking the constant term, $2c = 8$ which is correct.

This gives $(x^3 - x^2 - 2x + 8) \div (x + 2) = x^2 - 3x + 4$

Exercise 5.1

1 Multiply $(x^3 + 2x^2 - 3x - 4)$ by $(x + 1)$.

2 Multiply $(x^3 - 2x^2 + 3x + 2)$ by $(x - 1)$.

3 Multiply $(2x^3 - 3x^2 + 5)$ by $(2x - 1)$.

4 Multiply $(x^2 + 2x - 3)$ by $(x^2 - 2x + 3)$.

5 Multiply $(2x^2 - 3x + 4)$ by $(2x^2 - 3x - 4)$.

6 Simplify $(x^2 - 3x + 2)^2$.

7 Divide $(x^3 - 3x^2 + x + 1)$ by $(x - 1)$.

8 Divide $(x^3 - 3x^2 + x + 2)$ by $(x - 2)$.

9 Divide $(x^4 - 1)$ by $(x + 1)$.

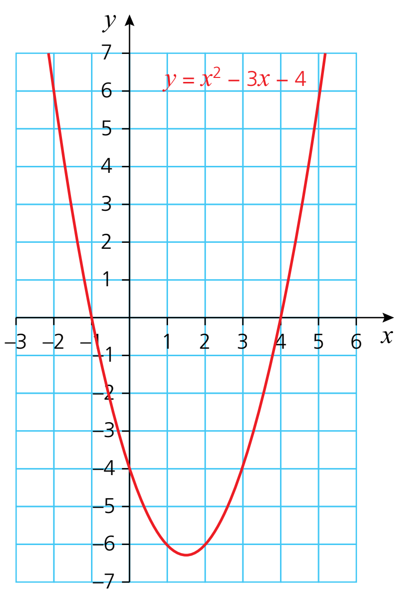
10 Divide $(x^2 - 16)$ by $(x + 2)$.

Solving cubic equations

When a polynomial can be factorised, you can find the points where the corresponding curve crosses the x-axis either as whole numbers or simple fractions.

For example, $y = x^2 - 3x - 4$ factorises to give $y = (x + 1)(x - 4)$.

The graph of this equation is a curve that crosses the x-axis at the points where $y = 0$. These values, $x = -1$ and $x = 4$, are called the **roots** of the equation $x^2 - 3x - 4 = 0$.



For a polynomial of the form $y = f(x)$, the roots are the solutions of $f(x) = 0$.

Worked example

a Draw the graph of $y = x^3 - 5x^2 + 2x + 8$.

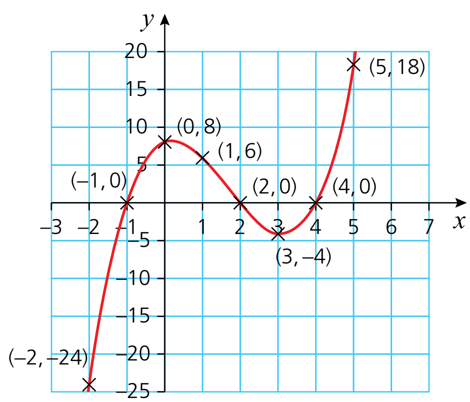
b Hence solve the equation $x^3 - 5x^2 + 2x + 8 = 0$.

Solution

a Start by setting up a table of values.

x	−2	−1	0	1	2	3	4	5
y	−24	0	8	6	0	−4	0	18

Then plot the curve.



The solution is '$x = -1$ or $x = 2$ or $x = 4$' but the roots are '−1 and 2 and 4'.

b The graph shows that the curve crosses the x-axis at the values −1, 2 and 4, giving the solution as $x = -1$, $x = 2$ or $x = 4$.

In some cases, a graph will not find all the roots but will allow you to find one or possibly two roots, or show you that there is only one root. The roots may not be whole numbers and may not even be rational as shown in the following examples.

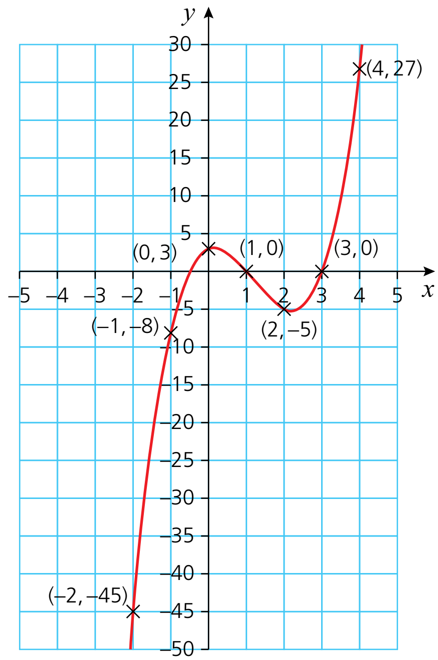
Worked example

Draw the graph of $y = 2x^3 - 7x^2 + 2x + 3$ and hence solve the equation $2x^3 - 7x^2 + 2x + 3 = 0$.

Solution

As before, start by setting up a table of values and then draw the curve.

x	−2	−1	0	1	2	3	4
y	−45	−8	3	0	−5	0	27



The graph shows that the curve crosses the x-axis at 1, at 3 and again between –1 and 0. You can find this root using trial and improvement.

−0.5 is chosen as x since it is half way between −1 and 0.

Let $x = -0.5$

$$\begin{aligned} f(0.5) &= 2(-0.5)^3 - 7(-0.5)^2 + 2(-0.5) + 3 \\ &= 0 \end{aligned}$$

In this case, you were lucky and found the final root, −0.5, with only one iteration.

So the roots of the equation are –0.5, 1 and 3.

Finding factors and the factor theorem

The equation in the example above has roots that are whole numbers or exact fractions. This implies that it could have been factorised. Roots at $-\frac{1}{2}$, 1 and 3 suggest the factorised form:

$$\left(x + \tfrac{1}{2}\right)(x - 1)(x - 3)$$

However multiplying the x terms from all the brackets should give $2x^3$ so one of the brackets must be multiplied by 2.

$$2x^3 - 7x^2 + 2x + 3 = (2x + 1)(x - 1)(x - 3)$$

It is not possible to factorise all polynomials. However, when a polynomial can be factorised, the solution to the corresponding equation follows immediately.

$$\begin{aligned} (2x+1)(x-1)(x-3) = 0 \quad &\Rightarrow \quad (2x+1) = 0 \text{ or } (x-1) = 0 \text{ or } (x-3) = 0 \\ &\Rightarrow \quad x = -0.5 \text{ or } x = 1 \text{ or } x = 3 \end{aligned}$$

This leads to an important result known as the **factor theorem**.

> If $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$ and $x = a$ is a root of the equation $f(x) = 0$.

Conversely, if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

It is not necessary to try all integer values when you are looking for possible factors. For example, with $x^3 - 3x^2 - 2x + 6 = 0$ you need only try the factors of 6 as possible roots, i.e. ±1, ±2, ±3 and ±6.

Worked example

a Show that $x = 2$ is a root of the equation $x^3 - 3x^2 - 4x + 12 = 0$ and hence solve the equation.

b Sketch the graph of $y = x^3 - 3x^2 - 4x + 12$.

Alternatively, you could factorise by long division.

Solution

a $f(2) = 2^3 - 3(2^2) - 4(2) + 12 = 0$

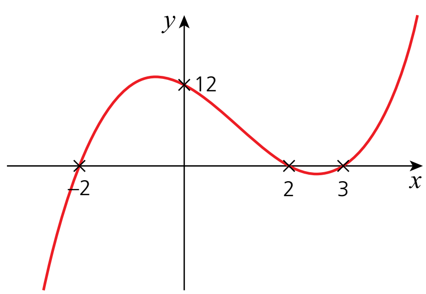
This implies that $x = 2$ is a root of the equation and hence $(x - 2)$ is a factor of $f(x)$.

Taking $(x - 2)$as a factor gives

$$\begin{aligned} x^3 - 3x^2 - 4x + 12 &= (x-2)(x^2 - x - 6) \\ &= (x-2)(x-3)(x+2) \end{aligned}$$

The solution to the equation is therefore $x = 2$, $x = 3$, or $x = -2$.

b The graph crosses the x-axis at $x = -2$, $x = 2$ and $x = 3$ and the y-axis at $y = 12$.



You will not be able to factorise the expression completely in all cases, but you may be able to find one factor by inspection as in the following example.

Worked example

Given that $f(x) = x^3 - x^2 - 4x + 12$

a Show that $(x - 3)$ is a factor of $f(x)$.

b Solve the equation $f(x) = 0$.

Solution

To show that $(x-3)$ is a factor, you need to show that $f(3) = 0$.

a $f(3) = 3^3 - 3^2 - 9(3) + 9$

$= 27 - 9 - 27 + 9 = 0$

This shows that $(x - 3)$ is a factor of $f(x)$.

b Once you have found one linear factor of a cubic expression, the remaining factor is quadratic. With practice you will be able to do this step by inspection.

Check by multiplying out that you agree with this answer.

$$x^3 - x^2 - 10x + 12 = (x - 3)(x^2 + 2x - 4)$$

How do you know it can't be factorised?

$x^2 + 2x - 4$ cannot be factorised so use the quadratic formula for the next step.

$$ax^2 + bx + c = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this example, $a = 1,\ b = 2,\ c = -4 \rightarrow x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(-4)}}{2(-1)}$

$$\rightarrow x = \frac{-2 \pm \sqrt{20}}{-2}$$

$$\rightarrow x = 1 \pm \sqrt{5}$$

The solution to the equation is therefore $x = 3$ or $x = 1 \pm \sqrt{5}$.

Using the factor theorem to solve a cubic equation

This is very similar to earlier work except that the first step is to find a linear factor by inspection.

Worked example

a Work systematically to find a linear factor of $x^3 - 5x^2 - 2x + 24$.

b Solve the equation $x^3 - 5x^2 - 2x + 24 = 0$.

c Sketch the graph of $y = x^3 - 5x^2 - 2x + 24$.

d Sketch $y = |x^3 - 5x^2 - 2x + 24|$ on a separate set of axes.

Solution

a Let $f(x) = x^3 - 5x^2 - 2x + 24$.

Start by working systematically through all factors of 24 until you find one giving $f(x) = 0$.

$f(1) = 1 - 5 - 2 + 24 = 18$

$f(-1) = -1 - 5 + 2 + 24 = 20$

$f(2) = 8 - 20 - 4 + 24 = 8$

$f(-2) = -8 - 20 + 4 + 24 = 0$

This shows that $(x+2)$ is a linear factor of $x^3 - 5x^2 - 2x + 24$.

b Factorising by inspection:

$$x^3 - 5x^2 - 2x + 24 = (x+2)(x^2 + ax + 12)$$

The second bracket starts with x^2 to get the x^3 term and finishes with 12 since $2 \times 12 = 24$.

Looking at the x^2 term on both sides: $-5x^2 = 2x^2 + ax^2$

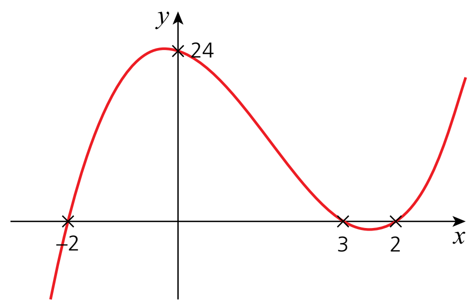
$\rightarrow a = -7$

$$x^3 - 5x^2 - 2x + 24 = 0 \rightarrow (x+2)(x^2 - 7x + 12) = 0$$

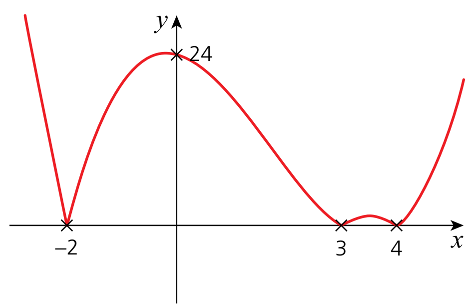
$$\rightarrow (x+2)(x-3)(x-4) = 0$$

$$\rightarrow x = -2,\ x = 3 \text{ or } x = 4$$

c The graph is a cubic curve with a positive x^3 term that crosses the x-axis at -2, 3 and 4 and crosses the y-axis when $y = 24$.



d To sketch the curve $y = |x^3 - 5x^2 - 2x + 24|$, first sketch the curve as above, then reflect in the x-axis any part of the curve which is below it.



Exercise 5.2

1 Determine whether the following linear functions are factors of the given polynomials:

a $x^3 - 8x + 7; \quad (x-1)$ **b** $x^3 + 8x + 7; \quad (x+1)$

c $2x^3 + 3x^2 - 4x - 1; \quad (x-1)$ **d** $2x^3 - 3x^2 + 4x + 1; \quad (x+1)$

2 Use the factor theorem to find a linear factor of each of the following functions. Then factorise each function as a product of three linear factors and sketch its graph.

a $x^3 - 7x - 6$ **b** $x^3 - 7x + 6$

c $x^3 + 5x^2 - x - 5$ **d** $x^3 - 5x^2 - x + 5$

Exercise 5.2 (cont)

3 Factorise each of the following functions completely:
   a $x^3 + x^2 + x + 1$
   b $x^3 - x^2 + x - 1$
   c $x^3 + 3x^2 + 3x + 2$
   d $x^3 - 3x^2 + 3x - 2$

4 For what value of a is $(x - 2)$ a factor of $x^3 - 2ax + 4$?

5 For what value of c is $(2x + 3)$ a factor of $2x^3 + cx^2 - 4x - 6$?

6 The expression $x^3 - 6x^2 + ax + b$ is exactly divisible by $(x - 1)$ and $(x - 3)$.
   a Find two simultaneous equations for a and b.
   b Hence find the values of a and b.

The remainder theorem

Worked example

Is there an integer root to the equation $x^3 - 2x^2 + x + 1 = 0$?

Solution

Since the first term is x^3 and the last term is +1, the only possible factors are $(x + 1)$ and $(x - 1)$.

This leads us to the **remainder theorem.**

$f(1) = 1$ and $f(-1) = -3$ so there is no integer root.

Any polynomial can be divided by another polynomial of lesser order using either long division or inspection. However, there will sometimes be a remainder. The steps for algebraic long division are very similar to those for numerical long division as shown below.

Look at $(x^3 - 2x^2 + x + 1) \div (x + 1)$.

Taking the first term from each (the dividend and the divisor) gives $x^3 \div x = x^2$, the first term on the top in the quotient.

$-3x^2 \div x$ gives $-3x$.

Similarly $4x \div x = 4$

$$\begin{array}{rl}
 & x^2 - 3x + 4 \\
x+1 & \overline{)\,x^3 - 2x^2 + x + 1} \\
- & \underline{x^3 + x^2} \\
 & \quad -3x^2 + x \\
- & \quad \underline{-3x^2 - 3x} \\
 & \qquad\qquad 4x + 1 \\
- & \qquad\qquad \underline{-4x + 4} \\
 & \qquad\qquad\quad -3
\end{array}$$

This result can be written as:

$$x^3 - 2x^2 + x + 1 = (x + 1)(x^2 - 3x + 4) - 3.$$

Substituting $x = -1$ into both sides gives a remainder of -3.

This means that $f(-1)$ will always be the remainder when a function $f(x)$ is divided by $(x + 1)$.

Generalising this gives the **remainder theorem**.

For any polynomial $f(x)$, $f(a)$ is the remainder when $f(x)$ is divided by $(x - a)$.

$$f(x) = (x - a)g(x) + f(a)$$

Worked example

Find the remainder when $f(x) = 2x^3 + 3x - 5$ is divided by $(x - 2)$.

Solution

Using the remainder theorem, the remainder is f(2).

$$f(2) = 2(2)^3 + 3(2) - 5 = 17$$

Worked example

When $2x^3 - 3x^2 + ax - 5$ is divided by $x - 2$, the remainder is 7. Find the value of a.

Solution

To find the remainder, substitute $x = 2$ into $2x^3 - 3x^2 + ax - 5$.

$$\begin{aligned} 2(2)^3 - 3(2)^2 + a(2) - 5 &= 7 \\ \rightarrow 16 - 12 + 2a - 5 &= 7 \\ \rightarrow -1 + 2a &= 7 \\ a &= 4 \end{aligned}$$

Exercise 5.3

1 For each function, find the remainder when it is divided by the linear factor shown in brackets:

a $x^3 + 2x^2 - 3x - 4$; $(x - 2)$

b $2x^3 + x^2 - 3x - 4$; $(x + 2)$

c $3x^3 - 3x^2 - x - 4$; $(x - 4)$

d $3x^3 + 3x^2 + x + 4$; $(x + 4)$

2 When $f(x) = x^3 + ax^2 + bx + 10$ is divided by $(x + 1)$, there is no remainder. When it is divided by $(x - 1)$, the remainder is 4. Find the values of a and b.

3 The equation $f(x) = x^3 + 4x^2 + x - 6$ has three integer roots. Solve $f(x) = 0$.

4 $(x - 2)$ is a factor of $x^3 + ax^2 + a^2x - 14$. Find all possible values of a.

5 When $x^3 + ax + b$ is divided by $(x - 1)$, the remainder is –12. When it is divided by $(x - 2)$, the remainder is also –12. Find the values of a and b and hence solve the equation $x^3 + ax + b = 0$.

6 Sketch each curve by first finding its points of intersection with the axes:

a $y = x^3 + 2x^2 - x - 2$

b $y = x^3 - 4x^2 + x + 6$

c $y = 4x - x^3$

d $y = 2 + 5x + x^2 - 2x^3$

Past–paper questions

1 The polynomial $f(x) = ax^3 - 15x^2 + bx - 2$ has a factor of $2x - 1$ and a remainder of 5 when divided by $x - 1$.

(i) Show that $b = 8$ and find the value of a. [4]

(ii) Using the values of a and b from part **(i)**, express $f(x)$ in the form $(2x - 1)\,g(x)$, where $g(x)$ is a quadratic factor to be found. [2]

(iii) Show that the equation $f(x) = 0$ has only one real root. [2]

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2 A function f is such that $f(x) = 4x^3 + 4x^2 + ax + b$. It is given that $2x - 1$ is a factor of both $f(x)$ and $f'(x)$.

(i) Show that $b = 2$ and find the value of a. [5]

Using the values of a and b from part **(i)**,

(ii) find the remainder when $f(x)$ is divided by $x + 3$, [2]

(iii) express $f(x)$ in the form $f(x) = (2x - 1)(px^2 + qx + r)$, where p, q and r are integers to be found, [2]

(iv) find the values of x for which $f(x) = 0$. [2]

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3 It is given that $f(x) = 6x^3 - 5x^2 + ax + b$ has a factor of $x + 2$ and leaves a remainder of 27 when divided by $x - 1$.

(i) Show that $b = 40$ and find the value of a. [4]

(ii) Show that $f(x) = (x + 2)(px^2 + qx + r)$, where p, q and r are integers to be found. [2]

(iii) Hence solve $f(x) = 0$. [2]

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Learning outcomes

Now you should be able to:

★ multiply two polynomials when the degree (i.e. the highest power) of at least one of them is greater than 2
★ divide one polynomial by another when the division gives no remainder
★ solve a cubic equation by first drawing the graph
★ know and use the remainder and factor theorems
★ find factors of polynomials
★ solve cubic equations.

Key points

- ✔ An expression of the form $ax^3 + bx^2 + cx + d$ where $a \neq 0$ is called a **cubic expression**.
- ✔ The graph of a cubic expression can be plotted by first calculating the value of y for each value of x in the range.
- ✔ The solution to a cubic equation is the set of values for which the corresponding graph crosses the x-axis.
- ✔ The **factor theorem** states: if $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$ and $x = a$ is a root of the equation $f(x) = 0$.
- ✔ The **remainder theorem** states: for any polynomial $f(x)$, $f(a)$ is the remainder when $f(x)$ is divided by $(x - a)$. This can be generalised to $f(x) = (x - a)g(x) + f(a)$.

6 Simultaneous equations

There is no branch of mathematics, however abstract, which may not someday be applied to the phenomena of the real world.

Nikolai Lobachevsky (1792 – 1856)

Sometimes, in Mathematics, it is necessary to use two or more variables to describe a situation. In such cases, you need more than one equation to find the individual values of the variables: two equations for two variables, three for three variables and so on. Although the focus of this chapter is on solving two equations to find the values of two variables, the techniques introduced can be extended to cover situations involving more than two variables.

The following problem can be solved using simultaneous equations.

A school has two netball teams and there is great rivalry between them. Last season, each team played 16 games and their positions in the league were tied with 26 points each. A team is awarded points for a win or a draw only. Team A wins 6 games and draws 8; team B wins 8 games and draws 2. How many points are awarded for:

a a win **b** a draw?

Let w denote the number of points for a win and d denote the number of points for a draw. Each team's results can be expressed as an equation, but unlike equations you have met previously, each one contains two variables.

$$6w + 8d = 26$$
$$8w + 2d = 26$$

In this case, since the values of w and d will be small integers, you could find the answer by trial and improvement, but in this chapter we will look at some structured methods for solving simultaneous equations.

Simultaneous equations can be solved graphically or algebraically using elimination or substitution. Graphical methods often give only approximate answers and so the main focus of this is algebraic methods.

Solving linear simultaneous equations

You will have already used graphs to solve simultaneous equations. The example below shows that this method does not always give an accurate solution. This is why it is not the focus of this chapter.

Worked example

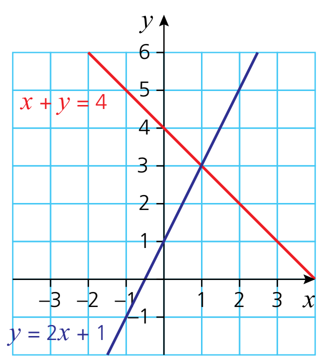
Solve the following pairs of simultaneous equations graphically and comment on your answers.

a $x + y = 4$
$y = 2x + 1$

b $x + y = 4$
$y = 4x + 2$

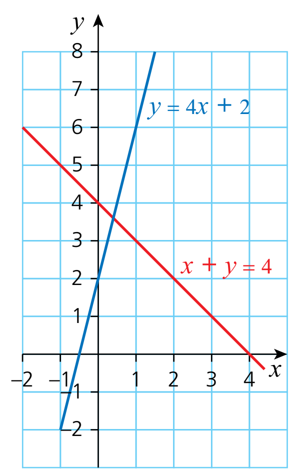
Solution

a



Here you can see that the two lines intersect at the point (1, 3) so the solution is $x = 1,\ y = 3$.

b



It is not clear from the graph exactly where these two lines intersect.

The elimination method

Worked example

Solve the following simultaneous equations by elimination.

$$2x + y = 10$$
$$x - y = 2$$

Solution

Since one equation contains $+y$ and one contains $-y$ adding them will eliminate y.

The rule Same Signs, Subtract (and opposite signs, add) is useful here.

$$\begin{aligned} 2x + y &= 10 \\ x - y &= 2 \\ \hline 3x \quad &= 12 \\ \hline \Rightarrow \quad x &= 4 \end{aligned}$$

Substitute this value into one of the equations.

You would generally choose the simpler one, in this case $x - y = 2$.

$$\begin{aligned} 4 - y &= 2 \\ \Rightarrow \quad y &= 2 \end{aligned}$$

Check these values in the other equation.

$$\begin{aligned} 2x + y &= 8 + 2 \\ &= 10 \end{aligned}$$

The solution to the simultaneous equations is $x = 4,\ y = 2$.

Sometimes you need to multiply one of the equations before you can eliminate a variable, as in the example below.

Worked example

Solve the following simultaneous equations by elimination.

$$3x + y = 13$$

$$x + 2y = 11$$

Solution

Start by multiplying the first equation by 2 so that the coefficient of y is the same in both equations.

Subtract to eliminate y

$$\begin{aligned} 6x + 2y &= 26 \\ x + 2y &= 11 \\ \hline 5x \quad &= 15 \\ \hline \Rightarrow x &= 3 \end{aligned}$$

An alternative starting point is to multiply the second equation by 3 so that the coefficient of x is the same in both equations. Try it and show that it gives the same answer.

Substitute this into the second equation (since it is the simpler one).

$$3+2y=11$$
$$\Rightarrow 2y=8$$
$$\Rightarrow y=4$$

The solution is therefore $x=3,\ y=4$.

Sometimes it is necessary to manipulate both of the original equations in order to eliminate one of the variables easily, as in the following example.

Worked example

Solve the following simultaneous equations by elimination.

$$3x+2y=1$$
$$2x+3y=4$$

Solution

Here you need to multiply each equation by a suitable number so that either the coefficients of x or the coefficients of y are the same.

$$3x+2y=1 \quad\Rightarrow\quad 9x+6y=3$$
$$2x+3y=4 \quad\Rightarrow\quad 4x+6y=8$$

3× the first equation

$$9x+6y=3$$

2× the second equation

$$4x+6y=8$$

Subtracting

$$\Rightarrow\quad 5x=-5$$
$$\Rightarrow\quad x=-1$$

You could choose either one.

Substitute $x=-1$ into the first of the two original equations.

$$-3+2y=1$$
$$\Rightarrow\quad 2y=4$$

Remember to check these values in the second of the original equations.

$$\Rightarrow\quad y=2$$

The solution is therefore $x=-1,\ y=2$.

The substitution method

Worked example

Solve the following simultaneous equations by substitution:

$$3x-y=-10$$
$$x=2-y$$

Solution

Substitute the expression for x from the second equation into the first.

$$3(2 - y) - y = -10$$
$$\Rightarrow \quad 6 - 3y - y = -10$$
$$\Rightarrow \quad 16 = 4y$$
$$\Rightarrow \quad y = 4$$

Since it is the simpler one

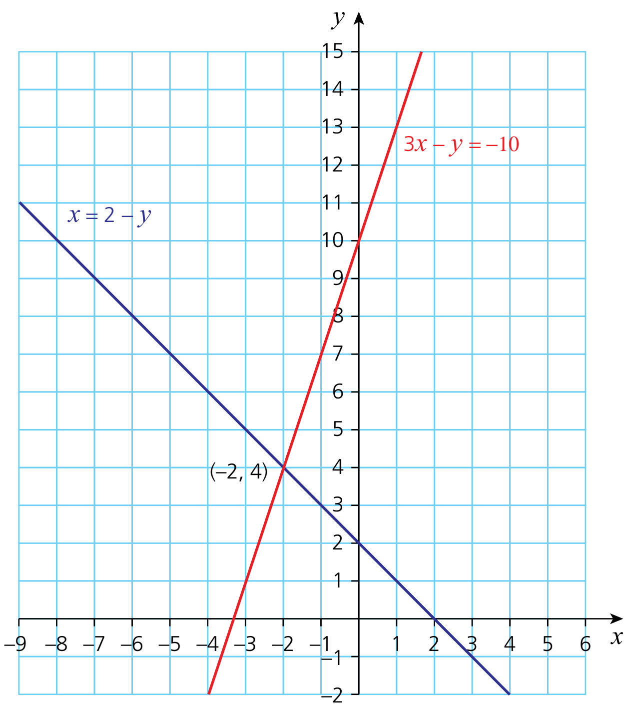
Substitute $y = 4$ into the second equation.

$$x = 2 - 4$$
$$\Rightarrow x = -2$$

It is a good idea to check these values using the equation you did not substitute into.

The solution is therefore $x = -2$, $y = 4$.

Since each of these equations can be represented by a straight line, solving them simultaneously gives the coordinates of their point of intersection.



Sometimes simultaneous equations may arise in everyday problems as in the following example.

Worked example

A taxi firm charges a fixed amount plus so much per kilometre. A journey of three kilometres costs \$4.60 and a journey of seven kilometres costs \$9.40. How much does a journey of six kilometres cost?

Solution

Let $\$f$ be the fixed amount and $\$m$ be the cost per kilometre. Writing this information as a pair of simultaneous equations:

$$f + 3m = 4.6$$
$$f + 7m = 9.4$$

Subtracting the first equation from the second:

$$4m = 4.8$$
$$\Rightarrow m = 1.2$$

Substituting into the first equation:

At this stage you must remember to answer the question as it is set.

$$f + 3 \times 1.2 = 4.6$$
$$\Rightarrow \quad f = 1.0$$

A journey of six kilometres will cost 1.0 + 6(1.2) = \$8.20.

Exercise 6.1

1 Solve the following pairs of simultaneous equations graphically:

a $y = x + 2$
$y = 2x - 3$

b $x + 2y = 3$
$2x - y = -4$

Use the substitution method to solve the simultaneous equations in Questions 2–5.

2 a $2x + y = 13$
$y = 2x + 1$

b $x + 2y = 13$
$x = 2y + 1$

3 a $3x + 4y = 2$
$y = 4x + 10$

b $4x + 3y = 2$
$x = 4y + 10$

4 $x - 3y = -2$
$y = 3x - 2$

5 $x + 4y = -13$
$x = 3y + 1$

Use the elimination method to solve the simultaneous equations in questions 6–9.

6 a $x + y = 4$
$x - y = 2$

b $x + 2y = 4$
$x - 2y = 2$

7 a $3x + y = 9$
$2x - y = 1$

b $3x + 2y = 9$
$x - y = 0.5$

8 $2x + 3y = -4$
$4x + 2y = 0$

9 $5x - 2y = -23$
$3x + y = -5$

10 3 pencils and 4 rulers cost \$5.20. 5 pencils and 2 rulers cost \$4. Find the cost of 6 pencils and a ruler.

11 At the cinema, 3 packets of popcorn and 2 packets of nuts cost \$16 and 2 packets of popcorn and 1 packet of nuts cost \$9. What is the cost of one packet of each?

Exercise 6.1 (cont)

12 Two adults and one child paid \$180 to go to the theatre and one adult and three children paid \$190. What it is the cost for two adults and five children?

13 A shop is trying to reduce their stock of books by holding a sale. \$20 will buy either 8 paperback and 4 hardback books or 4 paperbacks and 7 hardbacks. How much change would I get from \$40 if I bought 10 paperbacks and 10 hardbacks?

Solving non-linear simultaneous equations

The substitution method is particularly useful when one of the equations represents a curve, as in the following example.

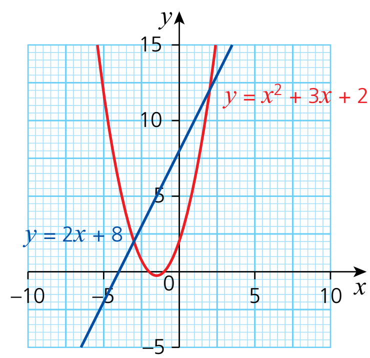
Worked example

a Sketch the graphs of $y = x^2 + 3x + 2$ and $2x = y - 8$ on the same axes.

b Use the method of substitution to solve these equations simultaneously

Solution

a



Substituting $x^2 + 3x + 2$ for y in the second equation

Collecting all the terms on one side

b $y = x^2 + 3x + 2$

$2x = y - 8$

$2x = (x^2 + 3x + 2) - 8$

$\Rightarrow 0 = x^2 + x - 6$

$\Rightarrow (x + 3)(x - 2) = 0$

$\Rightarrow x = -3$ or $x = 2$

Take each of these values in turn, and substitute into the **linear** equation to find the corresponding values of y.

When $x = -3$, $-6 = y - 8 \quad \Rightarrow \quad y = 2$.

This means that one possible solution is $x = -3,\ y = 2$.

When $x = 2$, $4 = y - 8 \quad \Rightarrow y = 12$.

This gives the other solution as $x = 2,\ y = 12$.

The full solution is therefore $x = -3,\ y = 2$ or $x = 2,\ y = 12$.

The original equations represent a curve and a line, so the two solutions give the coordinates of their points of intersection as in the graph above.

Note

1 It is equally acceptable to start by substituting for x from the second equation into the first. This gives $2x + 8 = x^2 + 3x + 2$ and leads to the same result.

2 Once you have found values of one variable, you must substitute into the **linear** equation. If you substitute into the non-linear equation, you **could** find other 'rogue' values appearing erroneously as solutions (but not in this case).

In the previous example, it would also have been possible to solve the two equations by simply drawing graph because the solution had integer values. However, this is often not the case so this method will not always give an accurate answer.

Worked example

a Use an algebraic method to find the points of intersection of the curve $x^2 + y^2 = 5$ and the line $y = x + 1$.

b Given that the curve is the circle with centre the origin and radius $\sqrt{5}$, illustrate your answer on a graph.

Solution

a In this case you should use the substitution method.

Substituting from the line equation into the equation for the curve

$$x^2 + (x+1)^2 = 5$$
$$\Rightarrow x^2 + (x^2 + 2x + 1) = 5$$
$$\Rightarrow 2x^2 + 2x - 4 = 0$$

Dividing by 2 to simplify

$$\Rightarrow x^2 + x - 2 = 0$$
$$\Rightarrow (x+2)(x-1) = 0$$
$$\Rightarrow x = -2 \text{ or } x = 1$$

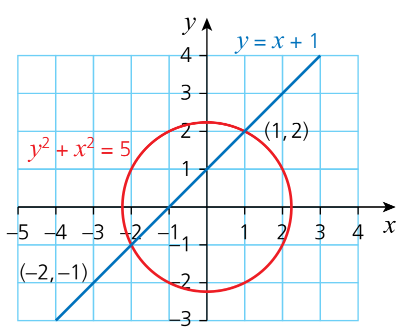
Substituting each of these values in turn into the **line** equation

When $x = -2$, $y = -2 + 1 = -1$.

When $x = 1$, $y = 1 + 1 = 2$.

The points of intersection are therefore $(-2, -1)$ and $(1, 2)$.

b



Exercise 6.2

1 Solve this pair of simultaneous equations algebraically:

$x^2 + y^2 = 13$
$x = 2$

2 Solve this pair of simultaneous equations algebraically and sketch a graph to illustrate your solution:

$y = x^2 - x + 8$
$y = 5x$

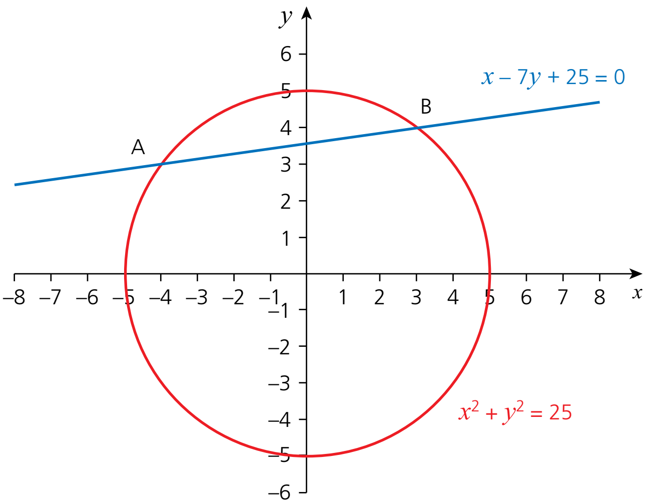
3 Solve this pair of simultaneous equations:

$xy = 4$
$y = x - 3$

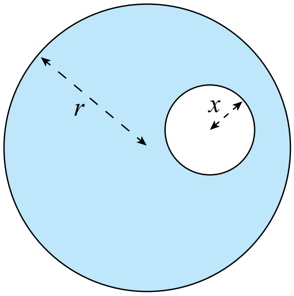
4 Solve this pair of simultaneous equations:

$y = 8x^2 - 2x - 10$
$4x + y = 5$

5 The diagram shows the circle $x^2 + y^2 = 25$ and the line $x - 7y + 25 = 0$. Find the coordinates of A and B.



6 The diagram shows a circular piece of card of radius r cm, from which a smaller circle of radius x cm has been removed. The area of the remaining card is 209π cm^2. The circumferences of the two circles add up to 38π. Write this information as a pair of simultaneous equations and hence find the values of r and x.



7 a Solve this pair of simultaneous equations:

$y = x^2 + 1$
$y = 2x$

b Why is there only one solution? Illustrate this using a sketch.

8 **a** Explain what happens when you try to solve this pair of simultaneous equations:

$$y = 2x^2 - 3x + 4$$
$$y = x - 1$$

b Illustrate your explanation with a sketch graph.

Past–paper questions

1 The line $y = 2x + 10$ intersects the curve $2x^2 + 3xy - 5y + y^2 = 218$ at the points A and B.
Find the equation of the perpendicular bisector of AB. [9]

Cambridge O Level Additional Mathematics 4037
Paper 23 Q10 November 2011
Cambridge IGCSE Additional Mathematics 0606
Paper 23 Q10 November 2011

2 The curve $y = xy + x^2 - 4$ intersects the line $y = 3x - 1$ at the points A and B. Find the equation of the perpendicular bisector of the line AB. [8]

Cambridge O Level Additional Mathematics 4037
Paper 11 Q5 June 2015
Cambridge IGCSE Additional Mathematics 0606
Paper 11 Q5 June 2015

3 Find the set of values of k for which the line $y = 3x - k$ does not meet the curve $y = kx^2 + 11x - 6$. [6]

Cambridge O Level Additional Mathematics 4037
Paper 23 Q3 November 2013
Cambridge IGCSE Additional Mathematics 0606
Paper 23 Q3 November 2013

Learning outcomes

Now you should be able to:

★ solve simple simultaneous equations in two unknowns by elimination or substitution.

Key points

Simultaneous equations may be solved using these methods.

✔ **Graphically:** This method may be used for any two simultaneous equations. The advantage is that it is generally easy to draw graphs, although it can be time-consuming. The disadvantage is that it may not give an answer to the level of accuracy required.

✔ **Elimination:** This is the most useful method when solving two linear simultaneous equations.

✔ **Substitution:** This method is best for one linear and one non-linear equation. You start by isolating one variable in the linear equation and then substituting it into the non-linear equation.

7 Logarithmic and exponential functions

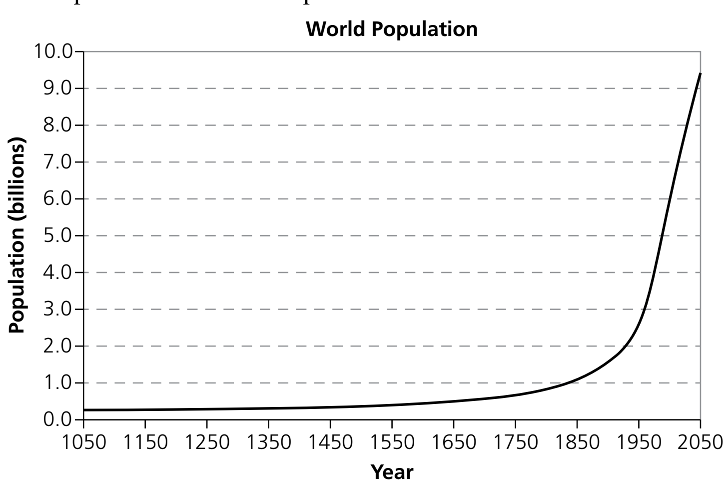
To forget one's ancestors is to be a brook without a source, a tree without a root.

Chinesepr overb

Discussion point

You have two parents and each of them has (or had) two parents so you have four grandparents. Going back you had $2^3 = 8$ great grandparents, $2^4 = 16$ great great grandparents and so on going backwards in time. Assuming that there is one generation every 30 years, and that all your ancestors were different people, estimate how many ancestors you had living in the year 1700. What about the year 1000?

The graph below shows an estimate of the world population over the last 1000 years. Explain why your answers are not realistic. What assumption has caused the problem?



Recent DNA analysis shows that almost everyone in Europe is descended from just seven women. Arriving at different times during the last 45 000 years, they survived wolves, bears and ice ages to form different clans that eventually became today's population. Another 26 maternal lineages have been uncovered on other continents.

Researching family history is a popular hobby and there are many Internet sites devoted to helping people find out details of their ancestry. Most of us only know about parents, grandparents, and possibly great-grandparents, but it is possible to go back much further.

Worked example

Assuming that a new generation occurs, on average, every 30 years, how many direct ancestors will be on your family tree if you go back 120 years? What about if you were able to go back 300 years?

Solution

30 years ago, you would have information about your two parents.

Each of these would have had two parents, so going back a further 30 years there are also four grandparents, another 30 years gives eight great-grandparents and so on.

If you tabulate these results, you can see a sequence starting to form.

Number of years	Number of people
30	2
60	$4 = 2^2$
90	$8 = 2^3$
120	$16 = 2^4$

This is a **geometric sequence** of numbers. You will meet these sequences in Chapter 12.

For each period of 30 years, the number of direct ancestors is double the number in the previous generation. After 120 years, the total number of ancestors is $2 + 4 + 8 + 16 = 30$.

300 years ago is ten periods of 30 years, so following the pattern, there are $2^{10} = 1024$ direct ancestors in this generation.

In practice, family trees are much more complicated, since most families have more than one child. It gets increasingly difficult the further back in time you research.

Discussion point

How many years would you expect to need to go back to find over 1 billion direct ancestors?

What date would that be?

Look at the graph on the previous page and say why this is not a reasonable answer.

Where has the argument gone wrong?

You may have answered the discussion point by continuing the pattern in the table at the top of the page. Or you may have looked for the smallest value of n- for which 2^n is greater than 1 billion. You can find this by trial and error but, as you will see, it is quicker to use logarithms to solve equations and inequalities like this.

Logarithms

Logarithm is another word for **index** or **power**.

For example, if you want to find the value of x such that $2^x = 8$, you can do this by checking powers of 2. However, if you have $2^x = 12$, for example, it is not as straightforward and you would probably need to resort to trial and improvement.

The equation $2^3 = 8$ can also be written as $\log_2 8 = 3$. The number 2 is referred to as the **base** of the logarithm.

Read this as 'log to base 2 of 8 equals 3'.

Similarly, $2^x = 12$ can be written as $\log_2 12 = x$.

In general,

$$a^x = y \Leftrightarrow x = \log_a y.$$

Most calculators have three buttons for logarithms.

- **log** which uses 10 as the base.
- **ln** which has as its base the number 2.718…, denoted by the letter e, which you will meet later in the chapter.
- **log** $_{\blacksquare}\square$ which allows you to choose your own base.

Worked example

Find the logarithm to base 2 of each of these numbers. Do not use a calculator.

a 32 **b** $\frac{1}{4}$ **c** 1 **d** $\sqrt{2}$

Solution

This is equivalent to being asked to find the power when the number is written as a power of 2.

a $32 = 2^5$, so $\log_2 32 = 5$

b $\frac{1}{4} = 2^{-2}$, so $\log_2 \frac{1}{4} = -2$

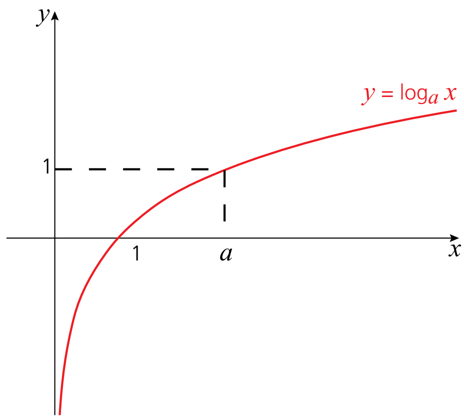
c $1 = 2^0$, so $\log_2 1 = 0$

$\log_n 1 = 0$ for all positive values of n.

d $\sqrt{2} = 2^{\frac{1}{2}}$, so $\log_2 \sqrt{2} = \frac{1}{2}$

Graphs of logarithms

The graph of $y = \log_a x$ has the same general shape for all values of the base a where $a > 1$.



The graph has the following properties:

- The curve only exists for positive values of x.
- The gradient of the graph is always positive. As the value of x increases, the gradient of the curve decreases.
- It crosses the x-axis at $(1, 0)$.
- The line $x = 0$ is an asymptote, i.e. the curve approaches it ever more closely but never actually touches or crosses it.
- The graph passes through the point $(a, 1)$.
- $\log_a x$ is negative for $0 < x < 1$.

Graphs of other logarithmic functions are obtained from this basic graph by applying one or more transformations – translations, stretches or reflections – as shown in the following examples.

Note

- A **translation** moves the graph – horizontally, vertically or in both directions – to a different position. It does not change in shape.
  When a > 0:
  - replacing x by $(x - a)$ moves the graph a units to the right (the positive direction)
  - replacing x by $(x + a)$ moves the graph a units to the left (the negative direction)
  - replacing y by $(y - a)$ moves the graph a units upwards (the positive direction)
  - replacing y by $(y + a)$ moves the graph a units downwards (the negative direction).
- A **reflection** gives a mirror image. In this book only reflections in the coordinate axes are considered.
  - Replacing x by $(-x)$ reflects the graph in the y-axis.
  - Replacing y by $(-y)$ reflects the graph in the x-axis.

Worked example

Sketch each pair of graphs and describe the transformation shown. In each pair, join (2, log 2) and (3, log 3) to their images.

a $y = \log x$ and $y = \log(x-3)$

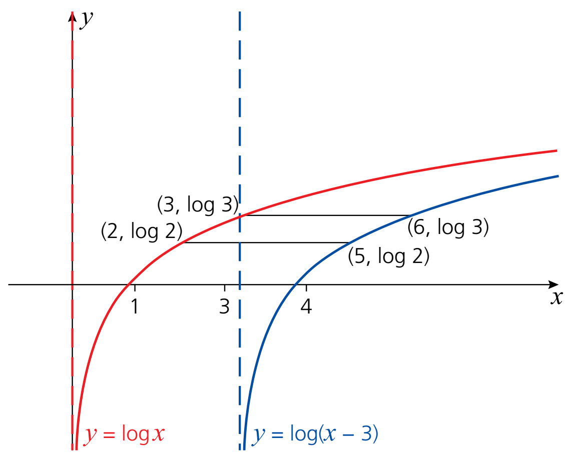
b $y = \log x$ and $y = \log(x+2)$

c $y = \log x$ and $y = -\log x$

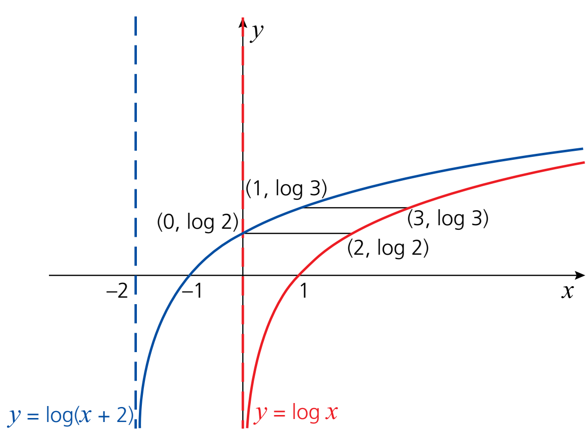
d $y = \log x$ and $y = \log(-x)$

Solution

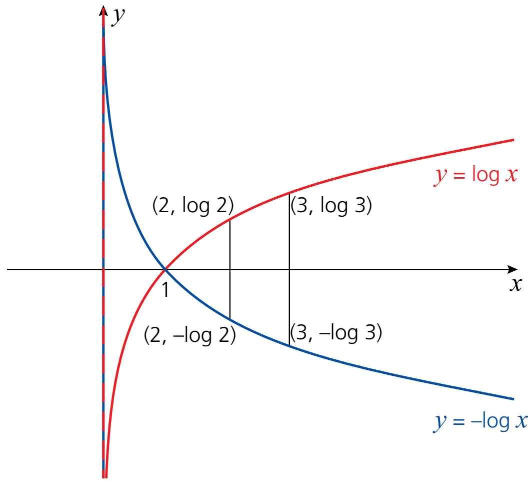
a The graph of $y = \log(x-3)$ is a translation of the graph of $y = \log x$ 3 units to the right.



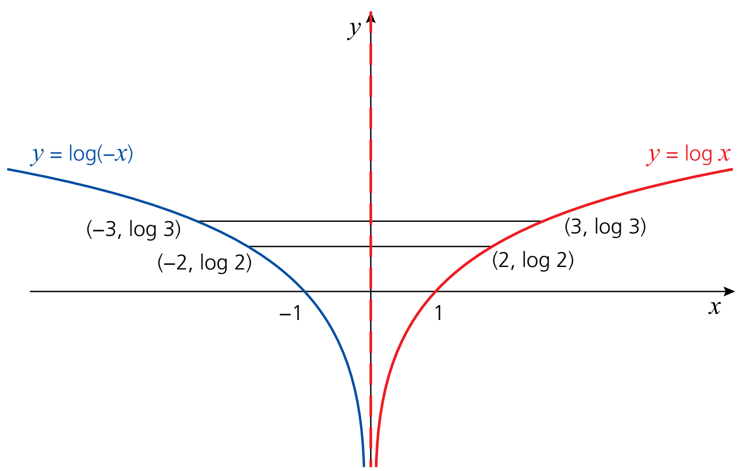
b The graph of $y = \log(x+2)$ is a translation of the graph of $y = \log x$ 2 units to the left.



c The graph of $y = -\log x$ (which is the same as $-y = \log x$) is a reflection of the graph of $y = \log x$ in the x-axis.



d The graph of $y = \log(-x)$ is a reflection of the graph of $y = \log x$ in the y-axis.



Worked example

You are given the curve of $y = \log x$ and told that $\log 3 = 0.48$ (2 d.p.).

a Sketch the graph of $y = \log 3 + \log x$.

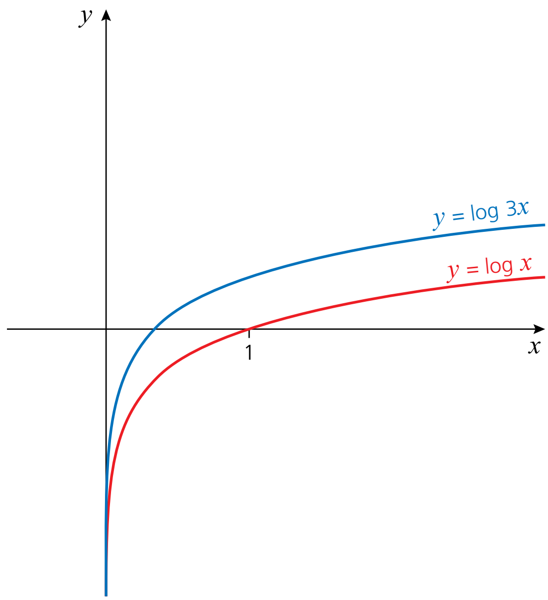
b What is the relationship between the graphs of $y = \log x$ and $y = \log 3 + \log x$?

c Sketch the graphs of $y = \log x$ and $y = \log 3x$ on the same axes.

d What do you notice?

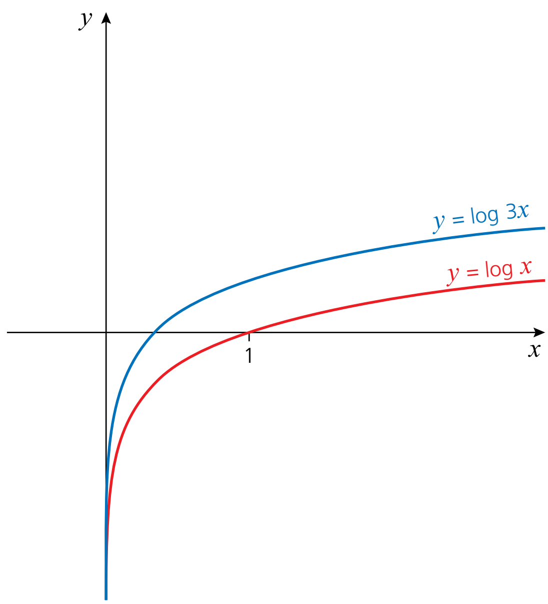
Solution

a



b The graph of $y = \log 3 + \log x$ is a translation of the graph of $y = \log x$ upwards by a distance of log 3.

c



You can use graphing software to show that the graph of $y = \log 3x$ is the same as the graph of $y = \log 3 + \log x$. This confirms one of the 'laws of logarithms' introduced below.

d The graph of $y = \log 3x$ looks the same as the graph of $y = \log 3 + \log x$.

Laws of logarithms

If a logarithmic expression is true for any base, the base is often omitted.

There are a number of rules for manipulating logarithms. They are derived from the rules for manipulating indices. These laws are true for all logarithms to any positive base.

Operation	Law for indices	Law for logarithms
Multiplication	$a^x \times a^y = a^{x+y}$	$\log_a xy = \log_a x + \log_a y$
Division	$a^x \div a^y = a^{x-y}$	$\log_a \frac{x}{y} = \log_a x - \log_a y$
Powers	$(a^x)^n = a^{nx}$	$\log_a x^n = n\log_a x$
Roots	$(a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}$	$\log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$
Logarithm of 1	$a^0 = 1$	$\log_a 1 = 0$
Reciprocals	$\frac{1}{a^x} = a^{-x}$	$\log_a \frac{1}{x} = \log_a 1 - \log_a x = -\log_a x$
Log to its own base	$a^1 = a$	$\log_a a = 1$

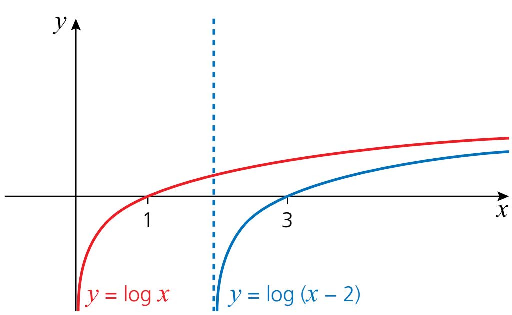
You can use these laws, together with the earlier work on translations, to help you sketch the graphs of a range of logarithmic expressions by breaking them down into small steps as shown below.

Worked example

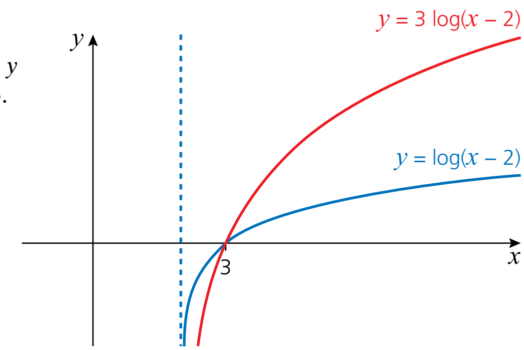
Sketch the graph of $y = 3\log(x-2)$.

Solution

Transforming the graph of the curve $y = \log x$ into $y = 3\log(x-2)$ involves two stages. Translating the graph of $y = \log x$ two units to the right gives the graph of $y = \log(x-2)$.



Multiplying log(x – 2) by 3 stretches the new graph in the y direction by a **scale factor** of 3.



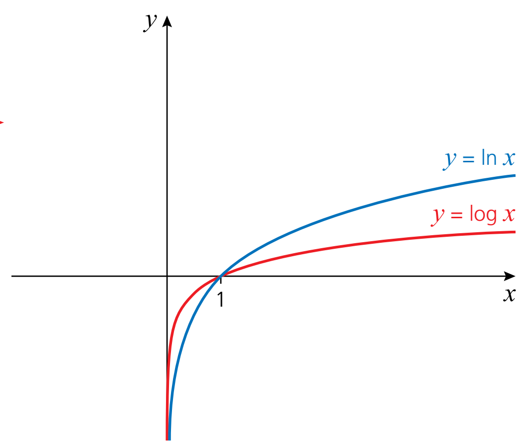
Logarithms to different bases

All the graphs you have met so far in the chapter could have been drawn to any base a greater than 1.

- When a logarithm is to the base 10 it can be written either as $\log_{10}$ or as lg. So, for example lg7 means $\log_{10}7$.
- Base e is the other common base for logarithms.

Notice that when you use a different base for the logarithm, the graph has a similar shape and still passes through the point (1, 0).

The graphs of logarithms with a base number that is not 10 are very similar to the graphs of logarithms with base 10.



Change of base of logarithms

It is sometimes useful to change the base of a logarithm.

$$x = \log_a b \Leftrightarrow a^x = b$$
$$\Leftrightarrow \log_c a^x = \log_c b$$
$$\Leftrightarrow x \log_c a = \log_c b$$
$$\Leftrightarrow x = \frac{\log_c b}{\log_c a}$$

Some calculators can manipulate logarithms to any positive base. Check whether yours is one of them.

Using logarithms to solve equations

Logarithms can be used to solve equations involving powers, to any level of accuracy.

This question does not ask for any particular base. In this case base 10 is used but you could alternatively have used base e. These are the two bases for logarithms on nearly all calculators.

Worked example

Solve the equation $3^x = 2000$.

Solution

Taking logarithms to the base 10 of both sides:

$$\lg 3^x = \lg 2000 \quad \Rightarrow \quad x \lg 3 = \lg 2000$$

$$\Rightarrow x = \frac{\lg 2000}{\lg 3} = 6.92 \text{ (3 s.f.)}$$

Logarithms can also be used to solve more complex equations.

Worked example

Solve the equation $4e^{3x} = 950$.

Solution

$$4e^{3x} = 950$$
$$\Rightarrow \quad e^{3x} = 237.5$$
$$\Rightarrow \quad 3x = \ln 237.5$$
$$\Rightarrow \quad 3x = 5.470167\ldots$$
$$\Rightarrow \quad x = 1.82 \text{ (3 s.f.)}$$

Taking the logarithms to base e of both sides

When there is a term of the form e^x, it is easier to use logarithms to base e, i.e. the ln button on your calculator.

Worked example

Use logarithms to solve the equation $3^{5-x} = 2^{5+x}$. Give your answer correct to 3 s.f.

Solution

No base is mentioned, so you can use logarithms to any base. Using base 10:

$$3^{5-x} = 2^{5+x}$$
$$\Rightarrow \quad \lg 3^{5-x} = \lg 2^{5+x}$$
$$\Rightarrow \quad (5-x)\lg 3 = (5+x)\lg 2$$
$$\Rightarrow \quad 5\lg 3 - x\lg 3 = 5\lg 2 + x\lg 2$$
$$\Rightarrow \quad 5\lg 3 - 5\lg 2 = x\lg 2 + x\lg 3$$
$$\Rightarrow \quad 5(\lg 3 - \lg 2) = x(\lg 2 + \lg 3)$$
$$\Rightarrow \quad x = \frac{5(\lg 3 - \lg 2)}{(\lg 2 + \lg 3)}$$
$$\Rightarrow \quad x = 1.13$$

Note that any base will yield the same answer. Using base 2:

$$3^{5-x} = 2^{5+x}$$
$$\Rightarrow \quad \log_2 3^{5-x} = \log_2 2^{5+x}$$
$$\Rightarrow \quad (5-x)\log_2 3 = (5+x)\log_2 2$$

Remember, $\log_2 2 = 1$.

$$\Rightarrow \quad 5\log_2 3 - x\log_2 3 = 5 + x$$
$$\Rightarrow \quad 5\log_2 3 - 5 = x + x\log_2 3$$
$$\Rightarrow \quad 5(\log_2 3 - 1) = x(1 + \log_2 3)$$
$$\Rightarrow \quad x = \frac{5(\log_2 3 - 1)}{1 + \log_2 3}$$
$$\Rightarrow \quad x = 1.13 \text{ (3 s.f.)}$$

Discussion point

Did you find one of these methods easier than the other?
If so, which one?

Using logarithms to solve inequalities

Logarithms are also useful to solve inequalities occurring, for example, in problems involving interest or depreciation.

When an inequality involves logs, it is often better to solve it as an equation first and then address the inequality. If you choose to solve it as an inequality, you may need to divide by a negative quantity and will therefore need to reverse the direction of the inequality sign. Both methods are shown in the example below.

Worked example

A second-hand car is bought for \$20 000 and is expected to depreciate at a rate of 15% each year. After how many years will it first be worth less than \$10 000?

Solution

The rate of depreciation is 15% so after one year the car will be worth 85% of the initial cost.

At the end of the second year, it will be worth 85% of its value at the end of Year 1, so $(0.85)^2 \times \$20000$.

Continuing in this way, its value after n years will be $(0.85)^n \times \$20000$.

Method 1: Solving as an equation

Solving the equation $(0.85)^n \times 20000 = 5000$

$$\Rightarrow \quad (0.85)^n = 0.5$$

$$\Rightarrow \quad \lg 0.85^n = \lg 0.5$$

$$\Rightarrow \quad n \lg 0.85 = \lg 0.5$$

$$\Rightarrow \quad n = \frac{\lg 0.5}{\lg 0.85}$$

$$\Rightarrow \quad n = 4.265\ldots$$

The car will be worth \$10 000 after 4.265 years, so it is **5 years** before it is first worth less than \$10 000.

Remember that lg 0.85 is negative and when you divide an inequality by a negative number, you must change the direction of the inequality.

Method 2: Solving as an inequality

Solving the inequality $(0.85)^n \times 20\,000 < 10\,000$

$$\Rightarrow \quad (0.85)^n < 0.5$$

$$\Rightarrow \quad \lg 0.85^n < \lg 0.5$$

$$\Rightarrow \quad n \lg 0.85 < \lg 0.5$$

$$\Rightarrow \quad n > \frac{\lg 0.5}{\lg 0.85}$$

$$\Rightarrow n > 4.265\ldots$$

The car will be worth less than \$10 000 after 5 years.

Exercise 7.1

In some of the following questions you are instructed not to use your calculator for the working, but you may use it to check your answers.

1 By first writing each of the following equations using powers, find the value of y without using a calculator:
a $y = \log_2 8$ **b** $y = \log_3 1$ **c** $y = \log_5 25$ **d** $y = \log_2 \frac{1}{4}$

2 $3^2 = 9$ can be written using logarithms as $\log_3 9 = 2$. Using your knowledge of indices, find the value of each of the following without using a calculator:
a $\log_2 16$ **b** $\log_3 81$ **c** $\log_5 125$ **d** $\log_4 \frac{1}{64}$

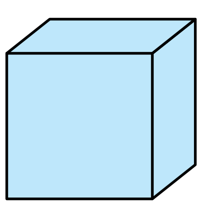
Remember that lg means $\log_{10}$.

3 Find the following without using a calculator:
a $\lg 100$ **b** lg(one million)
c $\lg \frac{1}{1000}$ **d** $\lg(0.000\,001)$

4 Using the rules for manipulating logarithms, rewrite each of the following as a single logarithm. For example, $\log 6 + \log 2 = \log(6 \times 2) = \log 12$.
a $\log 3 + \log 5$ **b** $3 \log 4$
c $\log 12 - \log 3$ **d** $\frac{1}{2} \log 25$
e $2 \log 3 + 3 \log 2$ **f** $4 \log 3 - 3 \log 4$
g $\frac{1}{2} \log 4 + 4 \log \frac{1}{2}$

5 Express each of the following in terms of $\log x$:
a $\log x^5 - \log x^2$ **b** $\log x^3 + 3 \log x$ **c** $5 \log \sqrt{x} - 3 \log \sqrt[3]{x}$

6 This cube has a volume of $800\,\text{cm}^3$.



a Use logarithms to calculate the side length correct to the nearest millimetre.
b What is the surface area of the cube?

Exercise 7.1 (cont)

7 Starting with the graph of $y = \ln x$, list the transformations required, in order when more than one is needed, to sketch each of the graphs. Use the transformations you have listed to sketch each graph.

a $y = 3\ln x$ **b** $y = \ln(x+3)$ **c** $y = 3\ln 2x$

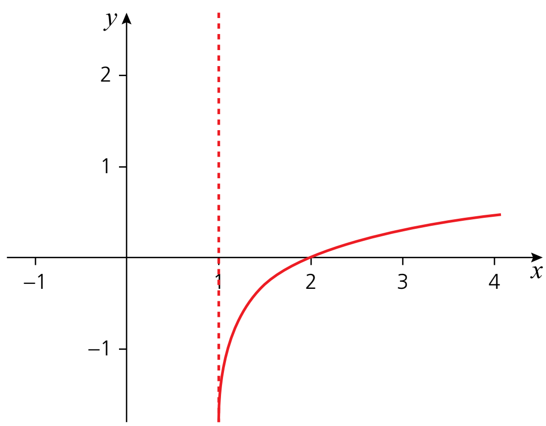
d $y = 3\ln x + 2$ **e** $y = -3\ln(x+1)$ **f** $y = \ln(2x+4)$

8 Match each equation from **i** to **vi** with the correct graph **a** to **f**.

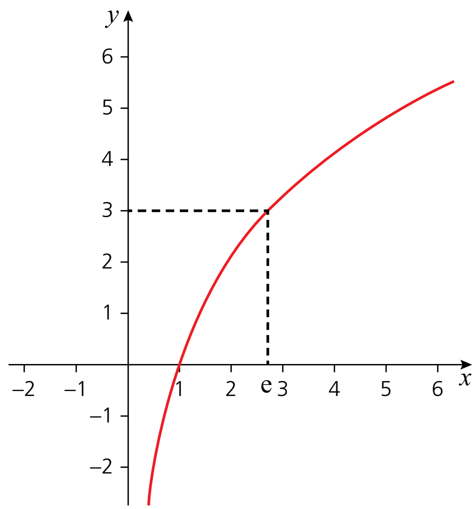
i $y = \log(x+1)$ **ii** $y = \log(x-1)$ **iii** $y = -\ln x$

iv $y = 3\ln x$ **v** $y = \log(2-x)$ **vi** $y = \ln(x+2)$

a



b



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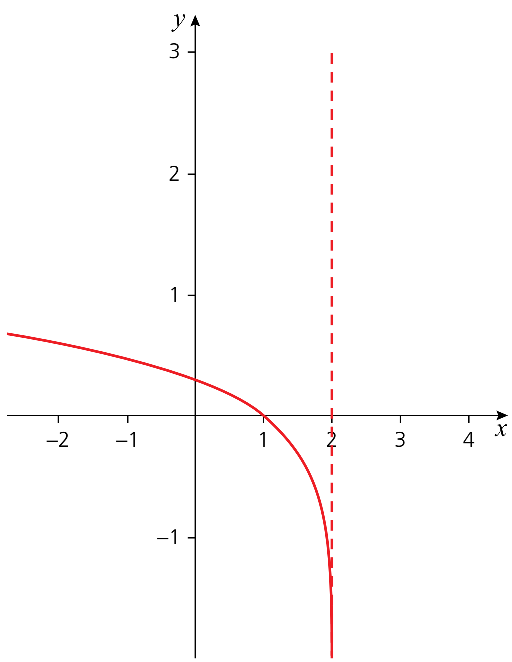
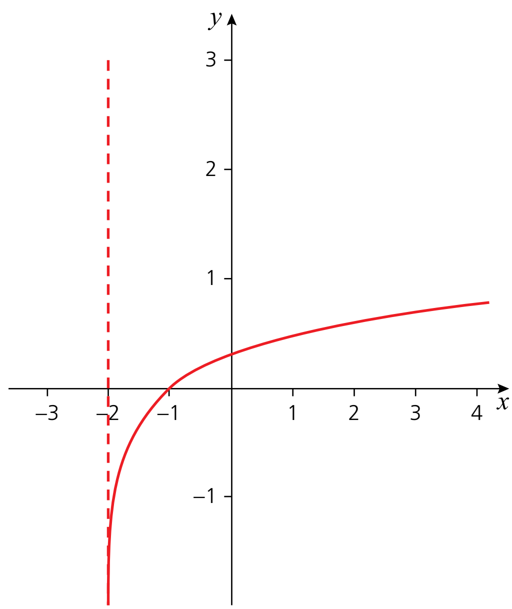
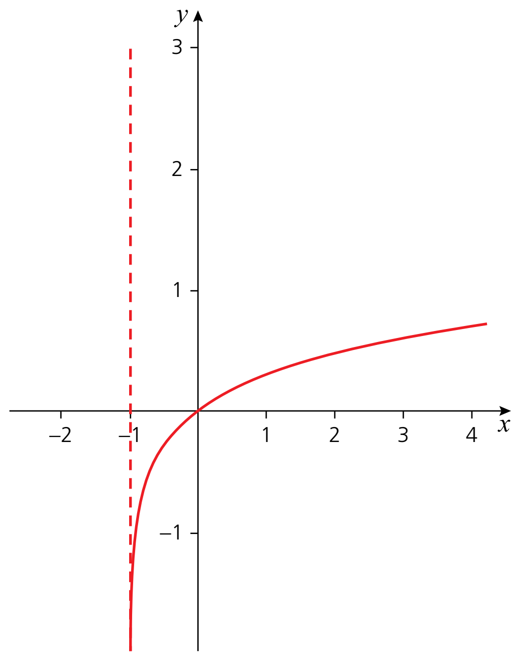


d



e



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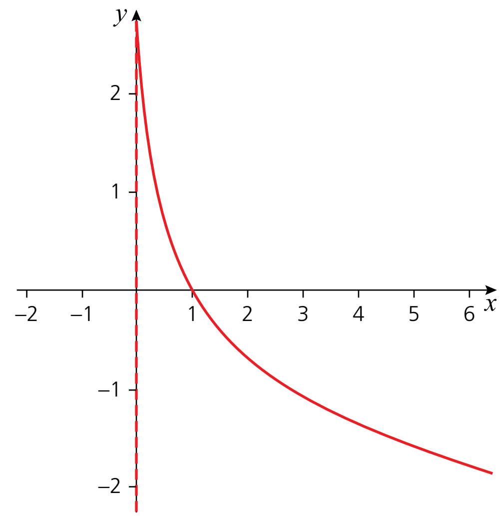


9 Solve the following equations for x, given that $\ln a = 3$:

a $a^{2x} = e^3$

b $a^{3x} = e^2$

c $a^{2x} - 3a^x + 2 = 0$

10 Before photocopiers were commonplace, school examination papers were duplicated using a process where each copy produced was only $c\%$ as clear as the previous copy. The copy was not acceptable if the writing was less than 50% as clear as the original. What is the value of c if the machine could produce only 100 acceptable copies from the original?

11 Use logarithms to solve the equation $5^{2x-1} = 4^{x+3}$. Give the value of x correct to 3 s.f.

12 a \$20 000 is invested in an account that pays interest at 2.4% per annum. The interest is added at the end of each year. After how many years will the value of the account first be greater than \$25 000?

b What percentage interest should be added each month if interest is to be accrued monthly?

c How long would the account take to reach \$25 000 if the interest was added:

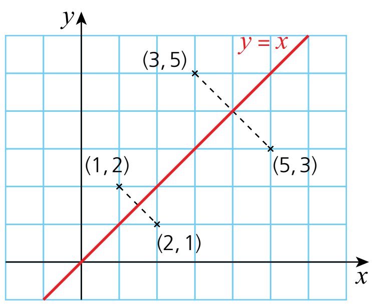
i every month

ii every day?

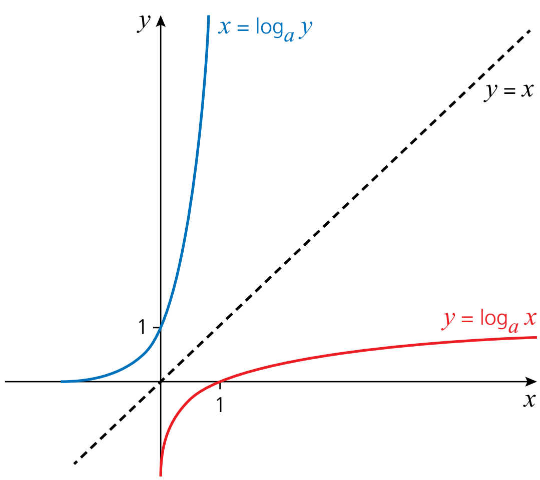
Exponential functions

The expression $y = \log_a x$ can be written as $x = a^y$. Therefore, the graphs of these two expressions are identical.

For any point, interchanging the x- and y-coordinates has the effect of reflecting the original point in the line $y = x$, as shown below.



Interchanging x and y for the graph $y = \log_a x$ (shown in red) gives the curve $x = \log_a y$ (shown in blue).



When rewritten with y as the subject of the equation, $x = \log_a y$ becomes $y = a^x$.

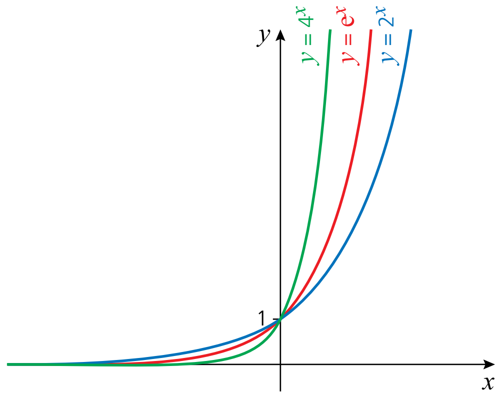
The function $y = a^x$ is called an **exponential function** and is the **inverse** of the logarithm function.

The most commonly used exponential function, known as **the exponential function**, is e^x, where e is the base of the logarithmic function $\ln x$ and is approximately equal to 2.718. You can manipulate exponential functions using the same rules as any other functions involving powers.

- $e^{a+b} = e^a \times e^b$
- $e^{a-b} = e^a \div e^b$

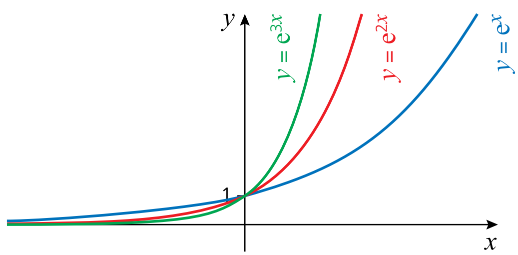
Graphs of e^x and associated exponential functions

The graph of $y = e^x$ has a similar shape to the graph of $y = a^x$ for positive value of a. The difference lies in the steepness of the curve.

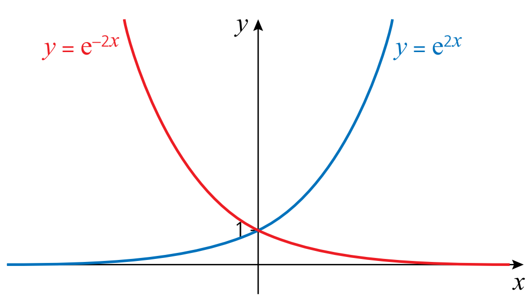


As the base number increases (i.e. 2, e and 4 in the equations above), the curve becomes steeper for positive values of x. All the y-values are positive and all the curves pass through and 'cross over' at the point (0, 1).

For positive integer values of n, curves of the form $y = e^{nx}$ are all related as shown below. Notice again, that the graphs all pass through the point (0, 1) and, as the value of n increases, the curves become steeper.

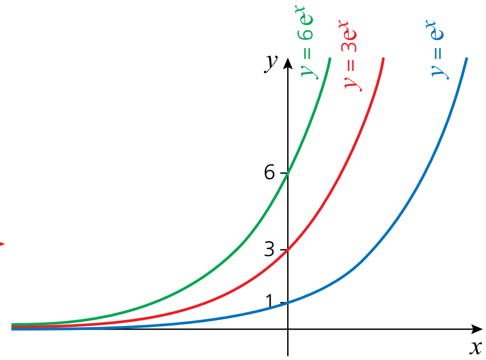


The graph of $y = e^{-x}$ is a reflection in the y-axis of the graph of $y = e^x$. The graphs of $y = e^{nx}$ and $y = e^{-nx}$ are related in a similar way for any integer value of n.



The family of curves $y = ke^x$, where k is a positive integer, is different set of transformations of the curve $y = e^x$. These represent stretches of the curve $y = e^x$ in the y-direction.

Notice that the curve $y = ke^x$ crosses the y-axis at $(0, k)$.



Similarly, for a fixed value of n, graphs of the family $y = ke^{nx}$ are represented by stretches of the graph $y = e^{nx}$ by scale factor k in the y-direction.

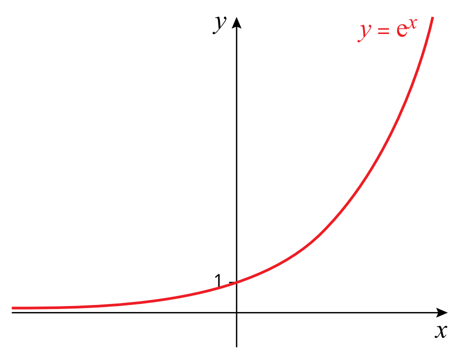
One additional transformation gives graphs of the form $y = ke^{nx} + a$.

Worked example

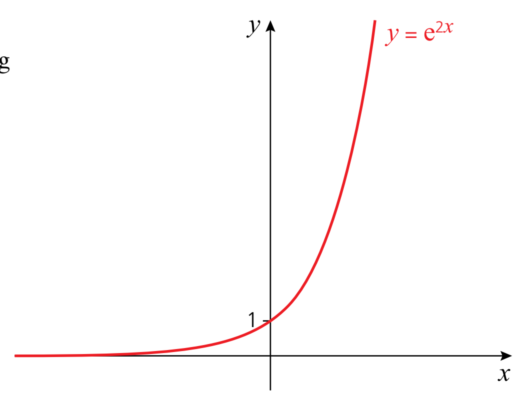
Sketch the graph of $y = 3e^{2x} + 1$.

Solution

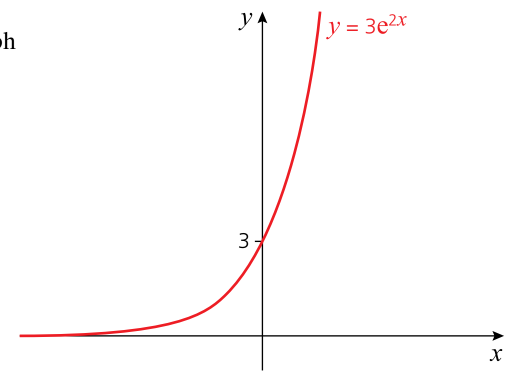
Start with $y = e^x$.



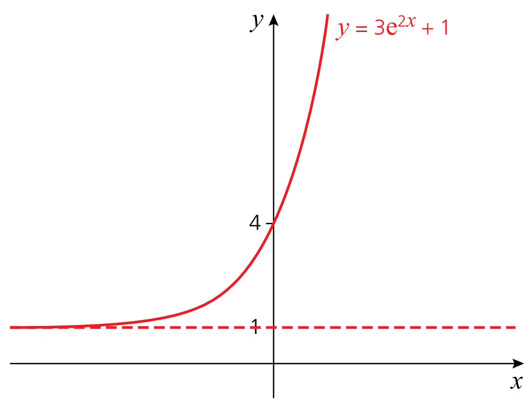
Transform to $y = e^{2x} = (e^x)^2$. The y values are squared, giving smaller values for $x < 0$ (where $y < 1$) and larger values for $x > 0$.



Stretch in the y-direction with a scale factor of 3 to give the graph of $y = 3e^{2x}$.



Translate 1 unit upwards to give $y = 3e^{2x} + 1$.



Worked example

a Solve the equation $e^{2x} - 5e^{x} + 6 = 0$

b Hence solve the equation $e^{4x} - 5e^{2x} + 6 = 0$

Solution

a $e^{2x} - 5e^{x} + 6 = 0$

$\Rightarrow (e^{x} - 2)(e^{x} - 3) = 0$

$\Rightarrow e^{x} = 2$ or $e^{x} = 3$

$\Rightarrow x = \ln 2$ or $x = \ln 3$

$\Rightarrow x = 0.693$ or $x = 1.099$ (3 d.p.)

b $e^{4x} - 5e^{2x} + 6 = (e^{2x} - 2)(e^{2x} - 3)$

So, either $2x = \ln 2 \quad \Rightarrow x = 0.347$ (3 d.p.)

or $2x = \ln 3 \quad \Rightarrow x = 0.549$ (3 d.p.)

Exponential growth and decay

The word 'exponential' is often used to refer to things that increase or decrease at a very rapid rate.

$y = e^x$ is called **the** exponential function.

Any function of the form $y = a^x$ is referred to as an exponential function. When $x > 0$, the function $y = a^x$ is referred to as **exponential growth**; when $x < 0$ it is **exponential decay**.

Worked example

During the growth of an organism, a cell divides into two approximately every 6 hours. Assuming that the process starts with a single cell, and none of the cells die, how many cells will there be after 1 week?

Solution

It is possible to work this out without any special formulae:

2 cells after 6 hours

4 cells after 12 hours

8 cells after 18 hours…

However as the numbers get larger, the working becomes more tedious.

Notice the pattern here using 6 hours as 1 time unit.

2^1 cells after 1 time unit

2^2 cells after 2 time units

2^3 cells after 3 time units…

1 day of 24 hours is 4 time units, so 1 week of 7 days is 28 time units. So after 1 week there will be $2^{28} = 268\,435\,456$ cells.

Worked example

A brand of 'invisible' ink fades rapidly once it is applied to paper. After each minute the intensity is reduced by one quarter. It becomes unreadable to the naked eye when the intensity falls below 5% of the original value.

a What is the intensity, as a percentage of the original value, after 3 minutes?

b After how many minutes does it become unreadable to the naked eye? Give your answer to the nearest whole number.

Solution

a After 1 minute it is $\frac{3}{4}$ of the original value.

After 2 minutes it is $\frac{3}{4}\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^2$ of the original value.

After 3 minutes it is $\left(\frac{3}{4}\right)^3 = \frac{27}{64}$ or approximately 42% of the original value.

It would be very tedious to continue the method in used above until the ink becomes unreadable.

b Using the pattern developed above:

After t minutes it is approximately $\left(\frac{3}{4}\right)^t$ of the original value.

The situation is represented by: $\left(\frac{3}{4}\right)^t < \frac{5}{100}$ $5\% = \frac{5}{100}$

Using logarithms to solve the inequality as an equation:

$$\lg\left(\frac{3}{4}\right)^t = \lg\frac{5}{100} \Rightarrow t\lg\left(\frac{3}{4}\right) = \lg\left(\frac{5}{100}\right)$$

$$\Rightarrow t\lg 0.75 = \lg 0.05$$

$$\Rightarrow t = \frac{\lg 0.05}{\lg 0.75}$$

$$\Rightarrow t = 10.4$$

Since the question asks for the time as a whole number of minutes, and the time is increasing, the answer is 11 minutes.

Exercise 7.2

It is a good idea to check the graphs you draw in Questions 1–4 using any available graphing software.

1 For each set of graphs:
i Sketch the graphs on the same axes.
ii Give the coordinates of any points of intersection with the axes.
a $y = e^x$, $y = e^x + 1$ and $y = e^{x+1}$
b $y = e^x$, $y = 2e^x$ and $y = e^{2x}$
c $y = e^x$, $y = e^x - 3$ and $y = e^{x-3}$

2 Sketch the graphs of $y = e^{3x}$ and $y = e^{3x} - 2$.

3 Sketch the graphs of $y = e^{2x}$, $y = 3e^{2x}$ and $y = 3e^{2x} - 1$.

4 Sketch each curve and give the coordinates of any points where it cuts the y-axis.
a $y = 2 + e^x$ **b** $y = 2 - e^x$ **c** $y = 2 + e^{-x}$ **d** $y = 2 - e^{-x}$

5 Solve the following equations:
a $5e^{0.3t} = 65$ **b** $13e^{0.5t} = 65$ **c** $e^{t+2} = 10$ **d** $e^{t-2} = 10$

6 The value, $\$V$, of an investment after t years is given by the formula $V = Ae^{0.03t}$, where $\$A$ is the initial investment.
a How much, to the nearest dollar, will an investment of \$4000 be worth after 3 years?
b To the nearest year, how long will I need to keep an investment for it to double in value?

7 The path of a projectile launched from an aircraft is given by the equation $h = 5000 - e^{0.2t}$, where h is the height in metres and t is the time in seconds.
a From what height was the projectile launched?
The projectile is aimed at a target at ground level.
b How long does it take to reach the target?

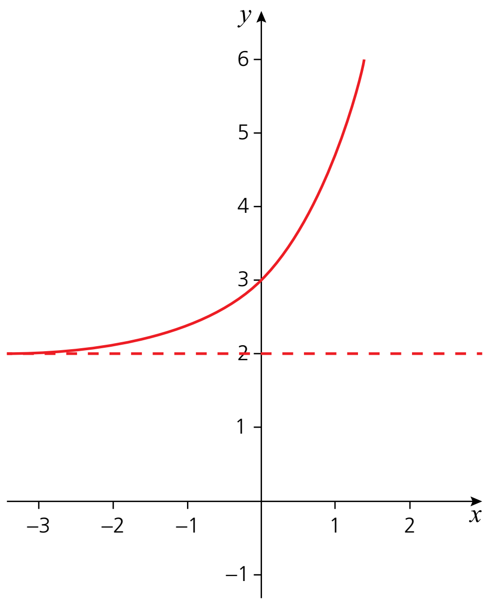
Exercise 7.2 (cont)

8 Match each equation from **i** to **vi** to the correct graph **a** to **f**.

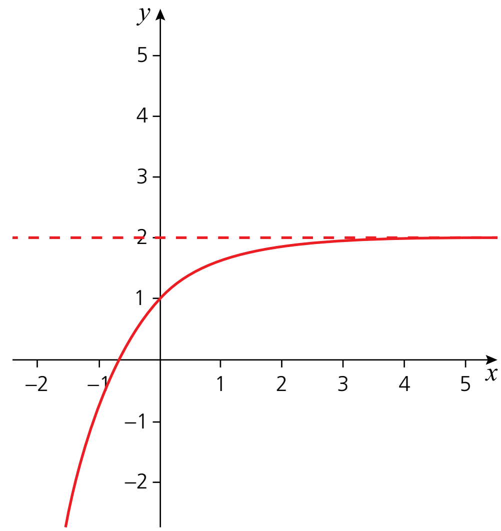
i $y = e^{2x}$ **ii** $y = e^{x} + 2$ **iii** $y = 2 - e^{x}$

iv $y = 2 - e^{-x}$ **v** $y = 3e^{-x} - 5$ **vi** $y = e^{-2x} - 1$

a



b



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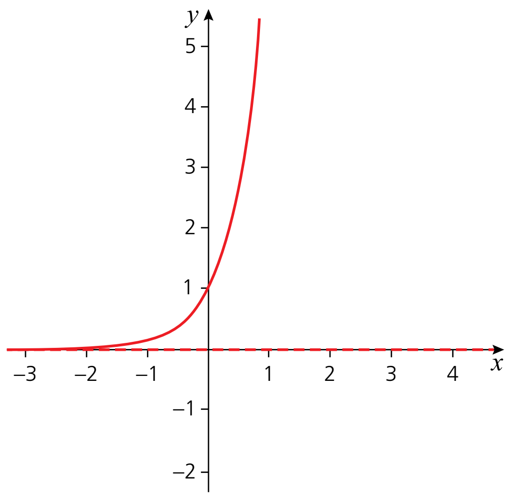
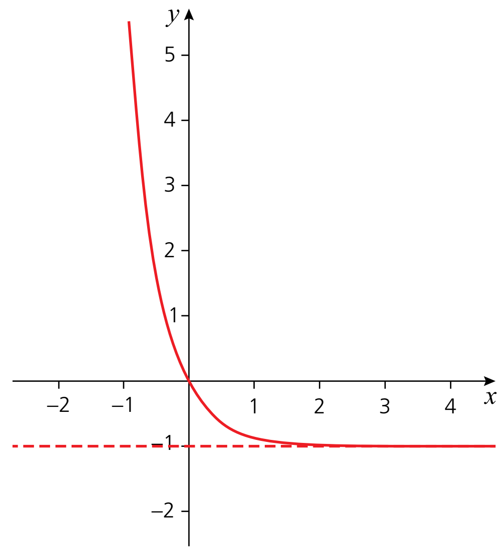


d



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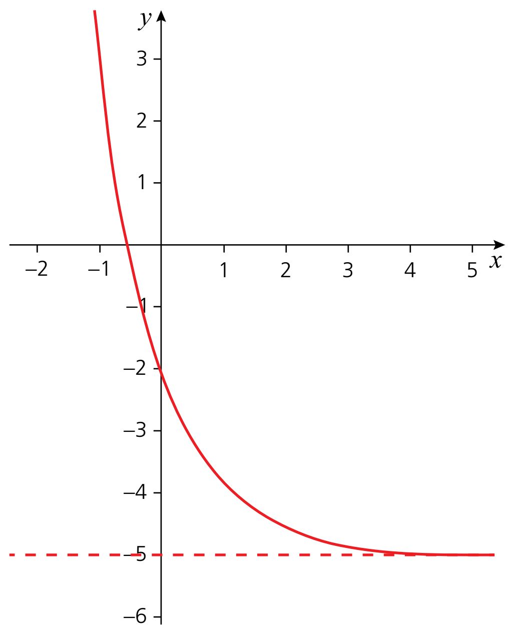
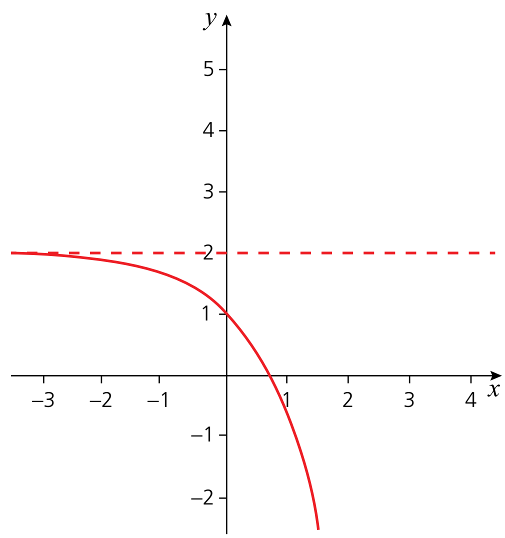


f



9 A radioactive substance of mass 100 g is decaying such that after t days the amount remaining, M, is given by the equation $M = 100e^{-0.002t}$.

a Sketch the graph of M against t.

b What is the half-life of the substance (i.e. the time taken to decay to half the initial mass)?

10 When David started his first job, he earned $15 per hour and was promised an annual increment (compounded) of 3.5%.

a What is his hourly rate be in his 5th year?

After 5 years he was promoted. His hourly wage increased to $26 per hour, with the same compounded annual increment.

b For how many more years will he need to work before his hourly rate reaches $30 per hour?

11 a Solve $2\left(3^{2x}\right) - 5\left(3^{x}\right) + 2 = 0$

b Solve $e^{x}e^{x+1} = 10$

c Solve $2^{2x} - 5\left(2^{x}\right) + 4 = 0$

Past–paper questions

1 Given that $\log_a pq = 9$ and $\log_a p^2q = 15$, find the value of

(i) $\log_a p$ and of $\log_a q$, [4]

(ii) $\log_p a + \log_q a$. [2]

Cambridge O Level Additional Mathematics 4037
Paper 12 Q4 November 2012
Cambridge IGCSE Additional Mathematics 0606
Paper 12 Q4 November 2012

2 Solve the simultaneous equations

$\log_3 a = 2\log_3 b$,

$\log_3 (2a - b) = 1$. [5]

Cambridge O Level Additional Mathematics 4037
Paper 13 Q5 November 2010
Cambridge IGCSE Additional Mathematics 0606
Paper 13 Q5 November 2010

3 The number of bacteria B in a culture, t days after the first observation, is given by

$B = 500 + 400e^{0.2t}$

(i) Find the initial number present. [1]

(ii) Find the number present after 10 days. [1]

(iv) Find the value of t when $B = 10\,000$. [3]

Cambridge O Level Additional Mathematics 4037
Paper 22 Q5 November 2014
(Part question: part (iii) omitted)
Cambridge IGCSE Additional Mathematics 0606
Paper 22 Q5 November 2014
(Part question: part (iii) omitted)

Learning outcomes

Now you should be able to:

★ recognise simple properties and graphs of the logarithmic and exponential functions including $\ln x$ and e^x and graphs of $ke^{nx} + a$ and $k\ln(ax + b)$ where n, k, a and b are integers

★ recognise and use the laws of logarithms (including change of base of logarithms)

★ solve equations of the form $a^x = b$.

Key points

- **Logarithm** is another word for **index** or **power**.
- The laws for logarithms are valid for all bases greater than 0 and are related to those for indices.

Operation	Law for indices	Law for logarithms
Multiplication	$a^x \times a^y = a^{x+y}$	$\log_a xy = \log_a x + \log_a y$
Division	$a^x \div a^y = a^{x-y}$	$\log_a \frac{x}{y} = \log_a x - \log_a y$
Powers	$(a^x)^n = a^{nx}$	$\log_a x^n = n\log_a x$
Roots	$(a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}$	$\log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$
Logarithm of 1	$a^0 = 1$	$\log_a 1 = 0$
Reciprocals	$\frac{1}{a^x} = a^{-x}$	$\log_a \frac{1}{x} = \log_a 1 - \log_a x = -\log_a x$
Log to its own base	$a^1 = a$	$\log_a a = 1$

- The graph of $y = \log x$: is only defined for $x > 0$
  has the y-axis as an asymptote
  has a positive gradient
  passes through $(0, 1)$ for all bases.
- Notation.
  The logarithm of x to the base a is written $\log_a x$.
  The logarithm of x to the base 10 is written $\lg x$ or $\log x$.
  The logarithm of x to the base e is written $\ln x$.
- An **exponential function** is of the form $y = a^x$.
- The exponential function is the inverse of the log function.
  $y = \log_a x \Leftrightarrow a^y = x$
- For $a > 0$, the graph of $y = a^x$: has the x-axis as an asymptote has a positive gradient passes through $(0, 1)$.
- For $a > 0$, the graph of $y = a^{-x}$: has the x-axis as an asymptote has a negative gradient passes through $(0, 1)$.

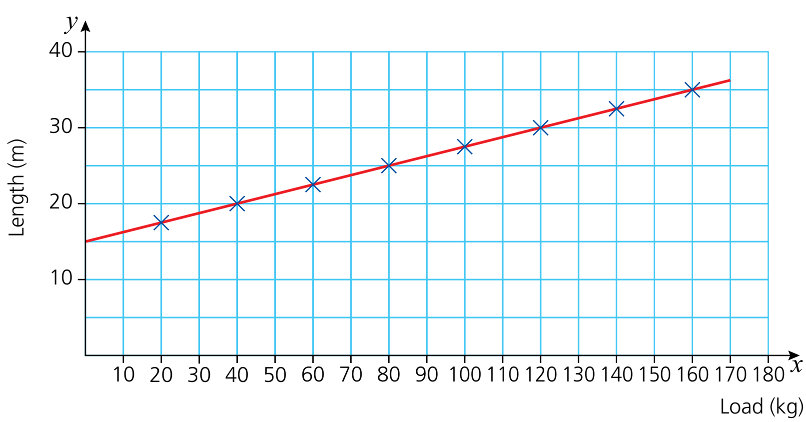
8 Straight line graphs

Every new body of discovery is mathematical in form because there is no other guidance we can have.

Charles Darwin (1809 – 1882)

Discussion point

If you do a bungee jump, you will want to be certain that the rope won't stretch too far.
In an experiment a rope is tested by hanging different loads on it and measuring its length. The measurements are plotted on this graph.



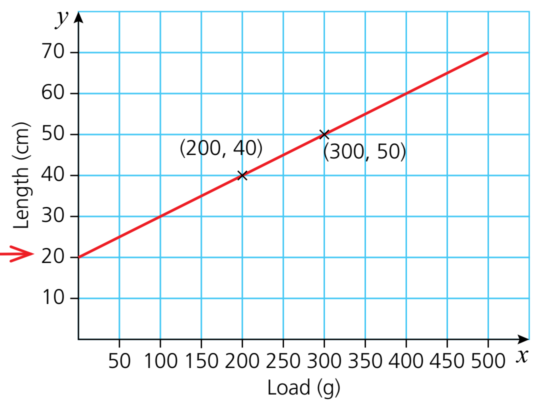
You will have met straight line graphs frequently in abstract algebraic problems, but they can also be used to find information in a practical situation such as this.

What does the graph tell you about the rope?

When a load of 200 g is attached to a spring, its stretched length is 40 cm. With a load of 300 g, its length is 50 cm. Assuming that the extension is proportional to the load, draw a graph to show the relationship between the load and the length of the spring and use it to find the natural length of the spring.

Since the load is the variable that can be directly controlled, it is plotted on the horizontal axis and the length of the spring on the vertical axis. Plotting the points (200, 40) and (300, 50) and joining them with a straight line gives the graph below.

The graph shows that the natural length of the spring, i.e. the length when there is no load attached, is 20 cm.



The information that the extension is proportional to the load tells you that the graph of the relationship will be a straight line.

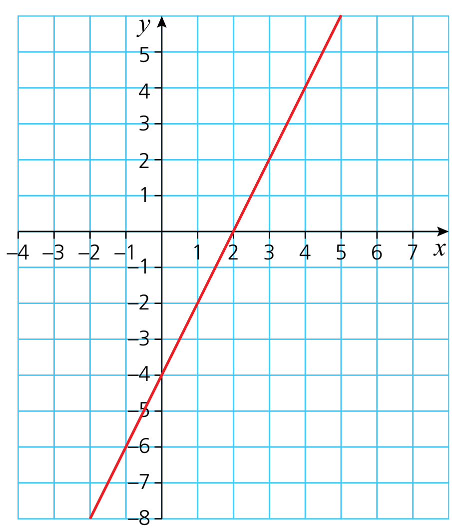
The straight line $y = mx + c$

When the equation of a straight line is written in the form $y = mx + c$, m represents the gradient of the line and the line crosses the y–axis at $(0, c)$.

You can use this to find the equation of a straight line given the graph.

Worked example

Find the equation of this straight line.



The gradient of the line joining the points (x_1, y_1) and (x_2, y_2) is given by Gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

Solution

The line crosses the y-axis at $(0, -4)$ so $c = -4$.

To find the gradient of the line, choose two points on the line and call them (x_1, y_1) and (x_2, y_2). The points of intersection with the axes, $(0, -4)$ and $(2, 0)$, are obvious choices.

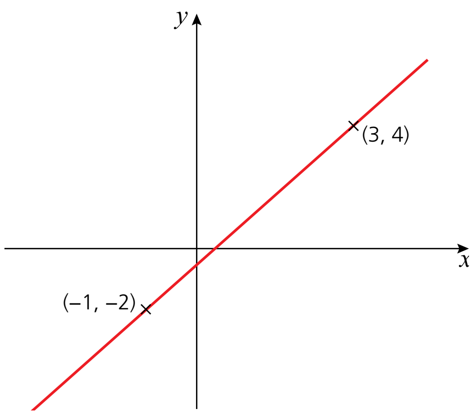
Using the gradient formula:

Gradient $(m) = \frac{0-(-4)}{2-0} = 2$

So the equation of the line is $y = 2x - 4$.

Worked example

Find the equation of the line shown.



Solution

Substitute each pair of coordinates into the equation $y = mx + c$.

Point $(-1, -2)$: $\quad -2 = m(-1) + c \qquad (1)$

Point $(3, 4)$: $\quad 4 = m(3) + c \qquad (2)$

Subtract equation (1) from equation (2).

$4 - (-2) = (3m + c) - (-m + c)$

$\Rightarrow \quad 6 = 3m + c + m - c$

$\Rightarrow \quad 6 = 4m$

$\Rightarrow \quad m = 1.5$

Equation 2 is the more straightforward equation because it has no negative signs.

Substitute $m = 1.5$ into equation (2).

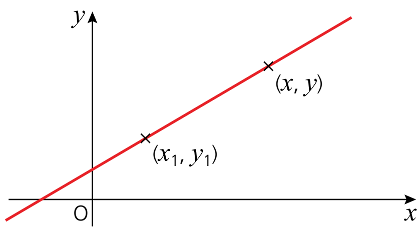
$4 = 3(1.5) + c$

$\Rightarrow \quad c = -0.5$

So the equation of the line is $y = 1.5x - 0.5$.

As well as $y = mx + c$, there are several other formulae for the equation of a straight line. One that you are likely to find useful deals with the situation where you know the gradient of the line, m, and the coordinates of one point on it, (x_1, y_1).

The equation is $y - y_1 = m(x - x_1)$.



Midpoint of a line

When a line has a fixed length, the midpoint, i.e. the point half way between the two ends of the line, has as its coordinates the average of the individual x- and y-coordinates.

Worked example

The midpoint of the line joining the points (x_1, y_1) and (x_2, y_2) is given by midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

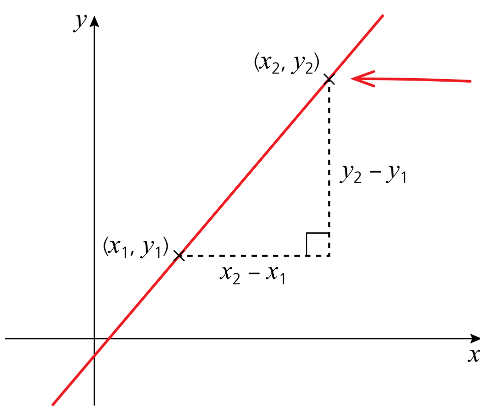
Find the midpoint of the line joining (2, 5) and (4, 13).

Solution

The coordinates of the midpoint are $\left(\frac{2+4}{2}, \frac{5+13}{2}\right) = (3, 9)$.

Length of a line

To find the length of the line joining two points, use Pythagoras' theorem.



The length of the line joining the points (x_1, y_1) and (x_2, y_2) is given by length $= \sqrt{((x_2 - x_1)^2 + (y_2 - y_1)^2)}$

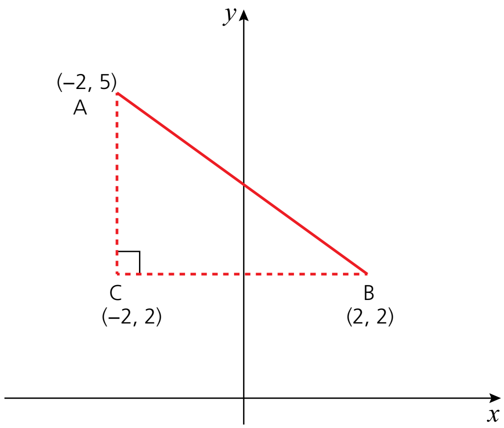
Worked example

Work out the length of the line joining the points A(−2, 5) and B(2, 2).

Solution

You can either:

- sketch the triangle and then use Pythagoras' theorem or
- use the formula given above.



From the triangle: $AC = 3$ units

$BC = 4$ units

So $AB^2 = 3^2 + 4^2$

$= 25$

$AB = 5$ units

Alternatively, substituting directly into the formula (without drawing a diagram) gives:

$$\text{length} = \sqrt{\left((2-(-2))^2 + (2-5)^2\right)}$$

$$= 5 \text{ units}$$

Parallel lines

Two lines are **parallel** if they have the same gradient. If you are given the equations of two straight line graphs in the form $y = mx + c$, you can immediately identify whether or not the lines are parallel. For example, $y = 3x - 7$ and $y = 3x + 2$ are parallel since they both have a gradient of 3.

If one or both of the equations are given in a different form, you will need to rearrange them in order to find out whether or not they are parallel.

Worked example

Show that the two lines $y = \frac{1}{2}x - 4$ and $x - 2y - 6 = 0$ are parallel.

Solution

Start by rearranging the second equation into the form $y = mx + c$.

$$x - 2y - 6 = 0 \Rightarrow x - 6 = 2y$$

$$\Rightarrow 2y = x - 6$$

$$\Rightarrow y = \frac{1}{2}x - 3$$

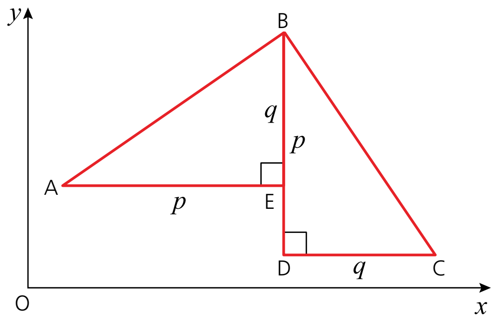
Both lines have a gradient of $\frac{1}{2}$ so are parallel.

Perpendicular lines

Two lines are **perpendicular** if they intersect at an angle of 90°.

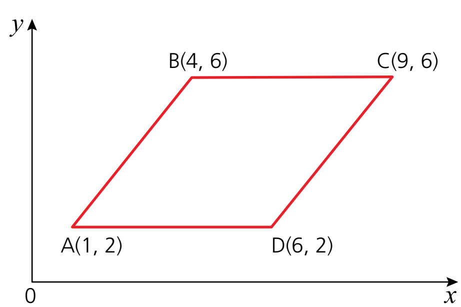
Activity

The diagram shows two congruent right-angled triangles where p and q can take any value.



1 Copy the diagram onto squared paper.
2 Explain why $\angle ABC = 90°$.
3 Calculate the gradient of AB (m_1) and the gradient of BC (m_2).
4 Show that $m_1 m_2 = -1$.

Worked example



a Explain why ABCD is a rhombus.
b Show that the diagonals AC and BD are perpendicular. (This result is always true for a rhombus.)

Solution

a A rhombus is a parallelogram with all sides equal in length.

AD and BC are both parallel to the x-axis and have length 5 units.

gradient of AB = gradient DC $= \dfrac{\text{increase in } y}{\text{increase in } x} = \dfrac{4}{3}$

$AB = DC = \sqrt{3^2 + 4^2} = 5$ units

So ABCD is a rhombus.

b Using the formula gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

gradient of AC $= \frac{6-2}{9-1} = \frac{1}{2}$

gradient of BD $= \frac{2-6}{6-4} = -2$

$\frac{1}{2} \times (-2) = -1$ so diagonals AC and BD are perpendicular.

Exercise 8.1

1 For each of the following pairs of points A and B, calculate:
- **i** the gradient of the line AB
- **ii** the gradient of the line perpendicular to AB
- **iii** the length of AB
- **iv** the coordinates of the midpoint of AB.
- **a** A(4, 3) B(8, 11)
- **b** A(5, 3) B(10, –8)
- **c** A(6, 0) B(8, 15)
- **d** A(–3, –6) B(2, –7)

2 A(0, 5), B(4, 1) and C(2, 7) are the vertices of a triangle. Show that the triangle is right angled:
- **a** by working out the gradients of the sides
- **b** by calculating the lengths of the sides.

3 A(3, 5), B(3, 11) and C(6, 2) are the vertices of a triangle.
- **a** Work out the perimeter of the triangle.
- **b** Sketch the triangle and work out its area using AB as the base.

4 A quadrilateral PQRS has vertices at P(–2, –5), Q(11, – 7), R(9, 6) and S(–4, 8).
- **a** Work out the lengths of the four sides of PQRS.
- **b** Find the coordinates of the midpoints of the diagonals PR and QS.
- **c** Without drawing a diagram, show that PQRS cannot be a square. What shape is PQRS?

5 The points A, B and C have coordinates (2, 3), (6, 12) and (11, 7) respectively.
- **a** Draw the triangle ABC.
- **b** Show by calculation that the triangle is isosceles and write down the two equal sides.
- **c** Work out the midpoint of the third side.
- **d** By first calculating appropriate lengths, calculate the area of triangle ABC.

6 A triangle ABC has vertices at A(3, 2), B(4, 0) and C(8, 2).
- **a** Show that the triangle is right angled.
- **b** Find the coordinates of point D such that ABCD is a rectangle.

7 P(–2, 3), Q(1, q) and R(7, 0) are collinear points (i.e. they lie on the same straight line).
- **a** Find the value of Q.
- **b** Write down the ratio of the lengths PQ : QR.

8 A quadrilateral has vertices A(–2, 8), B(–5, 5), C(5, 3) and D(3, 7).

a Draw the quadrilateral.
b Show by calculation that it is a trapezium.
c ABCE is a parallelogram. Find the coordinates of E.

9 In each part, find the equation of the line through the given point that is:

i parallel and **ii** perpendicular to the given line.

a $y = 2x + 6$; $(5, -3)$
b $x + 3y + 5 = 0$; $(-4, 7)$
c $2x = 3y + 1$; $(-1, -6)$

10 Find the equation of the perpendicular bisector of the line joining each pair of points.

a (2, 3) and (8, –1)
b (–7, 3) and (1, 5)
c (5, 6) and (4, –3)

11 P is the point (2, –1) and Q is the point (8, 2).

a Write the equation of the straight line joining P and Q.
b Find the coordinates of M, the midpoint of PQ.
c Write the equation of the perpendicular bisector of PQ.
d Write down the coordinates of the points where the perpendicular bisector crosses the two axes.

Relationships of the form $y = ax^n$

When you draw a graph to represent a practical situation, in many cases your points will lie on a curve rather than a straight line. When the relationships are of the form $y = ax^n$ or $y = Ab^x$, you can use logarithms to convert the curved graphs into straight lines. Although you can take the logarithms to any positive base, the forms log and ln are used in most cases.

Worked example

The data in the table were obtained from an experiment. y represents the mass in grams of a substance (correct to 2 d.p.) after a time t minutes.

t	4	9	14	19	24	29
y	3.00	4.50	5.61	6.54	7.35	8.08

Saira wants to find out if these values can be modelled by the function $y = at^n$.

a By taking logarithms to the base 10 of both sides, show that the model can be written as $\log y = n \log t + \log a$.

b Explain why, if the model is valid, plotting the graph of $\log y$ against $\log t$ will result in a straight line.

c Plot the graph of $\log y$ against $\log t$ and use it to estimate the values of a and t. Hence express the relationship in the form $y = at^n$.

d Assuming that this relationship continues for at least the first hour, after how long would there be 10 g of the substance?

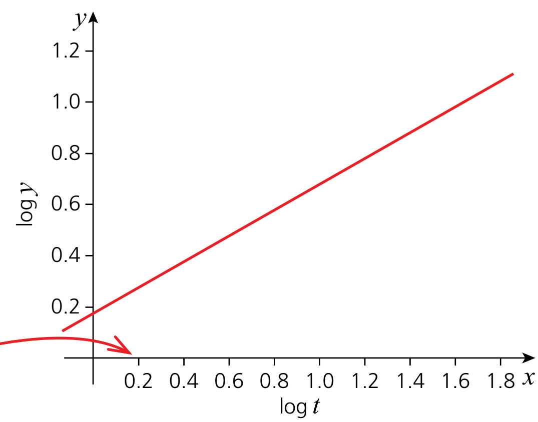
Solution

a $y = at^n \Rightarrow \lg y = \lg at^n$
$\Rightarrow \lg y = \lg a + \lg t^n$
$\Rightarrow \lg y = \lg a + n \lg t$
$\Rightarrow \lg y = n \lg t + \lg a$

b Comparing this with the equation $Y = mX + c$ gives $Y = \log y$ and $X = \log t$. This shows that if the model is valid, the graph of $\log y$ (on the vertical axis) against $\log t$ will be a straight line with gradient n and intercept on the vertical axis at $\log a$.

c

t	4	9	14	19	24	29
y	3.00	4.50	5.61	6.54	7.35	8.08
log *t*	0.60	0.95	1.15	1.28	1.38	1.46
log *y*	0.48	0.65	0.75	0.82	0.87	0.91



NOTE: You cannot have a break in the horizontal axis because this would lead to an incorrect point of intersection with the y-axis.

Using the points (0, 0.18) and (1.5, 0.92), the gradient of the line is:

gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

$\frac{0.92 - 0.18}{1.5 - 0} = 0.493$

This is approximately equal to 0.5, so $n = 0.5$.

Since the values from the graph are only approximate, the results should only be given to 1 or 2 d.p.

Using intercept on the y-axis $= \lg a$

$0.18 = \lg a$

$\Rightarrow a = 1.513 \approx 1.5$

Therefore the relationship is $y = 1.5t^{0.5}$ or $y = 1.5\sqrt{t}$.

Do not go beyond the values in the table unless the question tells you to. If it doesn't, you cannot be sure that the relationship you have found is valid outside of known bounds.

d There will be 10 g of the substance when $y = 10$

$\Rightarrow \quad 10 = 1.5\sqrt{t}$

$\Rightarrow 100 = 2.25t$

$\Rightarrow \quad t = \frac{100}{2.25}$

$= 44.44$

So, there will be 10 g after about 44 minutes.

Relationships of the form $y = Ab^x$

These are often referred to as exponential relationships since the variable is the power.

Worked example

The table shows the temperature, θ, recorded in degrees Celsius to the nearest degree, of a cup of coffee t minutes after it is poured and milk is added.

t	0	4	8	12	16	20
θ	80	63	50	40	32	25

Seb is investigating whether the relationship between temperature and time can be modelled by an equation of the form $\theta = Ab^t$.

a By taking logarithms to base e of both sides, show that the model can be written as $\ln\theta = \ln A + t\ln b$.

b Explain why, if the model is valid, plotting the graph of $\ln\theta$ against t will result in a straight line.

c Plot the graph of $\ln\theta$ against t and use it to estimate the values of A and b. Hence express the relationship in the form $\theta = Ab^t$.

d Why will this relationship not continue indefinitely?

Solution

a $\theta = Ab^t \Rightarrow \ln\theta = \ln Ab^t$

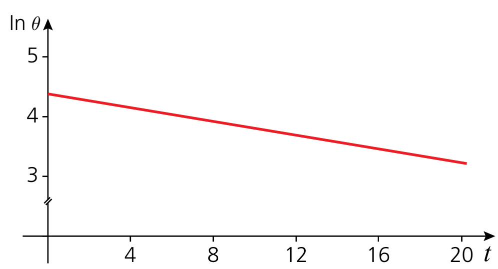
$$\Rightarrow \ln\theta = \ln A + \ln b^t$$

$$\Rightarrow \ln\theta = \ln A + t\ln b$$

b Rewriting $\ln\theta = \ln A + t\ln b$ as $\ln\theta = (\ln b)t + \ln A$ and comparing it with the equation $y = mx + c$ shows that plotting $\ln\theta$ against t will give a straight line with gradient $\ln b$ and intercept on the vertical axis at $\ln A$.

c

t	0	4	8	12	16	20
θ	80	63	50	40	32	25
$\ln\theta$	4.38	4.14	3.91	3.69	3.47	3.22



Using the points (0, 4.40) and (20, 3.22), the gradient of the line is:

$$\frac{3.22 - 4.38}{20 - 0} = -0.058$$

$-0.058 = \ln b \;\Rightarrow\; b = 0.94$ (2 d.p.)

The intercept on the vertical axis is at $\ln \theta = 4.40$.

From the table, this corresponds $\theta = 80$.

Therefore, the relationship is $\theta = 80 \times 0.94^t$.

d The relationship will not continue indefinitely since the coffee will not cool below room temperature.

Exercise 8.2

1 Match the equivalent relationships.

i $y = pr^x$ **ii** $y = rp^x$
iii $y = px^r$ **iv** $y = xp^r$
a $\log y = \log p + r \log x$ **b** $\log y = \log r + x \log p$
c $\lg y = \lg p + x \lg r$ **d** $\lg y = \lg x + r \lg p$

2 For each of the following models, k, a and b are constants. Use logarithms to base e to rewrite them in the form $y = mx + c$, stating the expressions equal to x, y, m and c in each case.

a $y = ka^x$ **b** $y = kx^a$
c $y = ak^x$ **d** $y = ax^k$

3 The table below shows the area, A, in square centimetres of a patch of mould t days after it first appears.

t	1	2	3	4	5	6
A	1.8	2.6	3.6	5.1	7.2	10.3

It is thought that the relationship between A and t is of the form $A = kb^t$.

a Show that the model can be written as $\log A = (\log b)t + \log k$.
b Plot the graph of $\log A$ against t and explain why it supports the assumption that $A = kb^t$.
c Use your graph to estimate the values of b and k.
d Estimate: **i** the time when the area of the mould was $6\,\text{cm}^2$
ii the area of the mould after 4.5 days.

4 It is thought that the relationship between two variables, a and b, is of the form $b = Pa^n$.
An experiment is conducted to test this assumption. The results are shown in the table.

a	2	4	6	8	10	12
b	9.8	12.1	13.7	14.9	16.0	16.9

a Show that the model can be written as $\ln b = n \ln a + \ln P$.
b Plot the graph of $\ln b$ against $\ln a$ and say why this supports the assumption $b = Pa^n$.
c Estimate the values of n and p.

5 With the exception of one, all the results in table satisfy, to one decimal place, the relationship $y = ax^n$.

x	1.3	1.6	1.9	2.2	2.5	2.8
y	5.8	8.2	14.9	21.5	29.6	39.4

a Use a suitable logarithmic method to find the values of a and n.
b If the values of x are correct, identify the incorrect value of y and estimate the correct value to 1 d.p.

6 The population P (in thousands) of a new town is modelled by the relationship $P = ka^t$ where t is the time in years. Its growth over the first five years is shown in the table below.

Year (t)	1	2	3	4	5
Population (P)	3.6	4.3	5.2	6.2	7.5

a Explain why you would expect that the graph of ln P against t to be a straight line.
b Draw up a table of values, plot the graph and use it to estimate values for a and k to 1 d.p.
c Using these values, calculate an estimate for the population after 20 years. How accurate is this likely to be?

Past–paper questions

1 **Solutions to this question by accurate drawing will not be accepted.**
The points $A(p, 1)$, $B(1, 6)$, $C(4, q)$ and $D(5, 4)$, where p and q are constants, are the vertices of a kite $ABCD$. The diagonals of the kite, AC and BD, intersect at the point E. The line AC is the perpendicular bisector of BD. Find

(i) the coordinates of E, [2]
(ii) the equation of the diagonal AC, [3]
(iii) the area of the kite $ABCD$. [3]

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Paper 21 Q9 June 2014
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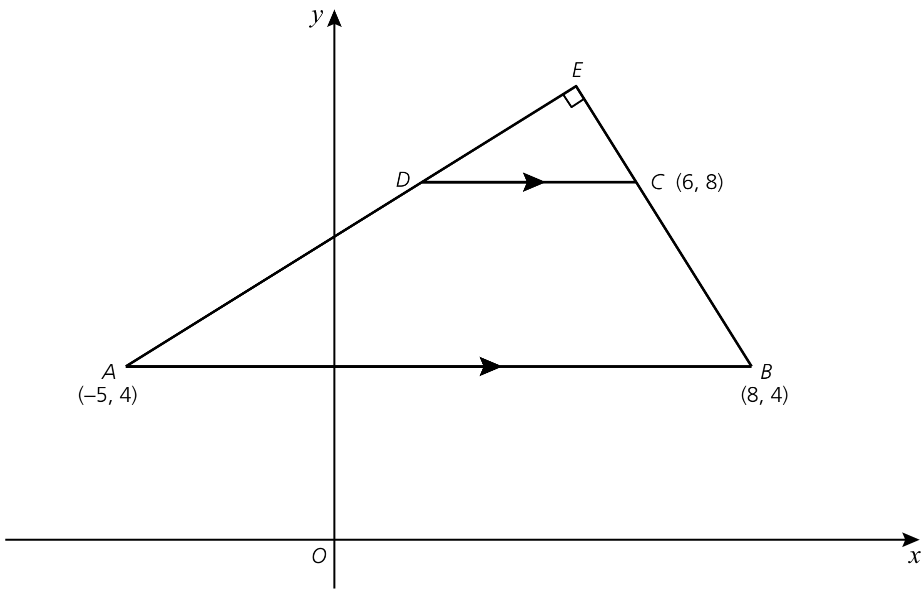
Paper 21 Q9 June 2014

2 Solutions to this question by accurate drawing will not be accepted.



The vertices of the trapezium $ABCD$ are the points $A(-5, 4)$, $B(8, 4)$, $C(6, 8)$ and D. The line AB is parallel to the line DC. The lines AD and BC are extended to meet at E and angle $AEB = 90°$.

(i) Find the coordinates of D and of E. [6]

(ii) Find the area of the trapezium $ABCD$. [2]

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Paper 12 Q7 November 2012
Cambridge IGCSE Additional Mathematics 0606
Paper 12 Q7 November 2012

3 Solutions to this question by accurate drawing will not be accepted.
The points $A(-3, 2)$ and $B(1, 4)$ are vertices of an isosceles triangle ABC, where angle $B = 90°$.

(i) Find the length of the line AB. [1]

(ii) Find the equation of the line BC. [3]

(iii) Find the coordinates of each of the two possible positions of C. [6]

Cambridge O Level Additional Mathematics 4037
Paper 12 Q10 November 2013
Cambridge IGCSE Additional Mathematics 0606
Paper 12 Q10 November 2013

Learning outcomes

Now you should be able to:

- ★ interpret the equation of a straight line graph in the form $y = mx + c$
- ★ solve questions involving midpoint and length of a line
- ★ know and use the condition for two lines to be parallel or perpendicular, including finding the equation of perpendicular bisectors
- ★ transform given relationships, including $y = ax^n$ and $y = Ab^x$, to straight line form and hence determine unknown constants by calculating the gradient or intercept of the transformed graph.

Key points

- ✔ An equation of the form $y = mx + c$ represents a straight line that has gradient m and intersects the y-axis at $(0, c)$.
- ✔ The midpoint of the line joining the points (x_1, y_1) and (x_2, y_2) is given by:

  midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- ✔ The length of the line joining the points (x_1, y_1) and (x_2, y_2) is given by:

  length $= \sqrt{\left((x_2 - x_1)^2 + (y_2 - y_1)^2\right)}$.
- ✔ Two lines are parallel if they have the same gradient.
- ✔ Two lines are perpendicular if they intersect at an angle of 90°.
- ✔ When the gradients of two parallel lines are given by m_1 and m_2, $m_1 m_2 = -1$.
- ✔ Logarithms can be used to describe the relationship between two variables in the following cases:

  i $y = ax^n$ Taking logs, $y = ax^n$ is equivalent to $\log y = \log a + n \log x$. Plotting $\log y$ against $\log x$ gives a straight line of gradient n that intersects the vertical axis at the point $(0, \log a)$.

  ii $y = Ab^x$ Taking logs, $= Ab^x$ is equivalent to $\log y = \log A + x \log b$. Plotting $\log y$ against x gives a straight line of gradient $\log b$ that intersects the vertical axis at the point $(0, \log A)$.

9 Circular measure

A circle is the reflection of eternity. It has no beginning and no end.

Maynard James Keenan (1964 –)

Discussion point

This is the Singapore Flyer. It has a radius of 75 metres. It takes about 30 minutes to complete one rotation, travelling at a constant speed. How fast do the capsules travel?

The tradition of measuring angles in degrees, and there being 360 degrees in one revolution, is thought to have come about because much of early mathematics was connected to astronomy, and the shepherd-astronomers of Sumeria believed that there were 360 days in a year.

The following notation is used in this chapter:

C represents the **circumference** of the circle – the distance round the circle.

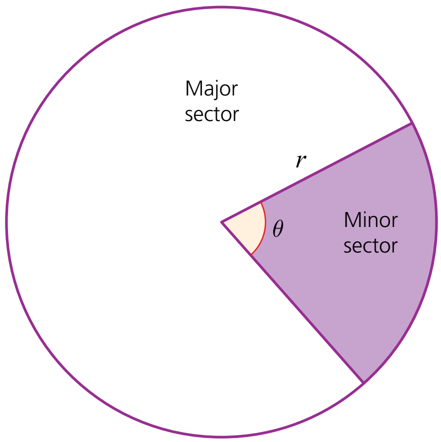
r represents the **radius** of the circle – the distance from the centre to any point on the circumference.

θ (the Greek letter theta) is used to represent the **angle** that an arc subtends at the centre of the circle.

A represents **area** – this may be the area of a whole circle or a sector.

Arc length and area of a sector

A **sector** of a circle looks similar to a piece of cake – it is the shape enclosed by an arc of the circle and two radii. If the angle at the centre is less than 180° it is called a **minor sector**, and if it is between 180° and 360° it is called a **major sector.**



Using ratios:

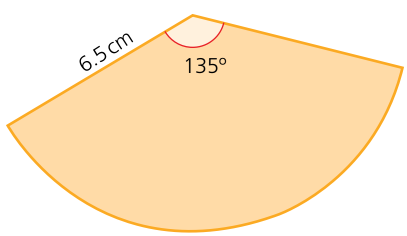
$$\frac{\text{arc length}}{\text{circumference of the circle}} = \frac{\text{area of the sector}}{\text{area of the circle}} = \frac{\theta}{360}$$

Worked example

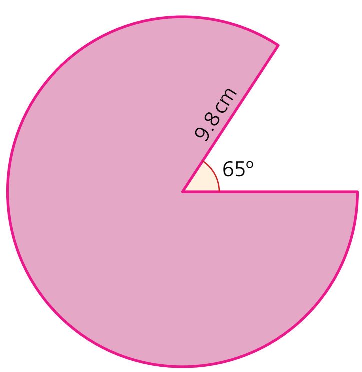
For each sector, calculate:

i the arc length   ii the area   iii the perimeter.

a



b



Solution

a **i** $\frac{\text{arc length}}{2\pi r} = \frac{\theta}{360}$ $\rightarrow$ arc length $= \frac{135}{360} \times 2 \times \pi \times 6.5$

$= 15.3\,\text{cm}$ (3 s.f.)

ii $\frac{\text{area}}{\pi r^2} = \frac{\theta}{360}$ $\rightarrow$ area $= \frac{135}{360} \times \pi \times 6.5^2$

$= 49.8\,\text{cm}^2$ (3 s.f.)

iii perimeter = arc length + 2 × radius

$= 15.3 + 2(6.5) = 28.3\,\text{cm}$ (3 s.f.)

b The angle of this sector is 360 – 65 = 295°

i $\frac{\text{arc length}}{2\pi r} = \frac{\theta}{360}$ $\rightarrow$ arc length $= \frac{295}{360} \times 2 \times \pi \times 9.8$

$= 50.5\,\text{cm}$ (3 s.f.)

ii $\frac{\text{area}}{\pi r^2} = \frac{\theta}{360}$ $\rightarrow$ area $= \frac{295}{360} \times \pi \times 9.8^2$

$= 247\,\text{cm}^2$ (3 s.f.)

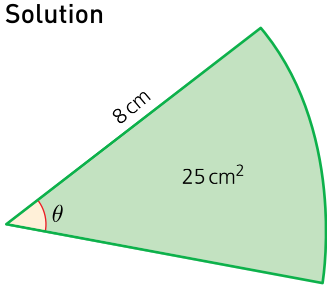
iii perimeter = arc length + 2 × radius

$= 50.5 + 2(9.8) = 70.1\,\text{cm}$ (3 s.f.)

Worked example

A sector of a circle of radius 8 cm has an area of 25 cm². Work out the angle at the centre.

Solution



Using sector area $= \frac{x}{360} \times \pi r^2$

$25 = \frac{\theta}{360} \times \pi \times 8^2$

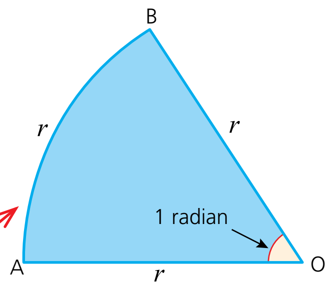
$\Rightarrow \theta = \frac{25 \times 360}{\pi \times 64}$

$= 44.8°$ (3 s.f.)

Radian measure

Radian measure is used extensively in mathematics because it simplifies many angle calculations. One radian (rad) is the angle in a sector when the arc length is equal to the radius. 1 rad is approximately 57.3°.

1 radian can also be written as 1^c.



Since the circumference of a circle is of length $2\pi r$, there are 2π arcs of length r round the circumference. This means that there are 2π radians in 360°.

Degrees	Radians
360	2π
180	π
90	$\frac{\pi}{2}$
60	$\frac{\pi}{3}$
45	$\frac{\pi}{4}$
30	$\frac{\pi}{6}$

1 degree is the same as $\frac{\pi}{180}$ radians, therefore:

- multiply by $\frac{\pi}{180}$ to convert degrees to radians
- multiply by $\frac{180}{\pi}$ to convert radians to degrees.

Note

- An angle given as a fraction of π is assumed to be in radians.
- If an angle is a simple fraction of 180°, its equivalent value in radians is usually expressed as a fraction of π.

Worked example

a Express the following in radians: **i** 75° **ii** 49°

b Express the following in degrees: **i** $\frac{\pi}{10}$ radians **ii** 1.25 radians

Solution

a i $75° = 75 \times \frac{\pi}{180} = \frac{5\pi}{12}$ radians

ii $49° = 49 \times \frac{\pi}{180} = 0.855$ radians (3 s.f.)

b **i** $\frac{\pi}{10}$ radians $= \frac{\pi}{10} \times \frac{180}{\pi} = 18°$

ii 1.25 radians $= 1.25 \times \frac{180}{\pi} = 71.6°$ (3 s.f.)

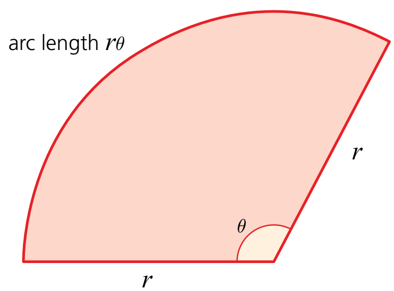
Using your calculator

Your calculator has modes for degrees and for radians, so always make sure that it is on the correct setting for any calculations that you do. There is usually a button marked DRG for **d**egrees, **r**adians and **g**rad (you will not use grad at this stage).

To find the value of sin 2.3^c, set your calculator to the radian mode and enter sin 2.3 followed by = or EXE, depending on your calculator. You should see the value 0.74570… on your screen.

Arc length and area of a sector in radians

Using the definition of a radian, an angle of 1 radian at the centre of a circle corresponds to an arc length equal to the radius r of the circle. Therefore an angle of θ radians corresponds to an arc length of $r\theta$.



It is accepted practice to write $r\theta$, with the Greek letter at the end, rather than θr.

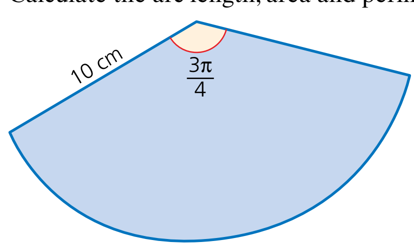
The area of this sector is the fraction $\frac{\theta}{2\pi}$ of the area of the circle (since 2π is the radian equivalent of 360°).

This gives the formula:

$$\text{area of a sector} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta.$$

Worked example

Calculate the arc length, area and perimeter of this sector.



Solution

arc length $= 10 \times \frac{3\pi}{4}$ Using arc length $= r\theta$

$= \frac{15\pi}{2}$ cm

sector area $= \frac{1}{2} \times 10^2 \times \frac{3\pi}{4}$ Using area of sector $= \frac{1}{2}r^2\theta$

$= \frac{75\pi}{2}$ cm^2

Using perimeter = arc length + 2 × radius

perimeter $= \frac{15\pi}{2} + 2 \times 10 = \frac{15\pi}{2} + 20$ cm

Exercise 9.1

1 Express each angle in radians, leaving your answer in terms of π if appropriate:

a 120° **b** 540° **c** 22° **d** 150° **e** 37.5°

2 Express each angle in degrees, rounding your answer to 3 s.f. where necessary:

a $\frac{2\pi}{3}$ **b** $\frac{5\pi}{9}$ **c** 3^c **d** $\frac{\pi}{7}$ **e** $\frac{3\pi}{8}$

3 The table gives information about some sectors of circles.

Copy and complete the table. Leave your answers as a multiple of π where appropriate.

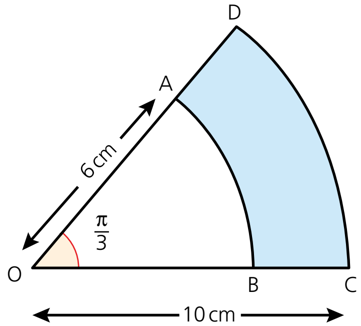
Radius, r (cm)	Angle at centre in degrees	Angle at centre in radians	Arc length, s (cm)	Area, A (cm^2)
8	120			
10			5	
	60		4	
6				12
	75			20

4 The table gives information about some sectors of circles. Copy and complete the table. Leave your answers as a multiple of π where appropriate.

Radius, r (cm)	Angle at centre in radians	Arc length, s (cm)	Area, A (cm^2)
10	$\frac{\pi}{3}$		
12		24	
	$\frac{\pi}{4}$	16	
5			25
	$\frac{3\pi}{5}$		40

Exercise 9.1 (cont)

5 OAB is a sector of a circle of radius 6 cm. ODC is a sector of a circle radius 10 cm. Angle AOB is $\frac{\pi}{3}$.

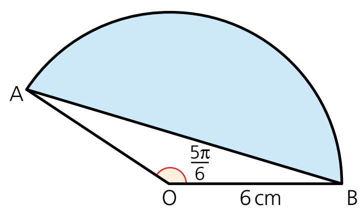


Express in terms of π:

a the area of ABCD

b the perimeter of ABCD.

6



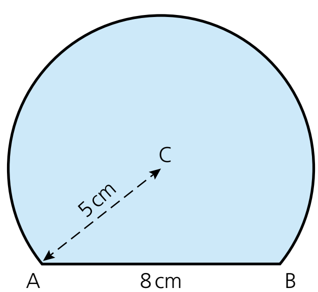
The shaded area is called a **segment** of a circle.

a Work out the area of the sector AOB.

b Calculate the area of the triangle AOB.

c Work out the shaded area.

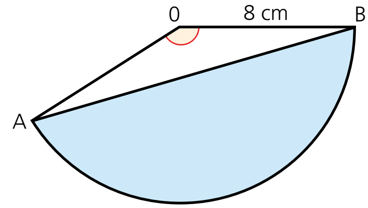
7 The diagram shows the cross-section of a paperweight. The paperweight is a sphere of radius 5 cm with the bottom cut off to create a circular flat base with diameter 8 cm.



a Calculate the angle ACB in radians.

b Work out the area of cross-section of the paperweight.

8 The perimeter of the sector in the diagram is $6\pi + 16$ cm.



Calculate:

a angle AOB

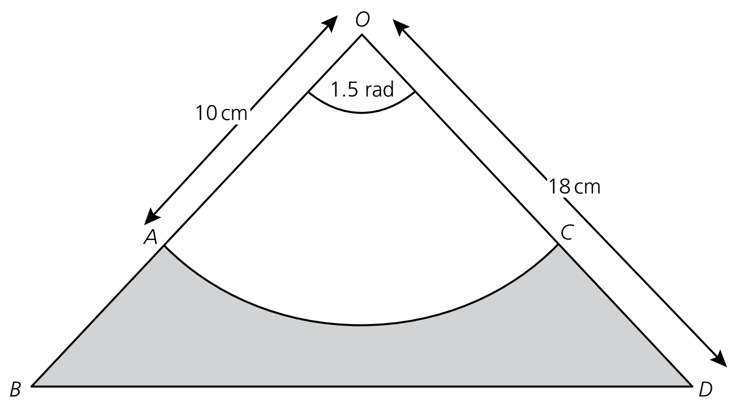
b the exact area of sector AOB

c the exact area of the triangle AOB

d the exact area of the shaded segment.

Past–paper questions

1



The diagram shows an isosceles triangle OBD in which $OB = OD = 18$ cm and angle $BOD = 1.5$ radians. An arc of the circle, centre O and radius 10 cm, meets OB at A and OD at C.

(i) Find the area of the shaded region. [3]

(ii) Find the perimeter of the shaded region. [4]

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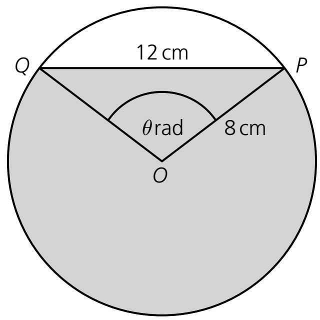
2 The diagram shows a circle, centre O, radius 8 cm. Points P and Q lie on the circle such that the chord $PQ = 12$ cm and angle $POQ = \theta$ radians.



(i) Show that $\theta = 1.696$, correct to 3 decimal places. [2]
(ii) Find the perimeter of the shaded region. [3]
(iii) Find the area of the shaded region. [3]

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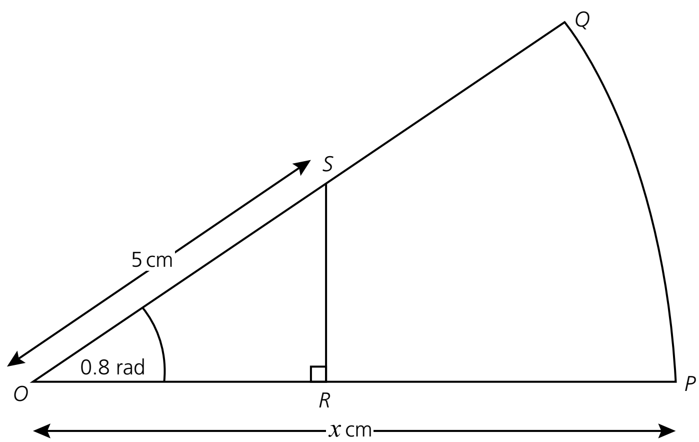
3



The diagram shows a sector OPQ of a circle with centre O and radius x cm. Angle POQ is 0.8 radians. The point S lies on OQ such that $OS = 5$ cm. The point R lies on OP such that angle ORS is a right angle. Given that the area of triangle ORS is one-fifth of the area of sector OPQ, find

(i) the area of sector OPQ in terms of x and hence show that the value of x is 8.837 correct to 4 significant figures, [5]
(ii) the perimeter of $PQSR$, [3]
(iii) the area of $PQSR$. [2]

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Paper 22 Q11 November 2014
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Paper 22 Q11 November 2014

Learning outcomes

Now you should be able to:

★ solve problems involving the arc length and sector area of a circle, including knowledge and use of radian measure.

Key points

✔ Angles are measured either in degrees or radians.
$180° = \pi$ radians

✔ The angle at the centre of the circle subtended by an arc that is the same length as the radius is 1 radian.

✔ The formulae for area of a circle ($A = \pi r^2$) and circumference of a circle ($C = 2\pi$) are the same whether the angle is measured in degrees or radians.

✔ You will need to learn these formulae.

	Radians
Area	πr^2
Circumference	$2\pi r$
Arc length (θ at centre)	$r\theta$
Sector area (θ at centre)	$\frac{1}{2}r^2\theta$

10 Trigonometry

The laws of nature are written in the language of mathematics ... the symbols are triangles, circles and other geometrical figures without whose help it is impossible to comprehend a single word.

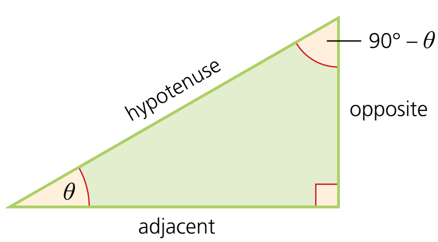
Galileo Galilei (1564–1642)

Discussion point

How can you estimate the angle the sloping sides of this pyramid make with the horizontal?

Using trigonometry in right-angled triangles

The simplest definitions of the trigonometrical functions are given in terms of the ratios of the sides of a right-angled triangle, for values of the angle θ between 0° and 90°.



In a triangle:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

Sin is an abbreviation of sine, cos of cosine and tan of tangent. The previous diagram shows that:

$$\sin\theta = \cos(90° - \theta) \text{ and } \cos\theta = \sin(90° - \theta).$$

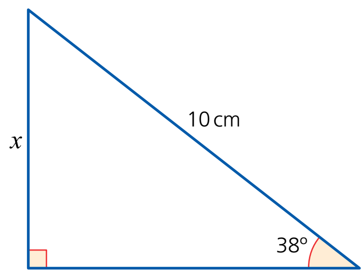
The Greek letter θ (theta) is often used to denote an angle. The Greek letters α (alpha) and β (beta) are also commonly used for this purpose.

Taking the first letters of each part gives the word 'sohcahtoa', which may help you to remember the formula.

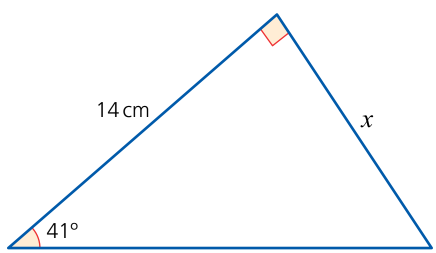
Worked example

Work out the length of x in each triangle. Give your answers correct to three significant figures.

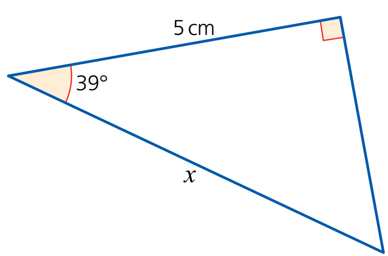
a



b



c



Solution

a $\frac{x}{10} = \sin 38°$

$\Rightarrow \quad x = 10\sin 38°$

$x = 6.16\text{ cm}$

b $\frac{x}{14} = \tan 41°$

$\Rightarrow \quad x = 14\tan 41°$

$\Rightarrow \quad x = 12.2\text{ cm}$

c $\frac{5}{x} = \cos 39°$

$\Rightarrow \quad 5 = x\cos 39°$

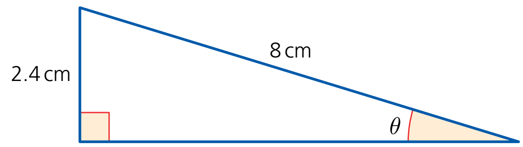
$\Rightarrow \quad x = \frac{5}{\cos 39°}$

$\Rightarrow \quad x = 6.43\text{ cm}$

Worked example

Work out the angle marked θ in each triangle. Give your answers correct to one decimal place.

a



Solution

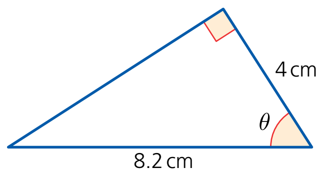
a $\sin\theta = \frac{2.4}{8}$

$\Rightarrow \quad \theta = \sin^{-1} 0.3$

$\Rightarrow \quad \theta = 17.5°$

$\sin^{-1} 0.3$ is shorthand notation for 'the angle θ where $\sin\theta = 0.3$'. $\cos^{-1} 0.3$ and $\tan^{-1} 0.3$ are similarly defined.

b

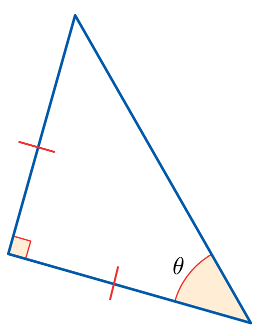


b $\cos\theta = \frac{4}{8.2}$

$\Rightarrow \quad \theta = \cos^{-1}\frac{4}{8.2}$

$\Rightarrow \quad \theta = 60.8°$

c



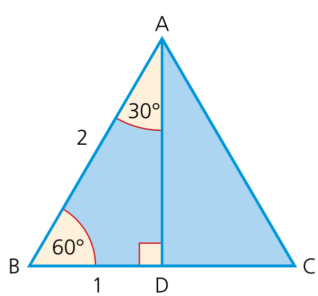
c The opposite and adjacent sides are equal, so $\tan\theta = 1$ $\Rightarrow \theta = 45°$.

Special cases

Certain angles occur frequently in mathematics and you will find it helpful to know the value of their trigonometrical functions.

The angles 30° and 60°

Triangle ABC is an equilateral triangle with side 2 units, and AD is a line of symmetry.



Using Pythagoras' theorem

$$AD^2 + 1^2 = 2^2 \Rightarrow AD = \sqrt{3}.$$

From triangle ABD,

$$\sin 60° = \frac{\sqrt{3}}{2}; \qquad \cos 60° = \frac{1}{2}; \qquad \tan 60° = \sqrt{3};$$

$$\sin 30° = \frac{1}{2}; \qquad \cos 30° = \frac{\sqrt{3}}{2}; \qquad \tan 30° = \frac{1}{\sqrt{3}}.$$

Worked example

Without using a calculator, find the value of $\sin^2 30° + \sin 60° \cos 30°$.
(Note that $\sin^2 30°$ means $(\sin 30°)^2$.)

Solution

$$\begin{aligned}\sin^2 30° + \sin 60° \cos 30° &= \left(\frac{1}{2}\right)^2 + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} \\ &= \frac{1}{4} + \frac{3}{4} \\ &= 1\end{aligned}$$

Note

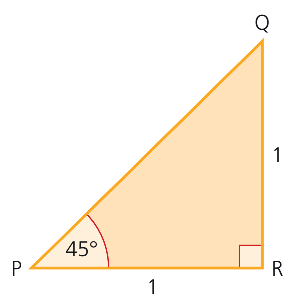
The equivalent results using radians are

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}; \qquad \cos\frac{\pi}{3} = \frac{1}{2}; \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

$$\sin\frac{\pi}{6} = \frac{1}{2}; \qquad \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}; \qquad \tan\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

The angle 45°

PQR is a right-angled isosceles triangle with equal sides of length 1 unit.



Using Pythagoras' theorem, $PQ = \sqrt{2}$.

This gives

$$\sin 45° = \frac{1}{\sqrt{2}}; \qquad \cos 45° = \frac{1}{\sqrt{2}}; \qquad \tan 45° = 1.$$

Note

In radians

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$; $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$; $\tan\frac{\pi}{4} = 1$

Worked example

Without using a calculator find the value of $\sin^2\frac{\pi}{4} + \cos^2\frac{\pi}{4}$.

When an angle is given in terms of π like this, it is in radians. $\frac{\pi}{4}$ radians = 45°

Solution

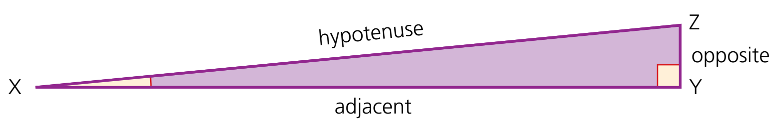
$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

So $\sin^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2}$

$= 1$

The angles 0° and 90°

Although you cannot have an angle of 0° in a triangle (because one side would be lying on top of another), you can still imagine what it might look like. In the diagram, the hypotenuse has length 1 unit and the angle at X is very small.



If you imagine the angle at X becoming smaller and smaller until it is zero, you can deduce that

$\sin 0° = \frac{0}{1} = 0$; $\cos 0° = \frac{1}{1} = 1$; $\tan 0° = \frac{0}{1} = 0$.

If the angle at X is 0°, then the angle at Z is 90°, and so you can also deduce that

Remember that in radians 90° is $\frac{\pi}{2}$.

$\sin 90° = \frac{1}{1} = 1$; $\cos 90° = \frac{0}{1} = 0$.

However when you come to find tan 90°, there is a problem. The triangle suggests this has value $\frac{1}{0}$, but you cannot divide by zero.

If you look at the triangle XYZ, you will see that what we actually did was to draw it with angle X not zero but just very small, and to argue:

'We can see from this what will happen if the angle becomes smaller and smaller so that it is effectively zero.'

In this case we are looking at the limits of the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ as the angle θ approaches zero. The same approach can be used to look again at the problem of tan 90°.

If the angle X is not quite zero, then the side ZY is also not quite zero, and tan Z is 1 (XY is almost 1) divided by a very small number and so is large. The smaller the angle X, the smaller the side ZY and so the larger the value of tan Z. We conclude that in the limit when angle X becomes zero and angle Z becomes 90°, tan Z is infinitely large, and so we say

Read these arrows as 'tends to'.

as $Z \to 90°$, $\tan Z \to \infty$ (infinity).

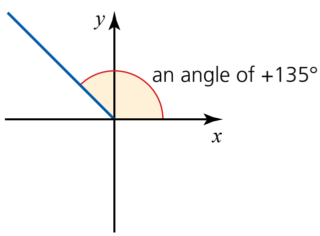
You can see this happening in the table of values below.

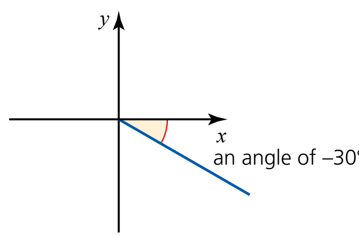
Z	**tan Z**
80°	5.67
89°	57.29
89.9°	572.96
89.99°	5729.6
89.999°	57 296

When Z actually equals 90°, we say that tan Z is **undefined**.

Positive and negative angles

Unless given in the form of bearings, angles are measured from the x-axis (as shown below). Anticlockwise is taken to be positive and clockwise to be negative.

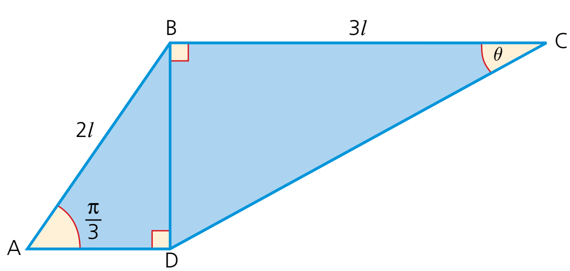




Worked example

In the diagram, angles ADB and CBD are right angles, angle BAD = $\frac{\pi}{3}$, AB = $2l$ and BC = $3l$.

Calculate the value of θ in radians.



Solution

First, find an expression for BD.

In triangle ABD, $\frac{BD}{AB} = \sin\frac{\pi}{3}$ $AB = 2l$

$$\Rightarrow \quad BD = 2l\sin\frac{\pi}{3}$$
$$= 2l \times \frac{\sqrt{3}}{2}$$
$$= \sqrt{3}l$$

In triangle BCD, $\tan\theta = \frac{BD}{BC}$

$$= \frac{\sqrt{3}l}{3l}$$
$$= \frac{1}{\sqrt{3}}$$
$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{\pi}{6}$$

Exercise 10.1

1 In the triangle PQR, PQ = 29 cm, QR = 21 cm and PR = 20 cm.

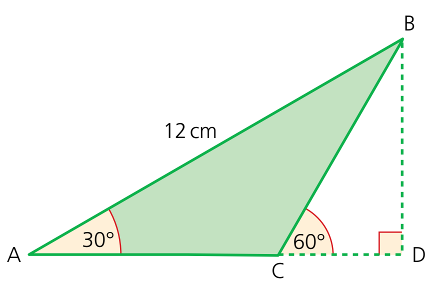
- **a** Show that the triangle is right-angled.
- **b** Write down the values of $\sin Q$, $\cos Q$ and $\tan Q$, leaving your answers as fractions.
- **c** Use your answers to part **b** to show that:
  - **i** $\sin^2 Q + \cos^2 Q = 1$
  - **ii** $\tan Q = \frac{\sin Q}{\cos Q}$

2 Without using a calculator, show that $\cos 60°\sin 30° + \sin 60°\cos 30° = 1$

3 Without using a calculator, show that $\cos^2 60°\cos^2 45° = \cos^2 30°$

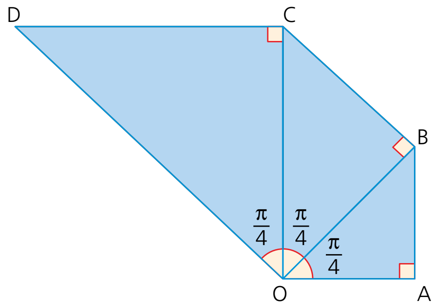
4 Without using a calculator, show that $3\cos^2\frac{\pi}{3} = \sin^2\frac{\pi}{3}$

5 In the diagram, AB = 12 cm, angle BAC = 30°, angle BCD = 60° and angle BDC = 90°.



- **a** Calculate the length of BD.
- **b** Show that $AC = 4\sqrt{3}$ cm.

6 In the diagram, OA = 1 cm, angle AOB = angle BOC = angle COD = $\frac{\pi}{4}$ and angle OAB = angle OBC = angle OCD = $\frac{\pi}{2}$.



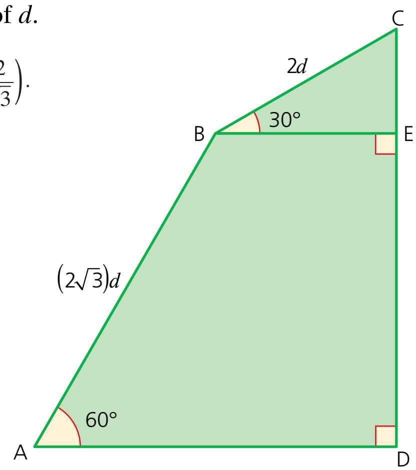
a Find the length of OD, giving your answer in the form $a\sqrt{2}$.

b Show that the perimeter of the pentagon OABCD is $4+3\sqrt{2}$ cm.

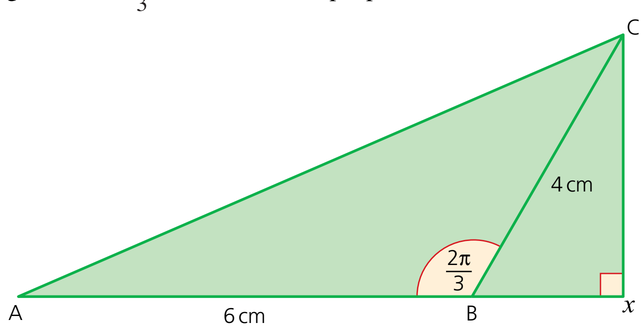
7 In the diagram, ABED is a trapezium with right angles at E and D, and CED is a straight line. The lengths of AB and BC are $(2\sqrt{3})d$ and $2d$ respectively, and angles BAD and CBE are 60° and 30° respectively.

a Find the length of CD in terms of d.

b Show that angle CAD = $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$.



8 In the diagram, ABC is a triangle in which AB = 6 cm, BC = 4 cm and angle ABC = $\frac{2\pi}{3}$. The line CX is perpendicular to the line ABX.



Exercise 10.1 (cont)

a Work out the exact length of BX.

b Show that angle CAB $= \tan^{-1}\left(\frac{\sqrt{3}}{4}\right)$

c Show that the exact length of AC is $\sqrt{76}$ cm.

Reciprocal trigonometrical functions

As well as sin, cos and tan there are three more trigonometrical ratios that you need to be able to use. These are the reciprocals of the three functions you have already met: cosecant (cosec), secant (sec) and cotangent (cot).

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}\left(= \frac{\cos\theta}{\sin\theta}\right)$$

Each of these functions is undefined for certain values of θ. For example, cosec θ is undefined for $\theta = 0°, 180°, 360°\ldots$ since sin θ is zero for these values.

Worked example

Solve the following equations for $0° < x < 90°$ rounding your answers to one decimal place where necessary.

a $\sec x = 2$ **b** $\operatorname{cosec} x = 2$ **c** $\cot x = 2$

Solution

a $\sec x = 2 \Rightarrow \frac{1}{\cos x} = 2$

$\Rightarrow \cos x = \frac{1}{2}$

$\Rightarrow x = 60°$

b $\operatorname{cosec} x = 2 \Rightarrow \frac{1}{\sin x} = 2$

$\Rightarrow \sin x = \frac{1}{2}$

$\Rightarrow x = 30°$

c $\cot x = 2 \Rightarrow \frac{1}{\tan x} = 2$

$\Rightarrow \tan x = \frac{1}{2}$

$\Rightarrow x = 26.6°$

Exercise 10.2

'Exact form' means give the answer using fractions and surds.

1 Write each value in exact form. Do not use a calculator.

a **i** $\sin 30°$ **ii** $\cos 30°$ **iii** $\tan 30°$

b **i** $\operatorname{cosec} 30°$ **ii** $\sec 30°$ **iii** $\cot 30°$

2 Write each value in exact form. Do not use a calculator.

a **i** $\sin 45°$ **ii** $\cos 45°$ **iii** $\tan 45°$

b **i** $\operatorname{cosec} 45°$ **ii** $\sec 45°$ **iii** $\cot 45°$

3 Write each value in exact form. Do not use a calculator.

a **i** $\sin \frac{\pi}{3}$ **ii** $\cos \frac{\pi}{3}$ **iii** $\tan \frac{\pi}{3}$

b **i** $\operatorname{cosec} \frac{\pi}{3}$ **ii** $\sec \frac{\pi}{3}$ **iii** $\cot \frac{\pi}{3}$

4 In the triangle ABC, angle $A = 90°$ and $\sec B = 2$.

a Work out the size of angles B and C.

b Find $\tan B$.

c Show that $1 + \tan^2 B = \sec^2 B$.

5 In the triangle ABC, angle $A = 90°$ and $\operatorname{cosec} B = 2$.

a Work out the size of angles B and C.

AC = 2 units

b Work out the lengths of AB and BC.

6 Given that $\sin\theta = \frac{3}{4}$ and θ is acute, find the values of $\sec\theta$ and $\cot\theta$.

7 In the triangle LMN, angle $M = \frac{\pi}{2}$ and $\cot N = 1$.

a Find the angles L and N.

b Find $\sec L$, $\operatorname{cosec} L$ and $\tan L$.

c Show that $1 + \tan^2 L = \sec^2 L$.

8 Malini is 1.5 m tall. At 8 o'clock one evening, her shadow is 6 m long. Given that the angle of elevation of the sun at that moment is α:

a show that $\cot\alpha = 4$,

b find the value of α.

9 **a** For what values of α are $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ all positive? Give your answers in both degrees and radians.

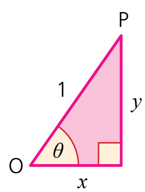
b Are there any values of α for which $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ are all negative? Explain your answer.

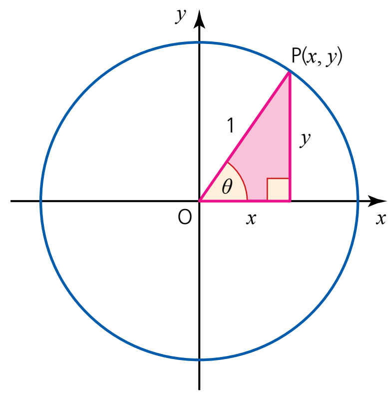
c Are there any values of α for which $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ are all equal? Explain your answer.

Trigonometrical functions for angles of any size

Is it possible to extend the use of the trigonometrical functions to angles greater than 90°, like sin 120°, cos 275° or tan 692°? The answer is yes – provided you change the definition of sine, cosine and tangent to one that does not require the angle to be in a right-angled triangle. It is not difficult to extend the definitions, as follows.

First look at the right-angled triangle below, which has hypotenuse of unit length.





This provides the definitions:

$$\sin\theta = \frac{y}{1} = y; \qquad \cos\theta = \frac{x}{1} = x; \qquad \tan\theta = \frac{y}{x}.$$

Now think of the angle θ being situated at the origin, as in the diagrams above, and allow θ to take any value. The vertex marked P has coordinates (x, y) and can now be anywhere on the unit circle.

This shows that the definitions above can be applied to *any* angle θ, whether it is positive or negative, and whether it is less than or greater than 90°:

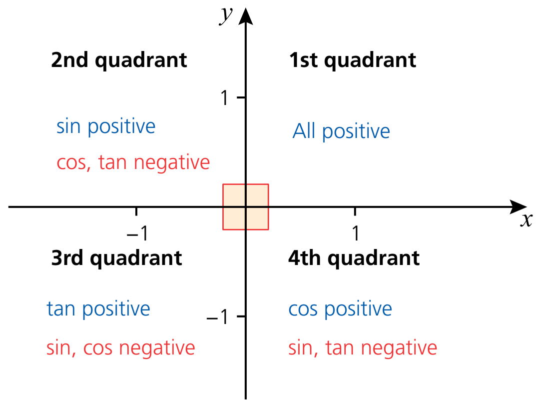
$$\sin\theta = y, \qquad \cos\theta = x, \qquad \tan\theta = \frac{y}{x}.$$

For some angles, x or y (or both) will take a negative value, so the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ will vary accordingly.

Worked example

The x- and y-axes divide the plane into four regions called quadrants. Draw a diagram showing the quadrants for values of x and y from –1 to 1. Label each quadrant to show which of the trigonometrical functions are positive and which are negative in each quadrant.

Solution

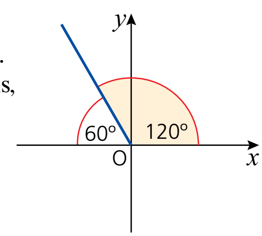


Worked example

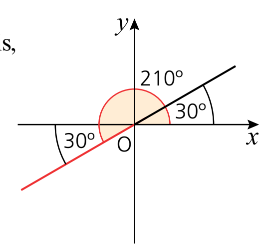
Find the value of: **a** $\sin 120°$ **b** $\cos 210°$ **c** $\tan 405°$.

Solution

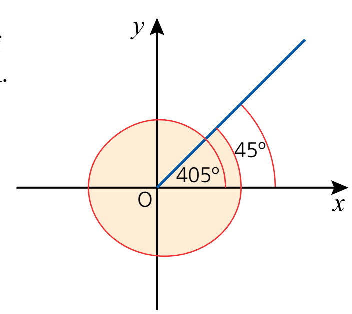
a 120° is in the second quadrant, so sin 120° is positive. The line at 120° makes an angle of 60° with the x-axis, so $\sin 120° = +\sin 60° = \frac{\sqrt{3}}{2}$



b 210° is in the third quadrant, so cos 210° is negative. The line at 210° makes an angle of 30° with the x-axis, so $\cos 210° = -\cos 30° = -\frac{\sqrt{3}}{2}$

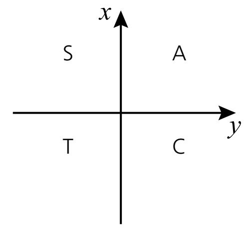


c 405° is in the first quadrant, so tan 405° is positive. The line at 405° makes an angle of 45° with the x-axis, so $\tan 405° = \tan 45° = 1$.



Note

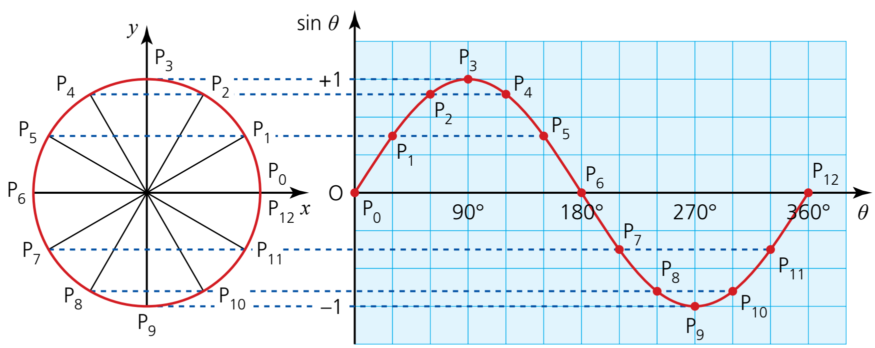
Look at this diagram. It gives you a useful aid for remembering the values for which sin, cos and tan are positive and negative.
A means all are positive.
S means sin is positive but the other two, cos and tan, are negative, and so on.
Starting from C and working anticlockwise this spells 'CAST'. Consequently, it is often referred to as 'The CAST Rule'.



Graphs of trigonometrical functions

The sine and cosine graphs

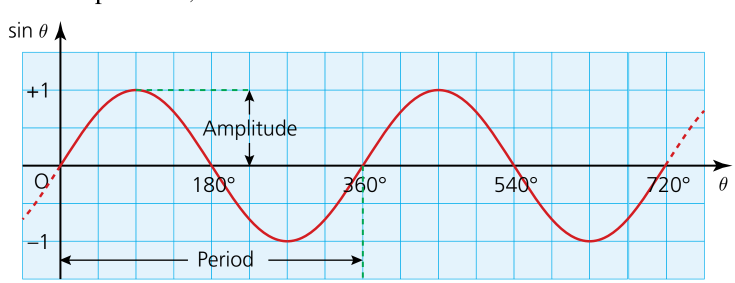
The diagram on the left below shows angles at intervals of 30° in the unit circle. The resulting coordinate, θ and y, are plotted relative to the axes in the diagram on the right. They have been joined with a continuous curve to give the graph of sin θ for $0° \leqslant \theta \leqslant 360°$. The resulting wave is called the sine curve.



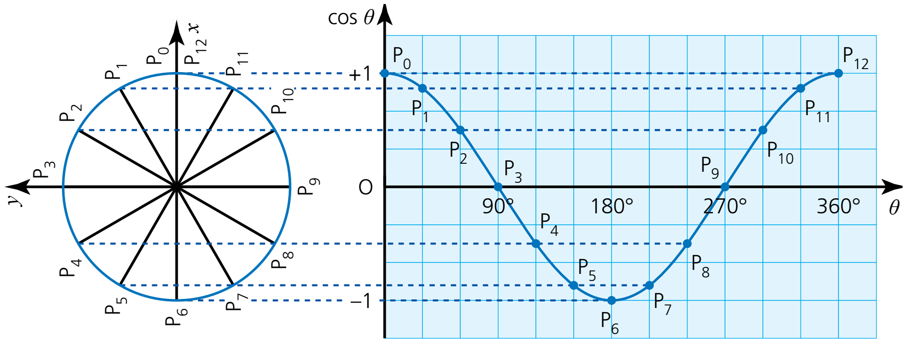
The angle 390° gives the same point P_1 on the circle as the angle 30°, the angle 420° gives point P_2 and so on. You can see that for angles from 360° to 720° the sine wave will simply repeat itself, as shown below. This is true also for angles from 720° to 1080° and so on.

Since the curve repeats itself every 360° the sine function is described as **periodic**, with **period** 360° or 2π radians.

The **amplitude** of such a curve is the largest displacement from the central position, i.e. the horizontal axis.

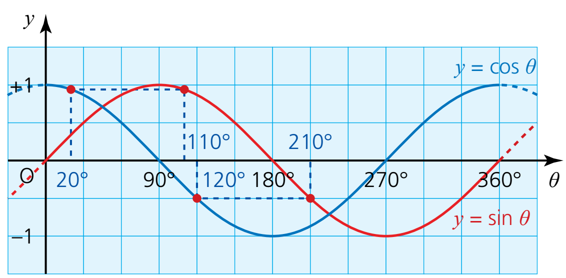


You can transfer the x-coordinates on to a set of axes in a similar way to obtain the graph of cos θ. This is most easily illustrated if you first rotate the circle through 90° anticlockwise. The diagram shows the circle in this new orientation, together with the resulting graph.



The cosine curve repeats itself for angles in the interval $360° \leqslant \theta \leqslant 720°$. This shows that the cosine function is also periodic with a period of 360°.

Notice that the graphs of sin θ and cos θ have exactly the same shape. The cosine graph can be obtained by translating the sine graph 90° to the left, as shown below.



The diagram shows that, for example,

$$\cos 20° = \sin 110°, \cos 90° = \sin 180°, \cos 120° = \sin 210°, \text{etc.}$$

In general:

$$\cos\theta \equiv \sin(\theta + 90°), \text{ and in radians } \cos\theta = \sin\left(\theta + \frac{\pi}{2}\right).$$

Discussion point

1 What do the graphs of sin θ and cos θ look like for negative angles?
2 Draw the curve of sin θ for $0° \leqslant \theta \leqslant 90°$.
Using only reflections, rotations and translations of this curve, how can you generate the curves of sin θ and cos θ for $0° \leqslant \theta \leqslant 360°$?

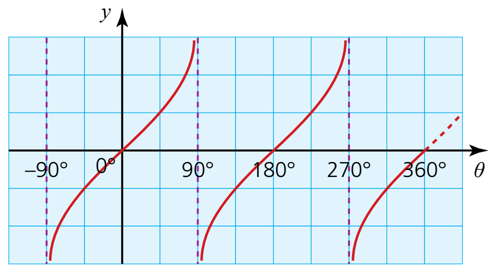
The tangent graph

The value of tan θ can be worked out from the definition $\tan\theta = \frac{y}{x}$ or by using $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

You have already seen that $\tan\theta$ is undefined for $\theta = 90°$. This is also the case for all other values of θ for which $\cos\theta = 0$, namely $270°, 450°, \ldots,$ and $-90°, -270°, \ldots$

The graph of $\tan\theta$ is shown below.
The dotted lines $\theta = \pm 90°$ and $\theta = 270°$ are **asymptotes**.

Asymptotes are not part of the curve. The branches get increasingly close to them but never actually touch them.



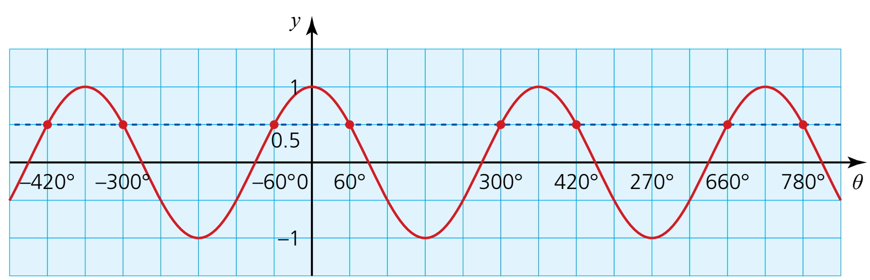
Note

The graph of $\tan\theta$ is periodic, like those for $\sin\theta$ and $\cos\theta$, but in this case the period is 180°. Again, the curve for $0 \leqslant \theta < 90°$ can be used to generate the rest of the curve using rotations and translations.

Solving trigonometrical equations using graphs

You can use these graphs when you solve trigonometric equations.

If you use the inverse function on your calculator to solve the equation $\cos\theta = 0.5$, the answer is given as 60°. However, the graph of $y = \cos\theta$ shows that this equation has infinitely many roots.



This graph of $y = \cos\theta$ shows that the roots for $\cos\theta = 0.5$ are:

$$\theta = \ldots, -420°, -300°, -60°, 60°, 300°, 420°, 660°, 780°, \ldots$$

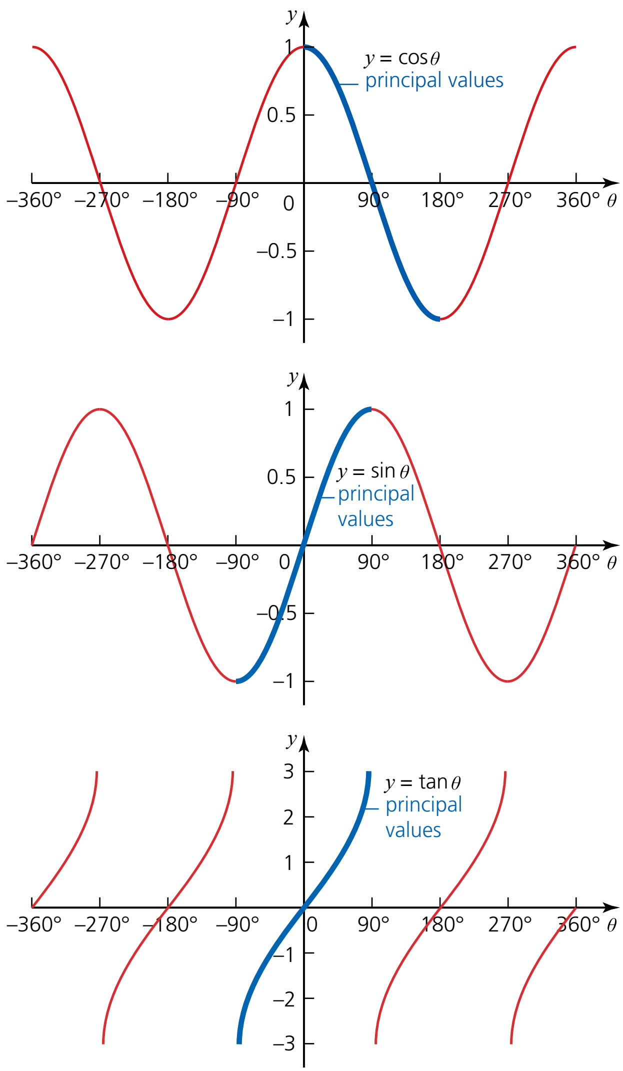
The functions cosine, sine and tangent are all many-one mappings, so their inverse mappings are one-many. In other words, the problem 'Find cos 60°' has only one solution (0.5), whilst 'Find θ such that $\cos\theta = 0.5$' has infinitely many solutions.

Remember that a function has to be either one-one or many-one. This means that in order to define inverse functions for cosine, sine and tangent, a restriction must be placed on the domain of each so that it becomes a one-one mapping. This is why your calculator will always give the value of the solution between:

$$0° \leqslant \theta \leqslant 180° \quad \text{(cos)}$$
$$-90° \leqslant \theta \leqslant 90° \quad \text{(sin)}$$
$$-90° < \theta < 90° \quad \text{(tan).}$$

Your calculator only gives one of the infinitely many roots. This is called the **principal value.**

The following diagrams are the graphs of cosine, sine and tangent together with their principal values. The graphs show you that the principal values cover the whole of the range (y values) for each function.



Discussion point

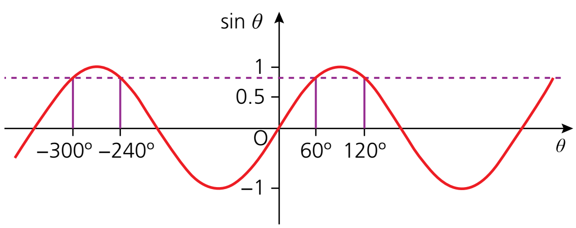
How are the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ changed if θ is measured in radians rather than degrees?

Worked example

Find values of θ in the interval $-360° \leqslant \theta \leqslant 360°$ for which $\sin\theta = \frac{\sqrt{3}}{2}$.

Solution

$\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°$. The graph of $\sin\theta$ is shown below.



So, the values of θ are $-300°, -240°, 60°, 120°$.

Worked example

Solve the equation $2\tan\theta + 1 = 0$ for $-180° \leqslant \theta \leqslant 180°$.

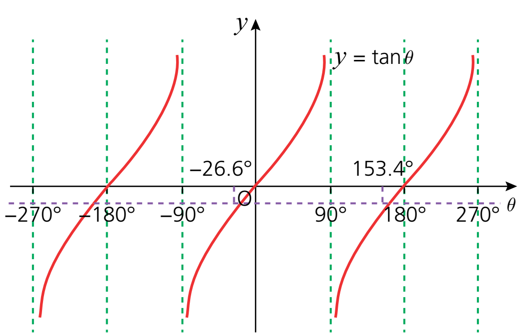
Solution

$$2\tan\theta + 1 = 0 \Rightarrow \tan\theta = -\frac{1}{2}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(-\frac{1}{2}\right)$$

Using a calculator

$$= -26.6° \text{ (1 d.p.)}$$



From the graph, the other answer in the range is:

$$\theta = -26.6° + 180° = 153.4°$$

So, the values of θ are $-26.6°, 153.4°$.

Exercise 10.3

1 **a** Sketch the curve $y = \cos x$ for $0° \leqslant x \leqslant 360°$.
b Solve the equation $\cos x = 0.5$ for $0° \leqslant x \leqslant 360°$, and illustrate the two roots on your sketch.
c State two other roots of $\cos x = 0.5$, given that x is no longer restricted to values between $0°$ and $360°$.
d Write down, without using your calculator, the value of $\cos 240°$.

2 **a** Sketch the curve of $y = \sin x$ for $-2\pi \leqslant x \leqslant 2\pi$.
b Solve the equation $\sin x = 0.6$ for $-2\pi \leqslant x \leqslant 2\pi$, and illustrate all the roots on your sketch.
c Sketch the curve $y = \cos x$ for $-2\pi \leqslant x \leqslant 2\pi$.
d Solve the equation $\cos x = 0.8$ for $-2\pi \leqslant x \leqslant 2\pi$, and illustrate all the roots on your sketch.
e Explain why some of the roots of $\sin x = 0.6$ are the same as those for $\cos x = 0.8$, and why some are different.

3 Solve the following equations for $0° \leqslant x \leqslant 2\pi$.
a $\tan x = \sqrt{3}$ **b** $\sin x = 0.5$ **c** $\cos x = -\frac{\sqrt{3}}{2}$
d $\tan x = \frac{1}{\sqrt{3}}$ **e** $\cos x = -0.7$ **f** $\cos x = 0.3$
g $\sin x = -\frac{1}{3}$ **h** $\sin x = -1$

4 Write the following as integers, fractions, or using square roots. You should not need your calculator.
a $\sin 45°$ **b** $\cos 60°$ **c** $\tan 45°$
d $\sin 120°$ **e** $\cos 150°$ **f** $\tan 180°$
g $\sin 405°$ **h** $\cos(-45°)$ **i** $\tan 225°$

5 In this question all the angles are in the interval $-180°$ to $180°$. Give all answers correct to one decimal place.
a Given that $\cos\alpha < 0$ and $\sin\alpha = 0.5$, find α.
b Given that $\tan\beta = 0.3587$ and $\sin\beta < 0$, find β.
c Given that $\cos\gamma = 0.0457$ and $\tan\gamma > 0$, find γ.

6 **a** Draw a sketch of the graph $y = \sin x$ and use it to demonstrate why $\sin x = \sin(180° - x)$.
b By referring to the graphs of $y = \cos x$ and $y = \tan x$, state whether the following are true or false.
i $\cos x = \cos(180° - x)$ **ii** $\cos x = -\cos(180° - x)$
iii $\tan x = \tan(180° - x)$ **iv** $\tan x = -\tan(180° - x)$

7 **a** For what values of α are $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ all positive given that $0° \leqslant \alpha \leqslant 360°$?
b Are there any values of α for which $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ are all negative? Explain your answer.
c Are there any values of α for which $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ are all equal? Explain your answer.

8 Solve the following equations for $0° \leqslant \theta \leqslant 360°$.
a $\cos(\theta - 20°) = \frac{1}{2}$ **b** $\tan(\theta + 10°) = \frac{\sqrt{3}}{3}$ **c** $\sin(\theta + 80°) = \frac{\sqrt{2}}{2}$
d $\tan 2\theta = \sqrt{3}$ **e** $\sin\left(\frac{1}{2}\theta\right) = \frac{1}{2}$ **f** $\cos 2\theta = -\frac{\sqrt{3}}{2}$
g $\sin 3\theta = \frac{1}{2}$ **h** $\sin 2\theta = 0$ **i** $\tan 3\theta = 1$

9 Solve the following equations for $-2\pi \leqslant x \leqslant 2\pi$.
a $10\sin x = 1$ **b** $2\cos x - 1 = 0$ **c** $\tan x + 2 = 0$
d $5\sin x + 2 = 0$ **e** $\cos^2 x = 1 - \sin x$

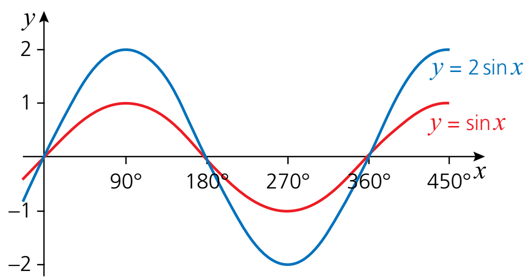
Transformations of trigonometrical graphs

Now that you are familiar with the graphs of the sine, cosine and tangent functions, you can see how to transform these graphs.

$y = a \sin x$ where a is a positive integer

How are the graphs of $y = \sin x$ and $y = 2 \sin x$ related?

To investigate this question, start by drawing graphs of the two functions using a graphical calculator or graph-drawing package.



Looking at the graphs:

- $y = \sin x$ has an amplitude of 1 unit and a period of 360°.
- $y = 2\sin x$ has an amplitude of 2 units and a period of 360°.

The graphs of $y = \sin x$ and $y = 2\sin x$ illustrate the following general result.

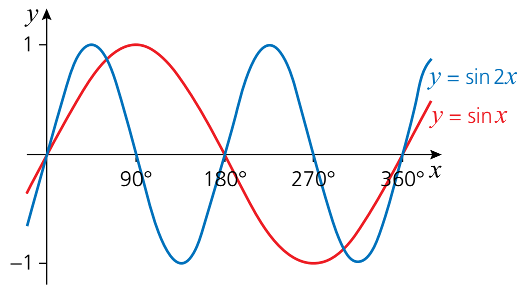
> The graph of $y = a \sin x$ is a sine curve that has an amplitude of a units and a period of 360°.
>
> The transformation is a **stretch of scale factor *a* parallel to the *y*-axis**.

$y = \sin bx$ where b is a simple fraction or integer

What is the relationship between the graph of $y = \sin x$ and $y = \sin 2x$?

Again, start by drawing the graphs. The table of values, calculated to one decimal place, is given below, but you can also draw the graphs using suitable software.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin x$	0	0.5	0.9	1.0	0.9	0.5	0	–0.5	–0.9	–1.0	–0.9	–0.5	0
$\sin 2x$	0	0.9	0.9	0	–0.9	–0.9	0	0.9	0.9	0	–0.9	–0.9	0



Looking at the graphs:

- $y = \sin x$ has an amplitude of 1 unit and a period of 360°.
- $y = \sin 2x$ has an amplitude of 1 unit and a period of 180°.

Similarly, the graph of $y = \sin 3x$ has a period of $360 \div 3 = 120°$, the graph of $y = \sin\left(\frac{x}{2}\right)$ has a period of $360 \div \frac{1}{2} = 720$, and so on.

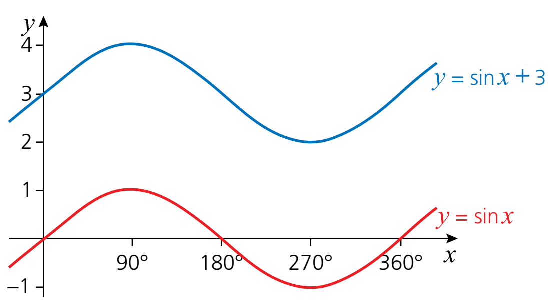
The graphs of $y = \sin x$ and $y = \sin 2x$ illustrate the following general results.

> The graph of $y = \sin bx$ is a sine curve that has amplitude 1 unit and period $\left(\frac{360}{b}\right)°$.
>
> The transformation is a **stretch of scale factor $\frac{1}{b}$ parallel to the x-axis**.

$y = \sin x + c$ where c is an integer

How are the graphs of $y = \sin x$ and $y = \sin x + 3$ related?

Again, start by drawing the graphs.



Looking at the graphs, $y = \sin x + 3$ has the same amplitude and period as $y = \sin x$ but is 3 units above it.

Similarly, the graph of $y = \sin x - 2$ is 2 units below the graph of $y = \sin x$.

The graphs of $y = \sin x$ and $y = \sin x + 3$ illustrate the following general result.

> The graph of $y = \sin x + c$ is the same shape as the graph of $y = \sin x$ but is translated vertically upwards through c units.
>
> The transformation is **a translation of** $\begin{pmatrix} 0 \\ c \end{pmatrix}$.

Combining transformations

The graph of $y = a \sin bx + c$ is a transformation of the graph of $y = \sin x$ effected by:

- a **stretch** parallel to the y-axis, **scale factor** $\boldsymbol{a}$
- a **stretch** parallel to the x-axis, **scale factor** $\boldsymbol{\frac{1}{b}}$
- a **translation** parallel to the y-axis of c units.

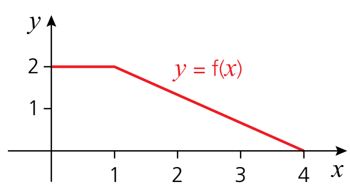
Discussion point

When drawing the graph of $y = a \sin bx + c$ using a series of transformations of the graph $y = \sin x$, why is it necessary to do the translation last?

All the transformations in this section have been applied to the graph of $y = \sin x$. The same rules can be applied to the graphs of all trigonometric functions, and to other graphs as well as those shown in these examples.

Worked example

The diagram shows the graph of a function $y = \mathrm{f}(x)$.

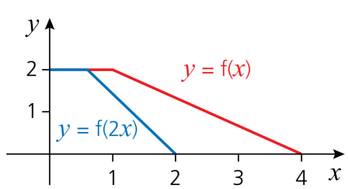


Sketch the graph of each of these functions.

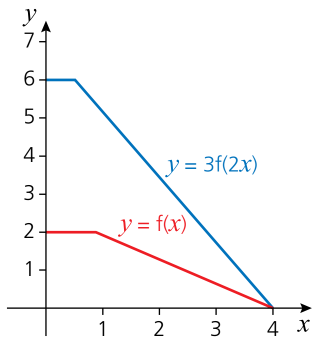
a $y = \mathrm{f}(2x)$ **b** $y = 3\mathrm{f}(x)$ **c** $y = 3\mathrm{f}(2x)$

Solution

a $y = \mathrm{f}(2x)$ is obtained from $y = \mathrm{f}(x)$ by applying a stretch of scale factor $\frac{1}{2}$ parallel to the x-axis.

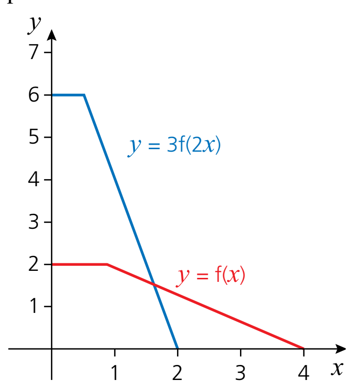


b $y = 3\text{f}(x)$ is obtained from $y = \text{f}(x)$ by applying a stretch of scale factor 3 parallel to the y-axis.



The order of the transformations is not important in this example because the two directions are independent.

c $y = 3\text{f}(2x)$ is obtained from $y = \text{f}(x)$ by applying a stretch of scale factor $\frac{1}{2}$ parallel to the x-axis and a stretch of scale factor 3 parallel to the y-axis.



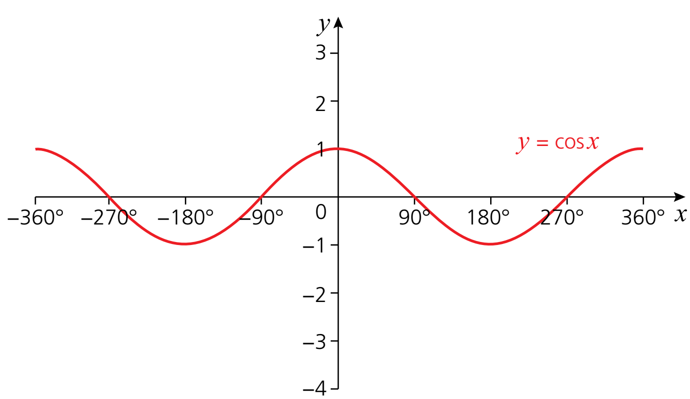
Worked example

Starting with the graph of $y = \cos x$

i State the transformations that can be used to sketch each curve.

ii Sketch each curve for $0° \leqslant x \leqslant 360°$.

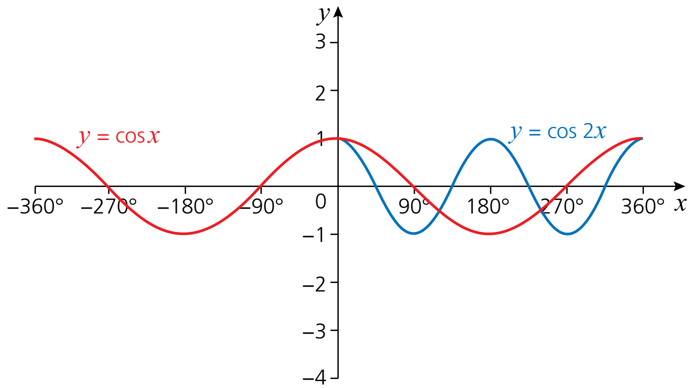
a $y = \cos 2x$ **b** $y = 3\cos 2x$ **c** $y = 3\cos 2x - 1$



Solution

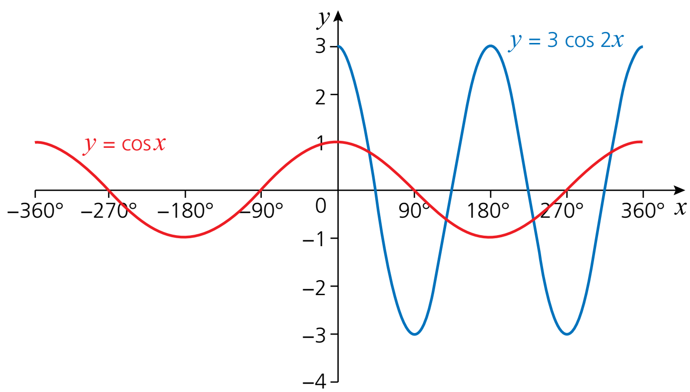
a i The graph of $y = \cos 2x$ is a stretch of $y = \cos x$ by scale factor $\frac{1}{2}$ in the x-direction.

ii



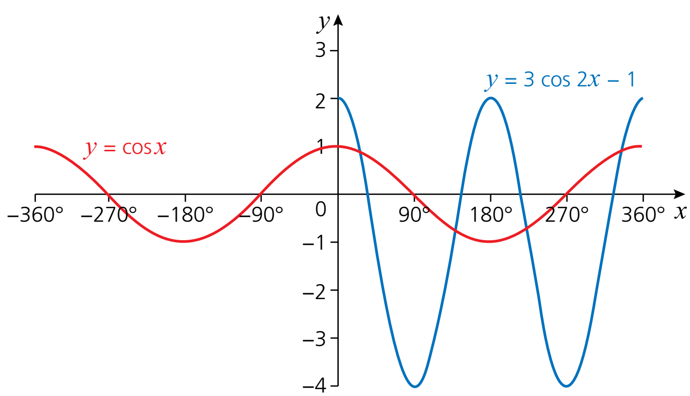
b i The graph of $y = 3\cos 2x$ is a stretch of $y = \cos x$ by scale factor $\frac{1}{2}$ in the x-direction and by scale factor 3 in the y-direction.

ii



c i The graph of $y = 3\cos 2x - 1$ is a stretch of $y = \cos x$ by scale factor $\frac{1}{2}$ in the x-direction and scale factor 3 in the y-direction, followed by a translation of 1 unit vertically downwards.

ii



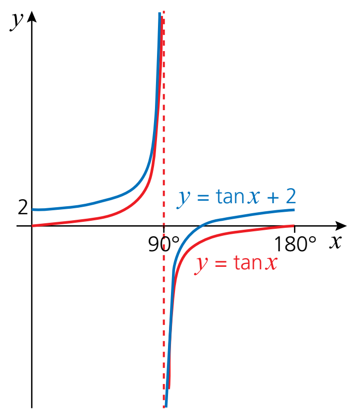
The curve $y = \tan x$ can also be translated and stretched. However, because $y = \tan x$ has no finite boundary in the y-direction, it is not as straightforward to show such stretches when the graphs approach the asymptotes.

Worked example

a Sketch the curve $y = \tan x$ for $0° \leqslant x \leqslant 180°$.

b On the same axes, sketch the curve $y = \tan x + 2$.

Solution



Exercise 10.4

1 For each transformation (i) to (iv):

a Sketch the graph of $y = \sin x$ and on the same axes sketch its image under the transformation.

b State the amplitude and period of the transformed graph.

c What do you notice about your answers for **iii** and **iv**?

i a stretch, scale factor 2, parallel to the y-axis

ii a translation of 1 unit vertically downwards

iii a stretch, scale factor 2, parallel to the y-axis followed by a translation of 1 unit vertically downwards.

iv a translation of 1 unit vertically downwards followed by a stretch of scale factor 2 parallel to the y-axis

2 **a** Apply each set of transformations to the graph of $y = \cos x$.

b Sketch the graph of $y = \cos x$ and the transformed curve on the same axes.

c State the amplitude and period of the transformed graph.

d What do you notice about your answers for **iii** and **iv**.

i a stretch, scale factor 2, parallel to the x-axis.

ii a translation of 180° in the negative x-direction

iii a stretch, scale factor 2, parallel to the x-axis followed by a translation of 180° in the negative x-direction

iv a translation of 180°in the negative x direction followed by a stretch of scale factor 2 parallel to the x-axis

Exercise 10.4 (cont)

3 **a** Apply these transformations to the graph of $y = \sin x$.

b Sketch the graph of $y = \sin x$ and the transformed curve on the same axes.

c State the amplitude and period of the transformed graph.

d What do you notice about your answers for **iii** and **iv**?

i a stretch, scale factor 2, parallel to the x-axis

ii a translation of 1 unit vertically upwards

iii a stretch, scale factor 2, parallel to the x-axis followed by a translation of 1 unit vertically upwards

iv a translation of 1 unit vertically upwards followed by a stretch, scale factor 2, parallel to the x-axis

4 State the transformations needed, in the correct order, to transform the first graph to the second graph.

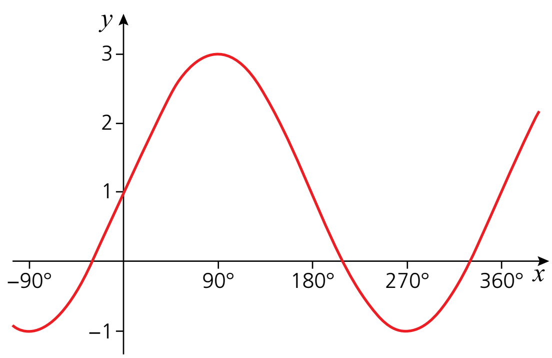
a $y = \tan x, \quad y = 3\tan 2x$

b $y = \tan x, \quad y = 2\tan x + 1$

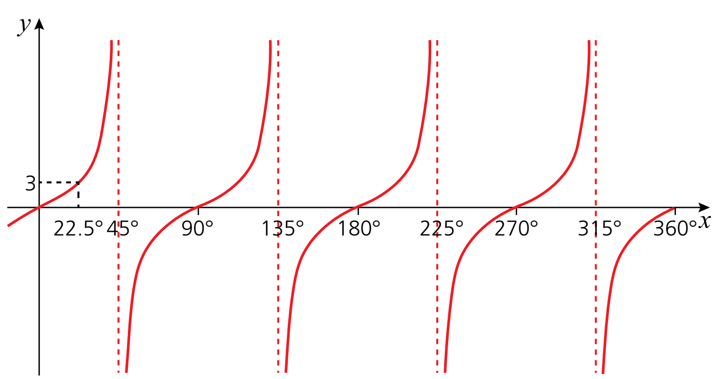
c $y = \tan x, \quad y = 2\tan(x - 180°)$

d $y = \tan x, \quad y = 3\tan\left(x + \frac{\pi}{2}\right) + 3$

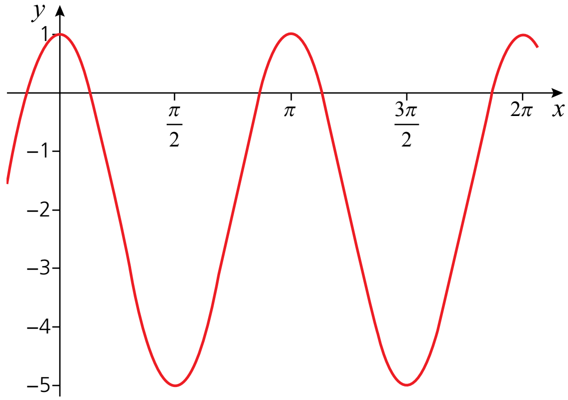
5 State the transformations required, in the correct order, to obtain this graph from the graph of $y = \sin x$.



6 State the transformations required, in the correct order, to obtain this graph from the graph of $y = \tan x$.

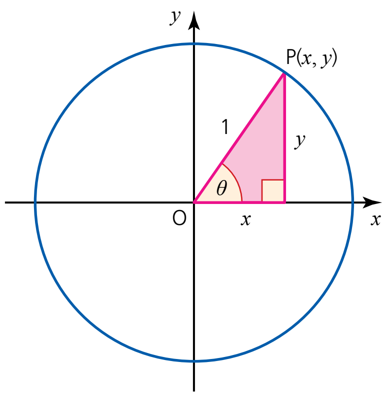


7 State the transformations required, in the correct order, to obtain this graph from the graph of $y = \cos x$.



Identities and equations

sin, cos and tan



Look at the diagram of the unit circle. It shows you that

$$x = \cos\theta,\ y = \sin\theta \text{ and } \frac{y}{x} = \tan\theta.$$

It follows that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

However, it more accurate to use the identity sign here because the relationship is true for all values of θ, so

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

Remember that an equation is only true for certain values of the variable, called the **solution** of the equation.

For example, $\tan\theta = 1$ is an equation: in the range $0° \leqslant \theta \leqslant 360°$, it is only true when $\theta = 45°$ or $225°$.

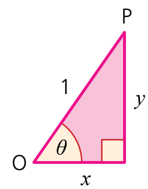
In this book, as in mathematics generally, an equals sign is often used where it would be more correct to use an identity sign. The identity sign is kept for situations where it is particularly important to emphasise that the relationship is an identity and not an equation.

By contrast, an **identity** is true for *all* values of the variable. For example,

$$\tan 45° \equiv \frac{\sin 45°}{\cos 45°}, \quad \tan 75° \equiv \frac{\sin 75°}{\cos 75°}, \quad \tan(-300°) \equiv \frac{\sin(-300°)}{\cos(-300°)}, \tan\frac{\pi}{6} = \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}}$$

and so on for all values of the angle.

The identity below is found by applying Pythagoras' theorem to any point P(x, y) on the unit circle.



$$y^2 + x^2 \equiv \text{OP}^2$$
$$(\sin\theta)^2 + (\cos\theta)^2 \equiv 1.$$

This is written as:

$$\sin^2\theta + \cos^2\theta \equiv 1.$$

You can use the identities $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta \equiv 1$ to prove other identities are true.

There are two methods you can use to prove an identity; you can use either method or a mixture of both.

Method 1

When both sides of the identity look equally complicated you can work with both the left-hand side (LHS) and the right-hand side (RHS) and show that LHS – RHS = 0 (as shown in the example below).

Worked example

Prove the identity $\cos^2\theta - \sin^2\theta \equiv 1 - 2\sin^2\theta$.

Solution

You need to show that $\cos^2\theta - \sin^2\theta - 1 + 2\sin^2\theta \equiv 0$.

Both sides look equally complicated, so show

LHS – RHS $= \cos^2\theta - \sin^2\theta - 1 + 2\sin^2\theta$

Simplifying:

$$\equiv \cos^2\theta + \sin^2\theta - 1$$
$$\equiv 1 - 1$$
$$\equiv 0 \text{ as required}$$

Using $\sin^2\theta + \cos^2\theta = 1$

Method 2

When one side of the identity looks more complicated than the other side, you can work with this side until you end up with the same as the simpler side, as shown in the next example. In this case you show LHS = RHS.

Worked example

Prove the identity $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{1}{\sin\theta} \equiv \dfrac{1}{\tan\theta}$.

Solution

$$\text{LHS} = \frac{\sin\theta}{1-\cos\theta} - \frac{1}{\sin\theta}$$

$$\equiv \frac{\sin^2\theta - (1-\cos\theta)}{(1-\cos\theta)\sin\theta}$$

Since $\sin^2\theta + \cos^2\theta = 1$, $\sin^2\theta = 1 - \cos^2\theta$

$$\equiv \frac{(1-\cos^2\theta) + \cos\theta - 1}{(1-\cos\theta)\sin\theta}$$

$$\equiv \frac{\cos\theta - \cos^2\theta}{(1-\cos\theta)\sin\theta}$$

$$\equiv \frac{\cos\theta(1-\cos\theta)}{(1-\cos\theta)\sin\theta}$$

$$\equiv \frac{\cos\theta}{\sin\theta} = \text{RHS}$$

$$\equiv \frac{1}{\tan\theta} \text{ as required}$$

cosec, sec and cot

The reciprocal of the relationship $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ is

$$\frac{1}{\tan\theta} \equiv 1 \div \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \quad \cot\theta \equiv \frac{\cos\theta}{\sin\theta}.$$

Similarly, dividing the relationship $\sin^2\theta + \cos^2\theta \equiv 1$ through by $\sin^2\theta$ gives

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta}$$

$$\Rightarrow \quad \mathbf{cosec}^2\theta \equiv 1 + \cot^2\theta$$

This is usually presented as $\quad \mathbf{cosec}^2\theta \equiv 1 + \cot^2\theta$

If instead you divide $\sin^2\theta + \cos^2\theta \equiv 1$ by $\cos^2\theta$ you get

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

$$\Rightarrow \quad \tan^2\theta + 1 \equiv \sec^2\theta$$

This is usually presented as $\quad \sec^2\theta \equiv 1 + \tan^2\theta$

Worked example

a Show that $\sec^2 x - \csc^2 x \equiv \tan^2 x - \cot^2 x$.

b Prove that $\frac{\sec\theta}{\tan\theta} \equiv \csc\theta$.

Solution

a Start with the left-hand side since this looks more complicated.

Using $\tan^2 x + 1 = \sec^2 x$ and $1 + \cot^2 x = \csc^2 x$

$$\begin{aligned}\text{LHS} \equiv \sec^2 x - \csc^2 x &\equiv (1 + \tan^2 x) - (1 + \cot^2 x)\\ &\equiv 1 + \tan^2 x - 1 - \cot^2 x\\ &\equiv \tan^2 x - \cot^2 x \equiv \text{RHS}\end{aligned}$$

b Again, start with the left-hand side since this looks more complicated.

It is often more straightforward to go back to the basic trigonometric functions $\sin\theta$ and $\cos\theta$.

$$\begin{aligned}\text{LHS} \equiv \frac{\sec\theta}{\tan\theta} &\equiv \sec\theta \div \tan\theta\\ &\equiv \frac{1}{\cos\theta} \div \frac{\sin\theta}{\cos\theta}\\ &\equiv \frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta}\\ &\equiv \frac{1}{\sin\theta}\\ &\equiv \csc\theta \equiv \text{RHS}\end{aligned}$$

You can also use this approach to solve equations involving the reciprocal functions. This involves using the definitions of the functions to find equivalent equations using sin, cos and tan. You will usually be given a range of values within which your solution must lie, so a sketch graph is useful to ensure that you find all possible values.

Worked example

Solve the following equations for

a $\csc\theta = 2$ for $0° \leqslant \theta \leqslant 360°$

b $\sec^2\theta + 2\tan^2\theta = 4$ for $0 \leqslant \theta \leqslant 2\pi$

Solution

a $\csc x = 2 \Rightarrow \frac{1}{\sin x} = 2$

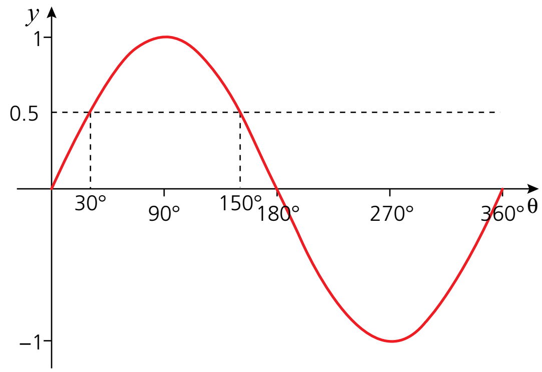
$$\Rightarrow 1 = 2\sin x$$

$$\Rightarrow \sin x = \frac{1}{2}$$

$$\Rightarrow x = 30°$$

This is called the **principal value**.

To find any other values in the interval $0° < \theta < 360°$, sketch the graph.



The graph shows that $\sin\theta$ is also $\frac{1}{2}$ when $\theta = 150°$.

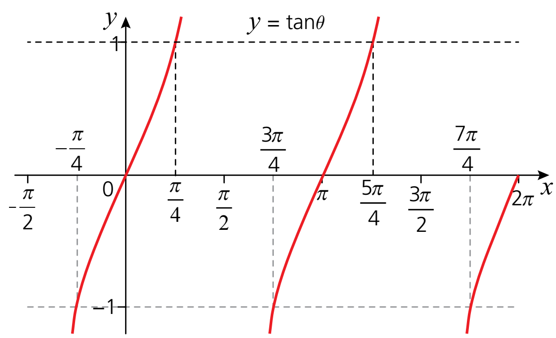
The solution is therefore $\theta = 30°$ or $\theta = 150°$.

b $\sec^2\theta + 2\tan^2\theta = 4 \Rightarrow (1 + \tan^2\theta) + 2\tan^2\theta = 4$

$$\Rightarrow \quad 1 + 3\tan^2\theta = 4$$
$$\Rightarrow \quad 3\tan^2\theta = 3$$
$$\Rightarrow \quad \tan^2\theta = 1$$
$$\Rightarrow \quad \tan\theta = \pm 1$$

$\tan^{-1} 1 = \frac{\pi}{4}$ and $\tan^{-1}(-1) = -\frac{\pi}{4}$

To find the values in the interval $0 \leqslant \theta \leqslant 2\pi$, sketch the graph.



The graph shows the value of $\tan\theta$ is also 1 when $\theta = \frac{5\pi}{4}$.

The principal value for $\tan\theta = -1$ is outside the required range. The graph shows that the values in the required range are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

The solution is therefore $\theta = \frac{\pi}{4}, \theta = \frac{3\pi}{4}, \theta = \frac{5\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Exercise 10.5

1 Prove that $\sin^2\theta - \cos^2\theta = 3 - 2\sin^2\theta - 4\cos^2\theta$.

2 Prove that $1 + \dfrac{1}{\tan^2\theta} = \dfrac{1}{\sin^2\theta}$.

3 Prove that $4\cos^2\theta + 5\sin^2\theta = \sin^2\theta + 4$.

4 Prove that $\dfrac{1-(\sin\theta - \cos\theta)^2}{\sin\theta\cos\theta} = 2$.

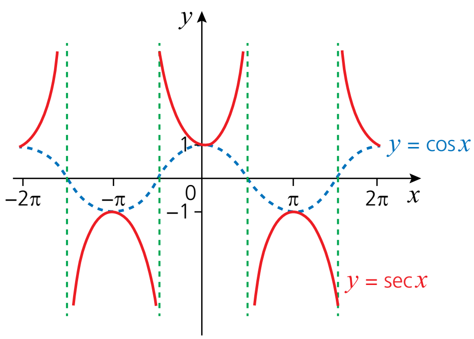
5 Solve the equation $\operatorname{cosec}^2\theta + \cot^2\theta = 2$ for $0° < x < 180°$.

6 Show that $\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A - \sin A} = \sec^2 A$.

7 Solve the equation $\sec^2\theta = 4$ for $0 \leqslant \theta \leqslant 2\pi$.

8 Prove the identity $\cot\theta + \tan\theta \equiv \sec\theta\operatorname{cosec}\theta$.

9 This is the graph of $y = \sec\theta$.



a Solve the equation $\sec\theta = 1$ for $-2\pi \leqslant \theta \leqslant 2\pi$.

b What happens if you try to solve $\sec\theta = 0.5$?

10 Solve the equation $\tan\theta = \sec\theta$ for $-360° \leqslant \theta \leqslant 360°$.

11 a Show that $12\sin^2 x + \cos x - 1 = 11 + \cos x - 12\cos^2 x$.

b Solve $12\sin^2 x + \cos x - 1 = 0$ for $0° \leqslant x \leqslant 360°$.

12 a Show that $\sin^2\theta + 1 - \sin\theta - \cos^2\theta = 2\sin^2\theta - \sin\theta$.

b Solve $\sin^2\theta + 1 - \sin\theta - \cos^2\theta = 0$ for $0 \leqslant \theta \leqslant 2\pi$.

Past-paper questions

1 (a) Solve $4\sin x = \operatorname{cosec} x$ for $0° \leqslant x \leqslant 360°$. [3]

(b) Solve $\tan^2 3y - 2\sec 3y - 2 = 0$ for $0° \leqslant y \leqslant 180°$. [6]

(c) Solve $\tan\left(z - \dfrac{\pi}{3}\right) = \sqrt{3}$ for $0 \leqslant z \leqslant 2\pi$ radians. [3]

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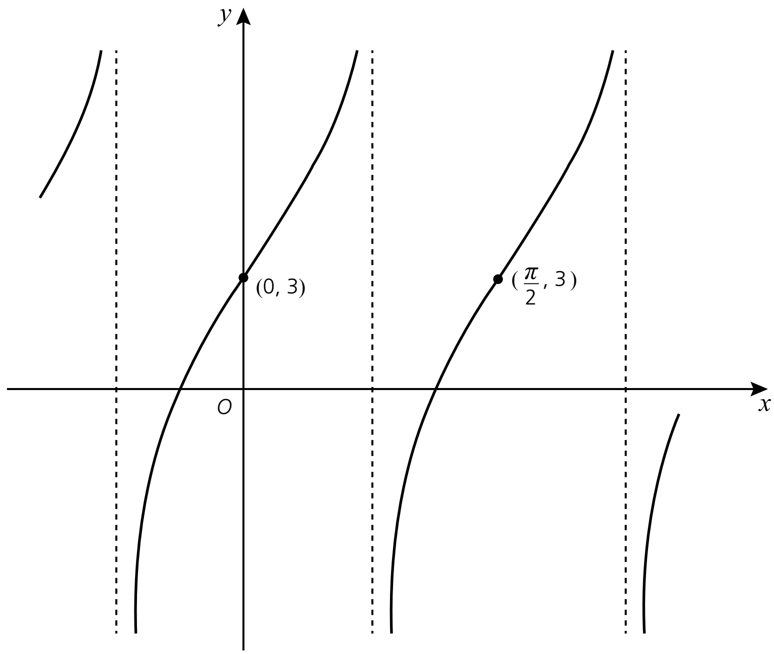

(a) **(i)** The diagram shows the graph of $y = A + C\tan(Bx)$ passing through the points $(0, 3)$ and $\left(\frac{\pi}{2}, 3\right)$. Find the value of A and of B. [2]

(ii) Given that the point $\left(\frac{\pi}{8}, 7\right)$ also lies on the graph, find the value of C. [1]

(b) Given that $f(x) = 8 - 5\cos 3x$, state the period and the amplitude of f. [2]

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Paper 23 Q4 November 2013

3 Show that $\dfrac{1}{1-\cos\theta} + \dfrac{1}{1+\cos\theta} = 2\text{cosec}^2\theta$. [3]

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Learning outcomes

Now you should be able to:

- recall and use the six trigonometric functions of angles of any magnitude (sine, cosine, tangent, secant, cosecant, cotangent)
- understand amplitude and periodicity and the relationship between graphs of related trigonometric functions, e.g. $\sin x$ and $\sin 2x$
- draw and use the graphs of

  $y = a\sin bx + c$
  $y = a\cos bx + c$
  $y = a\tan bx + c$

  where a is a positive integer, b is a simple fraction or integer (fractions will have a denominator 2, 3, 4, 6 or 8 only) and c is an integer
- use the relationships

  $\sin^2 A + \cos^2 A = 1$
  $\sec^2 A = 1 + \tan^2 A$
  $\operatorname{cosec}^2 A = 1 + \cot^2 A$
  $\dfrac{\sin A}{\cos A} = \tan A$
  $\dfrac{\cos A}{\sin A} = \cot A$
- solve simple trigonometric equations involving the six trigonometric functions and the above relationships (not including the general solution of trigonometric equations)
- prove simple trigonometric identities.

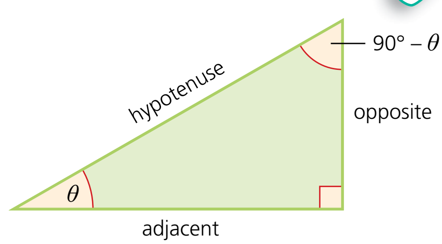
Key points

✔ In a right-angled triangle

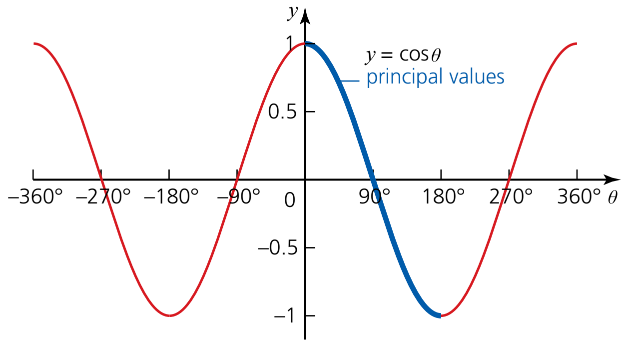
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

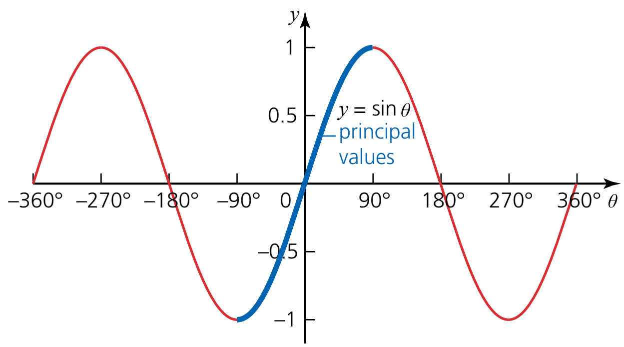
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

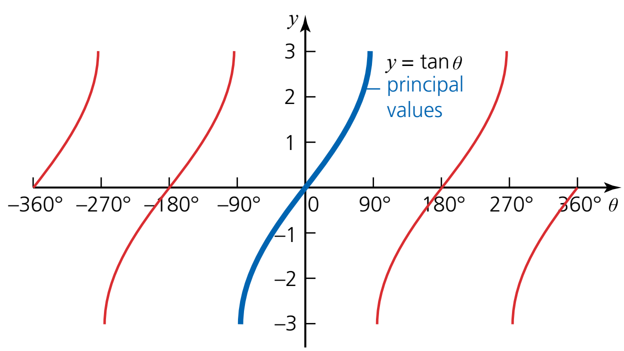
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$



✔ The graphs of the three main trigonometric functions have distinctive shapes as shown below.







If θ is in radians, the shapes of these curves are exactly the same but the scale on the horizontal axis goes from -2π to 2π instead of from –360° to 360°.

✔ The reciprocal trigonometric functions are defined as:

$$\text{cosec } \theta = \frac{1}{\sin\theta} \qquad \sec \theta = \frac{1}{\cos\theta} \qquad \cot \theta = \frac{1}{\tan\theta}$$

✔ The **amplitude** of an oscillating graph such as $y = \sin x$ or $y = \cos x$ is the largest displacement from the equilibrium position. For $y = \sin x$ or $y = \cos x$, the equilibrium position is the x-axis.

✔ The **period** of the oscillations is the interval over which the graph does one complete oscillation.

✔ The graph of $y = a \sin bx + c$ is a transformation of the graph of $y = \sin x$ by:

i a stretch of scale factor a in the y-direction

ii a stretch of scale factor $\frac{1}{b}$ in the x-direction

iii a translation of c units in the y-direction

Operation **(i)** must precede **(iii)** but otherwise the order of the transformations can be varied.

The same rules apply if sin is replaced by cos or tan.

✔ The following relationships are referred to as **trigonometric identities.**

$\sin^2 A + \cos^2 A \equiv 1$

$\sec^2 A \equiv 1 + \tan^2 A$

$\operatorname{cosec}^2 A \equiv 1 + \cot^2 A$

$\frac{\sin A}{\cos A} \equiv \tan A$

$\frac{\cos A}{\sin A} \equiv \cot A$

11 Permutations and combinations

It always seems impossible until it is done.

Nelson Mandela (1918 – 2013)

Discussion point

The combination lock has four numbers to be found and six choices for each number: 1, 2, 3, 4, 5 or 6. Suppose you have no idea what the code is, but you need to open the lock. It may seem like an impossible situation initially, but what if you try every possible combination of numbers systematically? How many possible combinations are there? Estimate how long it will take you to open the lock.

Factorials

Worked example

Winni is tidying her bookshelf and wants to put her five maths books together. In how many different ways can she arrange them?

Solution

There are 5 possible books that can go in the first space on the shelf.

There are 4 possible books for the second space.

There are 3 for the third space, 2 for the fourth and only 1 book left for the fifth space.

The total number of arrangements is therefore

5	$\times$	4	$\times$	3	$\times$	2	$\times$	1	$= 120$
Book 1		Book 2		Book 3		Book 4		Book 5	

This number, $5 \times 4 \times 3 \times 2 \times 1$, is called **5 factorial** and is written 5!

n must be a positive integer.

This example illustrates a general result. The number of ways of placing n different objects in a line is $n!$, where $n! = n \times (n-1) \times (n-2) \ldots \times 3 \times 2 \times 1$.

By convention, a special case is made for $n = 0$. The value of 0! is taken to be 1.

Worked example

Find the value of each of the following:

a 2! **b** 3! **c** 4! **d** 5! **e** 10!

Solution

a $2! = 2 \times 1 = 2$

b $3! = 3 \times 2 \times 1 = 6$

You can see that factorials go up very quickly in size.

c $4! = 4 \times 3 \times 2 \times 1 = 24$

d $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

e $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3\,628\,800$

Worked example

a Calculate $\frac{7!}{5!}$

b Calculate $\frac{5! \times 4! \times 3!}{6! \times 2!}$

Solution

a $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ and $5! = 5 \times 4 \times 3 \times 2 \times 1$

So $\frac{7!}{5!} = \frac{7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 42$

This result can be generalised as $\frac{n!}{m!} = \frac{n \times (n-1) \times (n-2) \times \ldots \times (m+1) \times m!}{m!}$ $= n \times (n-1) \times (n-2) \times \ldots \times (m+1)$

You can also write 7! as $7 \times 6 \times 5!$

$n > m$

Using this, $\frac{7!}{5!} = \frac{7 \times 6 \times 5!}{5!} = 7 \times 6 = 42$

b $\frac{5 \times 4 \times 3 \times 2 \times 1 \ \times \ 4 \times 3 \times 2 \times 1 \ \times \ 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \ \times \ 2 \times 1} = 12$

Worked example

a Find the number of ways in which all six letters in the word FOURTH can be arranged.

b In how many of these arrangements are the letters O and U next to each other?

Solution

a There are six choices for the first letter (F, O, U, R, T, H). Then there are five choices for the next letter, then four for the fourth letter and so on. So the number of arrangements of the letters is

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$

b The O and the U are to be together, so you can treat them as a single letter.

So there are five choices for the first letter '(F, OU, R, T or H)', four choices for the next letter and so on.

So the number of arrangements of these five 'letters' is

$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$

However $\boxed{\text{OU}}\ \boxed{\text{F}}\ \boxed{\text{R}}\ \boxed{\text{T}}\ \boxed{\text{H}}$

is different from $\boxed{\text{UO}}\ \boxed{\text{F}}\ \boxed{\text{R}}\ \boxed{\text{T}}\ \boxed{\text{H}}$

So each of the 120 arrangements can be arranged into two different orders.

The total number of arrangements with the O and U next to each other is

$2 \times 5! = 240$

Notice that the total number of ways of arranging the letters with the U and the O apart is $720 - 240 = 480$

Exercise 11.1

1 Calculate: **a** $7!$ **b** $\frac{9!}{7!}$ **c** $\frac{4! \times 6!}{7! \times 2!}$

2 Simplify: **a** $\frac{\boldsymbol{n}!}{(\boldsymbol{n}+1)!}$ **b** $\frac{(n-2)!}{(n-3)!}$

3 Simplify: **a** $\frac{(n+2)!}{n!}$ **b** $\frac{(n+1)!}{(n-1)!}$

4 Write in factorial notation:

a $\frac{9 \times 8 \times 7}{6 \times 5 \times 4}$ **b** $\frac{14 \times 15}{5 \times 4 \times 3 \times 2}$ **c** $\frac{(n+2)(n+1)n}{4 \times 3 \times 2}$

5 Factorise: **a** $6! + 7!$ **b** $n! + (n-1)!$

6 Write the number 42 using factorials only.

7 How many different four-letter arrangements can be formed from the letters P, Q, R and S if letters cannot be repeated?

8 How many different ways can seven books be arranged in a row on a shelf?

9 There are five drivers in a motoring rally.

How many different ways are there for the five drivers to finish?

Exercise 11.1 (cont)

10 There are five runners in a 60-metre hurdles race, one from each of the nations Japan, South Korea, Cambodia, Malaysia and Thailand.

How many different finishing orders are there?

11 Toben listens to 15 songs from a playlist. If he selects 'shuffle' so the songs are played in a random order, in how many different orders could the songs be played?

12 How many different arrangements are there of the letters in each word?

a ASK　**b** QUESTION　**c** SINGAPORE
d GOVERN　**e** VIETNAM　**f** MAJORITY

13 How many arrangements of the letters in the word ARGUMENT are there if:

a there are no restrictions on the order of the letters
b the first letter is an A
c the letters A and R must be next to each other
d the letters G and M must not be next to each other.

Permutations

In some situations, such as a race, the finishing order matters. An ordered arrangement of a number of people, objects or operations is called a **permutation**.

Worked example


Joyeeta

I should be one of the judges! When I saw the 10 contestants in the cookery competition, I knew which ones I thought were the best three. Last night they announced the results and I had picked the same three contestants in the same order as the judges!

What is the probability of Joyeeta's result?

Solution

The winner can be chosen in 10 ways.

The second contestant can be chosen in 9 ways.

The third contestant can be chosen in 8 ways.

Thus the total number of ways of placing three contestants in the first three positions is $10 \times 9 \times 8 = 720$. So the probability that Joyeeta's selection is correct is $\frac{1}{720}$.

In this example attention is given to the order in which the contestants are placed. The solution required a **permutation** of three objects from ten.

In general the number of permutations, ${}^{n}P_{r}$, of r objects from n is given by

$$ {}^{n}P_{r} = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1). $$

This can be written more compactly as

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!} $$

Worked example

Five people go to the theatre. They sit in a row with eight seats. Find how many ways can this be done if:

a they can sit anywhere

b all the empty seats are next to each other.

Solution

a The first person to sit down has a choice of eight seats.

The second person to sit down has a choice of seven seats.

The third person to sit down has a choice of six seats.

The fourth person to sit down has a choice of five seats.

The fifth person to sit down has a choice of four seats.

So the total number of arrangements is $8 \times 7 \times 6 \times 5 \times 4 = 6720$.

This is a permutation of five objects from eight, so a quicker way to work this out is:

number of arrangements $= {}^{8}P_{5} = 6720$.

b Since all three empty seats are to be together you can consider them to be a single 'empty seat', albeit a large one!

So there are six seats to seat five people.

So the number of arrangements is ${}^{6}P_{5} = 720$.

Combinations

In other situations, order is not important, for example, choosing five of eight students to go to the theatre. You are not concerned with the order in which people or objects are chosen, only with which ones are picked. A selection where order is not important is called a **combination**.

A maths teacher is playing a game with her students. Each student selects six numbers out of a possible 19 (numbers $1, 2, \ldots, 19$). The maths teacher then uses a random number machine to generate six numbers. If a student's numbers match the teacher's numbers then they win a prize.

Discussion point

You have the six winning numbers. Does it matter in which order the machine picked them?

The teacher says that the probability of an individual student picking the winning numbers is about 1 in 27 000. How can you work out this figure?

The key question is, how many ways are there of choosing six numbers out of 19?

If the order mattered, the answer would be $^{19}P_6$, or $19 \times 18 \times 17 \times 16 \times 15 \times 14$.

However, the order does not matter. The selection 1, 3, 15, 19, 5 and 18 is the same as 15, 19, 1, 5, 3, 18 and as 18, 1, 19, 15, 3, 5, and lots more. For each set of six numbers there are 6! arrangements that all count as being the same.

So, the number of ways of selecting six numbers, given that the order does not matter, is

$$\frac{19 \times 18 \times 17 \times 16 \times 15 \times 14}{6!}.$$

This is $\frac{^{19}P_6}{6!}$

This is called the number of **combinations** of 6 objects from 19 and is denoted by $^{19}C_6$.

Discussion point

Show that $^{19}C_6$ can be written as $\frac{19!}{6!\,13!}$.

Returning to the maths teacher's game, it follows that the probability of a student winning is $\frac{1}{^{19}C_6}$.

This is about 27 000.

Discussion point

How does the probability change if there are 29, 39 and 49 numbers to choose from?

This example shows a general result, that the number of ways of selecting r objects from n, when the order does not matter, is given by

$$^nC_r = \frac{n!}{r!(n-r)!} = \frac{^nP_r}{r!}$$

Discussion point

How can you prove this general result?

Another common notation for ${}^nC_r$ is $\binom{n}{r}$. Both notations are used in this book to help you become familiar with them.

Caution: The notation $\binom{n}{r}$ looks exactly like a column vector and so there is the possibility of confusing the two. However, the context will usually make the meaning clear.

Worked example

A student representative committee of five people is to be chosen from nine applicants. How many different selections are possible?

Solution

Number of selections $= \binom{9}{5} = \frac{9!}{5! \times 4!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126$

Worked example

In how many ways can a committee of five people be selected from five applicants?

Solution

Common sense tells us that there is only one way to make the committee, that is by appointing all applicants. So ${}^5C_5 = 1$. However, if we work from the formula

$$ {}^5C_5 = \frac{5!}{5!0!} = \frac{1}{0!} $$

for this to equal 1 requires the convention that 0! is taken to be 1.

Discussion point

Use the convention $0! = 1$ to show that ${}^nC_0 = {}^nC_n = 1$ for all values of n.

Exercise 11.2

1 **a** Find the values of **i** ${}^7P_3$ **ii** ${}^9P_4$ **iii** ${}^{10}P_8$
b Find the values of **i** ${}^7C_3$ **ii** ${}^9C_4$ **iii** ${}^{10}C_8$
c Show that, for the values of n and r in parts **a** and **b**, ${}^nC_r = \frac{{}^nP_r}{r!}$.

2 There are 15 competitors in a camel race. How many ways are there of guessing the first three finishers?

3 A group of 6 computer programmers is to be chosen to work the night shift from a set of 14 programmers. In how many ways can the programmers be chosen if the 6 chosen must include the shift-leader who is one of the 14?

Exercise 11.2 (cont)

4 Zaid decides to form a band. He needs a bass player, a guitarist, a keyboard player and a drummer. He invites applications and gets 6 bass players, 8 guitarists, 4 keyboard players and 3 drummers. Assuming each person applies only once, in how many ways can Zaid put the band together?

5 A touring party of cricket players is made up of 6 players from each of India, Pakistan and Sri Lanka and 3 from Bangladesh.
- **a** How many different selections of 11 players can be made for a team?
- **b** In one match, it is decided to have 3 players from each of India, Pakistan and Sri Lanka and 2 from Bangladesh. How many different team selections can now be made?

6 A committee of four is to be selected from ten candidates, five men and five women.
- **a** In how many distinct ways can the committee be chosen?
- **b** Assuming that each candidate is equally likely to be selected, determine the probabilities that the chosen committee contains:
  - **i** no women
  - **ii** two men and two women.

7 A committee of four is to be selected from four boys and six girls. The members are selected at random.
- **a** How many different selections are possible?
- **b** What is the probability that the committee will be made up of:
  - **i** all girls
  - **ii** more boys than girls?

8 A factory advertises six positions. Nine men and seven women apply.
- **a** How many different selections are possible?
- **b** How many of these include equal numbers of men and women?
- **c** How many of the selections include no men?
- **d** How many of the selections include no women?

9 A small business has 14 staff; 6 men and 8 women. The business is struggling and needs to make four members of staff redundant.
- **a** How many different selections are possible if the four staff are chosen at random?
- **b** How many different selections are possible if equal numbers of men and women are chosen?
- **c** How many different selections are possible if there are equal numbers of men and women remaining after the redundancies?

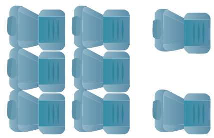
10 A football team consists of a goalkeeper, two defense players, four midfield players and four forwards. Three players are chosen to collect a medal at the closing ceremony of a competition.

How many selections are possible if one midfield player, one defense player and one forward must be chosen?

11 Find how many different numbers can be made by arranging all nine digits of the number 335 688 999 if:
- **i** there are no restrictions
- **ii** the number made is a multiple of 5.

12 Nimish is going to install 5 new game apps on her phone. She has shortlisted 2 word games, 5 quizzes and 16 saga games. Nimish wants to have at least one of each type of game. How many different selections of apps could Nimish possibly choose?

13 A MPV has seven passenger seats – one in the front, and three in each of the other two rows.



a In how many ways can all 8 seats be filled from a party of 12 people, assuming that they can all drive?

b In a party of 12 people, 3 are qualified drivers. They hire an MPV and a four-seater saloon car. In how many ways can the party fill the MPV given that one of the drivers must drive each vehicle?

14 Iram has 12 different DVDs of which 7 are films, 3 are music videos and 2 are documentaries.

a How many different arrangements of all 12 DVDs on a shelf are possible if the music videos are all next to each other?

b Iram makes a selection of 2 films, 2 music videos and 1 documentary. How many possible selections can be made?

15 A string orchestra consists of 15 violins, 8 violas, 7 cellos and 4 double basses. A chamber orchestra consisting of 8 violins, 4 violas, 2 cellos and 2 double basses is to be chosen from the string orchestra.

a In how many different ways can the chamber orchestra be chosen?

b Once the chamber orchestra is chosen, how many seating arrangements are possible if each instrument group has their own set of chairs?

c The violinists work in pairs. How many seating arrangements are possible for the violinists if they must sit with their partner?

16 An office car park has 12 parking spaces in a row. There are 9 cars to be parked.

a How many different arrangements are there for parking the 9 cars and leaving 3 empty spaces?

b How many different arrangements are there if the 3 empty spaces are next to each other?

Past-paper questions

1 A school council of 6 people is to be chosen from a group of 8 students and 6 teachers. Calculate the number of different ways that the council can be selected if

(i) there are no restrictions, [2]

(ii) there must be at least 1 teacher on the council and more students than teachers. [3]

After the council is chosen, a chairperson and a secretary have to be selected from the 6 council members.

(iii) Calculate the number of different ways in which a chairperson and a secretary can be selected. [1]

Cambridge O Level Additional Mathematics 4037
Paper 23 Q5 November 2011
Cambridge IGCSE Additional Mathematics 0606
Paper 23 Q5 November 2011

2 (a) (i) Find how many different 4-digit numbers can be formed from the digits 1, 3, 5, 6, 8 and 9 if each digit may be used only once. [1]
(ii) Find how many of these 4-digit numbers are even. [1]
(b) A team of 6 people is to be selected from 8 men and 4 women. Find the number of different teams that can be selected if
(i) there are no restrictions, [1]
(ii) the team contains all 4 women, [1]
(iii) the team contains at least 4 men. [3]

Cambridge O Level Additional Mathematics 4037
Paper 12 Q7 November 2013
Cambridge IGCSE Additional Mathematics 0606
Paper 12 Q7 November 2013

3 Arrangements containing 5 different letters from the word AMPLITUDE are to be made. Find
(a) (i) the number of 5-letter arrangements if there are no restrictions, [1]
(ii) the number of 5-letter arrangements which start with the letter A and end with the letter E. [1]

Cambridge O Level Additional Mathematics 4037
Paper 11 June 2012
(Part question: part b omitted)
Cambridge IGCSE Additional Mathematics 0606
Paper 11 June 2012
(Part question: part b omitted)

Learning outcomes

Now you should be able to:

★ recognise and distinguish between a permutation case and a combination case
★ recall and use the notation $n!$ (with $0! = 1$), and expressions for permutations and combinations of n items taken r at a time
★ answer simple problems on arrangement and selection.

Key points

- ✔ The number of ways of arranging n different objects in a line is $n!$ This is read as n factorial.
- ✔ $n! = n \times (n-1) \times (n-2) \ldots \times 3 \times 2 \times 1$ where n is a positive integer.
- ✔ By convention, $0! = 1$.
- ✔ The number of permutations of r objects from n is ${}^nP_r = \frac{n!}{(n-r)!}$
- ✔ The number of combinations of r objects from n is ${}^nC_r = \frac{n!}{(n-r)!r!}$
- ✔ The order matters for permutations, but not for combinations.

12 Series

I would like one grain of wheat to be put on the first square of my board, two on the second square, four on the third square, eight on the fourth and so on.

Attributed to Sissa ben Dahir (6th century)

Discussion point

The origin of the game of chess is uncertain, both in time and place. According to one legend it was invented by Sissa ben Dahir, Vizier to Indian king Shirham. The king asked Sissa ben Dahir what he would like for a reward, and his reply is quoted above. The king agreed without doing any calculations.

Given that one grain of wheat weighs about 50 mg, what mass of wheat would have been placed on the last square?

Definitions and notation

A **sequence** is a set of numbers in a given order, for example

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots.$$

Each of these numbers is called a **term** of the sequence. When the terms of a sequence are written algebraically, the position of any term in the sequence is usually shown by a subscript, so that a general sequence is written:

$u_1, u_2, u_3, \ldots$, with general term u_k.

For the previous sequence, the first term is $u_1 = \frac{1}{2}$, the second term is $u_2 = \frac{1}{4}$, and so on.

When the terms of a sequence are added together, for example,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

the resulting sum is called a **series**. The process of adding the terms together is called **summation** and indicated by the symbol $\sum$ (the Greek letter sigma), with the position of the first and last terms involved given as **limits**.

So $u_1 + u_2 + u_3 + u_4 + u_5$ is written $\sum_{k=1}^{k=5} u_k$ or $\sum_{k=1}^{5} u_k$.

In cases like this one, where there is no possibility of confusion, the sum is normally written more simply as $\sum_1^5 u_k$.

If all the terms are to be summed, it is usually denoted even more simply as $\sum_k u_k$, or even $\sum u_k$.

A sequence may have an infinite number of terms, in which case it is called an **infinite sequence**. The corresponding series is called an **infinite series**.

The phrase 'sum of a sequence' is often used to mean the sum of the terms of a sequence (i.e. the series).

Although the word **series** can describe the sum of the terms of any sequence in mathematics, it is usually used only when summing the sequence provides a useful or interesting overall result.

For example:

$$(1+t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4$$

This series has a finite number of terms (5).

$$\sqrt{11} = \frac{10}{3}\left[1 - \frac{1}{2}(0.01) - \frac{1}{8}(0.01)^2 - \frac{1}{16}(0.01)^3 \ldots\right]$$

This series has an infinite number of terms.

The binomial theorem

A special type of series is produced when a binomial (i.e. two-part) expression such as $(x + 1)$ is raised to a power. The resulting expression is often called a **binomial expansion**.

Expressions like these, consisting of integer powers of x and constants are called polynomials.

The simplest binomial expansion is $(x + 1)$ itself. This and other powers of $(x + 1)$ are given below.

$$
\begin{aligned}
(x+1)^1 &= 1x + 1\\
(x+1)^2 &= 1x^2 + 2x + 1\\
(x+1)^3 &= 1x^3 + 3x^2 + 3x + 1\\
(x+1)^4 &= 1x^4 + 4x^3 + 6x^2 + 4x + 1\\
(x+1)^5 &= 1x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1
\end{aligned}
$$

If you look at the coefficients on the right-hand side you will see that they form a pattern.

These numbers are called binomial coefficients.

$$
\begin{array}{ccccccccccc}
 & & & & & (1) & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

This is called **Pascal's triangle**, or the **Chinese triangle**. Each number is obtained by adding the two above it, for example

$$
\begin{array}{lc}
 & 4 \quad + \quad 6\\
\text{gives} & 10
\end{array}
$$

This pattern of coefficients is very useful when you need to write down the expansions of other binomial expressions. For example,

Notice how in each term the sum of the powers of x and y is the same as the power of $(x + y)$.

$$
\begin{aligned}
(x+y) &= 1x + 1y\\
(x+y)^2 &= 1x^2 + 2xy + 1y^2\\
(x+y)^3 &= 1x^3 + 3x^2y + 3xy^2 + 1y^3
\end{aligned}
$$

This is a binomial expression.

These numbers are binomial coefficients.

Worked example

Write out the binomial expansion of $(a + 3)^5$.

Solution

The binomial coefficients for power 5 are 1 5 10 10 5 1.

In each term, the sum of the powers of a and 3 must equal 5.

So the expansion is:

$$1 \times a^5 + 5 \times a^4 \times 3 + 10 \times a^3 \times 3^2 + 10 \times a^2 \times 3^3 + 5 \times a \times 3^4 + 1 \times 3^5$$

i.e. $a^5 + 15a^4 + 90a^3 + 270a^2 + 405a + 243$.

Worked example

Write out the binomial expansion of $(3x - 2y)^4$.

Solution

The binomial coefficients for power 4 are 1 4 6 4 1.

The expression $(3x - 2y)$ is treated as $(3x + (-2y))$.

So the expansion is

$$1 \times (3x)^4 + 4 \times (3x)^3 \times (-2y) + 6 \times (3x)^2 \times (-2y)^2 + 4 \times (3x) \times (-2y)^3 + 1 \times (-2y)^4$$

i.e. $81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$

Pascal's triangle (and the binomial theorem) had actually been discovered by Chinese mathematicians several centuries earlier, and can be found in the works of Yang Hui (around AD1270) and Chu Shi-kie (in AD1303). However, Pascal is remembered for his application of the triangle to elementary probability, and for his study of the relationships between binomial coefficients.

Tables of binomial coefficients

Values of binomial coefficients can be found in books of tables. It can be helpful to use these when the power becomes large, since writing out Pascal's triangle becomes progressively longer and more tedious, row by row. Note that since the numbers are symmetrical about the middle number, tables do not always give the complete row of numbers.

Worked example

Write out the full expansion of $(a + b)^8$.

Solution

The binomial coefficients for the power 8 are

1 8 28 56 70 56 28 8 1

and so the expansion is

$$a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8.$$

The formula for a binomial coefficient

You may need to find binomial coefficients that are outside the range of your tables. The tables may, for example, list the binomial coefficients for powers up to 20. What happens if you need to find the coefficient of x^{17} in the expansion of $(x + 2)^{25}$? Clearly you need a formula that gives binomial coefficients.

The first thing you need is a notation for identifying binomial coefficients. It is usual to denote the power of the binomial expression by n, and the position in the row of binomial coefficients by r, where r can take any value from 0 to n. So, for row 5 of Pascal's triangle

$n = 5$:	1	5	10	10	5	1
	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$

The general binomial coefficient corresponding to values of n and r is written as $\binom{n}{r}$. An alternative notation is ${}^nC_r$, which is said as 'N C R'.

Note that 0! is defined to be 1. You will see the need for this when you use the formula for $\binom{n}{r}$.

Thus $\binom{5}{3} = {}^5C_3 = 10$.

The next step is to find a formula for the general binomial coefficient $\binom{n}{r}$.

Real-world activity

The table shows an alternative way of laying out Pascal's triangle.

		Column (r)								
		0	**1**	**2**	**3**	**4**	**5**	**6**	**…**	**r**
Row (n)	**1**	1	1							
	2	1	2	1						
	3	1	3	3	1					
	4	1	4	6	4	1				
	5	1	5	10	10	5	1			
	6	1	6	15	20	15	6	1		
	…	…	…	…	…	…	…	…	…	
	…	…	…	…	…	…	…	…	…	…
	n	1	n	?	?	?	?	?	?	?

Show that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, by following the procedure below.

The numbers in column 0 are all 1.

To find each number in column 1 you multiply the 1 in column 0 by the row number, n.

1 Find, in terms of n, what you must multiply each number in column 1 by to find the corresponding number in column 2.

2 Repeat the process to find the relationship between each number in column 2 and the corresponding number in column 3.

3 Show that repeating the process leads to

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\times2\times3\times\ldots\times r} \text{ for } r \geqslant 1.$$

4 Show that this can also be written as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

and that it is also true for $r = 0$.

Worked example

Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to calculate these.

a $\binom{7}{0}$ **b** $\binom{7}{1}$ **c** $\binom{7}{2}$ **d** $\binom{7}{3}$

e $\binom{7}{4}$ **f** $\binom{7}{5}$ **g** $\binom{7}{6}$ **h** $\binom{7}{7}$

Solution

a $\binom{7}{0} = \frac{7!}{0!(7-0)!} = \frac{5040}{1 \times 5040} = 1$

b $\binom{7}{1} = \frac{7!}{1!6!} = \frac{5040}{1 \times 720} = 7$

c $\binom{7}{2} = \frac{7!}{2!5!} = \frac{5040}{2 \times 120} = 21$

d $\binom{7}{3} = \frac{7!}{3!4!} = \frac{5040}{6 \times 24} = 35$

e $\binom{7}{4} = \frac{7!}{4!3!} = \frac{5040}{24 \times 6} = 35$

f $\binom{7}{5} = \frac{7!}{5!2!} = \frac{5040}{120 \times 21} = 21$

g $\binom{7}{6} = \frac{7!}{6!1!} = \frac{5040}{720 \times 1} = 7$

h $\binom{7}{7} = \frac{7!}{7!0!} = \frac{5040}{5040 \times 1} = 1$

Note

Most scientific calculators have factorial buttons, e.g. $\boxed{x!}$. Many also have $\boxed{{}^nC_r}$ buttons. Find out how best to use your calculator to find binomial coefficients, as well as practising non-calculator methods.

Worked example

Notice how 19! was cancelled in working out $\binom{25}{6}$. Factorials become large numbers very quickly and you should keep a look-out for such opportunities to simplify calculations.

Find the coefficient of x^{19} in the expansion of $(x+3)^{25}$.

Solution

$$(x+3)^{25} = \binom{25}{0}x^{25} + \binom{25}{1}x^{24}3^1 + \binom{25}{2}x^{23}3^2 + \ldots + \binom{25}{8}6x^{19}3^6 + \ldots \binom{25}{25}3^{25}$$

So the required term is $\binom{25}{6} \times x^{19}\,3^6$

$$\binom{25}{6} = \frac{25!}{6!19!} = \frac{25 \times 24 \times 23 \times 22 \times 21 \times 20 \times \cancel{19!}}{6! \times \cancel{19!}}$$

$$= 177\,100.$$

So the coefficient of x^{19} is $177\,100 \times 3^6 = 129\,105\,900$.

The expansion of $(1 + x)^n$

When deriving the result for $\binom{n}{r}$ you found the binomial coefficients in the form

$$1 \quad n \quad \frac{n(n-1)}{2!} \quad \frac{n(n-1)(n-2)}{3!} \quad \frac{n(n-1)(n-2)(n-3)}{4!} \quad \ldots$$

This form is commonly used in the expansion of expressions of the type $(1 + x)^n$.

The first few terms

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{1\times 2} + \frac{n(n-1)(n-2)x^3}{1\times 2\times 3} + \frac{n(n-1)(n-2)(n-3)x^4}{1\times 2\times 3\times 4} + \ldots$$

The last few terms

$$+\frac{n(n-1)}{1\times 2}x^{n-2} + nx^{n-1} + 1x^n$$

Worked example

Use the binomial expansion to write down the first four terms, in ascending powers of x, of $(1 + x)^8$.

Solution

The power of x is the same as the largest number underneath.

$$(1+x)^8 = 1 + 8x + \frac{8\times 7}{1\times 2}x^2 + \frac{8\times 7\times 6}{1\times 2\times 3}x^3 + \ldots$$

Two numbers on top, two underneath.

Three numbers on top, three underneath.

$$= 1 + 8x + 28x^2 + 56x^3 + \ldots$$

An expression like $1 + 8x + 28x^2 + 56x^3$ ... is said to be in **ascending** powers of x, because the powers of x are increasing from one term to the next.

An expression like $x^8 + 8x^7 + 28x^6 + 56x^5$ ... is in **descending** powers of x, because the powers of x are decreasing from one term to the next.

Worked example

Use the binomial expansion to write down the first four terms, in ascending powers of x, of $(1 - 2x)^6$. Simplify the terms.

Solution

Think of $(1 - 2x)^6$ as $(1 + (-2x))^6$. Keep the brackets while you write out the terms.

$$(1+(-2x))^6 = 1 + 6(-2x) + \frac{6\times 5}{1\times 2}(-2x)^2 + \frac{6\times 5\times 4}{1\times 2\times 3}(-2x)^3 + \ldots$$

$$= 1 - 12x + 60x^2 - 160x^3 + \ldots$$

Notice how the signs alternate.

Exercise 12.1

1 Write out the following binomial expressions:

a $(1+x)^4$ **b** $(1+2x)^4$ **c** $(1+3x)^4$

2 Write out the following binomial expressions:

a $(2+x)^4$ **b** $(3+x)^4$ **c** $(4+x)^4$

3 Write out the following binomial expressions:

a $(x+y)^4$ **b** $(x+2y)^4$ **c** $(x+3y)^4$

4 Use a non-calculator method to calculate the following binomial coefficients. Check your answers using your calculator's shortest method.

a $\binom{5}{3}$ **b** $\binom{7}{2}$ **c** $\binom{7}{4}$

d $\binom{7}{5}$ **e** $\binom{5}{0}$ **f** $\binom{13}{3}$

5 Find the coefficients of the term shown for each expansion:

a x^4 in $(1+x)^6$ **b** x^5 in $(1+x)^7$ **c** x^6 in $(1+x)^8$

6 Find the first three terms, in ascending powers of x, in the expansion of $(3+kx)^5$.

7 Find the first three terms, in descending powers of x, in the expansion of $\left(3x-\frac{3}{x}\right)^6$.

8 **a** Simplify $(1+t)^3-(1-t)^3$.

b Show that $x^3-y^3=(x-y)\left(x^2+xy+y^2\right)$.

c Substitute $x=1+t$ and $y=1-t$ in the result in part **b** and show that your answer is the same as that for part **a**.

9 Find the coefficients of x^3 and x^4 for each of the following:

a $(1+x)(1-x)^6$ **b** $(1-x)(1+x)^6$

10 Write down the first four terms, in ascending powers of x, of the following binomial expressions:

a $(1-2x)^6$ **b** $(2-3x)^6$ **c** $(3-4x)^6$

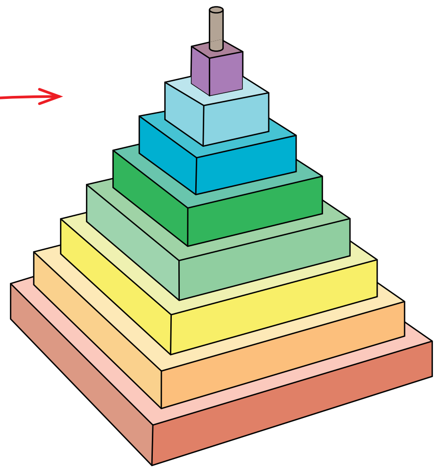
11 Find the first four terms, in descending powers of x, of the following binomial expressions:

a $\left(x^2+\frac{1}{x}\right)^5$ **b** $\left(x^2-\frac{1}{x}\right)^5$ **c** $\left(x^3+\frac{1}{x}\right)^5$ **d** $\left(x^3-\frac{1}{x}\right)^5$

12 The first three terms in the expansion of $(2-ax)^n$ in ascending powers of x are 32, –240 and 720. Find the values of a and n.

Arithmetic progressions

The smallest square shape in this toy has sides 1 cm long, and the lengths of the sides increase in steps of 1 cm.



Any ordered set of numbers, like the areas of the squares in this toy, form a sequence. In mathematics, we are particularly interested in sequences with a well-defined pattern, often in the form of an algebraic formula linking the terms. The area of the squares in the toy, in cm^2, are $1^2, 2^2, 3^2, 4^2, \ldots$ or $1, 4, 9, 16\ldots$.

A sequence in which the terms increase by the addition of a fixed amount (or decrease by the subtraction of a fixed amount) is described as an **arithmetic sequence** or **arithmetic progression (A. P.)**. The increase from one term to the next is called the **common difference**.

Thus the sequence $8 \quad 11 \quad 14 \quad 17\ldots$ is arithmetic with

$+3 \quad +3 \quad +3$

common difference 3. This sequence can be written algebraically as

$$u_k = 5 + 3k \text{ for } k = 1, 2, 3, \ldots$$

When $k = 1, u_1 = 5 + 3 = 8$

$k = 2, u_2 = 5 + 6 = 11$

$k = 3, u_3 = 5 + 9 = 14$ and so on.

This version has the advantage that the right-hand side begins with the first term of the sequence.

(You can also write this as $u_k = 8 + 3(k - 1)$ for $k = 1, 2, 3, \ldots$.)

As successive terms of an arithmetic progression increase (or decrease) by a fixed amount called the common difference, d, you can define each term in the sequence in relation to the previous term:

$$u_{k+1} = u_k + d.$$

When the terms of an arithmetic progression are added together, the sum is called an **arithmetic series**.

Notation

The following conventions are used in this book to describe arithmetic progressions and sequences:

- first term, $u_1 = a$
- number of terms $= n$
- last term, $u_n = l$
- common difference $= d$
- the general term, u_k, is that in position k (i.e. the kth term).

Thus in the arithmetic progression 7, 9, 11, 13, 15, 17, 19

$$a = 7, l = 19, d = 2 \text{ and } n = 7.$$

The terms are formed as follows:

$$\begin{aligned}
u_1 &= a & &= 7 \\
u_2 &= a + d & &= 7 + 2 &&= 9 \\
u_3 &= a + 2d & &= 7 + 2 \times 2 &&= 11 \\
u_4 &= a + 3d & &= 7 + 3 \times 2 &&= 13 \\
u_5 &= a + 4d & &= 7 + 4 \times 2 &&= 15 \\
u_6 &= a + 5d & &= 7 + 5 \times 2 &&= 17 \\
u_7 &= a + 6d & &= 7 + 6 \times 2 &&= 19
\end{aligned}$$

The 7th term is the 1st term (7) plus six times the common difference (2).

This shows that any term is given by the first term plus a number of differences. The number of differences is, in each case, one less than the number of the term. You can express this mathematically as

$$u_k = a + (k - 1)d.$$

For the last term, this becomes

$$l = a + (n - 1)d.$$

These are both general formulae so apply to any arithmetic progression.

Worked example

Find the 19th term in the arithmetic progression 20, 16, 12, …

Solution

In this case $a = 20$ and $d = -4$.

Using $u_k = a + (k - 1)d$, you obtain

$$\begin{aligned}
u_{19} &= 20 + (19 - 1) \times (-4) \\
&= 20 - 72 \\
&= -52.
\end{aligned}$$

The 19th term is -52.

Worked example

How many terms are there in the sequence 12, 16, 20, …, 556?

Solution

This is an arithmetic sequence with first term $a = 12$, last term $l = 556$ and common difference $d = 4$.

Using the result $l = a + (n-1)d$, you have

$$\begin{aligned} 556 &= 12 + 4(n-1) \\ \Rightarrow \quad 4n &= 556 - 12 + 4 \\ \Rightarrow \quad n &= 137 \end{aligned}$$

There are 137 terms.

Note

The relationship $l = a + (n-1)d$ may be rearranged to give

$$n = \frac{l-a}{d} + 1$$

This gives the number of terms in an A.P. directly if you know the first term, the last term and the common difference.

The sum of the terms of an arithmetic progression

When Carl Friederich Gauss (1777–1855) was at school he was always quick to answer mathematics questions. One day his teacher, hoping for half an hour of peace and quiet, told his class to add up all the whole numbers from 1 to 100. Almost at once the 10-year-old Gauss announced that he had done it and that the answer was 5050.

Gauss had not of course added the terms one by one. Instead he wrote the series down twice, once in the given order and once backwards, and added the two together:

$$\begin{aligned} S &= 1 + 2 + 3 + \ldots + 98 + 99 + 100 \\ S &= 100 + 99 + 98 + \ldots + 3 + 2 + 1. \end{aligned}$$

Adding, $\quad 2S = 101 + 101 + 101 + \ldots + 101 + 101 + 101.$

Since there are 100 terms in the series,

$$\begin{aligned} 2S &= 101 \times 100 \\ S &= 5050. \end{aligned}$$

The numbers 1, 2, 3, … , 100 form an arithmetic sequence with common difference 1. Gauss' method can be used for finding the sum of any arithmetic series.

It is common to use the letter S to denote the sum of a series. When there is any doubt as to the number of terms that are being summed, this is indicated by a subscript: S_5 indicates five terms, S_n indicates n terms.

Worked example

Find the value of $6 + 4 + 2 + \ldots + (-32)$.

Solution

This is an arithmetic progression, with common difference -2. The number of terms, n, can be calculated using

$$n = \frac{l-a}{d} + 1$$

$$n = \frac{-32-6}{-2} + 1$$
$$= 20$$

The sum S of the progression is then found as follows:

$$\begin{array}{rcccccc} S = & 6 + & 4 & + \ldots - & 30 & - & 32 \\ S = & -32 + & (-30) & - \ldots + & 4 & + & 6 \\ \hline 2S = & -26 + & (-26) & + \ldots + & (-26) & + & (-26). \end{array}$$

Since there are 20 terms, this gives $2S = -26 \times 20$, so $S = -26 \times 10 = -260$.

Generalising this method by writing the series in the conventional notation gives:

$$\begin{array}{rcccccccc} S_n = & [a] & + & [a+d] & + \ldots + & [a+(n-2)d] & + & [a+(n-1)d] \\ S_n = & [a+(n-1)d] & + & [a+(n-2)d] & + \ldots + & [a+d] & + & [a] \\ \hline 2S_n = & [2a+(n-1)d] & + & [2a+(n-1)d] & + \ldots + & [2a+(n-1)d] & + & [2a+(n-1)d] \end{array}$$

Since there are n terms, it follows that

$$S_n = \tfrac{1}{2}n[2a + (n-1)d].$$

This result can also be written as

$$S_n = \tfrac{1}{2}n(a + l).$$

Worked example

Find the sum of the first 100 terms of the progression

$$3\tfrac{1}{3},\ 3\tfrac{2}{3}, 4,\ \ldots$$

Solution

In this arithmetic progression

$$a = 3\tfrac{1}{3},\ d = \tfrac{1}{3} \text{ and } n = 100.$$

Using
$S_n = \frac{1}{2}n[2a + (n-1)d]$

$$S_n = \tfrac{1}{2} \times 100\left(6\tfrac{2}{3} + 99 \times \tfrac{1}{3}\right)$$
$$= 1983\tfrac{1}{3}.$$

Worked example

Tatjana starts a part-time job on a salary of \$10000 per year, and this increases by \$500 each year. Assuming that, apart from the annual increment, Tatjana's salary does not increase, find

a her salary in the 5th year

b the length of time she has been working to receive total earnings of \$122500.

Solution

Tatjana's annual salaries (in dollars) form the arithmetic sequence

$$10000, 10500, 11000, \ldots$$

with first term $a = 10000$, and common difference $d = 500$.

a Her salary in the 5th year is calculated using:

$$\begin{aligned} u_k &= a + (k-1)d \\ \Rightarrow \quad u_5 &= 10000 + (5-1) \times 500 \\ &= 12000. \end{aligned}$$

b The number of years that have elapsed when her total earnings are \$122500 is given by:

$$S = \frac{1}{2}n[2a + (n-1)d]$$

where $S = 122500$, $a = 10000$ and $d = 500$.

This gives $\quad 122\,500 = \frac{1}{2}n[2 \times 10000 + 500(n-1)]$.

This simplifies to the quadratic equation:

$$n^2 + 39n - 490 = 0.$$

Factorising,

$$(n-10)(n+49) = 0$$
$$\Rightarrow \quad n = 10 \text{ or } n = -49.$$

The root $n = -49$ is irrelevant, so the answer is $n = 10$.

Tatjana has earned a total of \$122 500 after 10 years.

Exercise 12.2

1 Are the following sequences arithmetic?
If so, state the common difference and the seventh term.

a 28, 30, 32, 34, …
b 1, 1, 2, 3, 5, 8, …
c 3, 9, 27, 81, …
d 5, 9, 13, 17, …
e 12, 8, 4, 0, …

2 The first term of an arithmetic sequence is –7 and the common difference is 4.

a Find the eighth term of the sequence.
b The last term of the sequence is 65. How many terms are there in the sequence?

3 The first term of an arithmetic sequence is 10, the seventh term is 46 and the last term is 100.
   a Find the common difference.
   b Find how many terms there are in the sequence.

4 There are 30 terms in an arithmetic progression.
   The first term is −4 and the last term is 141.
   a Find the common difference.
   b Find the sum of the terms in the progression.

5 The kth term of an arithmetic progression is given by

   $u_k = 12 + 4k$.

   a Write down the first three terms of the progression.
   b Calculate the sum of the first 12 terms of this progression.

6 Below is an arithmetic progression.

   $118 + 112 + \ldots + 34$

   a How many terms are there in the progression?
   b What is the sum of the terms in the progression?

7 The fifth term of an arithmetic progression is 32 and the tenth term is 62.
   a Find the first term and the common difference.
   b The sum of all the terms in this progression is 350.
      How many terms are there?

8 The ninth term of an arithmetic progression is three times the second term, and the first term is 5. The sequence has 20 terms.
   a Find the common difference.
   b Find the sum of all the terms in the progression.

9 a Find the sum of all the odd numbers between 150 and 250.
   b Find the sum of all the even numbers from 150 to 250 inclusive.
   c Find the sum of the terms of the arithmetic sequence with first term 150, common difference 1 and 101 terms.
   d Explain the relationship between your answers to parts **a**, **b** and **c**.

10 The first term of an arithmetic progression is 9000 and the tenth term is 3600.
   a Find the sum of the first 20 terms of the progression.
   b After how many terms does the sum of the progression become negative?

11 An arithmetic progression has first term −2 and common difference 7.
   a Write down a formula for the nth term of the progression. Which term of the progression equals 110?
   b Write down a formula for the sum of the first n terms of the progression. How many terms of the progression are required to give a sum equal to 2050?

12 Luca's starting salary in a company is \$45 000. During the time he stays with the company, it increases by \$1800 each year.
   a What is his salary in his sixth year?
   b How many years has Luca been working for the company when his total earnings for all his years there are \$531 000?

Exercise 12.2 (cont)

13 A jogger is training for a 5 km charity run. He starts with a run of 400 m, then increases the distance he runs in training by 100 m each day.

a How many days does it take the jogger to reach a distance of 5 km in training?

b What total distance will he have run in training by then?

14 A piece of string 20 m long is to be cut into pieces such that the lengths of the pieces form an arithmetic sequence.

a If the lengths of the longest and shortest pieces are 2 m and 50 cm respectively, how many pieces are there?

b If the length of the longest piece is 185 cm, how long is the shortest piece?

15 The ninth term of an arithmetic progression is 95 and the sum of the first four terms is –10.

a Find the first term of the progression and the common difference.

The nth term of the progression is 200.

b Find the value of n.

16 Following knee surgery, Adankwo has to do squats as part of her physiotherapy programme. Each day she must do 4 more squats than the day before. On the eighth day she did 31 squats. Calculate how many squats Adankwo completed:

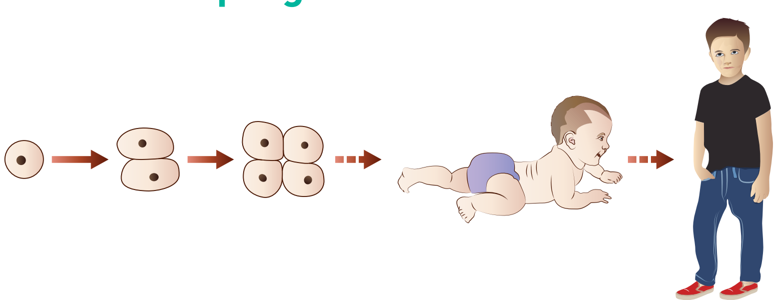
a on the first day

b in total by the end of the seventh day

c in total by the end of the nth day

d in total from the end of the nth day to the end of the $(2n)$th day. Simplify your answer.

Geometric progressions



A human being begins life as one cell, which divides into two, then four…

The terms of a **geometric sequence** or **geometric progression (G.P.)** are formed by multiplying one term by a fixed number, the **common ratio**, to obtain the next. This can be written inductively as:

$$u_{k+1} = ru_k \text{ with first term } u_1.$$

The sum of the terms of a geometric sequence is called a **geometric series**.

Notation

The following conventions are used in this book to describe geometric progressions:

- first term $u_1 = a$
- common ratio $= r$
- number of terms $= n$
- the general term, u_k, is that in position k (i.e. the kth term).

Thus in the geometric progression 2, 6, 18, 54, 162

$a = 2$, $r = 3$ and $n = 5$.

The terms of this sequence are formed as follows:

$$\begin{aligned} u_1 &= a &&= 2 \\ u_2 &= a \times r &&= 2 \times 3 &&= 6 \\ u_3 &= a \times r^2 &&= 2 \times 3^2 &&= 18 \\ u_4 &= a \times r^3 &&= 2 \times 3^3 &&= 54 \\ u_5 &= a \times r^4 &&= 2 \times 3^4 &&= 162. \end{aligned}$$

This shows that in each case the power of r is one less than the number of the term: $u_5 = ar^4$ and 4 is one less than 5. This can be written deductively as

$$u_k = ar^{k-1}.$$

For the last term this becomes

$$u_n = ar^{n-1}.$$

These are both general formulae so apply to any geometric sequence.

Given two consecutive terms of a geometric sequence, you can always find the common ratio by dividing the later term by the earlier term. For example, the geometric sequence … 7, 9, … has common ratio $r = \frac{9}{7}$.

Worked example

Find the ninth term in the geometric sequence 7, 28, 112, 448, …

Solution

In the sequence, the first term $a = 7$ and the common ratio $r = 4$.
Using $u_k = ar^{k-1}$

$$\begin{aligned} u_9 &= 7 \times 4^8 \\ &= 458\,752. \end{aligned}$$

Worked example

How many terms are there in the geometric sequence 3, 15, 75, … , 29 296 875?

Solution

Since it is a geometric sequence and the first two terms are 3 and 15, you can immediately write down

First term: $a = 3$
Common ratio: $r = 5$

The third term allows you to check you are right.

$$15 \times 5 = 75 \quad ✓$$

The nth term of a geometric sequence is ar^{n-1}, so in this case

$$3 \times 5^{n-1} = 29\,296\,875$$

Dividing by 3 gives

$$5^{n-1} = 9\,765\,625$$

Alternatively, you could find the solution by using trial and improvement and a calculator, since you know n must be a whole number.

Using logarithms, $\lg(5)^{(n-1)} = \lg 9765625$

$$\Rightarrow \quad (n-1)\lg 5 = \lg 9765625$$

$$\Rightarrow \quad n - 1 = \frac{\lg 9765625}{\lg 5} = 10$$

So $n = 11$ and there are 11 terms in the sequence.

Discussion point

How would you use a spreadsheet to solve the equation $5^{n-1} = 9\,765\,625$?

The sum of the terms of a geometric progression

This chapter began with the story of Sissa ben Dahir's reward for inventing chess. In the discussion point on page 184, you were asked how much grain would have been placed on the last square. This situation also gives rise to another question:

How many grains of wheat was the inventor actually asking for?

The answer is the geometric series with 64 terms and common ratio 2:

$$1 + 2 + 4 + 8 + \ldots + 2^{63}.$$

This can be summed as follows.

Call the series S:

$$S = 1 + 2 + 4 + 8 + \ldots + 2^{63}. \qquad ①$$

Now multiply it by the common ratio, 2:

$$2S = 2 + 4 + 8 + 16 + \ldots + 2^{64}. \qquad ②$$

Then subtract ① from ②:

$$
\begin{array}{lrcl}
② & 2S & = & \phantom{1 + {}} 2 + 4 + 8 + 16 + \ldots + 2^{63} + 2^{64} \\
① & S & = & 1 + 2 + 4 + 8 \qquad\quad + \ldots + 2^{63} \\
\text{Subtracting:} & S & = & -1 + 0 + 0 + 0 + \ldots + 2^{64}.
\end{array}
$$

The total number of wheat grains requested was therefore $2^{64} - 1$ (which is about 1.85×10^{19}).

Discussion point

How many tonnes of wheat is this, and how many tonnes would you expect there to be in China at any time?

(One hundred grains of wheat weigh about 2 grams. The world annual production of all cereals is about 1.8×10^9 tonnes.)

Note

The method shown above can be used to sum any geometric progression.

Worked example

Find the sum of $0.04 + 0.2 + 1 + \ldots + 78\,125$.

Solution

This is a geometric progression with common ratio 5.

Let $\quad S = 0.04 + 0.2 + 1 + \ldots + 78\,125. \qquad ①$

Multiplying by the common ratio, 5, gives:

$$5S = 0.2 + 1 + 5 + \ldots + 78\,125 + 390\,625. \qquad ②$$

Subtracting ① from ②:

$$
\begin{array}{rcl}
5S & = & \phantom{0.04 +{}} 0.2 + 1 + 5 + \ldots + 78\,125 + 390\,625 \\
S & = & 0.04 + 0.2 + 1 + 5 + \ldots + 78\,125 \\
\hline
4S & = & -0.04 + 0 + \ldots + 0 + 390\,625
\end{array}
$$

This gives $\quad 4S = 390\,624.96$

$\Rightarrow \quad S = 97\,656.24$

The same method can be applied to the general geometric progression to give a formula for its value:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}. \qquad ①$$

Multiplying by the common ratio, r, gives:

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n. \qquad ②$$

Subtracting ① from ②, as before, gives:

$$rS_n - S_n = ar^n - a$$

$$S_n(r-1) = a(r^n - 1)$$

so $$S_n = \frac{a(r^n-1)}{(r-1)}.$$

This can also be written as:

$$S_n = \frac{a(1-r^n)}{(1-r)}.$$

Worked example

a Solve the simultaneous equations $ar^2 = 6$
$ar^4 = 54$

b Find in each case the sum of the first five terms of the geometric progression.

Solution

a $ar^2 = 6 \Rightarrow a = \dfrac{6}{r^2}$

Substituting into $ar^4 = 54$ gives $\dfrac{6}{r^2} \times r^4 = 54$

$\Rightarrow r^2 = 9$

$\Rightarrow \quad r = \pm 3$

Substituting in $ar^2 = 6$ gives $a = \dfrac{2}{3}$ in both cases.

b When $r = +3$ terms are $\dfrac{2}{3}$, 2, 6, 18, 54 Sum $= 80\dfrac{2}{3}$

When $r = -3$ terms are $\dfrac{2}{3}$, -2, 6, -18, 54 Sum $= 40\dfrac{2}{3}$

Infinite geometric progressions

The progression $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is geometric, with common ratio $\frac{1}{2}$.

Summing the terms one by one gives $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{15}{16} \ldots$

Clearly the more terms you add, the nearer the sum gets to 2. In the limit, as the number of terms tends to infinity, the sum tends to 2.

$$\text{As } n \to \infty, S_n \to 2.$$

This is an example of a **convergent** series. The sum to infinity is a finite number.

You can see this by substituting $a = 1$ and $r = \frac{1}{2}$ in the formula for the sum of the series:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

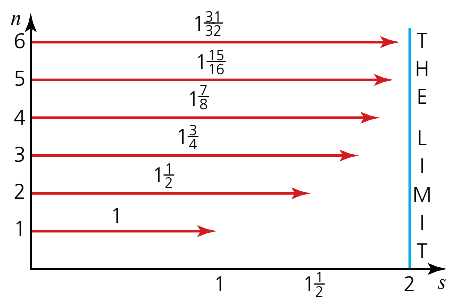
giving

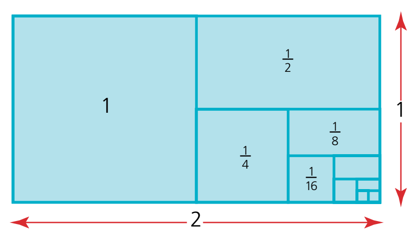
$$S_n = \frac{1 \times \left(1 - \left(\frac{1}{2}\right)^n\right)}{\left(1 - \frac{1}{2}\right)}$$

$$= 2 \times \left(1 - \left(\tfrac{1}{2}\right)^n\right).$$

The larger the number of terms, n, the smaller $\left(\frac{1}{2}\right)^n$ becomes and so the nearer S_n is to the limiting value of 2, as shown on the left. Notice that $\left(\frac{1}{2}\right)^n$ can never be negative, however large n becomes; so S_n can never exceed 2.

Notice how representing all of the terms of the geometric progression as in these diagrams shows that the sum can never exceed 2.





In the general geometric series $a + ar + ar^2 + \ldots$ the terms become progressively smaller in size if the common ratio r is between -1 and 1. In such cases, the geometric series is **convergent**.

If, on the other hand, the value of r is greater than 1 (or less than -1), the terms in the series become larger and larger in size and so the series is described as **divergent**.

A series corresponding to a value of r of exactly $+1$ consists of the first term a repeated over and over again. A sequence corresponding to a value of r of exactly -1 oscillates between $+a$ and $-a$. Neither of these is convergent.

It only makes sense to talk about the sum of an infinite series if it is convergent. Otherwise the sum is undefined.

The condition for a geometric series to converge, $-1 < r < 1$, ensures that as $n \to \infty$, $r^n \to 0$, and so the formula for the sum of a geometric series:

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

can be rewritten for an infinite series as:

$$S_\infty = \frac{a}{1-r}.$$

Worked example

Find the sum of the terms of the infinite progression 0.4, 0.04, 0.004, …

Solution

This is a geometric progression with $a = 0.4$ and $r = 0.1$.
Its sum is given by:

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{0.4}{1-0.1} \\ &= \frac{0.4}{0.9} \\ &= \frac{4}{9}. \end{aligned}$$

Note

You may have noticed that the sum of the series $0.4 + 0.04 + 0.004 + \ldots$ is $0.\dot{4}$, and that this recurring decimal is the same as $\frac{4}{9}$.

Worked example

The first three terms of an infinite geometric progression are 75, 45 and 27.

a Write down the common ratio.

b Find the sum of the terms of the progression.

Solution

a The common ratio is $\frac{45}{75} = \frac{3}{5}$.

using $S_\infty = \frac{a}{1-r}$ → **b** $S_\infty = \dfrac{75}{1-\frac{3}{5}} = 187.5$

Discussion point

A paradox

Consider the following arguments.

i

$$S = 1 - 2 + 4 - 8 + 16 - 32 + 64 - \ldots$$
$$\Rightarrow \quad S = 1 - 2(1 - 2 + 4 - 8 + 16 - 32 + \ldots)$$
$$= 1 - 2S$$
$$\Rightarrow \quad 3S = 1$$
$$\Rightarrow \quad S = \tfrac{1}{3}.$$

ii

$$S = 1 + (-2 + 4) + (-8 + 16) + (-32 + 64) + \ldots$$
$$\Rightarrow \quad S = 1 + 2 + 8 + 32 + \ldots$$

So S diverges towards $+\infty$.

iii

$$S = (1 - 2) + (4 - 8) + (16 - 32) + \ldots$$
$$\Rightarrow \quad S = -1 - 4 - 8 - 16 \ldots$$

So S diverges towards $-\infty$.

What is the sum of the series: $\frac{1}{3}$, $+\infty$, $-\infty$, or something else?

Exercise 12.3

1 Are the following sequences geometric?
If so, state the common ratio and calculate the seventh term.
 a $3, 6, 12, 24, \ldots$
 b $3, 6, 9, 12, \ldots$
 c $10, -10, 10, -10, 10, \ldots$
 d $1, 1, 1, 1, 1, 1, \ldots$
 e $15, 10, 5, 0, -5, \ldots$
 f $10,\ 5,\ \frac{5}{2},\ \frac{5}{4},\ \frac{5}{8},\ \ldots$
 g $2, 2.2, 2.22, 2.222, \ldots$

2 A geometric sequence has first term 5 and common ratio 2.
The sequence has seven terms.
 a Find the last term.
 b Find the sum of the terms in the sequence.

3 The first term of a geometric sequence of positive terms is 3 and the fifth term is 768.
 a Find the common ratio of the sequence.
 b Find the eighth term of the sequence.

4 A geometric sequence has first term $\frac{1}{16}$ and common ratio 4.
 a Find the fifth term.
 b Which is the first term of the sequence that exceeds 1000?

5 a Find how many terms there are in the following geometric sequence:
$7, 14, \ldots, 3584$.
 b Find the sum of the terms in this sequence.

Exercise 12.3 (cont)

6 a Find how many terms there are in the following geometric sequence: 100, 50, …, 0.390 625.
  b Find the sum of the terms in this sequence.

7 The fourth term of a geometric progression is 36 and the eighth term is 576. All the terms are positive.
  a Find the common ratio.
  b Find the first term.
  c Find the sum of the first ten terms.

8 The first three terms of an infinite geometric progression are 8, 4 and 2.
  a State the common ratio of this progression.
  b Calculate the sum to infinity of its terms.

9 The first three terms of an infinite geometric progression are 0.8, 0.08 and 0.008.
  a Write down the common ratio for this progression.
  b Find, as a fraction, the sum to infinity of the terms of this progression.
  c Find the sum to infinity of the geometric progression

$$0.8 - 0.08 + 0.008 - \ldots$$

  and hence show that $\frac{8}{11} = 0.\dot{7}\dot{2}$.

10 The first three terms of a geometric sequence are 100, 70 and 49.
  a Write down the common ratio of the sequence.
  b Which is the position of the first term in the sequence that has a value less than 1?
  c Find the sum to infinity of the terms of this sequence.
  d After how many terms is the sum of the sequence greater than 99% of the sum to infinity?

11 A geometric progression has first term 10 and its sum to infinity is 15.
  a Find the common ratio.
  b Find the sum to infinity if the first term is excluded from the progression.

12 The first four terms in an infinite geometric series are 216, 72, 24, 8.
  a What is the common ratio r?
  b Write down an expression for the nth term of the series.
  c Find the sum of the first n terms of the series.
  d Find the sum to infinity.
  e How many terms are needed for the sum to be greater than 323.999?

13 A tank is filled with 10 litres of water. Half the water is removed and replaced with anti-freeze and then thoroughly mixed. Half this mixture is then removed and replaced with anti-freeze. The process continues.
  a Find the first five terms in the sequence of amounts of water in the tank at each stage.
  b Find the first five terms in the sequence of amounts of anti-freeze in the tank at each stage.
  c Is either of these sequences geometric? Explain.

14 A pendulum is set swinging. Its first oscillation is through an angle of 20°, and each following oscillation is through 95% of the angle of the one before it.

a After how many swings is the angle through which it swings less than 1°?

b What is the total angle it has swung through at the end of its tenth oscillation?

15 A ball is thrown vertically upwards from the ground. It rises to a height of 15 m and then falls and bounces. After each bounce it rises vertically to $\frac{5}{8}$ of the height from which it fell.

a Find the height to which the ball bounces after the nth impact with the ground.

b Find the total distance travelled by the ball from the first throw to the tenth impact with the ground.

16 The first three terms of an arithmetic sequence, a, $a + d$ and $a + 2d$, are the same as the first three terms, a, ar, ar^2, of a geometric sequence ($a \neq 0$). Show that this is only possible if $r = 1$ and $d = 0$.

17 a, b and c are three consecutive terms in a sequence.

a Prove that if the sequence is an arithmetic progression then $a + c = 2b$.

b Prove that if the sequence is a geometric progression then $ac = b^2$.

18 a Solve the simultaneous equations $ar = 12$, $ar^5 = 3072$ (there are two possible answers).

b In each case, find the sum of the first ten terms of the geometric progression with first term a and common ratio r.

Past-paper questions

1 Find the values of the positive constants p and q such that, in the binomial expansion of $(p + qx)^{10}$, the coefficient of x^5 is 252 and the coefficient of x^3 is 6 times the coefficient of x^2. [8]

Cambridge O Level Additional Mathematics 4037
Paper 11 Q9 June 2012
Cambridge IGCSE Additional Mathematics 0606
Paper 11 Q9 June 2012

2 (i) Find the coefficient of x^3 in the expansion of $\left(1 - \frac{x}{2}\right)^{12}$. [2]

(ii) Find the coefficient of x^3 in the expansion of $(1 + 4x)\left(1 - \frac{x}{2}\right)^{12}$. [3]

Cambridge O Level Additional Mathematics 4037
Paper 21 Q2 June 2011
Cambridge IGCSE Additional Mathematics 0606
Paper 21 Q2 June 2011

3 (i) Find the first four terms in the expansion of $(2+x)^6$ in ascending powers of x. [3]

(ii) Hence find the coefficient of x^3 in the expansion of $(1+3x)(1-x)(2+x)^6$. [4]

Cambridge O Level Additional Mathematics 4037
Paper 21 Q7 June 2013
Cambridge IGCSE Additional Mathematics 0606
Paper 21 Q7 June 2013

Learning outcomes

Now you should be able to:

★ use the binomial theorem for expansion of $(a+b)^n$ for positive integer n

★ use the general term $\binom{n}{r}a^{n-r}b^r$, $0 \leqslant r \leqslant n$ (knowledge of the greatest term and properties of the coefficients is not required)

★ recognise arithmetic and geometric progressions

★ use the formulae for the nth term and for the sum of the first n terms to solve problems involving arithmetic or geometric progressions

★ use the condition for the convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression.

Key points

✔ An expression of the form $(ax+b)^n$ where n is an integer is called a **binomial expression**.

✔ **Binomial coefficients**, denoted by $\binom{n}{r}$ or ${}^nC_r$ can be found:

- using Pascal's triangle
- using tables
- using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

✔ The binomial expansion of $(1+x)^n$ can also be written as

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots nx^{n-1} + x^n$$

✔ A sequence is an ordered set of numbers, $u_1, u_2, u_3, \ldots, u_k, \ldots u_n$, where u_k is the general term.

✔ In an arithmetic sequence, $u_{k+1} = u_k + d$ where d is a fixed number called the **common difference**.

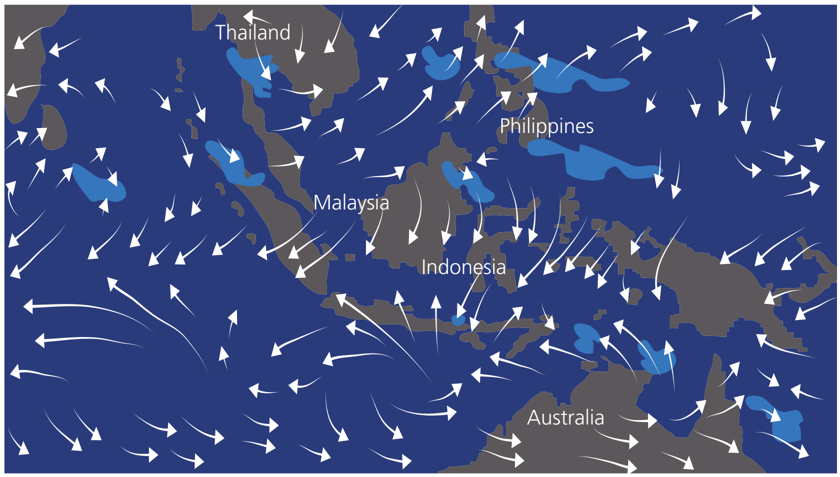
✔ In a geometric sequence, $u_{k+1} = ru_k$ where r is a fixed number called the **common ratio**.

- ✔ For an arithmetic progression with first term a, common difference d and n terms
  - the kth term $u_k = a + (k-1)d$
  - the last term $l = a + (n-1)d$
  - the sum of the terms $= \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]$
- ✔ For a geometric progression with first term a, common ratio r and n terms
  - the kth term $a_k = ar^{k-1}$
  - the last term $a_n = ar^{n-1}$
  - the sum of the terms $= \frac{a(r^n - 1)}{(r-1)}$ for $r > 1$ or $\frac{a(1-r^n)}{(1-r)}$ for $r < 1$
- ✔ For an infinite geometric series to converge, $-1 < r < 1$. In this case the sum of all terms is given by $\frac{a}{1-r}$.

13 Vectors in two dimensions

Thought is an idea in transit.

Pythagoras (C. 569 BC – 475 BC)



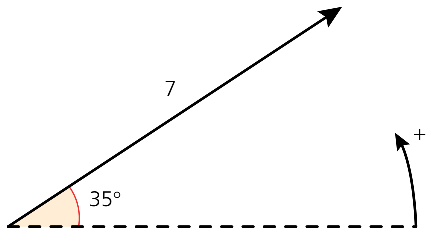
Discussion point

The lines on this weather map are examples of vectors. What do they tell you about the wind at any place?

Terminology and notation

The focus of this chapter is vectors in two dimensions. A **vector** is a quantity that has both magnitude and direction, for example, a **velocity** of $60\,\mathrm{km\,h^{-1}}$ in a southerly direction. In contrast, a **scalar** quantity has a magnitude but no direction attached to it, for example, a **speed** of $60\,\mathrm{km\,h^{-1}}$ where no direction is given.

A vector in two dimensions can be represented by drawing a straight line with an arrowhead to define the direction. The direction is often given as the angle the vector makes with the positive x-axis, with anticlockwise taken as positive.

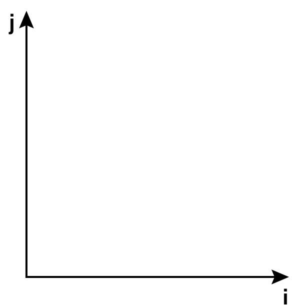


You need to be able to recognise and use a number of different conventions for writing vectors, as outlined below.

A vector joining the points A and B can be written as $\overrightarrow{AB}$ or just given a single letter, e.g. **r**.

In print, vectors are usually in bold type, for example, **a**, but when you are writing them by hand it is usual to underline them, as in $\underline{a}$, or put an arrow above as in $\overrightarrow{OA}$.

A **unit vector** is any vector of length 1. **i** and **j** are principal unit vectors of length 1 in the positive x and y directions respectively.



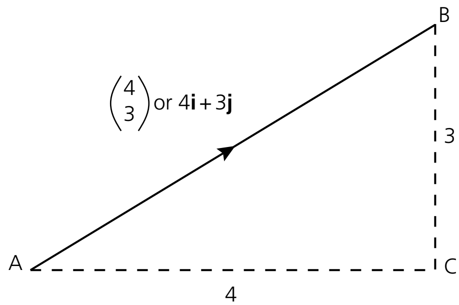
Position vectors start at the origin and are the vector equivalent of coordinates. For example, the vector joining the origin to the point (2, 5) is written as

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

This is a **column vector.**

This vector is in **component form:** the two components represent distances in the x and y directions.

or $2\mathbf{i} + 5\mathbf{j}$.



In the diagram, the vector $\overrightarrow{AB}$ is 4 units in the x direction and 3 units in the y direction and can be written either as the column vector $\begin{pmatrix}4\\3\end{pmatrix}$ or as $4\mathbf{i} + 3\mathbf{j}$.

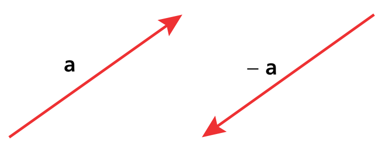
This can be found by using Pythagoras' theorem or, in this example, by recognising it as a 3, 4, 5 triangle.

The **magnitude** or **modulus** of a vector is its length and is denoted by vertical lines on either side of the vector, for example, $|\mathbf{a}|$ or $|\mathbf{OA}|$.

In the diagram above, $|\overrightarrow{AC}| = 4$, $|\overrightarrow{BC}| = 3$ and $|\overrightarrow{AB}| = 5$.

All of the vectors introduced so far have had both their x and y components in the positive directions, but this is not always the case.

The **negative** of a vector **a** is the vector **–a**. **–a** is parallel to **a** but in the opposite direction.



Worked example

Sketch the following vectors and find their magnitude:

a $\begin{pmatrix}-1\\3\end{pmatrix}$ **b** $2\mathbf{i} - 5\mathbf{j}$ **c** $\begin{pmatrix}-3\\-4\end{pmatrix}$

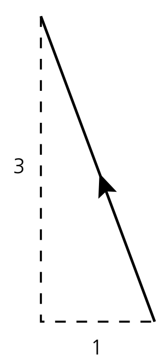
Solution

Using Pythagoras' theorem to find the magnitude of the vector

a $\left|\begin{pmatrix}-1\\3\end{pmatrix}\right| = \sqrt{(-1)^2 + 3^2}$

$= \sqrt{10}$

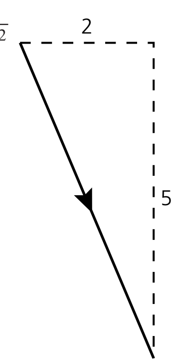
$= 3.16$ (3 s.f.)



b $|2\mathbf{i} - 5\mathbf{j}| = \sqrt{2^2 + (-5)^2}$

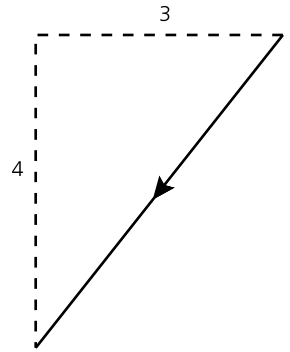
$= \sqrt{29}$

$= 5.39$ (3 s.f.)



c $\left|\begin{pmatrix}-3\\-4\end{pmatrix}\right| = \sqrt{((-3)^2 + (-4)^2)}$

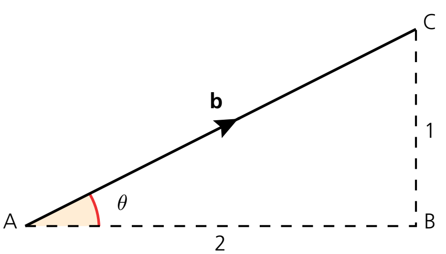
$= 5$



Worked example

Write the vector **b**

a as a column vector
b using the unit vectors **i** and **j**.



Solution

'distance to the right' first and 'distance up' below

a From the diagram, $\mathbf{b} = \begin{pmatrix}2\\1\end{pmatrix}$.

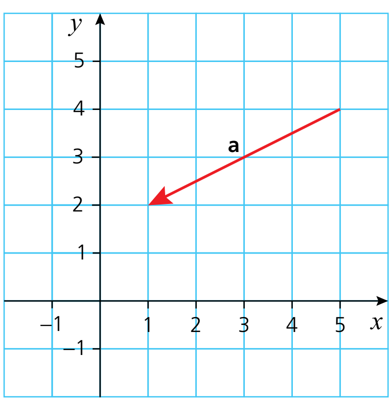
This is called component form.

b $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$

Exercise 13.1

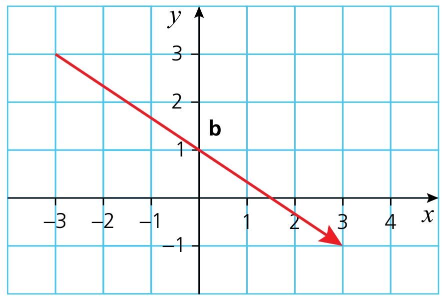
1 Express the following vectors in component form:

a

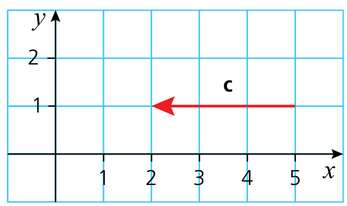


Exercise 13.1 (cont)

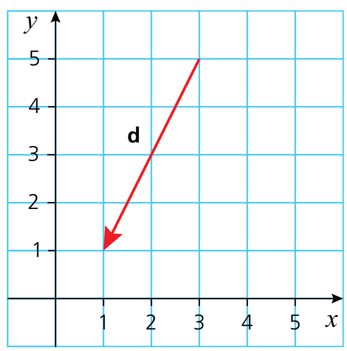
b



c



d



2 The coordinates of points P, Q, R and S are (–1, –2), (–2, 1), (1, 2) and (2, –1) respectively. The origin is the point O.

a Mark the points on a grid. Use equal scales on the two axes.

b Write as column vectors:

i $\overrightarrow{OR}$ **ii** $\overrightarrow{RO}$

c Write as column vectors:

i $\overrightarrow{PR}$ **ii** $\overrightarrow{QS}$

d Write down the lengths of the vectors:

i $\overrightarrow{PQ}$ **ii** $\overrightarrow{QR}$ **iii** $\overrightarrow{RS}$ **iv** $\overrightarrow{SP}$

e Describe the quadrilateral PQRS.

3 Draw diagrams to illustrate each of the following vectors:

a $2\mathbf{i}$ **b** $3\mathbf{j}$ **c** $2\mathbf{i} + 3\mathbf{j}$ **d** $2\mathbf{i} - 3\mathbf{j}$

4 For each of the following vectors

i draw a diagram **ii** find its magnitude.

a $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ **d** $\begin{pmatrix} 5 \\ -7 \end{pmatrix}$

5 A(–1, 4), B(2, 7) and C(5, 0) form the vertices of a triangle.
- **a** Draw the triangle on graph paper. Using your diagram, write the vectors representing the sides AB, BC and AC as column vectors.
- **b** Which is the longest side of the triangle?

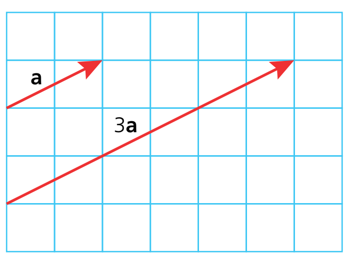
6 A, B, C and D have coordinates (–3, –4), (0, 2), (5, 6) and (2, 0) respectively.
- **a** Draw the quadrilateral ABCD on graph paper.
- **b** Write down the position vectors of the points A, B, C and D.
- **c** Write down the vectors $\overrightarrow{AB}$, $\overrightarrow{DC}$, $\overrightarrow{BC}$ and $\overrightarrow{AD}$.
- **d** Describe shape ABCD.

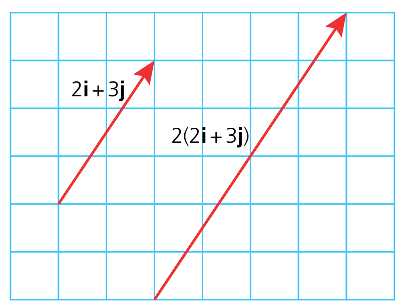
7 $\overrightarrow{AB} = \mathbf{j}$, $\overrightarrow{BC}$ is the vector $2\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{CD} = \mathbf{i}$.
- **a** Sketch the shape ABCD.
- **b** Write $\overrightarrow{AD}$ as a column vector.
- **c** Describe shape ABCD.

Multiplying a vector by a scalar

When a vector is multiplied by a scalar (i.e. a number) then its length is multiplied but its direction is unchanged.

For example, $3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a}$ gives a vector three times as long as **a** *in the same direction* and $2(2\mathbf{i} + \mathbf{j})$ gives a vector twice as long as $2\mathbf{i} + \mathbf{j}$ *in the same direction*.





Adding and subtracting vectors

To add vectors written in component form, simply add the *x*-components together and the *y*-components together as shown in the example below.

Worked example

Add the vectors $2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{i} - 2\mathbf{j}$

a using algebra **b** by graphing them.

Collecting like terms

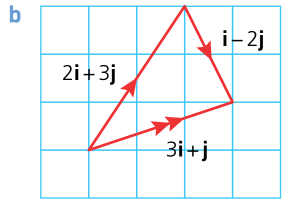
Solution

a $(2\mathbf{i} + 3\mathbf{j}) + (\mathbf{i} - 2\mathbf{j}) = (2 + 1)\mathbf{i} + (3 + (-2))\mathbf{j}$

$= 3\mathbf{i} + \mathbf{j}$

This is called the resultant vector.

b



The resultant vector is shown by using two arrows.

To subtract one vector from another, add the equivalent negative vector. So, in the same way that $5 - 2 = 5 + (-2) = 3$,

$$\begin{pmatrix}6\\5\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}6\\5\end{pmatrix} + \begin{pmatrix}-4\\-2\end{pmatrix}$$

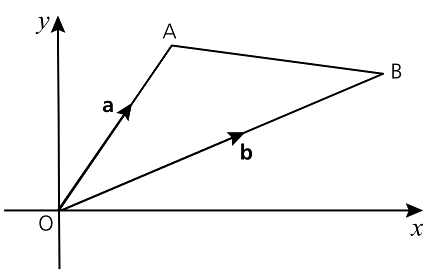
$$= \begin{pmatrix}6+(-4)\\5+(-2)\end{pmatrix}$$

$$= \begin{pmatrix}2\\3\end{pmatrix}$$

Alternatively, you can simply subtract the second component from the first component in each case, for example, $\begin{pmatrix}8\\-2\end{pmatrix} - \begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}8-(-1)\\(-2)-4\end{pmatrix} = \begin{pmatrix}9\\-6\end{pmatrix}$

A very important result involves subtracting vectors.

Look at this diagram:



The position vector of A is $\overrightarrow{OA} = \mathbf{a}$.

The position vector of B is $\overrightarrow{OB} = \mathbf{b}$.

What can you say about the vector $\overrightarrow{AB}$ joining point A to point B?

Vector addition gives $\mathbf{a} + \overrightarrow{AB} = \mathbf{b}$

So, $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Worked example

The point P has position vector $-5\mathbf{i} + 3\mathbf{j}$.

The point Q has position vector $7\mathbf{i} - 8\mathbf{j}$.

Find the vector $\overrightarrow{PQ}$.

Solution

$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$

$\overrightarrow{PQ} = 7\mathbf{i} - 8\mathbf{j} - (-5\mathbf{i} + 3\mathbf{j})$

$\overrightarrow{PQ} = 12\mathbf{i} - 11\mathbf{j}$

Zero and unit vectors

The **zero vector**, **0**, is the result of adding two vectors **a** and (**–a**). Since **a** + (**–a**) starts and finishes at the same place, **a** + (**–a**) = **0**.

Remember, in column form $\mathbf{i}$ is written as $\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{j}$ is written as $\begin{pmatrix}0\\1\end{pmatrix}$.

It can be written as $0\mathbf{i} + 0\mathbf{j}$ or as $\begin{pmatrix}0\\0\end{pmatrix}$.

To find the **unit vector** in the direction of any given vector, divide the given vector by its magnitude (length).

Worked example

Find unit vectors parallel to:

a $\begin{pmatrix}3\\4\end{pmatrix}$ **b** $2\mathbf{i} - 3\mathbf{j}$.

Solution

a $\left|\begin{pmatrix}3\\4\end{pmatrix}\right| = \sqrt{3^2 + 4^2}$

$= 5$

Remember to multiply each component of the vector by $\frac{1}{5}$.

The required unit vector is $\frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}0.6\\0.8\end{pmatrix}$

b $|2\mathbf{i} - 3\mathbf{j}| = \sqrt{2^2 + (-3)^2}$

$= \sqrt{13}$

This could also be written as $0.555\mathbf{i} - 1.11\mathbf{j}$ (3 s.f.) but it is often better to leave an answer in an exact form.

The required unit vector is $\frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{j})$.

Worked example

Find the unit vector in the direction of $2\mathbf{i} - 4\mathbf{j}$. Give your answer in simplest surd form.

Solution

$|2\mathbf{i} - 4\mathbf{j}| = \sqrt{2^2 + (-4)^2}$

$= \sqrt{20}$

$= 2\sqrt{5}$

The required unit vector is $\frac{1}{2\sqrt{5}}(2\mathbf{i} - 4\mathbf{j}) = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$.

Rationalising the denominator, this can also be written as $\frac{\sqrt{5}}{5}\mathbf{i} - \frac{2\sqrt{5}}{5}\mathbf{j}$.

Worked example

$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

Find: **a** $2\mathbf{a} + 3\mathbf{b}$ **b** $3\mathbf{a} - 2\mathbf{b}$

Solution

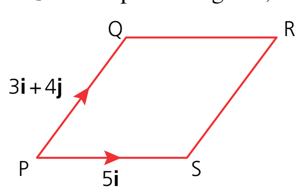
Notice that in $2\mathbf{a} + 3\mathbf{b}$, 2 and 3 are scalars multiplying the vectors **a** and **b**.

a $$2\mathbf{a} + 3\mathbf{b} = 2\begin{pmatrix} 3 \\ 4 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} + \begin{pmatrix} 3 \\ 21 \end{pmatrix} = \begin{pmatrix} 6+3 \\ 8+21 \end{pmatrix} = \begin{pmatrix} 9 \\ 29 \end{pmatrix}$$

b $$3\mathbf{a} - 2\mathbf{b} = 3\begin{pmatrix} 3 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} - \begin{pmatrix} 2 \\ 14 \end{pmatrix} = \begin{pmatrix} 9-2 \\ 12-14 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

Worked example

PQRS is a parallelogram, with $\overrightarrow{PQ} = 3\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{PS} = 5\mathbf{i}$.



a State the vectors that represent SR and QR.

b Find the lengths of the sides of the parallelogram, and hence identify the type of parallelogram.

Solution

a SR is parallel to PQ so is represented by the same vector, i.e. $3\mathbf{i} + 4\mathbf{j}$.

Similarly, QR is parallel to PS and so $\overrightarrow{QR} = 5\mathbf{i}$.

b $$PS = QR = |5\mathbf{i}| = 5 \text{ units}$$

$$PQ = SR = |3\mathbf{i} + 4\mathbf{j}| = \sqrt{3^2 + 4^2} = 5 \text{ units}$$

A parallelogram with all four sides equal is a rhombus.

There are many applications of vectors because many quantities have magnitude and direction. One of these is velocity and this is illustrated in the following example.

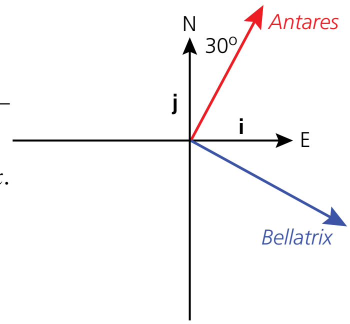
Worked example

Remember that bearings are measured clockwise from the north.

In this example, answers are given to 2 s. f. The unit vectors **i** and **j** are in the directions east and north.

The *Antares* is a sailing boat. It is travelling at a speed of 3 km h^{-1} on a bearing of 030°.

a Find the components of this boat's velocity in the directions east and north.



The *Bellatrix* is another boat. It has velocity 2**i** – 1.5**j** in km h^{-1}.

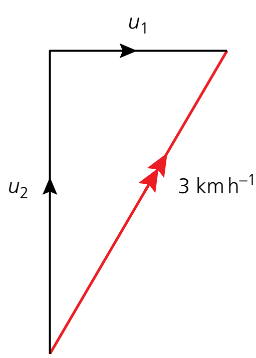
b Find the speed and direction of the *Bellatrix*.

Both boats start at the same place.

c How far apart are they after 2 hours?

Solution

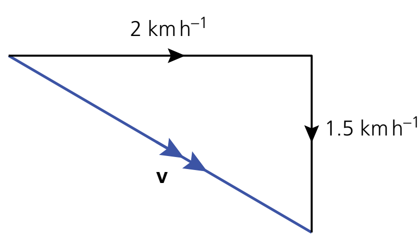
a The components of the velocity of the Antares are shown on this right-angled triangle.



They are	$u_1 = 3\sin 30° = 1.5$	in the direction east
and	$u_1 = 3\cos 30° = 2.6$	in the direction north

So the velocity of the *Antares* is 1.5**i** + 2.6**j** km h^{-1}.

b The velocity of the *Bellatrix* is shown in this triangle.



The boat's speed is the magnitude of its velocity, $|\mathbf{v}| = \sqrt{2^2 + 1.5^2} = 2.5$

The angle α in the diagram is given by $\tan\alpha = \frac{1.5}{2} \quad \Rightarrow \quad \alpha = 37°$

and so the compass bearing on which the boat is travelling is 90º + 37º =127º.

So the speed of the *Bellatrix* is 2.5 km h^{-1} and the direction is 127º.

c Assuming both boats started at the origin, their positions after 2 hours are

Antares $2 \times (1.5\mathbf{i} + 2.6\mathbf{j}) = 3\mathbf{i} + 5.2\mathbf{j}$

Bellatrix $2 \times (2\mathbf{i} - 1.5\mathbf{j}) = 4\mathbf{i} - 3\mathbf{j}$

So the displacement from the *Bellatrix* to the *Antares* is

$$(3\mathbf{i} + 5.2\mathbf{j}) - (4\mathbf{i} - 3\mathbf{j}) = -\mathbf{i} + 7.2\mathbf{j}$$

and the distance between the boats is

$\sqrt{(-1)^2 + 7.2^2} = 7.3$ km (to 2 s.f.).

In this example, the velocity of the *Antares* was given in speed-direction form and was converted into components form. For the *Bellatrix* the reverse process was followed, converting from components into speed and direction. You need to be able to do both these conversions.

Exercise 13.2

1 Simplify the following:

a $\begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}1\\5\end{pmatrix}$ b $\begin{pmatrix}3\\-1\end{pmatrix} + \begin{pmatrix}-1\\3\end{pmatrix}$ c $\begin{pmatrix}5\\0\end{pmatrix} + \begin{pmatrix}0\\5\end{pmatrix}$

2 Simplify the following:

a $(2\mathbf{i} + 3\mathbf{j}) - (3\mathbf{i} - 2\mathbf{j})$ b $3(2\mathbf{i} + 3\mathbf{j}) - 2(3\mathbf{i} - 2\mathbf{j})$

3 Given that $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = -\mathbf{i} + 5\mathbf{j}$, find the following vectors:

a $\mathbf{a} + \mathbf{b} + \mathbf{c}$ b $\mathbf{a} + \mathbf{b} - \mathbf{c}$ c $\mathbf{a} - \mathbf{b} + \mathbf{c}$

d $2\mathbf{a} + \mathbf{b} + 3\mathbf{c}$ e $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$ f $2(\mathbf{a} + \mathbf{b}) - 3(\mathbf{b} - \mathbf{c})$

g $2(2\mathbf{a} + \mathbf{b} - \mathbf{c}) - 3(\mathbf{a} - 2\mathbf{b} + \mathbf{c})$

4 Sketch each of the following vectors and find their moduli:

a $3\mathbf{i} + 4\mathbf{j}$ b $3\mathbf{i} - 4\mathbf{j}$ c $7\mathbf{i}$ d $-7\mathbf{i}$

e $5\mathbf{i} + 3\mathbf{j}$ f $2\mathbf{i} - 7\mathbf{j}$ g $6\mathbf{i} - 6\mathbf{j}$ h $\mathbf{i} + \mathbf{j}$

5 Write the vectors joining each pair of points

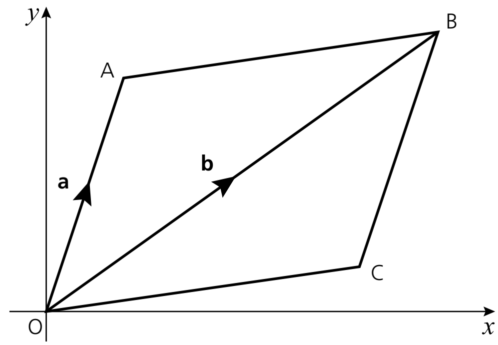
i in the form $a\mathbf{i} + b\mathbf{j}$ ii as a column vectors.

a (1, 4) to (3, 7) b (1, 3) to (2, –4)

c (0, 0) to (3, 5) d (–3, 7) to (7, –3)

e (–4, 2) to (0, 0) f (–5, –2) to (–1,0)

6

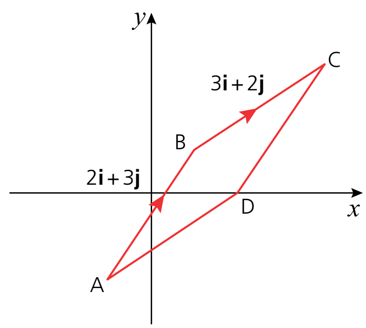


The diagram shows a parallelogram OABC with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
Write the following vectors in terms of **a** and **b**:

a $\overrightarrow{AB}$ **b** $\overrightarrow{BA}$ **c** $\overrightarrow{CB}$ **d** $\overrightarrow{BC}$

e $\overrightarrow{OB}$ **f** $\overrightarrow{BO}$ **g** $\overrightarrow{AC}$ **h** $\overrightarrow{CA}$

7



ABCD is a rhombus where A is the point (–1, –2).

a Write the vectors $\overrightarrow{AD}$ and $\overrightarrow{DC}$ as column vectors.

b Write the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ as column vectors.

c Use the properties of a parallelogram to find the coordinates of the point of intersection of the diagonals.

d Find the gradients of the diagonals and hence verify that the diagonals are perpendicular.

8 Find unit vectors parallel to each of the following:

a $3\mathbf{i} - 4\mathbf{j}$ **b** $5\mathbf{i} + 7\mathbf{j}$ **c** $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$

d $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$ **e** $5\mathbf{i}$ **f** $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

9 A is the point (–3, –2), B is the point (5, 4) and C is the point (2, 8).

a Sketch the triangle ABC.

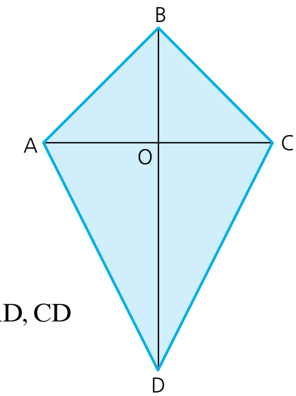
b Find the vectors representing the three sides AB, BC and CA in the form $x\mathbf{i} + y\mathbf{j}$.

c Find the lengths of each side of the triangle.

d What type of triangle is triangle ABC?

Exercise 13.2 (cont)

10 ABCD is a kite and AC and BD meet at the origin O. A is the point (–4, 0), B is (0, 4) and D is (0, –8).
The diagonals of a kite are perpendicular and O is the midpoint of AC.



a Find each of the following in terms of **i** and **j**:

i $\overrightarrow{OC}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{BC}$

iv $\overrightarrow{AD}$ **v** $\overrightarrow{CD}$ **vi** $\overrightarrow{AC}$

b Find the lengths of the lines OC, AB, BC, AD, CD and AC.

c State two descriptions that are common to the triangles AOB, BOC and ABC.

11 A (4, 4), B(24, 19) and C (48, 12) form the vertices of a triangle.

a Sketch the triangle.

b Write the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ as column vectors.

c Find the lengths of the sides of the triangle.

d What type of triangle is ABC?

12 Salman and Aloke are hiking on a flat level ground. Their starting point is taken as the origin and the unit vectors **i** and **j** are in the directions east and north. Salman walks with constant velocity 3**i** + 6**j** kilometres per hour. Aloke walks on a compass bearing of 300° at a steady speed of 6.5 kilometres per hour.

i Who is walking fastest and by how much?

ii How far apart are they after 1½ hours?

13 Ama has her own small aeroplane. One afternoon, she flies for 1 hour with a velocity of 120**i** + 160**j** km h^{-1} where **i** and **j** are unit vectors in the directions east and north.

Then she flies due north for 1 hour at the same speed. Finally, she returns to her starting point; flying in a straight line at the same speed.

Find, to the nearest degree, the direction in which she travels on the final leg of her journey and, to the nearest minute, how long it takes her.

Past-paper questions

1 In this question **i** is a unit vector due East and **j** is a unit vector due North.
At 12 00 hours, a ship leaves a port *P* and travels with a speed of 26 kmh^{-1} in the direction 5**i** + 12**j**.

(i) Show that the velocity of the ship is (10**i** + 24**j**) kmh^{-1}. [2]

(ii) Write down the position vector of the ship, relative to *P*, at 16 00 hours. [1]

(iii) Find the position vector of the ship, relative to *P*, *t* hours after 16 00 hours. [2]

At 16 00 hours, a speedboat leaves a lighthouse which has position vector $(120\mathbf{i} + 81\mathbf{j})$ km, relative to P, to intercept the ship. The speedboat has a velocity of $(-22\mathbf{i} + 30\mathbf{j})$ kmh^{-1}.

(iv) Find the position vector, relative to P, of the speedboat t hours after 16 00 hours. [1]

(v) Find the time at which the speedboat intercepts the ship and the position vector, relative to P, of the point of interception. [4]

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2 Relative to an origin O, the position vectors of the points A and B are $2\mathbf{i} - 3\mathbf{j}$ and $11\mathbf{i} + 42\mathbf{j}$ respectively.

(i) Write down an expression for $\overrightarrow{AB}$. [2]

The point C lies on AB such that $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$.

(ii) Find the length of $\overrightarrow{OC}$. [4]

The point D lies on $\overrightarrow{OA}$ such that $\overrightarrow{DC}$ is parallel to $\overrightarrow{OB}$.

(iii) Find the position vector of D. [2]

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Paper 21 Q8 June 2012
Cambridge IGCSE Additional Mathematics 0606
Paper 21 Q8 June 2012

3 Relative to an origin O, the position vectors of the points A and B are $\mathbf{i} - 4\mathbf{j}$ and $7\mathbf{i} + 20\mathbf{j}$ respectively. The point C lies on AB and is such that $\overrightarrow{AC} = \frac{2}{3}\overrightarrow{AB}$. Find the position vector of C and the magnitude of this vector. [5]

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Paper 21 Q3 June 2011
Cambridge IGCSE Additional Mathematics 0606
Paper 21 Q3 June 2011

Learning outcomes

Now you should be able to:

★ use vectors in any form, e.g. $\begin{pmatrix} a \\ b \end{pmatrix}$, $\overrightarrow{\text{LM}}$, $\mathbf{r}$, $p\mathbf{i} - q\mathbf{j}$
★ know and use position vectors and unit vectors
★ find the magnitude of a vector; add and subtract vectors and multiply vectors by scalars
★ compose and resolve velocities.

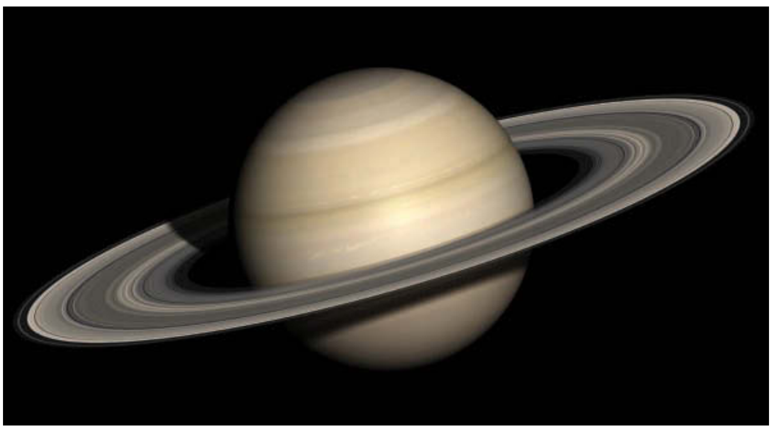
Key points

- ✔ A vector quantity has both a magnitude and a direction; a scalar quantity has magnitude only.
- ✔ Vectors are typeset in bold, for example **a**, or they may be written as lines with arrows along the top, for example $\overrightarrow{OA}$. When they are hand-written they are underlined, for example $\underline{a}$.
- ✔ The length of a vector is also referred to as its magnitude or modulus. The length of the vector **a** is written as $|\mathbf{a}|$ or a and can be found using Pythagoras' theorem.
- ✔ A unit vector has length 1. Unit vectors in the directions x and y are denoted by **i** and **j** respectively.
- ✔ A vector can be written in component form, $x\mathbf{i} + y\mathbf{j}$ or $\begin{pmatrix} x \\ y \end{pmatrix}$, as in magnitude-direction form – (r, θ).
- ✔ The position vector $\overrightarrow{OA}$ of a point A is the vector joining the origin to A.
- ✔ The vector $\overrightarrow{AB}$ is given by **b** – **a** where **a** and **b** are the positions vectors of A and B.

14 Differentiation

If I have seen further than others, it is by standing upon the shoulders of giants.

Isaac Newton (1642–1727)

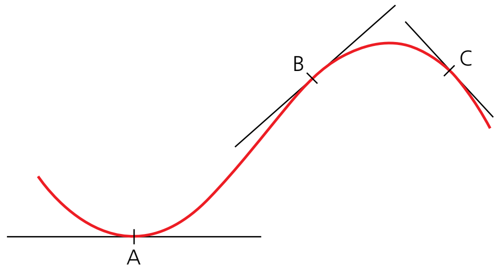


Discussion point

Look at the planet Saturn in the image above. What connection did Newton make between an apple and the motion of the planets?

In Newton's early years, mathematics was not advanced enough to enable people to calculate the orbits of the planets round the sun. In order to address this, Newton invented calculus, the branch of mathematics that you will learn about in this chapter.

The gradient function



The curve in the diagram has a zero gradient at A, a positive gradient at B and a negative gradient at C.

Although you can calculate the gradient of a curve at a given point by drawing a tangent at that point and using two points on the tangent to calculate its gradient, this process is time-consuming and the results depend on the accuracy of your drawing and measuring. If you know the equation of the curve, you can use **differentiation** to calculate the gradient.

Worked example

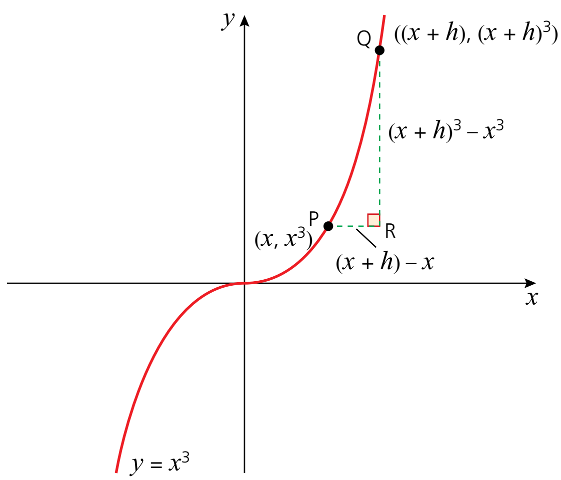
Work out the gradient of the curve $y = x^3$ at the general point (x, y).

Solution

Let P have the general value x as its x-coordinate, so P is the point (x, x^3).

Let the x-coordinate of Q be $(x + h)$ so Q is the point $((x + h), (x + h)^3)$.

Since it is on the curve $y = x^3$



The gradient of the chord PQ is given by

$$\begin{aligned}\frac{\mathrm{QR}}{\mathrm{PR}} &= \frac{(x+h)^3 - x^3}{(x+h) - x}\\ &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}\\ &= \frac{3x^2h + 3xh^2 + h^3}{h}\\ &= \frac{h(3x^2 + 3xh + h^2)}{h}\\ &= 3x^2 + 3xh + h^2\end{aligned}$$

As Q gets closer to P, h takes smaller and smaller values and the gradient approaches the value of $3x^2$, which is the gradient of the tangent at P.

The gradient of the curve $y = x^3$ at the point (x, y) is equal to $3x^2$.

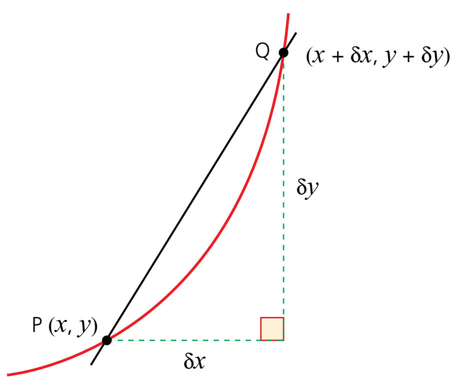
The gradient function is the gradient of the curve at the general point (x, y).

If the equation of the curve is written as $y = \mathrm{f}(x)$, then the **gradient function** is written as $\mathrm{f}'(x)$. Using this notation, the result in the previous example can be written as

$$\mathrm{f}(x) = x^3 \quad \Rightarrow \quad \mathrm{f}'(x) = 3x^2.$$

In the previous example, h was used to denote the difference between the x-coordinates of the points P and Q, where Q is close to P.

h is sometimes replaced by δx. The Greek letter δ (delta) is shorthand for 'a small change in' and so δx represents a small change in x, δy a small change in y and so on.



In the diagram the gradient of the chord PQ is $\frac{\delta y}{\delta x}$.

In the limit as δx tends towards 0, δx and δy both become infinitesimally small and the value obtained for $\frac{\delta y}{\delta x}$ approaches the gradient of the tangent at P.

$$\lim \frac{\delta y}{\delta x} \text{ is written as } \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Using this notation, you have a rule for differentiation.

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}$$

The gradient function, $\frac{dy}{dx}$, is sometimes called the **derivative** of y with respect to x. When you find it you have **differentiated** y with respect to x.

If the curve is written as $y = f(x)$, then the derivative is $f'(x)$.

If you are asked to differentiate a relationship in the form $y = f(x)$ in this book, this means differentiate with respect to x unless otherwise stated.

Note

There is nothing special about the letters x, y or f. If, for example, your curve represents time, t, on the horizontal axis and velocity, v, on the vertical axis, then the relationship could be referred to as $v = g(t)$. In this case v is a function of t and the gradient function is given by $\frac{dv}{dt} = g'(t)$.

The differentiation rule

Although it is possible to find the gradient from first principles which establishes a formal basis for differentiation, in practice you will use the differentiation rule introduced above;

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}.$$

You can also use this rule to differentiate (find the gradient of) equations that represent straight lines. For example, the gradient of the line $y = x$ is the same as $y = x^1$, so using the rule for differentiation, $\frac{dy}{dx} = 1 \times x^0 = 1$.

Lines of the form $y = c$ are parallel to the x-axis.

The gradient of the line $y = c$ where c is a constant is 0. For example, $y = 4$ is the same as $y = 4x^0$ so using the rule for differentiation, $\frac{dy}{dx} = 4 \times 0 \times x^{-1} = 0$. In general, differentiating any constant gives zero.

The rule can be extended further to include functions of the type $y = kx^n$ for any constant k, to give

This result is true for all powers of x, positive, negative and fractional.

$$y = kx^n \quad \Rightarrow \quad \frac{dy}{dx} = nkx^{n-1}.$$

You may find it helpful to remember the rule as

multiply by the power of x and reduce the power by 1.

Worked example

For each function, find the gradient function.

a $y = x^7$ **b** $u = 4x^3$ **c** $v = 5t^2$

d $y = 4x^{-3}$ **e** $P = 4\sqrt{t}$ **f** $y = \dfrac{4x^3 - 5}{x^2}$

Solution

a $y = x^7$

$$\Rightarrow \frac{dy}{dx} = 7x^6$$

b $u = 4x^3$

$$\Rightarrow \frac{du}{dx} = 4 \times 3x^2 = 12x^2$$

c $v = 5t^2$

$$\Rightarrow \frac{dv}{dt} = 5 \times 2t = 10t$$

d $y = 4x^{-3}$

$$\Rightarrow \frac{dy}{dx} = 4 \times (-3)x^{-3-1} = -12x^{-4}$$

e $P = 4t^{\frac{1}{2}}$

Using $\sqrt{t} = t^{\frac{1}{2}}$

$$\Rightarrow \frac{dP}{dt} = 4 \times \frac{1}{2}t^{\frac{1}{2}-1}$$

$$= \frac{2}{t^{\frac{1}{2}}}$$

$$= \frac{2}{\sqrt{t}}$$

f $y = \dfrac{4x^3 - 5}{x^2}$

$$\Rightarrow y = \frac{4x^3 - 5}{x^2} \Rightarrow y = \frac{4x^3}{x^2} - \frac{5}{x^2}$$

$$\Rightarrow y = 4x - 5x^{-2}$$

$$\frac{dy}{dx} = 4 + 10x^{-3}$$

$$= 4 + \frac{10}{x^3}$$

Sums and differences of functions

Many of the functions you will meet are sums or differences of simpler functions. For example, the function $(4x^3 + 3x)$ is the sum of the functions $4x^3$ and $3x$. To differentiate these functions, differentiate each part separately and then add the results together.

Worked example

Differentiate $y = 4x^3 + 3x$.

Solution

$$\frac{dy}{dx} = 12x^2 + 3$$

This example illustrates the general result that

$$y = f(x) + g(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) + g'(x).$$

Worked example

Given that $y = 2x^3 - 3x + 4$, find

a $\dfrac{dy}{dx}$

b the gradient of the curve at the point (2, 14).

Solution

a $\dfrac{dy}{dx} = 6x^2 - 3$

b At (2, 14), $x = 2$.

$$\frac{dy}{dx} = 6 \times (2)^2 - 3 = 21$$

Substituting $x = 2$ in the expression for $\frac{dy}{dx}$

Exercise 14.1

Differentiate the following functions using the rules

$$y = kx^n \quad \Rightarrow \quad \frac{dy}{dx} = nkx^{n-1}$$

and $$y = f(x) + g(x) \quad \Rightarrow \quad \frac{dy}{dx} = f'(x) + g'(x).$$

1 **a** $y = x^4$ **b** $y = 2x^3$ **c** $y = 5$ **d** $y = 10x$

2 **a** $y = x^{\frac{1}{2}}$ **b** $y = 5\sqrt{x}$ **c** $P = 7t^{\frac{3}{2}}$ **d** $y = \frac{1}{5}x^{\frac{5}{2}}$

3 **a** $y = 2x^5 + 4x^2$ **b** $y = 3x^4 + 8x$ **c** $y = x^3 + 4$ **d** $y = x - 5x^3$

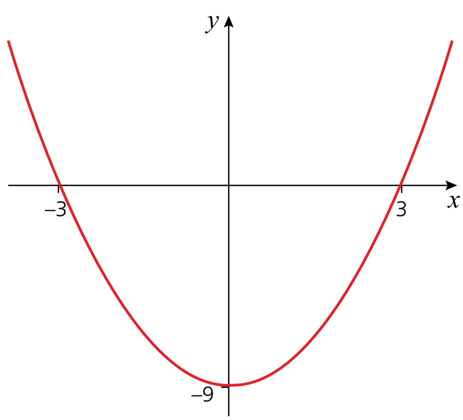
4 **a** $f(x) = \dfrac{1}{x^2}$ **b** $f(x) = \dfrac{6}{x^3}$

c $f(x) = 4\sqrt{x} - \dfrac{8}{\sqrt{x}}$ **d** $f(x) = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

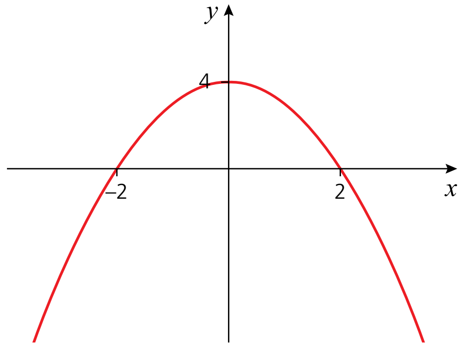
5 **a** $y = x(x - 1)$ **b** $y = (x + 1)(2x - 3)$

c $y = \dfrac{x^3 + 5x}{x^2}$ **d** $y = x\sqrt{x}$

6 Find the gradient of the curve $y = x^2 - 9$ at the points of intersection with the x- and y-axes.



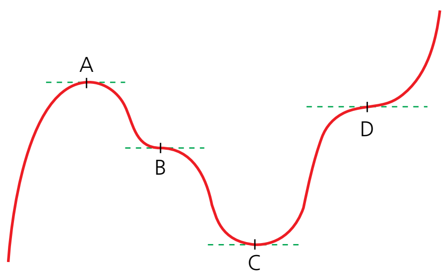
7 **a** Copy the curve of $y = 4 - x^2$ and draw the graph of $y = x - 2$ on the same axes.



b Find the coordinates of the points where the two graphs intersect.
c Find the gradient of the curve at the points of intersection.

Stationary points

A **stationary point** is a point on a curve where the gradient is zero. This means that the tangents to the curve at these points are horizontal. The diagram shows a curve with four stationary points: A, B, C and D.



The points A and C are **turning points** of the curve because as the curve passes through these points, it changes direction completely: at A the gradient changes from positive to negative and at C from negative to positive. A is called a **maximum** turning point, and C is a **minimum** turning point.

At B the curve does not turn: the gradient is negative both to the left and to the right of this point. B is a **stationary point of inflection**.

Discussion point

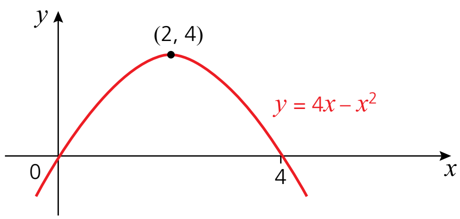
What can you say about the gradient to the left and right of D?

Note

Points where a curve ‘twists’ but doesn’t have a zero gradient are also called points of inflection. However, in this section you will look only at *stationary* points of inflection. The tangent at a point of inflection both touches and intersects the curve.

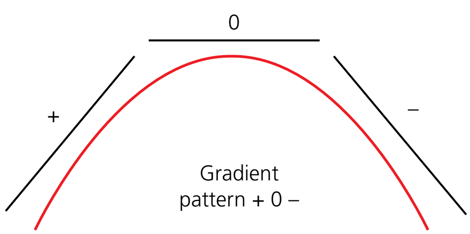
Maximum and minimum points

The graph shows the curve of $y = 4x - x^2$.



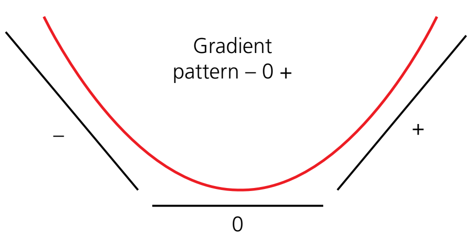
- The curve has a *maximum* point at (2, 4).
- The gradient $\frac{dy}{dx}$ at the maximum point is zero.
- The gradient is positive to the left of the maximum and negative to the right of it.

This is true for any maximum point as shown below.



For any minimum turning point, the gradient

- is zero at that point
- goes from negative to zero to positive.



Once you can find the position of any stationary points, and what type of points they are, you can use this information to help you sketch graphs.

Worked example

a For the curve $y = x^3 - 12x + 3$

i find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the values of x for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

ii classify the points on the curve with these values of x

iii find the corresponding values of y

iv sketch the curve.

b Why can you be confident about continuing the sketch of the curve beyond the x-values of the turning points?

c You did not need to find the coordinates of the points where the curve crosses the x-axis before sketching the graph. Why was this and under what circumstances would you find these points?

Solution

a i $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $3x^2 - 12 = 0$

$\Rightarrow \quad 3(x^2 - 4) = 0$

$\Rightarrow \quad 3(x + 2)(x - 2) = 0$

$\Rightarrow \quad x = -2$ or $x = 2$

Looking at the gradient pattern around $x = -2$

ii When $x = -3$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3(-3)^2 - 12 = 15.$

When $x = -1$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 3(-1)^2 - 12 = -9.$

The gradient pattern is + 0 –

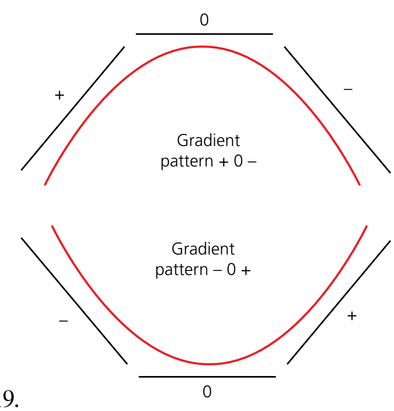
$\Rightarrow$ maximum turning point at $x = -2$.

When $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3(1)^2 - 12 = -9$

When $x = 3$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 3(3)^2 - 12 = 15$

The gradient pattern is – 0 +

$\Rightarrow$ minimum turning point at $x = +2$

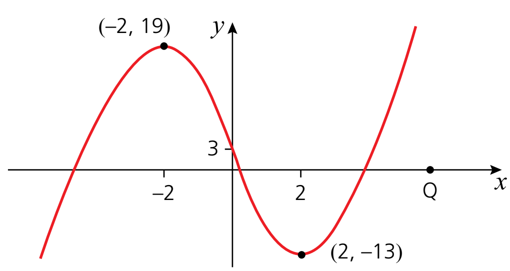


$(-2, 19)$ is a maximum and $(2, -13)$ a minimum.

iii When $x = -2$, $y = (-2)^3 - 12(-2) + 3 = 19.$
When $x = +2$, $y = (2)^3 - 12(2) + 3 = -13.$

The value of y when $x = 0$ tells you where the curve crosses the y-axis.

iv When $x = 0$, $y = (0)^3 - 12(0) + 3 = 3.$



b A cubic has at most 2 turning points and they have both been found. So the parts of the curve beyond them (to the left and to the right) just get steeper and steeper.

c The sketch is showing the shape of the curve and this is not affected by where it crosses the axes. However, you can see from the equation that it crosses the y-axis at $(0, 3)$ and it is good practice to mark this in.

Worked example

Find all the turning points on the graph of $y = t^4 - 2t^3 + t^2 - 2$ and then sketch the curve.

Solution

$$\frac{dy}{dt} = 4t^3 - 6t^2 + 2t$$

$$\begin{aligned} \frac{dy}{dt} = 0 \quad &\Rightarrow \quad 4t^3 - 6t^2 + 2t = 0 \\ &\Rightarrow \quad 2t(2t^2 - 3t + 1) = 0 \\ &\Rightarrow \quad 2t(2t - 1)(t - 1) = 0 \\ &\Rightarrow \quad t = 0 \text{ or } t = 0.5 \text{ or } t = 1 \end{aligned}$$

Turning points occur when $\frac{dy}{dt} = 0$.

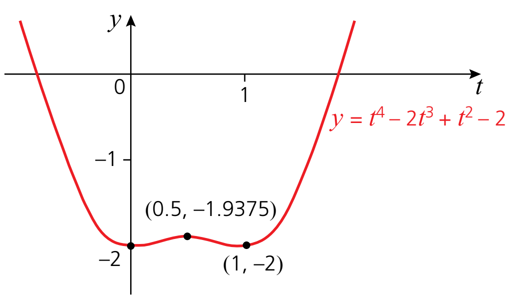
When $t = 0$, $\quad y = (0)^4 - 2(0)^3 + (0)^2 - 2 = -2$.

When $t = 0.5$, $\quad y = (0.5)^4 - 2(0.5)^3 + (0.5)^2 - 2 = -1.9375$.

When $t = 1$, $\quad y = (1)^4 - 2(1)^3 + (1)^2 - 2 = -2$.

Plotting these points suggests that $(0.5, -1.9375)$ is a maximum turning point and $(0, -2)$ and $(1, -2)$ are minima, but you need more information to be sure. For example when $t = -1$, $y = +2$ and when $t = 2$, $y = +2$ so you know that the curve goes above the horizontal axis on both sides.

You can find whether the gradient is positive or negative by taking a test point in each interval. For example, $t = 0.25$ in the interval $0 < t < 0.5$; when $t = 0.25$, $\frac{dy}{dt}$ is positive.



Exercise 14.2

You can use a graphic calculator to check your answers.

For each curve in questions 1 – 8:

i find $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$
ii classify the point(s) on the curve with these x-values
iii find the corresponding y-value(s)
iv sketch the curve.

1 $y = 1 + x - 2x^2$

2 $y = 12x + 3x^2 - 2x^3$

3 $y = x^3 - 4x^2 + 9$

4 $y = x^2\,(x - 1)^2$

5 $y = x^4 - 8x^2 + 4$

6 $y = x^3 - 48x$

7 $y = x^3 + 6x^2 - 36x + 25$

8 $y = 2x^3 - 15x^2 + 24x + 8$

9 The graph of $y = px + qx^2$ passes through the point $(3, -15)$. Its gradient at that point is -14.
  a Find the values of p and q.
  b Calculate the maximum value of y and state the value of x at which it occurs.

10 a Find the stationary points of the function $f(x) = x^2(3x^2 - 2x - 3)$ and distinguish between them.
  b Sketch the curve $y = f(x)$.

Using second derivatives

In the same way as $\frac{dy}{dx}$ or $f'(x)$ is the gradient of the curve $y = f(x)$, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $f''(x)$ represents the gradient of the curve $y = f'(x)$.

This is also written as $\frac{d^2y}{dx^2}$ and is called the **second derivative**. You can find it by differentiating the function $\frac{dy}{dx}$.

Note that $\frac{d^2y}{dx^2}$ is not the same as $\left(\frac{dy}{dx}\right)^2$.

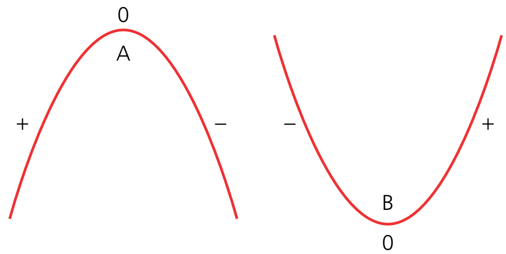
Worked example

Find $\frac{d^2y}{dx^2}$ for $y = 4x^3 + 3x - 2$.

Solution

$$\frac{dy}{dx} = 12x^2 + 3 \Rightarrow \frac{d^2y}{dx^2} = 24x$$

In many cases, you can use the second derivative to determine if a stationary point is a maximum or minimum instead of looking at the value of $\frac{dy}{dx}$ on either side of the turning point.



$\frac{d^2y}{d^2x} < 0$ at A $\qquad$ $\frac{d^2y}{dx^2} > 0$ at B

At A, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ showing that the gradient is zero and since $\frac{d^2y}{dx^2} < 0$, it is decreasing near that point, so must be going from positive to negative. This shows that A is a maximum turning point.

At B, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ showing that the gradient is zero and since $\frac{d^2y}{dx^2} > 0$, it is increasing near that point, so must be going from negative to positive. This shows that B is a minimum turning point.

Note that if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at the same point, you cannot make a decision about the type of turning point using this method.

Worked example

For $y = 2x^3 - 3x^2 - 12x + 4$

a Find $\frac{dy}{dx}$ and find the values of x when $\frac{dy}{dx} = 0$.

b Find the value of $\frac{d^2y}{dx^2}$ at each stationary point and hence determine its nature.

c Find the value of y at each of the stationary points.

d Sketch the curve $y = 2x^3 - 3x^2 - 12x + 4$.

Solution

a $\frac{dy}{dx} = 6x^2 - 6x - 12$

$= 6(x^2 - x - 2)$

$= 6(x + 1)(x - 2)$

So $\frac{dy}{dx} = 0$ when $x = -1$ and when $x = 2$.

b $\frac{d^2y}{dx^2} = 12x - 6$

When $x = -1$, $\frac{d^2y}{dx^2} = -18 \Rightarrow$ a maximum

When $x = 2$, $\frac{d^2y}{dx^2} = 18 \Rightarrow$ a minimum

c When $x = -1$, $y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 4$

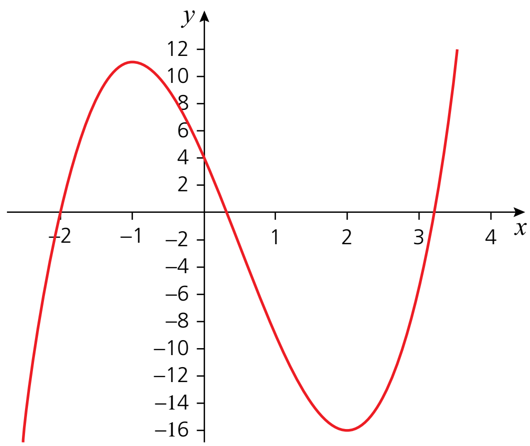
$= 11$

When $x = 2$, $y = 2(2)^3 - 3(2)^2 - 12(2) + 4$

$= -16$

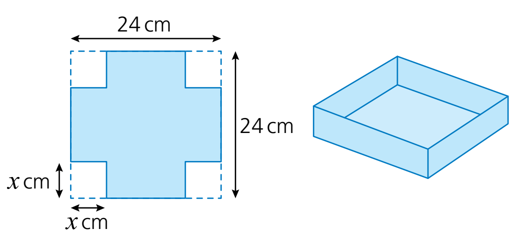
d The curve has a maximum turning point at $(-1, 11)$ and a minimum turning point at $(2, -16)$.

When $x = 0$, $y = 4$, so the curve crosses the y-axis at $(0, 4)$.



Worked example

Maria has made some sweets as a gift and makes a small box for them from a square sheet of card of side 24 cm. She cuts four identical squares of side x cm, one from each corner, and turns up the sides to make the box, as shown in the diagram.



a Write down an expression for the volume V of the box in terms of x.

b Find $\frac{dV}{dx}$ and the values of x when $\frac{dV}{dx} = 0$.

c Comment on this result.

d Find $\frac{d^2V}{dx^2}$ and hence find the depth when the volume is a maximum.

Solution

a The base of the box is a square of side $(24 - 2x)$ cm and the height is x cm, so

$V = (24 - 2x)^2 \times x$

Taking a factor of 2 out of each bracket

$= 4x(12 - x)^2\,\text{cm}^3$

b $V = 4x(144 - 24x + x^2)$

$= 576x - 96x^2 + 4x^3$

So $\dfrac{dV}{dx} = 576 - 192x + 12x^2$

$= 12(48 - 16x + x^2)$

$= 12(12 - x)(4 - x)$

So $\dfrac{dV}{dx} = 0$ when $x = 12$ and when $x = 4$.

c When $x = 12$ there is no box, since the piece of cardboard was only a square of side 24 cm.

Using $\dfrac{dV}{dx} = 576 - 192x + 12x^2$

d $\dfrac{d^2V}{dx^2} = -192 + 24x$

When $x = 4$, $\dfrac{d^2V}{dx^2} = -96$ which is negative.

Therefore the volume is a maximum when the depth $x = 4$ cm.

Exercise 14.3

1 Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for each of the following functions:

a $y = x^3 - 3x^2 + 2x - 6$ **b** $y = 3x^4 - 4x^3$

c $y = x^5 - 5x + 1$

2 For each of the following curves

i find any stationary points

ii use the second derivative test to determine their nature.

a $y = 2x^2 - 3x + 4$ **b** $y = x^3 - 2x^2 + x + 6$

c $y = 4x^4 - 2x^2 + 1$ **d** $y = x^5 - 5x$

3 For $y = 2x^3 - 3x^2 - 36x + 4$

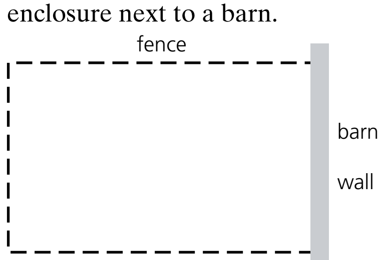
a Find $\dfrac{dy}{dx}$ and the values of x when $\dfrac{dy}{dx} = 0$.

b Find the value of $\dfrac{d^2y}{dx^2}$ at each stationary point and hence determine its nature.

c Find the value of y at each stationary point.

d Sketch the curve.

4 A farmer has 160 m of fencing and wants to use it to form a rectangular enclosure next to a barn.



Find the maximum area that can be enclosed and give its dimensions.

5 A cylinder has a height of h metres and a radius of r metres where $h + r = 3$.

a Find an expression for the volume of the cylinder in terms of r.
b Find the maximum volume.

6 A rectangle has sides of length x cm and y cm.
a If the perimeter is 24 cm, find the lengths of the sides when the area is a maximum, confirming that it is a maximum.
b If the area is 36 cm^2, find the lengths of the sides when the perimeter is a minimum, confirming that it is a minimum.

Equations of tangents and normals

Now that you know how to find the gradient of a curve at any point, you can use this to find the equation of the tangent at any given point on the curve.

Worked example

a Find the equation of the tangent to the curve $y = 3x^2 - 5x - 2$ at the point $(1, -4)$.

b Sketch the curve and show the tangent on your sketch.

Solution

Substituting $x = 1$ into this gradient function gives the gradient of the curve and therefore the tangent at this point.

a $y = 3x^2 - 5x - 2 \Rightarrow \dfrac{dy}{dx} = 6x - 5$

At $(1, -4)$, $\dfrac{dy}{dx} = 6 \times 1 - 5$

and so $m = 1$

So the equation of the tangent is given by

$$y - y_1 = (x - x_1)$$

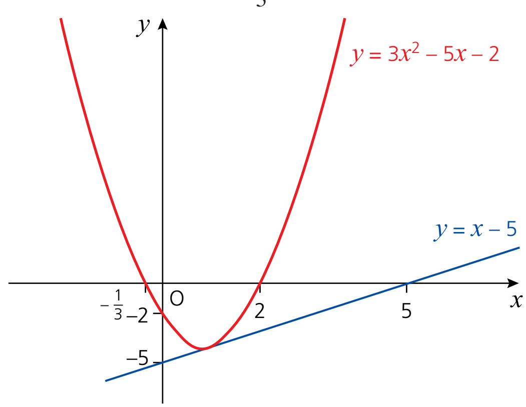
$x_1 = 1$, $y_1 = 4$ and $m = 1$

$$y - (-4) = 1(x - 1)$$
$$\Rightarrow \quad y = x - 5$$

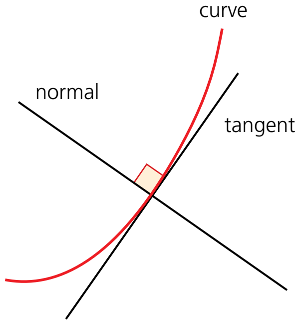
This is the equation of the tangent.

b $y = 3x^2 - 5x - 2$ is a ∪-shaped quadratic curve that crosses the crosses the y-axis when $y = -2$ and x-axis when $3x^2 - 5x - 2 = 0$.

$$3x^2 - 5x - 2 = 0 \Rightarrow (3x + 1)(x - 2) = 0$$
$$\Rightarrow x = -\frac{1}{3} \text{ or } x = 2$$



The **normal** to a curve at given point is the straight line that is at right angles to the tangent at that point, as shown below.



Remember that for perpendicular lines

$$m_1 m_2 = -1.$$

Worked example

Find the equation of the tangent and normal to the curve $y = 4x^2 - 2x^3$ at the point $(1, 2)$.

Draw a diagram showing the curve, the tangent and the normal.

Solution

$$y = 4x^2 - 2x^3 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 8x - 6x^2$$

At $(1, 2)$, the gradient is $\frac{\mathrm{d}y}{\mathrm{d}x} = 8 - 6 = 2$

The gradient of the tangent is $m_1 = 2$

It is slightly easier to use $y - y_1 = m(x - x_1)$ here than $y = mx + c$. If you substitute the gradient $m = 2$ and the point $(1, 2)$ into $y = mx + c$, you get $2 = 2 \times 1 + c$ and so $c = 0$. So the equation of the tangent is $y = 2x$.

So, using $\qquad y - y_1 = m(x - x_1)$

the equation of the tangent is $y - 2 = 2(x - 1)$

$$y = 2x$$

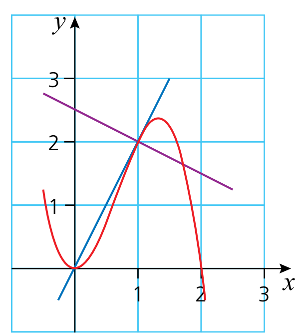
The gradient of the normal is $m_2 = -\frac{1}{m_1} = -\frac{1}{2}$

So, using $\qquad y - y_1 = m(x - x_1)$

the equation of the normal is $y - 2 = -\frac{1}{2}(x - 1)$

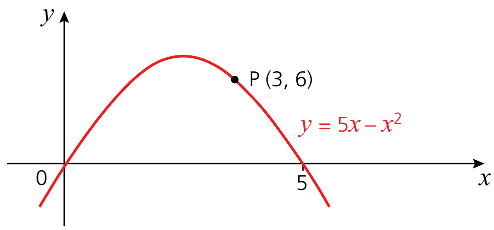
$$y = -\frac{x}{2} + \frac{5}{2}.$$

The curve, tangent and normal are shown on this graph.



Exercise 14.4

1 The sketch graph shows the curve of $y = 5x - x^2$.
The marked point, P, has coordinates (3, 6).



Find:

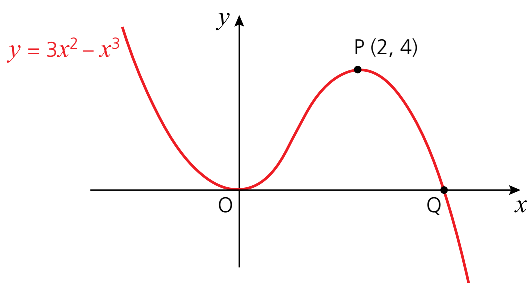
a the gradient function $\frac{dy}{dx}$

b the gradient of the curve at P

c the equation of the tangent at P

d the equation of the normal at P.

2 The sketch graph shows the curve of $y = 3x^2 - x^3$. The marked point, P, has coordinates (2, 4).



a Find:

i the gradient function $\frac{dy}{dx}$

ii the gradient of the curve at P

iii the equation of the tangent at P

iv the equation of the normal at P.

b The graph touches the x-axis at the origin O and crosses it at the point Q. Find:

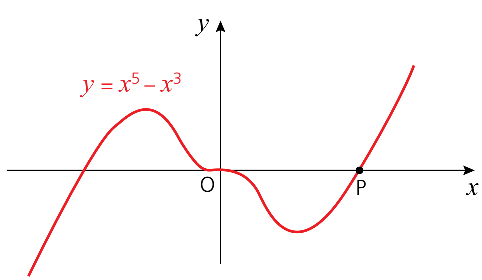
i the coordinates of Q

ii the gradient of the curve at Q

iii the equation of the tangent at Q.

c Without further calculation, state the equation of the tangent to the curve at O.

3 The sketch graph shows the curve of $y = x^5 - x^3$.



Exercise 14.4 (cont)

Find:

a the coordinates of the point P where the curve crosses the positive x-axis

b the equation of the tangent at P

c the equation of the normal at P.

The tangent at P meets the y-axis at Q and the normal meets the y-axis at R.

d Find the coordinates of Q and R and hence find the area of triangle PQR.

4 **a** Given that $f(x) = x^3 - 3x^2 + 4x + 1$, find $f'(x)$.

b The point P is on the curve $y = f(x)$ and its x-coordinate is 2.

  i Calculate the y-coordinate of P.

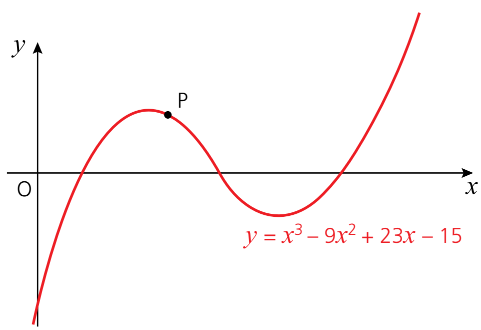
  ii Find the equation of the tangent at P.

  iii Find the equation of the normal at P.

c Find the values of x for which the curve has a gradient of 13.

5 The sketch graph shows the curve of $y = x^3 - 9x^2 + 23x - 15$.

The point P marked on the curve has its x-coordinate equal to 2.



Find:

a the gradient function $\frac{dy}{dx}$

b the gradient of the curve at P

c the equation of the tangent at P

d the coordinates of another point on the curve, Q, at which the tangent is parallel to the tangent at P

e the equation of the tangent at Q.

6 The point $(2, -8)$ is on the curve $y = x^3 - px + q$.

a Use this information to find a relationship between p and q.

b Find the gradient function $\frac{dy}{dx}$.

The tangent to this curve at the point $(2, -8)$ is parallel to the x-axis.

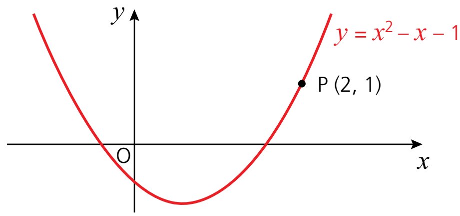
c Use this information to find the value of p.

d Find the coordinates of the other point where the tangent is parallel to the x-axis.

e State the coordinates of the point P where the curve crosses the y-axis.

f Find the equation of the normal to the curve at the point P.

7 The sketch graph shows the curve of $y = x^2 - x - 1$.



a Find the equation of the tangent at the point P(2, 1).

The normal at a point Q on the curve is parallel to the tangent at P.

b State the gradient of the tangent at Q.

c Find the coordinates of the point Q.

8 A curve has the equation $y = (x - 3)(7 - x)$.

a Find the gradient function $\frac{dy}{dx}$.

b Find the equation of the tangent at the point (6, 3).

c Find the equation of the normal at the point (6, 3).

d Which one of these lines passes through the origin?

9 A curve has the equation $y = 1.5x^3 - 3.5x^2 + 2x$.

a Show that the curve passes through the points (0, 0) and (1, 0).

b Find the equations of the tangents and normals at each of these points.

c Prove that the four lines in **b** form a rectangle.

Deriving these results from first principles is beyond the scope of this book.

Differentiating other functions of *x*

So far you have differentiated polynomials and other powers of *x*. Now this is extended to other expressions, starting with the three common trigonometrical functions. When doing this you will use the standard results that follow.

sin *x*, cos *x* and tan *x*

$$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$$

$$y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$$

$$y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$$

Recall $\sec x = \frac{1}{\cos x}$

When differentiating any trigonometric function, the angle must be in **radians**.

Worked example

Differentiate each of the following functions:

a $y = \sin x - \cos x$

b $y = 2\tan x + 3$

Solution

Using the results above

a $\frac{dy}{dx} = \cos x - (-\sin x)$

$= \cos x + \sin x$

b $y = 2\tan x + 3 \Rightarrow \frac{dy}{dx} = 2(\sec^2 x) + 0$

$= 2\sec^2 x$

Differentiating a constant always gives zero.

Worked example

a Sketch the graph of $y = \sin\theta$ for $0 \le \theta \le 2\pi$.

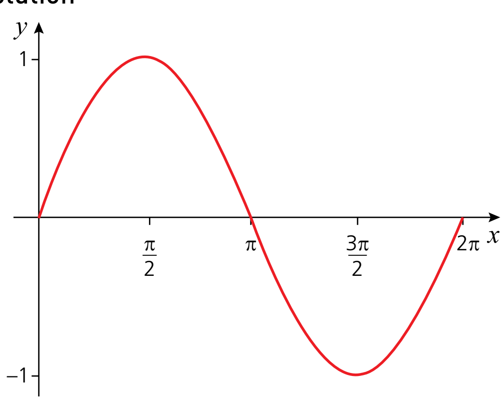
b **i** Find the value of $\frac{dy}{d\theta}$ when $\theta = \frac{\pi}{2}$.

ii At which other point does $\frac{dy}{d\theta}$ have this value?

c Use differentiation to find the value of $\frac{dy}{d\theta}$ when $\theta = \pi$.

Solution

a



b **i** The tangent to the curve when $\theta = \frac{\pi}{2}$ is horizontal, so $\frac{dy}{d\theta} = 0$.

ii The gradient is also 0 when $\theta = \frac{3\pi}{2}$.

c $y = \sin\theta \Rightarrow \frac{dy}{d\theta} = \cos\theta$

When $\theta = \pi$, $\frac{dy}{d\theta} = \cos\pi = -1$.

Worked example

a Find the turning point of the curve $y = \sin x - \cos x$ and determine its nature.

b Sketch the curve for $0 \leqslant x \leqslant \pi$.

This means decide if it is a maximum or minimum point.

Solution

a $y = \sin x - \cos x \quad \Rightarrow \quad \frac{dy}{dx} = \cos x + \sin x$

At the turning points $\cos x + \sin x = 0$

$\Rightarrow \quad \sin x = -\cos x$ Divide by $\cos x$

$\Rightarrow \quad \tan x = -1$

$\Rightarrow \quad x = -\frac{\pi}{4}$ (not in the required range)

or $\quad x = \frac{3\pi}{4}$

When $x = \frac{3\pi}{4}$, $y = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}$

$= \sqrt{2}$

The turning point is at $\left(\frac{3\pi}{4}, \sqrt{2}\right)$.

When $x = \frac{\pi}{2}$ (to the left), $y = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = 1$.

$1 < \sqrt{2}$

When $x = \pi$ (to the right), $y = \sin\pi - \cos\pi = 1$.

So the point $\left(\frac{3\pi}{4}, \sqrt{2}\right)$ is a maximum turning point.

Check where the curve crosses the axes.

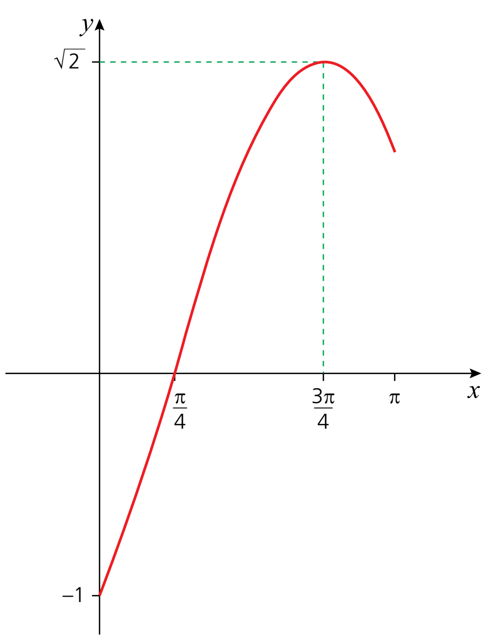
b When $x = 0$, $y = \sin 0 - \cos 0 = -1$.

When $y = 0$, $0 = \sin x - \cos x$

$\Rightarrow \sin x = \cos x$ Divide by $\cos x$

$\Rightarrow \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4}$



Worked example

For the curve $y = 2\cos\theta$ find:

a the equation of the tangent at the point where $\theta = \frac{\pi}{3}$

b the equation of the normal at the point where $\theta = \frac{\pi}{3}$.

Solution

a $y = 2\cos\theta \quad \Rightarrow \quad \frac{dy}{d\theta} = -2\sin\theta$

When $\theta = \frac{\pi}{3}$, $y = 2\cos\frac{\pi}{3}$
$= 1$

and $\frac{dy}{d\theta} = -2\sin\frac{\pi}{3}$
$= -\sqrt{3}$

Using $y = mx + c$

So the equation of the tangent is given by $y = -\theta\sqrt{3} + c$.

Substituting values for y and θ:

$$1 = -\left(\frac{\pi}{3}\right)\sqrt{3} + c \qquad \Rightarrow c = 1 + \frac{\pi\sqrt{3}}{3}$$

The equation of the tangent is therefore

$$y = -\theta\sqrt{3} + 1 + \frac{\pi\sqrt{3}}{3}.$$

b The gradient of the normal $= -1 \div \frac{dy}{d\theta}$

$$= -1 \div \left(-\sqrt{3}\right)$$
$$= \frac{1}{\sqrt{3}}$$

Using $y = mx + c$

The equation of the normal is given by $y = \frac{1}{\sqrt{3}}\theta + c$.

Substituting values for y and θ:

$$1 = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}\right) + c \quad \Rightarrow \quad c = 1 - \frac{\pi}{3\sqrt{3}}$$
$$= 1 - \frac{\pi\sqrt{3}}{9}$$

The equation of the normal is therefore

$$y = \frac{1}{\sqrt{3}}\theta + 1 - \frac{\pi}{3\sqrt{3}}.$$

e^x and $\ln x$

Again, deriving these results from first principles is beyond the scope of this book.

You met exponential and logarithmic functions in Chapter 7. Here are the standard results for differentiating them.

This is the only function where $y = \frac{dy}{dx}$.

$$y = e^x \Rightarrow \frac{dy}{dx} = e^x$$

$$y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

Worked example

Differentiate each of the following functions:

a $y = 5\ln x$

b $y = \ln(5x)$

c $y = 2e^x + \ln(2x)$

Solution

a $y = 5\ln x \quad \Rightarrow \quad \dfrac{dy}{dx} = 5\left(\dfrac{1}{x}\right)$

$\qquad = \dfrac{5}{x}$

b $y = \ln(5x) \quad \Rightarrow \quad y = \ln 5 + \ln x$

$\qquad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{x}$

ln 5 is a number so differentiating it gives zero.

c $y = 2e^x + \ln(2x) \quad \Rightarrow \quad \dfrac{dy}{dx} = 2e^x + \dfrac{1}{x}$

Part **b** shows an important result. Since $\ln(ax) = \ln a + \ln x$ for all values where $a > 0$,

$$y = \ln(ax) \Rightarrow \frac{dy}{dx} = \frac{1}{x}.$$

Worked example

a Find the turning point of the curve $y = 2x - \ln x$ and determine its nature.

b Sketch the curve for $0 < x \leqslant 3$.

Solution

a $y = 2x - \ln x \quad \Rightarrow \quad \dfrac{dy}{dx} = 2 - \dfrac{1}{x}$

$\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad 2 = \dfrac{1}{x}$

$\qquad \Rightarrow x \quad = 0.5$

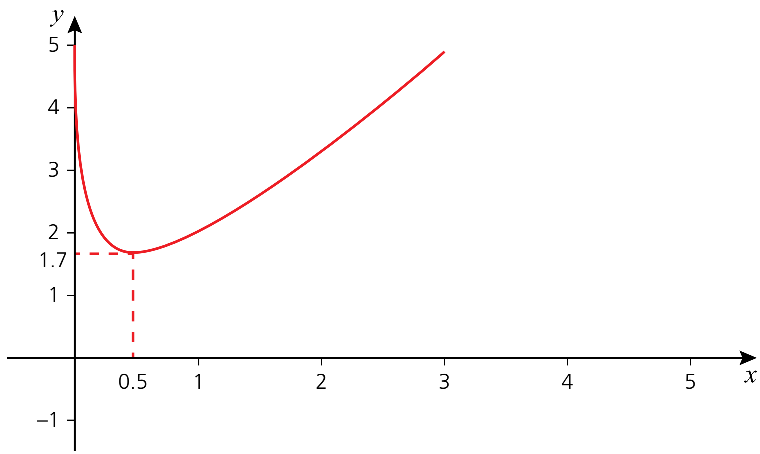
When $x = 0.5$, $2x - \ln x = 1.7$ (1 d.p.).

When $x = 0.3$ (to the left), $2x - \ln x = 1.8$ (1 d.p.).

When $x = 1.0$ (to the right), $2x - \ln x = 2$ (1 d.p.).

Therefore the point $(0.5, 1.7)$ is a minimum turning point.

b



Note

In this graph the y-axis is an asymptote. The curve gets nearer and nearer to it but never quite reaches it.

Notice that $\ln x$ is not defined for $x \leqslant 0$, and as $x \to 0$, $\ln x \to -\infty$ so $2x - \ln x \to +\infty$.

Worked example

For the curve $y = 2e^x + 5$ find the equation of:

a the tangent at the point where $x = -1$

b the normal at the point where $x = -1$.

Solution

a $y = 2e^x + 5 \quad \Rightarrow \quad \frac{dy}{dx} = 2e^x$

When $x = -1$, $\quad y = 2e^{-1} + 5$

$$= \frac{2}{e} + 5$$

$$\frac{dy}{dx} = 2e^{-1}$$

Using $y = mx + c$ →

So the equation of the tangent is given by $y = 2e^{-1}x + c$.

Substituting values for y and x:

$$\frac{2}{e} + 5 = 2e^{-1}(-1) + c$$

$$= -\frac{2}{e} + c$$

$$\Rightarrow c = \frac{4}{e} + 5$$

The equation of the tangent is therefore

$$y = \frac{2}{e}x + \frac{4}{e} + 5.$$

b The gradient of the normal $= -1 \div \frac{dy}{dx}$

$$= -1 \div \left(\frac{2}{e}\right)$$

$$= -\frac{e}{2}$$

Using $y = mx + c$ → The equation of the normal is given by $y = -\frac{e}{2}x + c$.

Substituting values for y and x:

$$\frac{2}{e} + 5 = -\frac{e}{2}(-1) + c$$

$$\Rightarrow c = \frac{2}{e} + 5 - \frac{e}{2}$$

The equation of the normal is therefore

$$y = -\frac{e}{2}x + \frac{2}{e} + 5 - \frac{e}{2}.$$

Differentiating products and quotients of functions

Deriving these results from first principles is beyond the scope of this book.

Sometimes you meet functions like $y = x^2e^x$, which are the product of two functions, x^2 and e^x. To differentiate such functions you use the **product rule**.

When $u(x)$ and $v(x)$ are two functions of x

» $y = uv \quad \Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

A shorthand form of $y = u(x) \times v(x)$

Worked example

Differentiate $y = (x^2 + 1)(2x - 3)$

a by expanding the brackets

b by using the product rule.

Solution

a $y = (x^2 + 1)(2x - 3)$

$$= 2x^3 - 3x^2 + 2x - 3$$

$$\Rightarrow \frac{dy}{dx} = 6x^2 - 6x + 2$$

b Let $u = (x^2 + 1)$ and $v = (2x - 3)$

$$\frac{du}{dx} = 2x \text{ and } \frac{dv}{dx} = 2$$

Product rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

So $\frac{dy}{dx} = (x^2 + 1)(2) + (2x - 3)(2x)$

$$= 2x^2 + 2 + 4x^2 - 6x$$

$$= 6x^2 - 6x + 2$$

In this example you had a choice of methods; both gave you the same answer. In the next example there is no choice; you must use the product rule.

Worked example

Differentiate each of the following functions:

a $y = x^2 e^x$ **b** $y = x^3 \sin x$ **c** $y = (2x^3 - 4)(e^x - 1)$

Solution

a Let $u = x^2$ and $v = e^x$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = e^x$

Product rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

So $\frac{dy}{dx} = x^2 e^x + e^x(2x)$

$= xe^x(x+2)$

b Let $u = x^3$ and $v = \sin x$

$\frac{du}{dx} = 3x^2$ and $\frac{dv}{dx} = \cos x$

Product rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

So $\frac{dy}{dx} = x^3 \cos x + (\sin x)(3x^2)$

$= x^2(x\cos x + 3\sin x)$

c Let $u = (2x^3 - 4)$ and $v = (e^x - 1)$

$\frac{du}{dx} = 6x^2$ and $\frac{dv}{dx} = e^x$

Product rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

So $\frac{dy}{dx} = (2x^3 - 4)(e^x) + (e^x - 1)(6x^2)$

$= 2x^3 e^x - 4e^x + 6x^2 e^x - 6x^2$

Sometimes you meet functions like $y = \frac{e^x}{x^2+1}$ where one function, in this case e^x, is divided by another, $x^2 + 1$. To differentiate such functions you use the **quotient rule**.

For $$y = \frac{u}{v}$$

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Worked example

Differentiate $y = \frac{x^3 + 3}{2x^2}$

a by simplifying first **b** by using the quotient rule.

Solution

a $y = \dfrac{x^3+3}{x^2}$

$= (x^3+3)x^{-2}$

$= x + 3x^{-2}$

So $\dfrac{dy}{dx} = 1 - 6x^{-3}$

b Let $u = x^3 + 3$ and $v = x^2$

$\dfrac{du}{dx} = 3x^2$ and $\dfrac{dv}{dx} = 2x$

Quotient rule: $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{x^2(3x^2) - (x^3+3)2x}{(x^2)^2}$

$= \dfrac{3x^4 - 2x^4 - 6x}{x^4}$

$= \dfrac{x^4 - 6x}{x^4}$

$= 1 - 6x^{-3}$

This quotient rule is longer in this case, but is useful when it is not possible to simplify first.

Worked example

Differentiate each of the following functions:

a $y = \dfrac{2x^3+3}{x^2-1}$

b $y = \dfrac{e^x}{x^2}$

Solution

a $y = \dfrac{2x^3+3}{x^2-1}$

Let $u = 2x^3 + 3$ and $v = x^2 - 1$

$\dfrac{du}{dx} = 6x^2$ and $\dfrac{dv}{dx} = 2x$

Quotient rule: $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{(x^2-1)6x^2 - (2x^3+3)2x}{(x^2-1)^2}$

$= \dfrac{6x^4 - 6x^2 - 4x^4 - 6x}{(x^2-1)^2}$

$= \dfrac{2x^4 - 6x^2 - 6x}{(x^2-1)^2}$

$= \dfrac{2x(x^3 - 3x - 3)}{(x^2-1)^2}$

b $y = \dfrac{e^x}{x^2}$

Let $u = e^x$ and $v = x^2$

$\dfrac{du}{dx} = e^x$ and $\dfrac{dv}{dx} = 2x$

Quotient rule: $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{x^2(e^x) - e^x(2x)}{(x^2)^2}$

$= \dfrac{x^2e^x - 2xe^x}{(x^2)^2}$

$= \dfrac{xe^x(x-2)}{x^4}$

$= \dfrac{e^x(x-2)}{x^3}$

Differentiating composite functions

Sometimes you will need to differentiate an expression that is a function of a function.

For example, look at $y = \sqrt{x^2+1}$; the function 'Take the positive square root of', denoted by $\sqrt{}$, is applied to the function (x^2+1). In such cases you use the **chain rule**.

You know how to differentiate $y = \sqrt{x}$ and you know how to differentiate $y = x^2+1$ but so far you have not met the case where two functions like this are combined into one.

The first step is to make a substitution.

Let $u = x^2 + 1$.

So now you have to differentiate $y = \sqrt{u}$ or $y = u^{\frac{1}{2}}$.

You know the right-hand side becomes $\frac{1}{2}u^{\frac{1}{2}-1}$ or $\frac{1}{2}u^{-\frac{1}{2}}$.

What about the left-hand side?

The differentiation is with respect to u rather than x and so you get $\frac{\mathrm{d}y}{\mathrm{d}u}$ rather than $\frac{\mathrm{d}y}{\mathrm{d}x}$ that you actually want.

To go from $\frac{\mathrm{d}y}{\mathrm{d}u}$ to $\frac{\mathrm{d}y}{\mathrm{d}x}$, you use the **chain rule**,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}.$$

You made the substitution $u = x^2 + 1$ and differentiating this gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x.$$

So now you can substitute for both $\frac{\mathrm{d}y}{\mathrm{d}u}$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$ in the chain rule and get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}u^{-\frac{1}{2}} \times 2x = xu^{-\frac{1}{2}}$$

This isn't quite the final answer because the right-hand side includes the letter u whereas it should be given entirely in terms of x.

Substituting $u = x^2 + 1$, gives $\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(x^2+1\right)^{-\frac{1}{2}}$ and this is now the answer.

However it can be written more neatly as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\sqrt{x^2+1}}.$$

Notice how the awkward function, $\sqrt{x^2+1}$, has reappeared in the final answer.

This is an important method and with experience you will find short cuts that will mean you don't have to write everything out in full as it has been here.

Worked example

Given that $y=(2x-3)^4$, find $\frac{dy}{dx}$.

Solution

Let $u=(2x-3)$ so $y=u^4$

$$\frac{dy}{du}=4u^3$$
$$=4(2x-3)^3$$
$$\frac{du}{dx}=2$$

Using $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$$\Rightarrow \frac{dy}{dx}=4(2x-3)^3\times 2$$
$$=8(2x-3)^3$$

You can use the chain rule in conjunction with the product rule or the quotient rule as shown in the following example.

Worked example

Find $\frac{dy}{dx}$ when $y=(2x+1)(x+2)^{10}$.

Solution

Let $u=(2x+1)$ and $v=(x+2)^{10}$

Using the chain rule to find $\frac{dv}{dx}$

Then $\frac{du}{dx}=2$ and $\frac{dv}{dx}=10(x+2)^9\times 1$

Using the product rule $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

$$\frac{dy}{dx}=(2x+1)\times 10(x+2)^9+(x+2)^{10}\times 2$$
$$=10(2x+1)(x+2)^9+2(x+2)^{10}$$
$$=2(x+2)^9[5(2x+1)+(x+2)]$$
$$=2(x+2)^9(11x+7)$$

Taking $2(x+2)^9$ out as a common factor

Exercise 14.5

1 Differentiate each of the following functions:

a $y=3\sin x-2\tan x$ **b** $y=5\sin\theta-6$ **c** $y=2\cos\theta-2\sin\theta$
d $y=4\ln x$ **e** $y=\ln 4x$ **f** $y=3e^x$
g $y=2e^x-\ln 2x$

2 Use the product rule to differentiate each of the following functions:

a $y=x\sin x$ **b** $y=x\cos x$ **c** $y=x\tan x$
d $y=e^x\sin x$ **e** $y=e^x\cos x$ **f** $y=e^x\tan x$

3 Use the quotient rule to differentiate each of the following functions:

a $y=\frac{\sin x}{x}$ **b** $y=\frac{x}{\sin x}$ **c** $y=\frac{\cos x}{x^2}$
d $y=\frac{x^2}{\cos x}$ **e** $y=\frac{x}{\tan x}$ **f** $y=\frac{\tan x}{x}$

Exercise 14.5 (cont)

4 Use the chain rule to differentiate each of the following functions:

a $y = (x+3)^4$ **b** $y = (2x+3)^4$ **c** $y = \left(x^2+3\right)^4$

d $y = \sqrt{x+3}$ **e** $y = \sqrt{2x+3}$ **f** $y = \sqrt{x^2+3}$

5 Use an appropriate method to differentiate each of the following functions:

a $y = \dfrac{\sin x}{1+\cos x}$ **b** $y = \dfrac{1+\cos x}{\sin x}$

c $y = \sin x(1+\cos x)$ **d** $y = \cos x(1+\sin x)$

e $y = \sin x(1+\cos x)^2$ **f** $y = \cos x(1+\sin x)^2$

6 Use an appropriate method to differentiate each of the following functions:

a $y = \mathrm{e}^x \ln x$ **b** $y = \dfrac{\mathrm{e}^x}{\ln x}$ **c** $y = \dfrac{\ln x}{\mathrm{e}^x}$

7 Use an appropriate method to differentiate each of the following functions:

a $\mathrm{e}^{-x} \sin x$ **b** $y = \dfrac{\mathrm{e}^{-x}}{\sin x}$ **c** $y = \dfrac{\sin x}{\mathrm{e}^{-x}}$

8 A curve has the equation $y = \sin x - \cos x$ where x is measured in radians.

a Show that the curve passes through the points $(0, -1)$ and $(\pi, 1)$.

b Find the equations of the tangents and normals at each of these points.

9 A curve has the equation $y = 2\tan x - 1$ where x is measured in radians.

a Show that the curve passes through the points $(0, -1)$ and $\left(\frac{\pi}{4}, 1\right)$.

b Find the equations of the tangents and normals at each of these points.

10 A curve has the equation $y = 2\ln x - 1$.

a Show that the curve passes through the point $(e, 1)$

b Find the equations of the tangent and normal at this point.

11 A curve has the equation $y = \mathrm{e}^x - \ln x$.

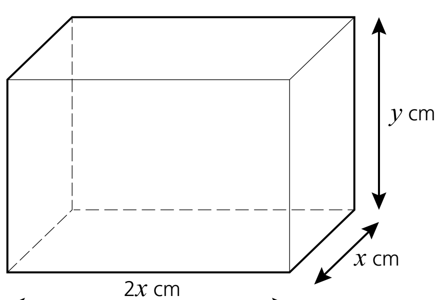
a Sketch the curves $y = \mathrm{e}^x$ and $y = \ln x$ on the same axes and explain why this implies that $\mathrm{e}^x - \ln x$ is always positive.

b Show that the curve $y = \mathrm{e}^x - \ln x$ passes through the point $(1, e)$.

c Find the equations of the tangent and normal at this point.

Past-paper questions

1 The diagram shows a cuboid with a rectangular base of sides x cm and $2x$ cm. The height of the cuboid is y cm and its volume is 72 cm^3.



(i) Show that the surface area A cm^2 of the cuboid is given by $A = 4x^2 + \dfrac{216}{x}$. [3]

(ii) Given that x can vary, find the dimensions of the cuboid when A is a minimum. [4]

(iii) Given that x increases from 2 to $2 + p$, where p is small, find, in terms of p, the corresponding approximate change in A, stating whether this change is an increase or a decrease. [3]

Cambridge O Level Additional Mathematics 4037
Paper 11 Q12–OR June 2011
Cambridge IGCSE Additional Mathematics 0606
Paper 11 Q12–OR June 2011

2 Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when

(i) $y = \cos 2x \sin\left(\dfrac{x}{3}\right)$, [4]

(ii) $y = \dfrac{\tan x}{1 + \ln x}$. [4]

Cambridge O Level Additional Mathematics 4037
Paper 21 Q10 June 2014
Cambridge IGCSE Additional Mathematics 0606
Paper 21 Q10 June 2014

3 **(a)** Differentiate $4x^3 \ln(2x + 1)$ with respect to x. [3]

(b) **(i)** Given that $y = \dfrac{2x}{\sqrt{x+2}}$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x+4}{(\sqrt{x+2})^3}$. [4]

Cambridge O Level Additional Mathematics 4037
Paper 12 November 2013
(Part question: part b (ii) and (iii) omitted)
Cambridge IGCSE Additional Mathematics 0606
Paper 12 November 2013
(Part question: part b (ii) and (iii) omitted)

Learning outcomes

Now you should be able to:

★ understand the idea of a derived function
★ use the notations $\mathrm{f}'(x)$, $\mathrm{f}''(x)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ $\left[= \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\right]$
★ apply differentiation to gradients, tangents and normals, stationary points, connected rates of change, small increments and approximations and practical maxima and minima problems
★ use the first and second derivative tests to discriminate between maxima and minima
★ use the derivatives of the standard functions x^n (for any rational n), $\sin x$, $\cos x$, $\tan x$, e^x, $\ln x$, together with constant multiples, sums and composite functions of these
★ differentiate products and quotients of functions.

Key points

- ✔ $y = kx^n \Rightarrow \frac{dy}{dx} = nkx^{n-1}$ and $y = c \Rightarrow \frac{dy}{dx} = 0$,
  where n is a positive integer and k and c are constants.
- ✔ $y = f(x) + g(x) \Rightarrow \frac{dy}{dx} = f'(x) + g'(x)$
- ✔ $\frac{dy}{dx} = 0$ at a stationary point. The nature of the stationary point can be determined by looking at the sign of the gradient immediately either side of it or by considering the sign of $\frac{d^2y}{dx^2}$.
  - If $\frac{d^2y}{dx^2} < 0$, the point is a maximum.
  - If $\frac{d^2y}{dx^2} > 0$, the point is a minimum.
  - If $\frac{d^2y}{dx^2} = 0$, the point could be a maximum, a minimum or a point of inflection. Check the values of $\frac{dy}{dx}$ on either side of the point to determine its nature.
- ✔ For the tangent and normal at (x_1, y_1)
  - the gradient of the tangent, $m_1 = \frac{dy}{dx}$
  - the gradient of the normal, $m_2 = -\frac{1}{m_1}$
  - the equation of the tangent is $y - y_1 = m_1(x - x_1)$
  - the equation of the normal is $y - y_1 = m_2(x - x_1)$.
- ✔ Derivatives of other functions:

Function	**Derivative** $\frac{dy}{dx}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x
$\ln x$	$\frac{1}{x}$

- ✔ The product rule $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.
- ✔ The quotient rule $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.
- ✔ The chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

15 Integration

Growth is painful. Change is painful. But nothing is as painful as staying stuck where you do not belong.

N.R. Narayana Murthy (1946 –)

Discussion point

Mita is a long-distance runner. She carries a speed meter, which tells her what her speed is at various times during a race.

Time (hours)	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
Speed (metres per second)	4.4	4.4	4.4	4.6	5.0	5.2	0

What race do you think she was running?

How would you estimate her time?

Integration is the process of getting from a differential equation to the general solution.

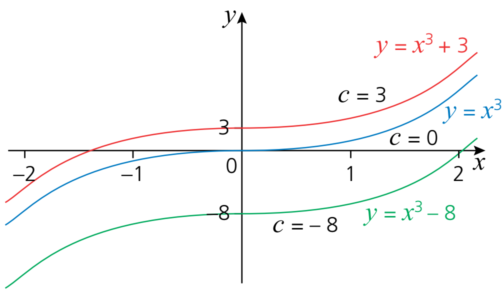
Integration involves using the rate of change of a quantity to find its total value at the end of an interval, for example using the speed of a runner to find the distance travelled at any time. The process is the reverse of differentiation.

Look at the **differential equation** $\frac{dy}{dx} = 3x^2$.

Since $\frac{dy}{dx} = 3x^2$ for x^3, $x^3 + 7$ and $x^3 - 3$, these expressions are all solutions of this equation.

The **general solution** of this differential equation is given as $y = x^3 + c$, where c is an **arbitrary constant** that can take any value (positive, negative or zero).

A solution containing an arbitrary constant gives a family of curves, as shown below. Each curve corresponds to a particular value of c.



Suppose that you are also given that the solution curve passes through the point (1, 4). Substituting these coordinates in $y = x^3 + c$ gives

$$4 = 1^3 + c \quad \Rightarrow \quad c = 3$$

So the equation of the curve is $y = x^3 + 3$. This is called the **particular solution.**

This example shows that if you know a point on a curve in the family, you can find the value of c and therefore the particular solution of a differential equation.

The rule for differentiation is usually given as

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}.$$

It can also be given as

$$y = x^{n+1} \quad \Rightarrow \quad \frac{dy}{dx} = (n+1)x^n$$

which is the same as

$$y = \frac{1}{n+1}x^{n+1} \quad \Rightarrow \quad \frac{dy}{dx} = x^n.$$

Reversing this gives you the rule for integrating x^n. This is usually written using the integral symbol, $\int$.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad \text{for} \quad n \neq -1$$

Note that if you are asked to integrate an expression f(x), this will mean integrate with respect to x unless otherwise stated.

The integral when $n = -1$ is a special case. If you try to apply the general rule, $n + 1$ is zero on the bottom line and so the expression you get is undefined. Instead you use the result that:

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$$

You will use this result later in the chapter.

Notice the use of dx on the left-hand side. This tells you that you are integrating with respect to x. So in this case you would read the left-hand side as 'The integral of x^n with respect to x'.

You may find it helpful to remember the rule as

- **add 1 to the power**
- **divide by the new power**
- **add a constant.**

Remember to include the arbitrary constant, c, until you have enough information to find a value for it.

Worked example

Integrate each of the following:

a x^6 b $5x^4$ c 7 d $4\sqrt{x}$

Solution

a $\frac{x^7}{7} + c$ b $5 \times \frac{x^5}{5} + c = x^5 + c$

c 7 can be thought of as $7x^0$ so applying the rule gives $7x + c$

d $4\sqrt{x} = 4x^{\frac{1}{2}}$ so applying the rule gives $\dfrac{4x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{8}{3}x^{\frac{3}{2}} + c$

Worked example

Given that $\frac{dy}{dx} = 6x^2 + 2x - 5$

a Find the general solution of this differential equation.

b Find the equation of the curve with this gradient function that passes through the point $(1, 7)$.

Solution

a $y = 6 \times \frac{x^3}{3} + 2 \times \frac{x^2}{2} - 5x + c$

By integration

$= 2x^3 + x^2 - 5x + c$

b Since $(1, 7)$ is a point on the graph

$7 = 2(1)^3 + 1^2 - 5 + c$

$\Rightarrow \quad c = 9$

$\Rightarrow \quad y = 2x^3 + x^2 - 5x + 9$

Worked example

Find f(x) given that f′(x) = $2x + 4$ and f(2) = –4.

Solution

$f'(x) = 2x + 4$

By integration

$f(x) = \frac{2x^2}{2} + 4x + c$

$= x^2 + 4x + c$

$f(2) = -4 \Rightarrow -4 = (2)^2 + 4(2) + c$

$\Rightarrow c = -16$

$\Rightarrow f(x) = x^2 + 4x - 16$

Worked example

A curve passes through (3, 5). The gradient of the curve is given by $\frac{dy}{dx} = x^2 - 4$.

a Find y in terms of x.

b Find the coordinates of any stationary points of the graph of y.

c Sketch the curve.

Solution

a $\frac{dy}{dx} = x^2 - 4 \Rightarrow y = \frac{x^3}{3} - 4x + c$

When $x = 3$, $5 = 9 - 12 + c$

$\Rightarrow c = 8$

So the equation of the curve is $y = \frac{x^3}{3} - 4x + 8$.

b $\frac{dy}{dx} = 0$ at all stationary points.

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x + 2)(x - 2) = 0$

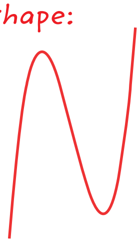
Substituting these values into the equation to find y

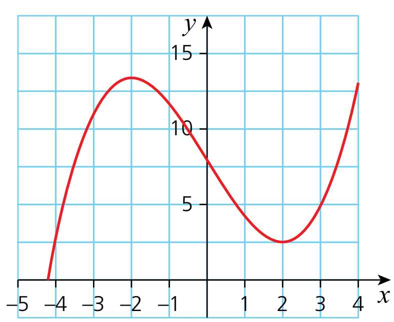
$\Rightarrow x = -2$ or $x = 2$

The stationary points are $(-2, 13\frac{1}{3})$ and $(2, 2\frac{2}{3})$.

c It crosses the y-axis at (0, 8).

The curve is a cubic with a positive x^3 term with two turning points, so it has this shape:





Worked example

Find $\int (x^3 - 2x^2)\,dx$.

Solution

$$\int (x^3 - 2x^2)\,dx = \frac{x^4}{4} - \frac{2x^3}{3} + c$$

Worked example

Find $\int (2x+1)(x-4)\,dx$

Solution

You need to multiply out the brackets before you can integrate.

$$\int (2x+1)(x-4)\,dx = \int (2x^2 - 7x - 4)\,dx$$

$$= \frac{2x^3}{3} - \frac{7x^2}{2} - 4x + c$$

Exercise 15.1

1 Find y in each of the following cases:

a $\frac{dy}{dx} = 4x + 2$

b $\frac{dy}{dx} = 6x^2 - 5x - 1$

c $\frac{dy}{dx} = 3 - 5x^3$

d $\frac{dy}{dx} = (x - 2)(3x + 2)$

2 Find $f(x)$ given that

a $f'(x) = 5x + 3$

b $f'(x) = x^4 + 2x^3 - x + 8$

c $f'(x) = (x - 4)(x^2 + 2)$

d $f'(x) = (x - 7)^2$

3 Find the following integrals:

a $\int 5\,dx$

b $\int 5x^3\,dx$

c $\int (2x - 3)\,dx$

d $\int (3x^3 - 4x + 3)\,dx$

4 Find the following integrals:

a $\int (3 - x)^2\,dx$

b $\int (2x + 1)(x - 3)\,dx$

c $\int (x + 1)^2\,dx$

d $\int (2x - 1)^2\,dx$

5 Find the equation of the curve $y = f(x)$ that passes through the specified point for each of the following gradient functions:

a $\frac{dy}{dx} = 2x - 3$; $(2, 4)$

b $\frac{dy}{dx} = 4 + 3x^3$ $(4, -2)$

c $\frac{dy}{dx} = 5x - 6$; $(-2, 4)$

d $f'(x) = x^2 + 1$; $(-3, -3)$

e $f'(x) = (x + 1)(x - 2)$; $(6, -2)$

f $f'(x) = (2x + 1)^2$; $(1, -1)$

Exercise 15.1 (cont)

6 Find the equation of the curve $y = f(x)$ that passes through the specified point for each of the following gradient functions:

a $\frac{dy}{dx} = 2\sqrt{x} - 1$; $(1, 1)$

b $f'(x) = x - \sqrt{x}$; $(4, 2)$

7 You are given that $\frac{dy}{dx} = 2x + 3$.

a Find $\int (2x + 3)\,dx$.

b Find the general solution of the differential equation.

c Find the equation of the curve with gradient function $\frac{dy}{dx}$ and that passes through $(2, -1)$.

d Hence show that $(-1, -13)$ lies on the curve.

8 The curve C passes through the point $(3, 21)$ and its gradient at any point is given by $\frac{dy}{dx} = 3x^2 - 4x + 1$.

a Find the equation of the curve C.

b Show that the point $(-2, -9)$ lies on the curve.

9 a Find $\int (4x - 1)\,dx$.

b Find the general solution of the differential equation $\frac{dy}{dx} = 4x - 1$.

c Find the particular solution that passes through the point $(-1, 4)$.

d Does this curve pass above, below or through the point $(2, 4)$?

10 The curve $y = f(x)$ passes through the point $(2, -4)$ and $f'(x) = 2 - 3x^2$. Find the value of $f(-1)$.

11 A curve, C, has stationary points at the points where $x = 0$ and where $x = 2$.

a Explain why $\frac{dy}{dx} = x^2 - 2x$ is a possible expression for the gradient of C. Give a different possible expression for $\frac{dy}{dx}$.

b The curve passes through the point $(3, 2)$.

Given that $\frac{dy}{dx}$ is $x^2 - 2x$, find the equation of C.

Definite integrals

So far, all the integrals you have met have been **indefinite integrals** such as $\int 3x^2 dx$; the resulting expressions for y have all finished with '$+ c$'. You may or may not have been given additional information to enable you to find a value for c.

By contrast, a **definite integral** has two limits.

$$\int_1^3 3x^2 dx$$

This is the upper limit.

This is the lower limit.

To find the value of a definite integral, you integrate it and substitute in the values of the limits. Then you subtract the value of the integral at the lower limit from the value of the integral at the upper limit.

Worked example

Find $\int_1^3 3x^2\,dx$.

Solution

Subtracting the value at $x = 1$ from the value at $x = 3$

$$\int 3x^2\,dx = x^3 + c$$

$$(3^3 + c) - (1^3 + c) = 26 \text{ so } \int_1^3 3x^2 dx = 26$$

Notice how the c is eliminated when you simplify this expression.

When evaluating definite integrals, it is common practice to omit the c and write

$$\int_1^3 3x^2\,dx = [x^3]_1^3 = [3^3] - [1^3] = 26.$$

The definite integral is defined as

$$\int_a^b f'(x)\,dx = [f(x)]_a^b = f(b) - f(a).$$

Worked example

'Evaluate' means 'find the numerical value of'.

Evaluate $\int_1^4 (x^2 + 3)\,dx$.

Solution

$$\int_1^4 (x^2 + 3)\,dx = \left[\frac{x^3}{3} + 3x\right]_1^4$$

$$= \left(\frac{4^3}{3} + 3\times 4\right) - \left(\frac{1^3}{3} + 3\times 1\right)$$

$$= 30$$

Worked example

Evaluate $\int_{-1}^3 (x + 1)(x - 3)\,dx$.

Solution

Notice how you need to expand $(x+1)(x+3)$ before integrating it.

$$\int_{-1}^3 (x + 1)(x - 3)\,dx = \int_{-1}^3 (x^2 - 2x - 3)\,dx.$$

$$= \left[\frac{x^3}{3} - x^2 - 3x\right]_{-1}^3$$

$$= \left(\frac{3^3}{3} - 3^2 - 3\times 3\right) - \left(\frac{(-1)^3}{3} - (-1)^2 - 3\times(-1)\right)$$

$$= -10\frac{2}{3}$$

Exercise 15.2

Evaluate the following definite integrals:

1 $\int_1^2 3x^2\,dx$

2 $\int_1^4 4x^3\,dx$

3 $\int_{-1}^1 6x^2\,dx$

4 $\int_1^5 4\,dx$

5 $\int_2^4 (x^2+1)\,dx$

6 $\int_{-2}^3 (2x+5)\,dx$

7 $\int_2^5 (4x^3-2x+1)\,dx$

8 $\int_5^6 (x^2-5)\,dx$

9 $\int_1^3 (x^2-3x+1)\,dx$

10 $\int_{-1}^2 (x^2+3)\,dx$

11 $\int_{-4}^{-1} (16-x^2)\,dx$

12 $\int_1^3 (x+1)(3-x)\,dx$

13 $\int_2^4 (3x(x+2))\,dx$

14 $\int_{-1}^1 (x+1)(x-1)\,dx$

15 $\int_{-1}^2 (x+4x^2)\,dx$

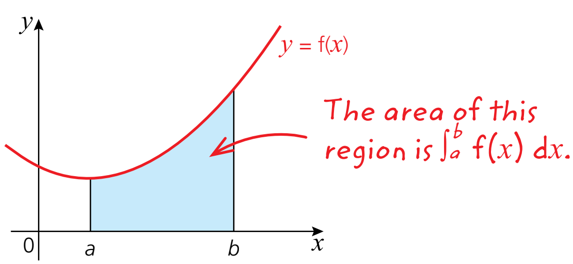
16 $\int_{-1}^1 x(x-1)(x+1)\,dx$

17 $\int_{-1}^3 (x^3+2)\,dx$

18 $\int_{-3}^1 (9-x^2)\,dx$

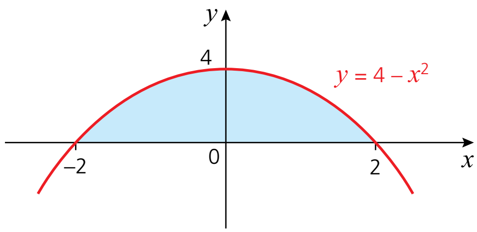
Finding the area between a graph and the x-axis

The area under the curve, $y = f(x)$ between $x = a$ and $x = b$, the shaded region in the graph below, is given by a definite integral: $\int_a^b f(x)\,dx$.



Worked example

Find the area of the shaded region under the curve $y = 4 - x^2$.



Solution

$$\text{Area} = \int_{-2}^{2} (4 - x^2)\ dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

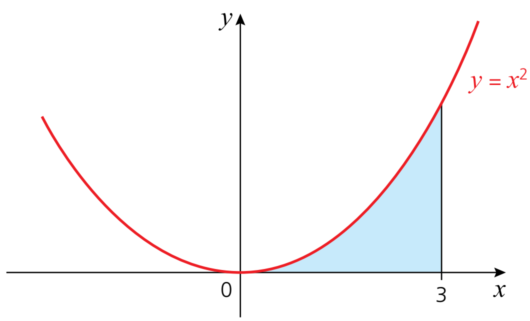
$$= \left[4 \times 2 - \frac{2^3}{3}\right] - \left[4 \times (-2) - \frac{(-2)^3}{3}\right]$$

$$= 10\frac{2}{3}\ \text{units}^2$$

Exercise 15.3

Find the area of each of the shaded regions:

1



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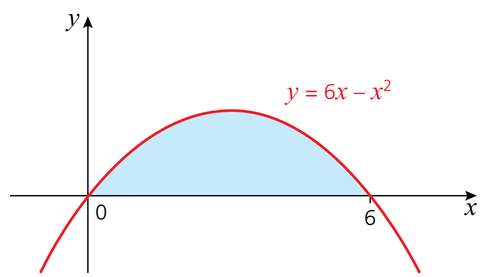

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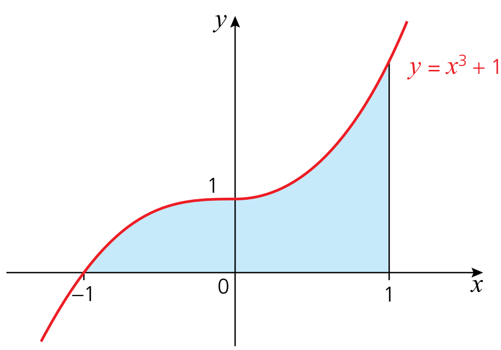

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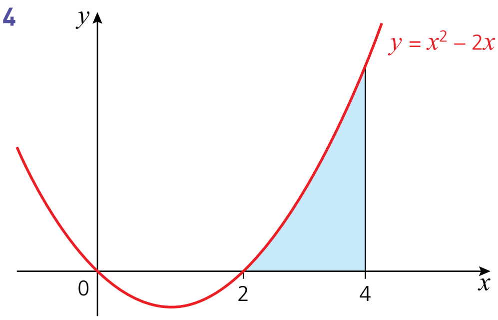
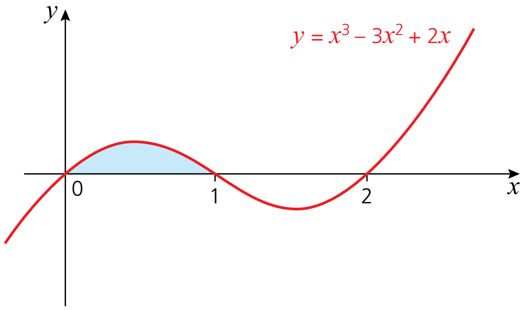


Exercise 15.3 (cont)

5



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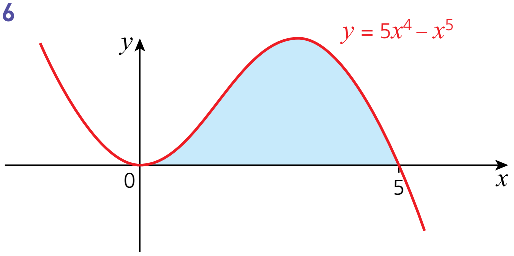

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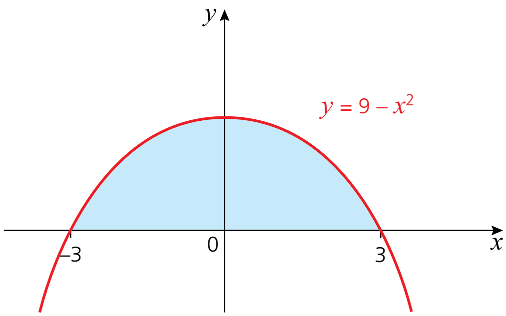

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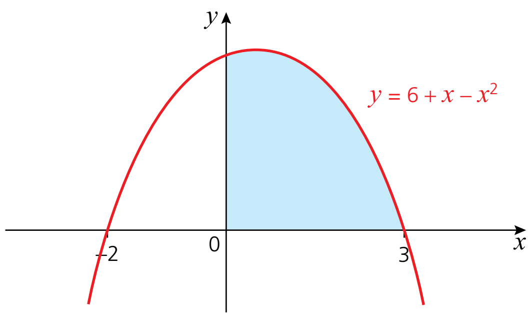

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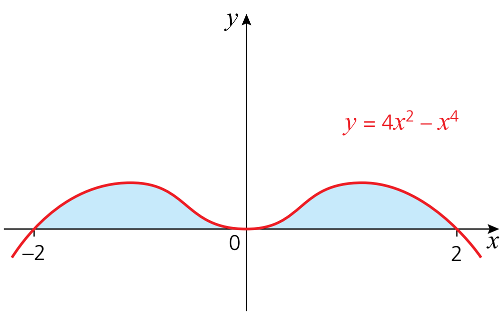

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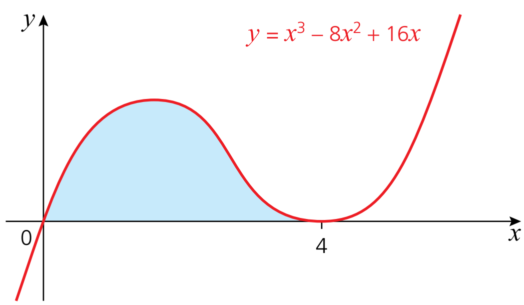
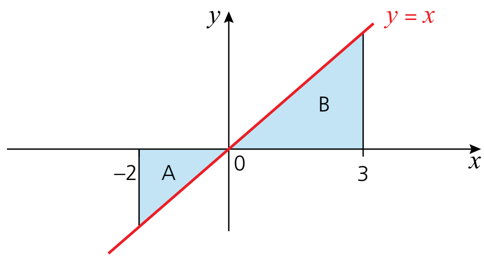


So far all the areas you have found have been above the x-axis. The next example involves a region that is below the x-axis.

Worked example

The diagram shows the line $y = x$ and two regions marked A and B.



a Calculate the areas of A and B using the formula for the area of a triangle.

b Evaluate $\int_{-2}^{0} x\,dx$ and $\int_{0}^{3} x\,dx$. What do you notice?

c Evaluate $\int_{-2}^{3} x\,dx$. What do you notice?

Solution

a Area of A $= \frac{1}{2} \times 2 \times 2 = 2$ square units.

Area of B $= \frac{1}{2} \times 3 \times 3 = 4.5$ square units.

b
$$\int_{-2}^{0} x\,dx = \left(\frac{x^2}{2}\right)_{-2}^{0}$$
$$= 0 - (2)$$
$$= -2$$

$$\int_{0}^{3} x\,dx = \left(\frac{x^2}{2}\right)_{0}^{3}$$
$$= 4.5 - 0$$
$$= 4.5$$

The areas have the same numerical values as the integral but when the area is below the x-axis, the integral is negative.

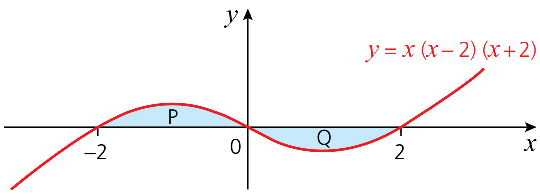
c
$$\int_{-2}^{3} x\,dx = \left(\frac{x^2}{2}\right)_{-2}^{3} = 4.5 - (2)$$
$$= 4.5 - (2)$$
$$= 2.5$$

The areas above and below the x-axis have cancelled each other out.

This example shows you how using integration gives a negative answer for the area of a region below the x-axis. In some contexts this will make sense and in others it won't, so you always have to be careful.

Worked example

The curve $y = x(x - 2)(x + 2)$ is drawn on the axes.



a Use integration to find the areas of each of the shaded regions P and Q.

b Evaluate $\int_{-2}^{2} x(x-2)(x+2)\,dx$.

c What do you notice?

Solution

a Area of P:

$$\int_{-2}^{0} x(x-2)(x+2)\,dx = \int_{-2}^{0} (x^3 - 4x)\,dx$$
$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0}$$
$$= 0 - \left[\frac{(-2)^4}{4} - 2\times(-2)^2\right]$$
$$= 4$$

So P has an area of 4 units^2.

Area of Q:

$$\int_{0}^{2} x(x-2)(x+2)\,dx = \int_{0}^{2} (x^3 - 4x)\,dx$$
$$= \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$
$$= \left[\frac{2^4}{4} - 2\times 2^2\right] - 0$$
$$= -4$$

So Q also has an area of 4 units^2.

The areas of P and Q are the same since the curve has rotational symmetry about the origin.

b

$$\int_{-2}^{2} x(x-2)(x+2)\,dx = \int_{-2}^{2} (x^3 - 4x)\,dx$$
$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{2}$$
$$= \left[\frac{2^4}{4} - 2\times 2^2\right] - \left[\frac{(-2)^4}{4} - 2\times(-2)^2\right]$$
$$= 0$$

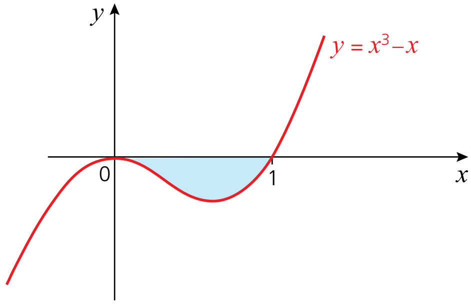
Always draw a sketch graph when you are going to calculate areas. This will avoid any cancelling out of areas above and below the x-axis.

c The areas of P and Q have ‘cancelled out’.

Exercise 15.4

1 The sketch shows the curve $y = x^3 - x$.

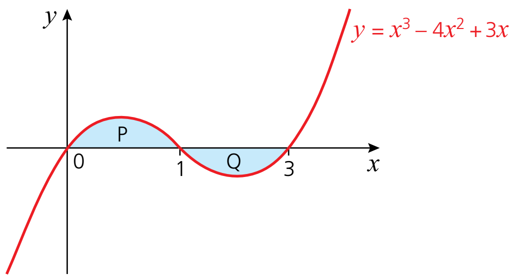
Calculate the area of the shaded region.



2 The sketch shows the curve $y = x^3 - 4x^2 + 3x$.

a Calculate the area of each shaded region.

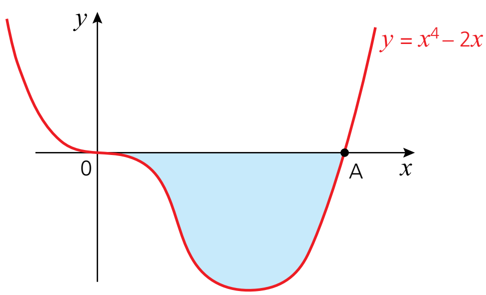
b State the area enclosed between the curve and the x-axis.



3 The sketch shows the curve $y = x^4 - 2x$.

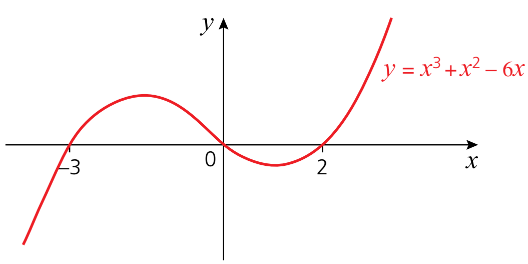
a Find the coordinates of the point A.

b Calculate the area of the shaded region.



4 The sketch shows the curve $y = x^3 + x^2 - 6x$.

Work out the area between the curve and the x-axis.



Exercise 15.4 (cont)

5 a Sketch the curve $y = x^2$ for $-3 < x < 3$.
b Shade the area bounded by the curve, the lines $x = -1$ and $x = 2$ and the x-axis.
c Find, by integration, the area of the region you have shaded.

6 a Sketch the curve $y = x^2 - 2x$ for $-1 < x < 3$.
b For what values of x does the curve lie below the x-axis?
c Find the area between the curve and the x-axis.

7 a Sketch the curve $y = x^3$ for $-3 < x < 3$.
b Shade the area between the curve, the x-axis and the line $x = 2$.
c Find, by integration, the area of the region you have shaded.
d Without any further calculation, state, with reasons, the value of $\int_{-2}^{2} x^3 \, dx$.

8 a Shade, on a suitable sketch, the region with an area given by $\int_{-1}^{2} (x^2 + 1) \, dx$.
b Evaluate this integral.

9 a Evaluate $\int_{1}^{4} (2x + 1) \, dx$.
b Interpret this integral on a sketch graph.

Integrating other functions of x

As with differentiation, there are a number of special cases when integrating. The proofs are beyond the scope of this book, but you will need to know and be able to use these results.

Differentiation ⇒	Basic integral ⇒	Generalised integral
$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$, for $n \neq -1$	$\int (ax+b)^n \, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$, for $n \neq -1$
$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$	$\int \cos x \, dx = \sin x + c$	$\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + c$
$y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$	$\int \sin x \, dx = -\cos x + c$	$\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + c$
$y = e^x \Rightarrow \frac{dy}{dx} = e^x$	$\int e^x \, dx = e^x + c$	$\int e^{(ax+b)} \, dx = \frac{1}{a}e^{(ax+b)} + c$
$y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln\lvert x\rvert + c$	$\int \frac{1}{ax+b} \, dx = \frac{1}{a}\ln\lvert ax+b\rvert + c$

Integrating $\frac{1}{x}$ (i.e. x^{-1}) does not follow the general rule since that would give $\frac{x^0}{0}$ which is undefined.

Worked example

Find the following indefinite integrals:

a $\int \frac{1}{2x-3}\,dx$ **b** $\int (2x-3)^4\,dx$ **c** $\int (2x-3)^{\frac{1}{2}}\,dx$

d $\int e^{2x-3}\,dx$ **e** $\int \sin(2x-3)\,dx$ **f** $\int \cos(2x-3)\,dx$

Solution

a Using $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

gives $\int \frac{1}{2x-3}\,dx = \frac{1}{2}\ln|2x-3| + c$

b Using $\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$

gives $\int (2x-3)^4\,dx = \frac{1}{2}\frac{(2x-3)^5}{5} + c = \frac{(2x-3)^5}{10} + c$

c Using $\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$

gives $\int (2x-3)^{\frac{1}{2}}\,dx = \frac{1}{2}\frac{(2x-3)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{3}(2x-3)^{\frac{3}{2}} + c$

d Using $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

gives $\int e^{2x-3}\,dx = \frac{1}{2}e^{2x-3} + c$

e Using $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

gives $\int \sin(2x-3)\,dx = -\frac{1}{2}\cos(2x-3) + c$

f Using $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

gives $\int \cos(2x-3)\,dx = \frac{1}{2}\sin(2x-3) + c$

Worked example

When integrating trigonometric functions, the angles must be in radians

Evaluate the following definite integrals:

a $\int_2^3 \frac{1}{2x+1}\,dx$ **b** $\int_2^3 (2x+1)^4\,dx$ **c** $\int_2^3 e^{2x+1}\,dx$

d $\int_0^1 (2x+1)^{\frac{1}{2}}\,dx$ **e** $\int_0^1 (2x+1)^{-2}\,dx$ **f** $\int_0^{\frac{\pi}{3}} \sin\left(2x+\frac{\pi}{6}\right)dx$

g $\int_0^{\frac{\pi}{3}} \cos\left(2x+\frac{\pi}{6}\right)dx$

Solution

Using $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|+c$

a
$$\int_2^3 \frac{1}{2x+1}\,dx = \left[\frac{1}{2}\ln|2x+1|\right]_2^3$$
$$= \frac{1}{2}\ln 7 - \frac{1}{2}\ln 5$$
$$= \frac{1}{2}(\ln 7 - \ln 5)$$
$$= \frac{1}{2}\ln\frac{7}{5}$$

Using $\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c$

b
$$\int_2^3 (2x+1)^4\,dx = \left[\frac{1}{2}\frac{(2x+1)^5}{5}\right]_2^3$$
$$= \frac{1}{2}\left[\frac{7^5}{5} - \frac{5^5}{5}\right]$$
$$= 1368.2$$

Using $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}+c$

c
$$\int_2^3 e^{2x+1}\,dx = \left[\frac{1}{2}e^{2x+1}\right]_2^3$$
$$= \frac{1}{2}(e^7 - e^5)$$

d
$$\int_0^1 (2x+1)^{\frac{1}{2}}\,dx = \left[\frac{1}{2}\frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$
$$= \frac{1}{3}\left[3^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$
$$= 1.40 \text{ (3s.f.)}$$

e
$$\int_0^1 (2x+1)^{-2}\,dx = \left[\frac{1}{2}\frac{(2x+1)^{-1}}{-1}\right]_0^1$$
$$= -\frac{1}{2}[3^{-1} - 1^{-1}]$$
$$= \frac{1}{3}$$

Using $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$

f
$$\int_0^{\frac{\pi}{3}} \sin\left(2x+\frac{\pi}{6}\right)dx = \left[-\frac{1}{2}\cos\left(2x+\frac{\pi}{6}\right)\right]_0^{\frac{\pi}{3}}$$
$$= -\frac{1}{2}\cos\frac{5\pi}{6} + \frac{1}{2}\cos\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2}$$

Using $\int \cos(ax+b)\,dx$

$= \frac{1}{a}\sin(ax+b) + c$

g $\int_0^{\frac{\pi}{3}} \cos\left(2x+\frac{\pi}{6}\right)dx = \left[\frac{1}{2}\sin\left(2x+\frac{\pi}{6}\right)\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\sin\frac{5\pi}{6} - \frac{1}{2}\sin\frac{\pi}{6}$

$= 0$

Exercise 15.5

1 Find the following indefinite integrals:

a $\int \frac{1}{3x+1}dx$ **b** $\int (3x+1)^4\,dx$ **c** $\int e^{3x+1}dx$

d $\int \sin(3x+1)dx$ **e** $\int \cos(3x+1)dx$ **f** $\int \frac{3}{x-3}dx$

g $\int (2x-1)^3\,dx$ **h** $\int 4e^{2x-3}dx$ **i** $\int 3\sin(3x)dx$

j $\int 4\cos\left(\frac{x}{2}\right)dx$ **k** $\int (x-2)^{\frac{3}{2}}\,dx$ **l** $\int (2x-1)^{\frac{3}{2}}\,dx$

2 Evaluate the following definite integrals:

a $\int_2^4 \frac{1}{3x+1}dx$ **b** $\int_2^4 (3x+1)^4\,dx$ **c** $\int_2^4 e^{3x+1}\,dx$

d $\int_0^{\frac{\pi}{3}} \sin\left(3x+\frac{\pi}{3}\right)dx$ **e** $\int_0^{\frac{\pi}{3}} \cos\left(3x+\frac{\pi}{3}\right)dx$ **f** $\int_4^8 \frac{4}{x-2}dx$

g $\int_{-1}^3 (2x+3)^4\,dx$ **h** $\int_0^2 10e^{-2x}\,dx$ **i** $\int_0^{\frac{\pi}{2}} \sin\left(2x-\frac{\pi}{4}\right)dx$

j $\int_0^{\frac{\pi}{2}} \cos\left(2x-\frac{\pi}{4}\right)dx$

Past–paper questions

1 (a) A curve is such that $\frac{dy}{dx} = ae^{1-x} - 3x^2$, where a is a constant. At the point $(1, 4)$, the gradient of the curve is 2.

(i) Find the value of a. [1]

(ii) Find the equation of the curve. [5]

(b) (i) Find $\int (7x+8)^{\frac{1}{3}}dx$. [2]

(ii) Hence evaluate $\int_0^8 (7x+8)^{\frac{1}{3}}dx$. [2]

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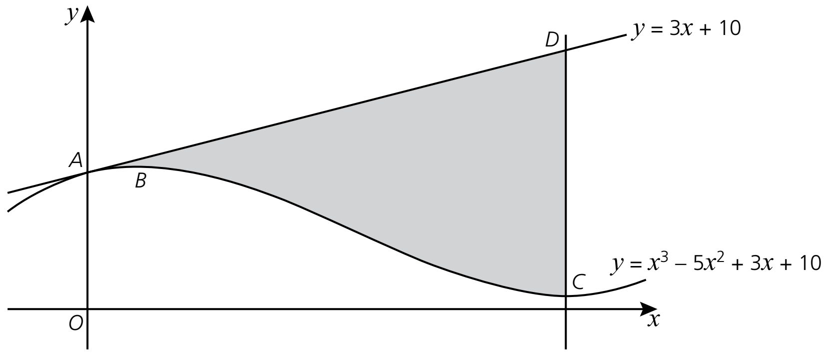
Cambridge IGCSE Additional Mathematics 0606
Paper 11 Q10 June 2011

2



The diagram shows parts of the line $y = 3x + 10$ and the curve $y = x^3 - 5x^2 + 3x + 10$. The line and the curve both pass through the point A on the y-axis. The curve has a maximum at the point B and a minimum at the point C. The line through C, parallel to the y-axis, intersects the line $y = 3x + 10$ at the point D.

(i) Show that the line AD is a tangent to the curve at A. [2]

(ii) Find the x-coordinate of B and of C. [3]

(iii) Find the area of the shaded region $ABCD$, showing all your working. [5]

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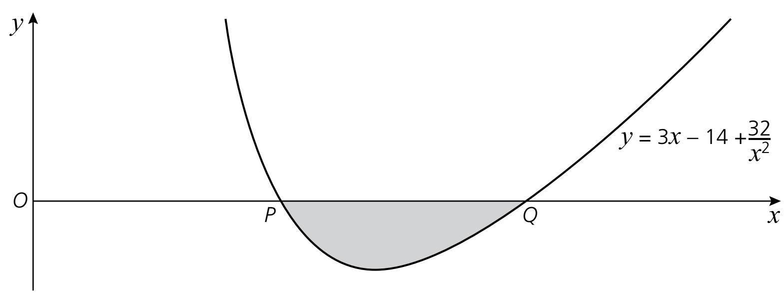
Paper 11 Q9 June 2015
Cambridge IGCSE Additional Mathematics 0606
Paper 11 Q9 June 2015

3 The diagram below shows part of the curve $y = 3x - 14 + \dfrac{32}{x^2}$ cutting the x-axis at the points P and Q.



(iii) State the x-coordinates of P and Q. [1]

(iv) Find $\int (3x - 14 + \dfrac{32}{x^2})\, dx$ and hence determine the area of the shaded region. [4]

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(Part question: parts (i) and (ii) omitted)
Cambridge IGCSE Additional Mathematics 0606
Paper 22 Q12 November 2014
(Part question: parts (i) and (ii) omitted)

Learning outcomes

Now you should be able to:

- ★ understand integration as the reverse process of differentiation
- ★ evaluate definite integrals and apply integration to the evaluation of plane areas
- ★ integrate sums of terms in powers of x including $\frac{1}{x}$ and $\frac{1}{ax+b}$
- ★ integrate functions of the form $(ax + b)^n$ for any rational n, $\sin(ax + b)$, $\cos(ax + b)$, e^{ax+b}

Key points

- ✔ $\frac{dy}{dx} = x^n \Rightarrow y = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$

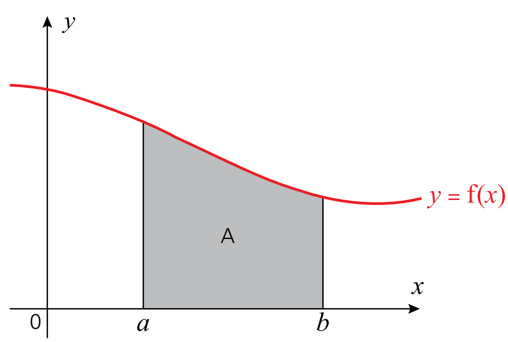
  This is an indefinite integral.

- ✔ $\int_a^b x^n \, dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}$ for $n \neq -1$

  This is a definite integral.

- ✔ The area of a region between a curve $y = f(x)$ and the x-axis is given by $\int_a^b y \, dx$.

  Area of A $= \int_a^b y \, dx = \int_a^b f(x) \, dx$



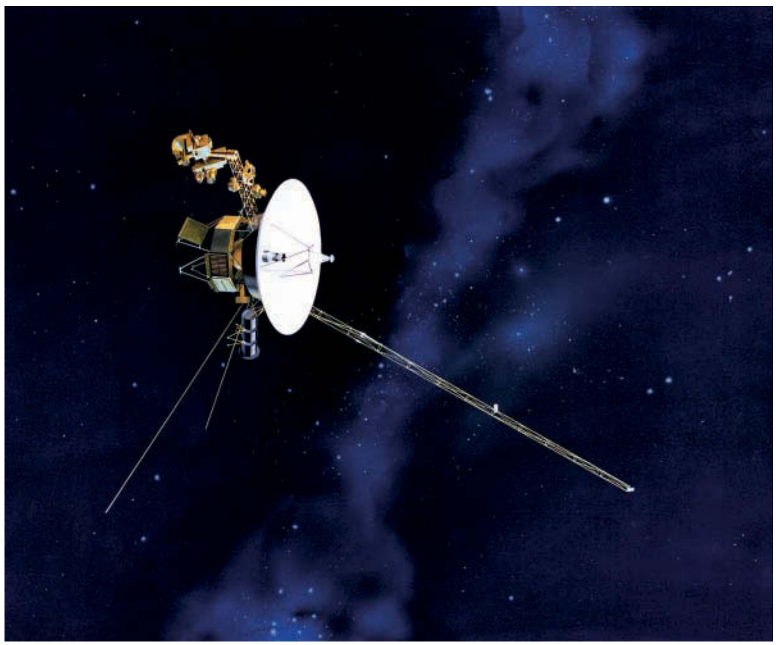
- ✔ Areas below the x-axis give rise to negative values for the integral.
- ✔ Integrals of other functions where c is a constant:

Function $y = \mathrm{f}(x)$	**Integral** $\int y \, \mathrm{d}x$
$\frac{1}{x}$	$\ln\lvert x \rvert + c$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b \rvert + c$
$(ax+b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$
e^{ax+b}	$\frac{1}{a}\mathrm{e}^{ax+b} + c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + c$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$

16 Kinematics

In future, children won't perceive the stars as mere twinkling points of light: they'll learn that each is a Sun, orbited by planets fully as interesting as those in our Solar system.

Martin Rees (1942 –)



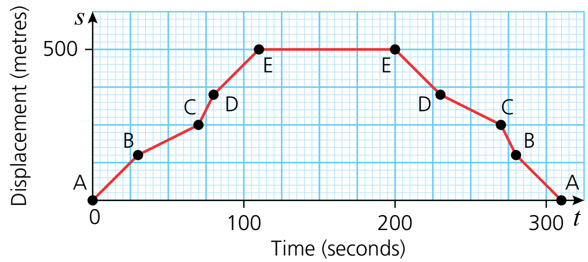
Discussion point

A spacecraft leaves the Earth on a journey to Jupiter. Its initial direction is directly towards Jupiter. Will it travel in a straight line?

Displacement and velocity

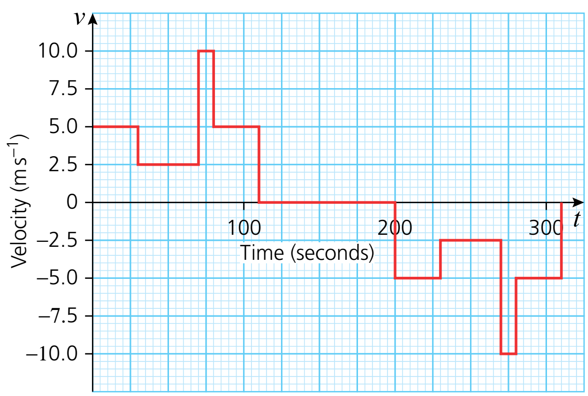
You know that $\frac{\mathrm{dy}}{\mathrm{dx}}$ represents the rate of change of y with respect to x. It gives the gradient of the x–y graph, where x is plotted on the horizontal axis and y on the vertical axis.

The following graph represents the distance, s metres, travelled by a cyclist along a country road in time, t seconds. Time is measured along the horizontal axis and distance from the starting point is measured on the vertical axis. When he reaches E the cyclist takes a short break and then returns home along the same road.



Speed is given by the gradient of the distance–time graph. In this graph the axes are labelled s and t, rather than y and x, so the gradient (representing the speed) is given by $\frac{ds}{dt}$.

A graph showing **displacement** (the distance from the starting point) looks quite different from one showing the total distance travelled.



Velocity is given by the gradient of a displacement–time graph.

Acceleration is $\left(\frac{\text{change in velocity}}{\text{time taken}}\right)$ and this is given by the gradient of a velocity–time graph.

Motion in a straight line

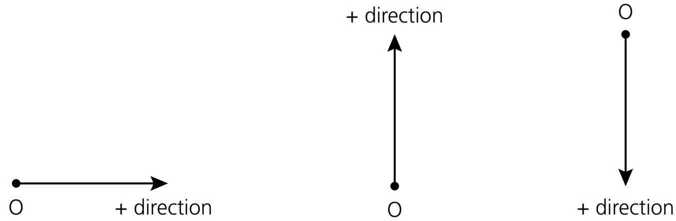
You will be treating each object as a **particle**, i.e. something with a mass but no dimension.

In the work that follows you will use displacement, which measures position, rather than distance travelled.

Before doing anything else you need to make two important decisions:

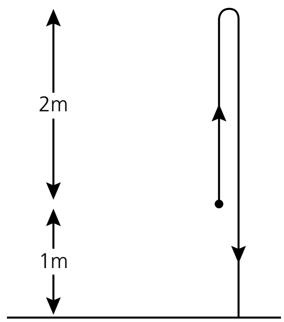
1 where you will take your origin
2 which direction you will take as positive.

Some options are:

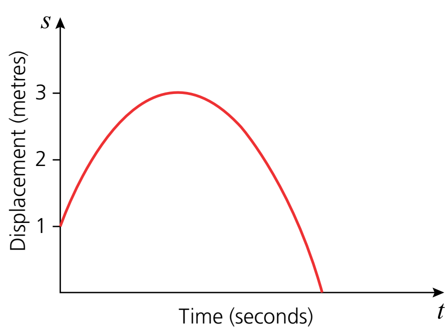


Think about the motion of a tennis ball that is thrown up vertically and allowed to fall to the ground, as in the diagram below. Assume that the ball leaves your hand at a height of 1 m above the ground and rises a further 2 m to the highest point. At this point the ball is *instantaneously at rest*.

This means it is about to change direction through 180°.



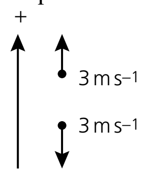
The displacement–time graph of the ball's flight is shown below. For this graph, displacement is measured from ground level with upwards as the positive direction.



Note

Be careful not to confuse the terms velocity and speed. Speed has magnitude (size) but no direction. Velocity has direction and magnitude. For example, taking upwards as the positive direction,

- a speed of 3 m s^{-1} upwards is a velocity +3 m s^{-1}
- a speed of 3 m s^{-1} downwards is a velocity of –3 m s^{-1}.



The table gives the terms that you will be using, together with their definitions, units and the letters that are commonly used to represent those quantities.

Quantity	Definition	S.I. unit	Unit symbol	Notation
Time	Measured from a fixed origin	second	s	t
Distance	Distance travelled in a given time	metre	m	x, y, s
Speed	Rate of change of distance	metre per second	m s^{-1}	$v = \frac{dx}{dt}$ etc.
Displacement	Distance from a fixed origin	metre	m	x, y, s, h
Velocity	Rate of change of displacement	metre per second	m s^{-1}	$v = \frac{ds}{dt}$
Acceleration	Rate of change of velocity	metre per second per second	m s^{-2}	$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ etc.

Like velocity, acceleration can also be either positive or negative. A negative acceleration is an object either moving in the positive direction and slowing down, or moving in the negative direction and speeding up.

Motion with variable acceleration: the general case

If the motion involves variable acceleration, you must use calculus. You should know and be able to use these relationships.

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

These relationships are used in the next two examples.

Worked example

a The displacement in metres, s, of a sports car from its initial position during the first 4 seconds is given by

$$s = 12t^2 - t^3.$$

Find:

i an expression for the velocity in terms of t
ii the initial velocity
iii the velocity after 4 seconds
iv an expression for the acceleration in terms of t
v the accelerations after 4 seconds.

b The national speed limit in Great Britain is 70 mph.
At the end of 4 seconds, would the driver of this sports car be breaking the British national speed limit?

Solution

a i $v = \dfrac{ds}{dt}$

$\quad = 24t - 3t^2$

ii When $t = 0$, $v = 0$

The initial velocity is $0\,\mathrm{m\,s^{-1}}$.

iii When $t = 4$, $v = 24 \times 4 - 3 \times 4^2$

$\qquad = 48$

The velocity, after 4 seconds is $48\,\mathrm{m\,s^{-1}}$.

iv $a = \dfrac{dv}{dt}$

$\quad = 24 - 6t$

v When $t = 4$, $a = 24 - 6 \times 4$

$\qquad = 0$

The acceleration after 4 seconds is $0\,\mathrm{m\,s^{-2}}$.

b $48\,\mathrm{m\,s^{-1}} = \dfrac{48 \times 60 \times 60}{1000}$

$\qquad = 172.8\,\mathrm{km\,h^{-1}}$

$172.8\,\mathrm{km\,h^{-1}} \approx \dfrac{5}{8} \times 172.8$

$\qquad = 108\,\mathrm{mph}$

The driver would be breaking the British speed limit.

Worked example

A particle travels in a straight line such that t seconds after passing through a fixed point O, its displacement s metres is given by $s = 5 + 2t^3 - 3t^2$.

a Find:

i expressions for the velocity and acceleration in terms of t
ii the times when it is at rest.

b Sketch the velocity–time graph.

c Find:
i how far it is from O when it is at rest
ii the initial acceleration of the particle.

Solution

a i $v = \frac{ds}{dt} = 6t^2 - 6t$

$a = \frac{dv}{dt} = 12t - 6$

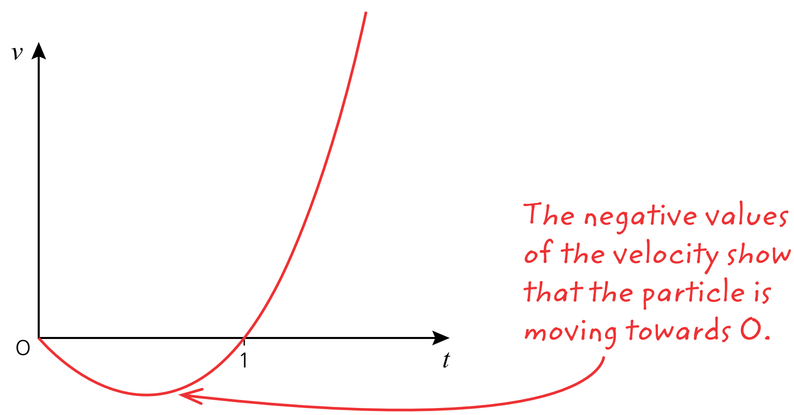
Notice that the acceleration varies with time.

ii The particle is at rest when $v = 0$.

$\Rightarrow \quad 6t^2 - 6t = 0$
$\Rightarrow \quad 6t(t - 1) = 0$
$\Rightarrow \quad t = 0$ or $t = 1$

So the particle is at rest initially and after 1 second.

b The graph of v against t is a ∪-shaped curve that crosses the t axis at $t = 0$ and $t = 1$.



c i When $t = 0, s = 5$.

When $t = 1, s = 5 + 2 - 3$
$= 4$.

The particle is at rest initially when it is 5 m from O and after 1 second when it is instantaneously at rest 4 m from O.

ii When $t = 0, a = -6$. The initial acceleration is $-6\ \mathrm{m\,s^{-2}}$.

Discussion point

How would you interpret the negative acceleration in the above example?

Exercise 16.1

1 In each of the following cases $t \geqslant 0$. The quantities are given in SI units, so distances are in metres and times in seconds.:

i find expressions for the velocity and acceleration at time t

ii use these expressions to find the initial position, velocity and acceleration

iii find the time and position when the velocity is zero.

a $s = 5t^2 - t + 3$

b $s = 3t - t^3$

c $s = t^4 - 4t - 6$

d $s = 4t^3 - 3t + 5$

e $s = 5 - 2t^2 + t$

2 A particle is projected in a straight line from a point O. After t seconds its displacement, s metres, from O is given by $s = 3t^2 - t^3$.

a Write expressions for the velocity and acceleration at time t.

b Find the times when the body is instantaneously at rest.

c What distance is travelled between these times?

d Find the velocity when $t = 4$ and interpret your result.

e Find the initial acceleration.

3 A ball is thrown upwards and its height, h metres, above ground after t seconds is given by $h = 1 + 4t - 5t^2$.

a From what height was the ball projected?

b Write an expression for the velocity of the ball at time t.

c When is the ball instantaneously at rest?

d What is the greatest height reached by the ball?

e After what length of time does the ball hit the ground?

f Sketch the graph of h against t.

g At what speed is the ball travelling when it hits the ground?

4 In the early stages of its motion the height of a rocket, h metres, is given by $h = \frac{1}{6}t^4$, where t seconds is the time after launch.

a Find expressions for the velocity and acceleration of the rocket at time t.

b After how long is the acceleration of the rocket $72\,\text{m}\,\text{s}^{-2}$?

c Find the height and velocity of the rocket at this time.

5 The velocity of a moving object at time t seconds is given by $v\,\text{m}\,\text{s}^{-1}$, where $v = 15t - 2t^2 - 25$.

a Find the times when the object is instantaneously at rest.

b Find the acceleration at these times.

c Find the velocity when the acceleration is zero.

d Sketch the graph of v against t.

Finding displacement from velocity and velocity from acceleration

In the previous section, you used the result $v = \frac{ds}{dt}$; in other words when s was given as an expression in t, you differentiated to find v. Therefore when v is given as an expression in t, integrating v gives an expression for s,

$$s = \int v \, dt.$$

Similarly, you can reverse the result $a = \frac{dv}{dt}$ to give

$$v = \int a \, dt.$$

Worked example

The acceleration is not constant.

A particle P moves in a straight line so that at time t seconds its acceleration is $(6t + 2)\,\text{m}\,\text{s}^{-2}$.

P passes through a point O at time $t = 0$ with a velocity of $3\,\text{m}\,\text{s}^{-1}$.

Find:

a the velocity of P in terms of t

b the distance of P from O when $t = 2$.

Solution

a $v = \int a \, dt$

$= \int (6t + 2) \, dt$

$= 3t^2 + 2t + c$

When $t = 0$, $v = 3$

$\Rightarrow \quad c = 3.$

c represents the initial velocity.

Therefore $v = 3t^2 + 2t + 3$.

b $s = \int v \, dt$

$= \int (3t^2 + 2t + 3) \, dt$

$= t^3 + t^2 + 3t + k$

k is the value of the displacement when $t = 0$.

When $t = 0$, $s = 0$

$\Rightarrow \quad k = 0$

$\Rightarrow \quad s = t^3 + t^2 + 3t$

When $t = 2$, $s = 8 + 4 + 6$

$= 18.$

When $t = 2$ the particle is 18 m from O.

Worked example

This tells you that the acceleration varies with time.

The acceleration of a particle $a\,\text{m s}^{-2}$, at time t seconds is given by $a = 6 - 2t$.

When $t = 0$, the particle is at rest at a point 4 m from the origin O.

a Find expressions for the velocity and displacement in terms of t.

b Find when the particle is next at rest, and its displacement from O at that time.

Solution

a $v = \int a\,\mathrm{d}t$

$= \int (6 - 2t)\,\mathrm{d}t$

$= 6t - t^2 + c$

When $t = 0$, $v = 0$ (given) $\Rightarrow$ $c = 0$

Therefore $v = 6t - t^2$.

$s = \int v\,\mathrm{d}t$

$= \int (6t - t^2)\,\mathrm{d}t$

$= 3t^2 - \frac{t^3}{3} + k$

When $t = 0$, $s = 4$ (given) $\Rightarrow$ $k = 4$

Therefore $s = 3t^2 - \frac{t^3}{3} + 4$.

b The particle is at rest when $v = 0$ $\Rightarrow$ $6t - t^2 = 0$

$\Rightarrow$ $t(6 - t) = 0$

$\Rightarrow$ $t = 0$ or $t = 6$

The particle is next at rest after 6 seconds.

When $t = 6$, $s = 3 \times 6^2 - \frac{6^3}{3} + 4$

$= 40$

The particle is 40 m from O after 6 seconds.

Worked example

A particle is projected along a straight line.

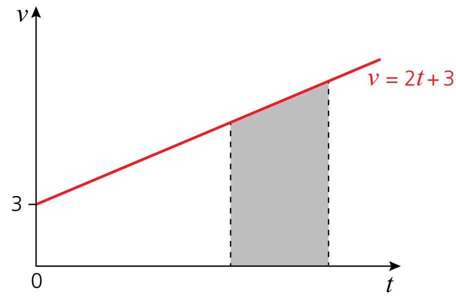
Its velocity, $v\,\text{m s}^{-1}$, after t seconds is given by $v = 2t + 3$.

a Sketch the graph of v against t.

b Find the distance the particle moves in the third second.

Solution

a $v = 2t + 3$ is a straight line with gradient 2 that passes through (0, 3).



b **Method 1**

The graph shows that the velocity is always positive, so the velocity and speed are the same. The distance travelled is equal to the area under the graph.

The third second starts when $t = 2$ and finishes when $t = 3$.

Using the formula for the area of a trapezium,

$$\begin{aligned} \text{distance} &= \tfrac{1}{2}(7 + 9) \times 1 \\ &= 8\,\text{m}. \end{aligned}$$

Method 2

The area under a graph can also be found using integration.

$$\begin{aligned} \text{Distance} &= \int_a^b v \ \mathrm{d}t \\ &= \int_2^3 (2t + 3)\ \mathrm{d}t \\ &= \left[t^2 + 3t\right]_2^3 \\ &= [9 + 9] - [4 + 6] \\ &= 8\,\text{m} \end{aligned}$$

Discussion point

- Which method did you prefer to use in the previous example?
- Which method would you need to use if v was given by $v = 3t^2 + 2$?
  - Is acceleration constant in this case? How can you tell?
- Could you have used the constant acceleration (*suvat*) equations?
- Can you use calculus when acceleration is constant?

Exercise 16.2

1 Find expressions for the velocity, v, and displacement, s, at time t in each of the following cases:
 a $a = 2 - 6t$; when $t = 0$, $v = 1$ and $s = 0$
 b $a = 4t$; when $t = 0$, $v = 4$ and $s = 3$
 c $a = 12t^2 - 4$; when $t = 0$, $v = 2$ and $s = 1$
 d $a = 2$; when $t = 0$, $v = 2$ and $s = 4$
 e $a = 4 + t$; when $t = 0$, $v = 1$ and $s = 3$

2 A particle P sets off from the origin, O, with a velocity of $9\,\mathrm{m\,s^{-1}}$ and moves along the x-axis.
At time t seconds, its acceleration is given by $a = (6t - 12)\,\mathrm{m\,s^{-2}}$.
 a Find expressions for the velocity and displacement at time t.
 b Find the time when the particle returns to its starting point.

3 A particle P starts from rest at a fixed origin O when $t = 0$.
The acceleration $a\,\mathrm{m\,s^{-2}}$ at time t seconds is given by $a = 6t - 6$.
 a Find the velocity of the particle after 1 second.
 b Find the time after leaving the origin when the particle is next instantaneously at rest, and the distance travelled to this point.

4 The speed, $v\,\mathrm{m\,s^{-1}}$, of a car during braking is given by $v = 30 - 5t$, where t seconds is the time since the brakes were applied.
 a Sketch a graph of v against t.
 b How long does the car take to stop?
 c How far does it travel while braking?

5 A particle P moves in a straight line, starting from rest at the point O. t seconds after leaving O, the acceleration, $a\,\mathrm{m\,s^{-2}}$, of P is given by $a = 4 + 12t$.
 a Find an expression for the velocity of the particle at time t.
 b Calculate the distance travelled by P in the third second.

6 The velocity $v\,\mathrm{m\,s^{-1}}$, of a particle P at time t seconds is given by $v = t^3 - 4t^2 + 4t + 2$.
P moves in a straight line.
 a Find an expression for the acceleration, $a\,\mathrm{m\,s^{-2}}$, in terms of t.
 b Find the times at which the acceleration is zero, and say what is happening between these times.
 c Find the distance travelled in the first three seconds.

Past–paper questions

1 A particle P moves in a straight line such that, t s after leaving a point O, its velocity v m s^{-1} is given by $v = 36t - 3t^2$ for $t \geqslant 0$.

(i) Find the value of t when the velocity of P stops increasing. [2]

(ii) Find the value of t when P comes to instantaneous rest. [2]

(iii) Find the distance of P from O when P is at instantaneous rest. [3]

(iv) Find the speed of P when P is again at O. [4]

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Paper 12 Q12 June 2013

2 A particle travels in a straight line so that, t s after passing through a fixed point O, its velocity, v ms^{-1}, is given by $v = 3 + 6 \sin 2t$.

(i) Find the velocity of the particle when $t = \frac{\pi}{4}$. [1]

(ii) Find the acceleration of the particle when $t = 2$. [3]
The particle first comes to instantaneous rest at the point P.

(iii) Find the distance OP. [5]

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Paper 23 Q9 November 2013
Cambridge IGCSE Additional Mathematics 0606
Paper 23 Q9 November 2013

3 A particle travels in a straight line so that, t s after passing through a fixed point O, its displacement s m from O is given by $s = \ln(t^2 + 1)$.

(i) Find the value of t when $s = 5$. [2]

(ii) Find the distance travelled by the particle during the third second. [2]

(iii) Show that, when $t = 2$, the velocity of the particle is 0.8 ms^{-1}. [2]

(iv) Find the acceleration of the particle when $t = 2$. [3]

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Paper 13 Q10 November 2010

Learning outcomes

Now you should be able to:

★ apply differentiation and integration to kinematics problems that involve displacement, velocity and acceleration of a particle moving in a straight line with variable or constant acceleration, and the use of x–t and v–t graphs.

Key points

Quantity	Definition	S.I. unit	Unit symbol	Notation
Time	Measured from a fixed origin	second	s	t
Distance	Distance travelled in a given time	metre	m	x, y, s
Speed	Rate of change of distance	metre per second	$\mathrm{m\,s^{-1}}$	$v = \frac{dx}{dt}$ etc.
Displacement	Distance from a fixed origin	metre	m	x, y, s, h
Velocity	Rate of change of displacement	metre per second	$\mathrm{m\,s^{-1}}$	$v = \frac{ds}{dt}$
Acceleration	Rate of change of velocity	metre per second per second	$\mathrm{m\,s^{-2}}$	$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ etc.

- ✔ For a **displacement–time** graph, the gradient is the velocity.
- ✔ For a **velocity–time** the gradient is the acceleration and the area under the graph is the displacement.
- ✔ For a **distance–time** graph the gradient is the speed.
- ✔ For general motion:
  - $s = \int v \, dt$ (Displacement is the area under a velocity–time graph.)
  - $v = \int a \, dt$

Mathematical notation

Miscellaneous symbols

$=$	is equal to
$\neq$	is not equal to
$\equiv$	is identical to or is congruent to
$\approx$	is approximately equal to
$\sim$	is distributed as
$\cong$	is isomorphic to
$\propto$	is proportional to
$<$	is less than
$\leqslant$	is less than or equal to
$>$	is greater than
$\geqslant$	is greater than or equal to
∞	infinity
$\Rightarrow$	implies
$\Leftarrow$	is implied by
$\Leftrightarrow$	implies and is implied by (is equivalent to)

Operations

$a + b$	a plus b
$a - b$	a minus b
$a \times b, ab$	a multiplied by b
$a \div b, \frac{a}{b}$	a divided by b
$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \ldots + a_n$
$\sqrt{a}$	the non-negative square root of a, for $a \in \mathbb{R}, a \geqslant 0$
$\sqrt[n]{a}$	the (real) nth root of a, for $a \in \mathbb{R}$, where $\sqrt[n]{a} \geqslant 0$ for $a \geqslant 0$
$\lvert a \rvert$	the modulus of a
$n!$	n factorial
$\binom{n}{r}$, ${}^{n}\mathrm{C}_{r}$	the binomial coefficient $\frac{n!}{r!(n-r)!}$ for $n, r \in \mathbb{Z}$ and $0 \leqslant r \leqslant n$

Functions

$\mathrm{f}(x)$	the value of the function f at x
$\mathrm{f}: \mathrm{A} \to \mathrm{B}$	f is a function under which each element of set A has an image in set B
$\mathrm{f}: x \mapsto y$	the function f maps the element x to the element y
f^{-1}	the inverse function of the one-one function f
gf	the composite function of f and g, which is defined by $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$
$\lim\limits_{x \to a} \mathrm{f}(x)$	the limit of $\mathrm{f}(x)$ as x tends to a
$\Delta x, \delta x$	an increment of x
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of y with respect to x
$\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$	the nth derivative of y with respect to x
$\mathrm{f}'(x), \mathrm{f}''(x), \ldots, \mathrm{f}^n(x)$	the first, second, …, nth derivatives of $\mathrm{f}(x)$ with respect to x
$\int y\,\mathrm{d}x$	the indefinite integral of y with respect to x
$\int_a^b y\,\mathrm{d}x$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
$\dot{x}, \ddot{x} \ldots$	the first, second, … derivatives of x with respect to t

Exponential and logarithmic functions

e	base of natural logarithms
$\mathrm{e}^x, \exp(x)$	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	natural logarithm of x
$\lg x, \log_{10} x$	logarithm of x to base 10

Circular functions

$\left.\begin{array}{r}\sin, \cos, \tan \\ \operatorname{cosec}, \sec, \cot\end{array}\right\}$	the circular functions
$\left.\begin{array}{r}\sin^{-1}, \cos^{-1}, \tan^{-1} \\ \operatorname{cosec}^{-1}, \sec^{-1}, \cot^{-1}\end{array}\right\}$	the inverse circular functions }

Vectors

$\mathbf{a}$	the vector $\mathbf{a}$
$\overrightarrow{\mathrm{AB}}$	the vector represented in magnitude and direction by the directed line segment AB
$\hat{\mathbf{a}}$	a unit vector in the direction of $\mathbf{a}$
$\mathbf{i}, \mathbf{j}$	unit vectors in the directions of the Cartesian coordinate axes
$\begin{pmatrix} x \\ y \end{pmatrix}$	the vector $x\mathbf{i} + y\mathbf{j}$
$\lvert \mathbf{a} \rvert, a$	the magnitude of $\mathbf{a}$

Answers

The questions, with the exception of those from past question papers, and all example answers that appear in this book were written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers which are contained in this publication.

Chapter 1 Functions

Discussion point Page 1

The amount of petrol, in litres, is measured and the cost is calculated from it.

Discussion point Page 2

The numbers not appearing as final digits are 2, 3, 7, 8.

Exercise 1.1 Page 10

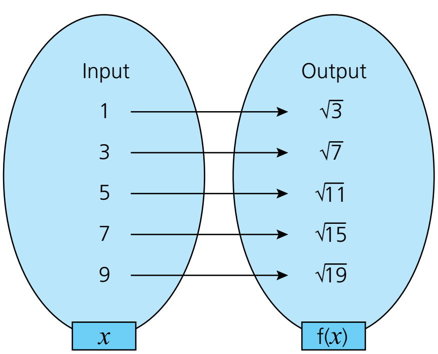
1 **a** 13
b –2
c 4
d 5.5 or $\frac{11}{2}$

2 **a** 36
b 16
c 4
d $\frac{25}{4}$

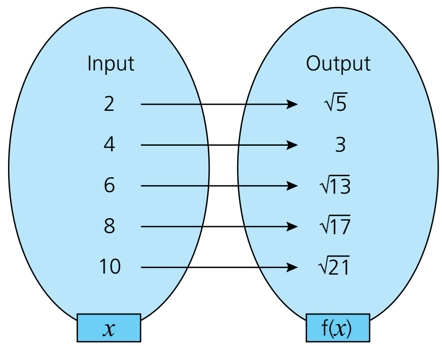
3 **a** 13
b 28
c 1
d $\frac{4}{3}$

4 **a** 4
b –2
c 2
d $\frac{13}{6}$

5 **a**



b



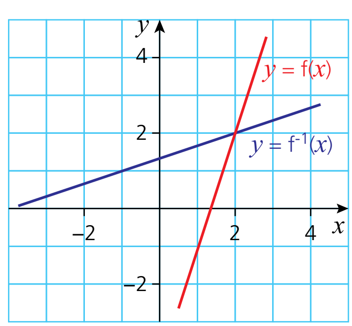
c Any numbers less than $(-\frac{1}{2})$ must be excluded as an input.

6 **a** $\{1, 4, 7, 10, 13\}$
b $\{-3, -\frac{5}{2}, -2, -\frac{3}{2}, -1\}$
c $\{y : y \geq 0\}$
d $\{y : y \geq 6\}$

7 **a** 0
b $x < 1$
c 1.5
d $\{x : x < -\sqrt{2} \text{ or } x > \sqrt{2}\}$

8 **a** $f^{-1}(x) = \frac{x+2}{7}$
b $g^{-1}(x) = \frac{2x-4}{3}$
c $h^{-1}(x) = \sqrt{x} + 1$ for $x \geq 0$
d $f^{-1}(x) = \sqrt{x-4}$ for $x \geq 4$

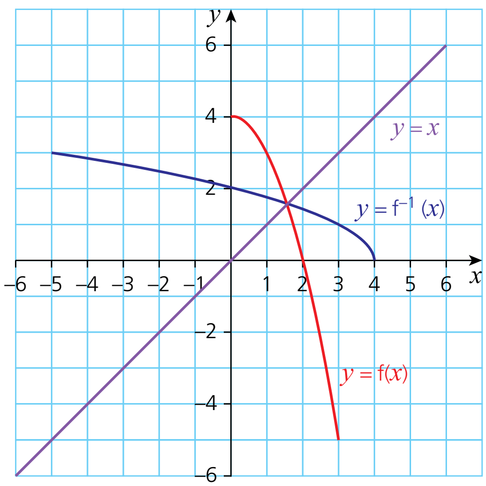
9 **a** $f^{-1}(x) = \frac{x+4}{3}$
b



10 a See the curve $y = f(x)$ on the graph in part c below.

b $f^{-1}(-5) = 3$
$f^{-1}(0) = 2$
$f^{-1}(3) = 1$
$f^{-1}(4) = 0$

c



Exercise 1.2 Page 14

1 a 14

b $3x^2 + 2$

c $4x^2$

d $12(x^2) + 2$

2 a $\sqrt{17}$

b -1

c $\sqrt{9 - 2x}$

d $4 - \sqrt{2x + 1}$

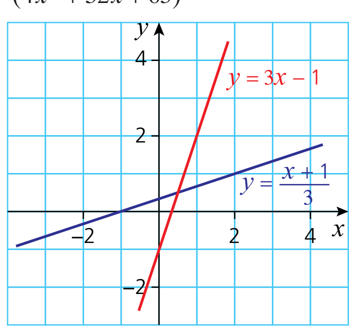
3 a $x + 8$

b $8x^4$

c $\dfrac{2x + 1}{2x + 3}$

d $\dfrac{1}{(4x^2 + 32x + 65)}$

4 a



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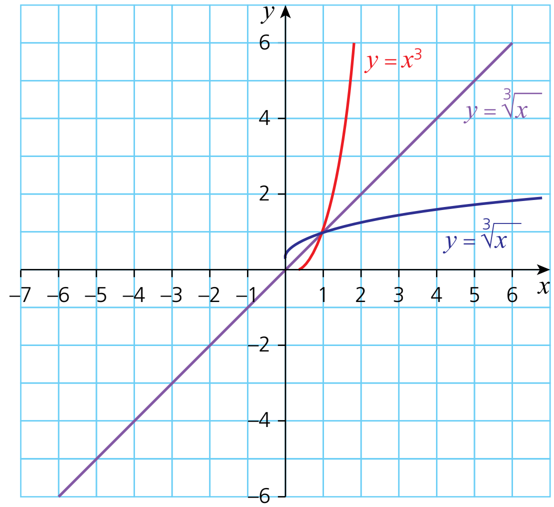


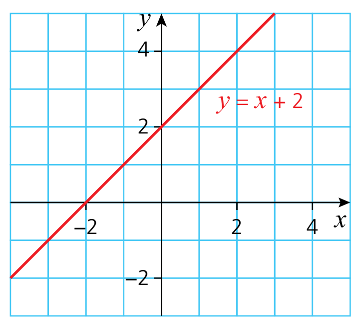
5 a -1 or 7

b -4 or 3

c -1 or $\frac{7}{3}$

d -4 or 0

6 a



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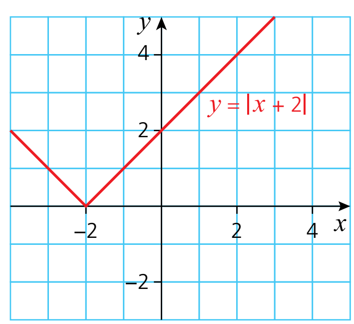

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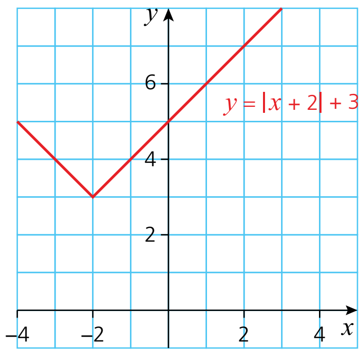
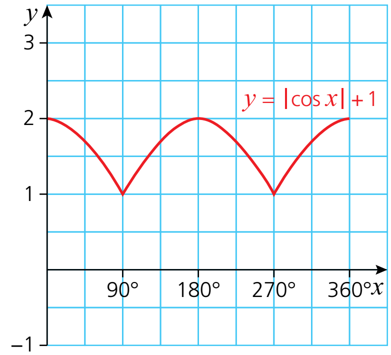
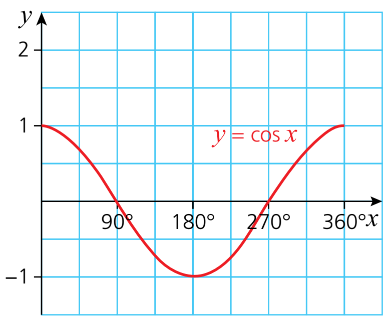


d



7 a



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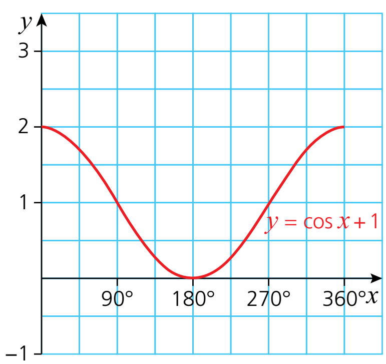

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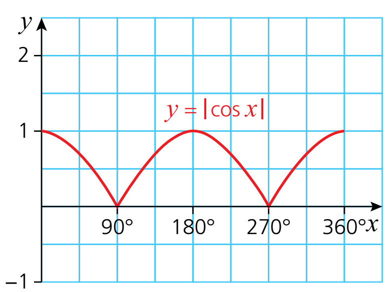
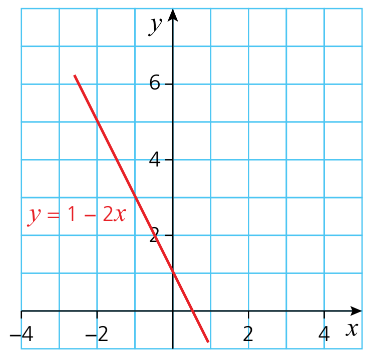


8 The equation for graph 2 is $y = |2x - 1|$
The equation for graph 3 is $y = |2x - 1| + 3$

9 a $y = x^2 + 1$

b $y = \sqrt{x - 1}$

10 a i



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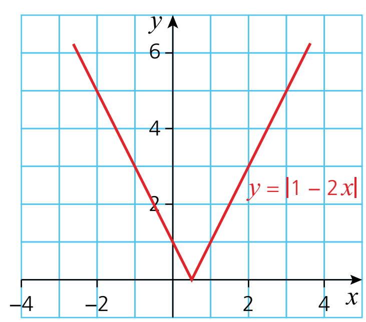

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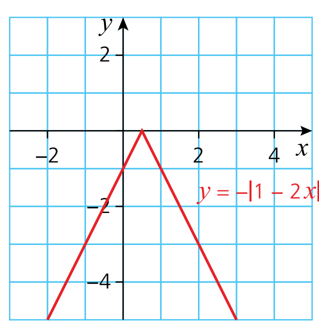

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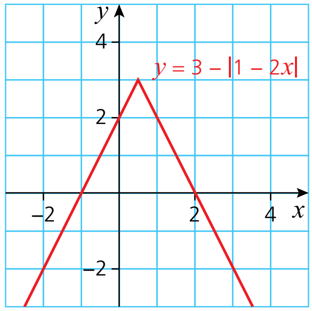

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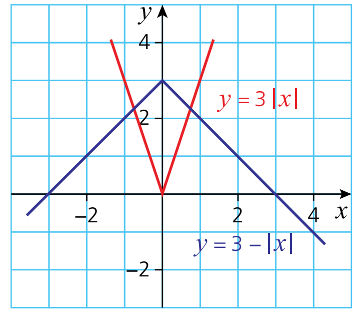

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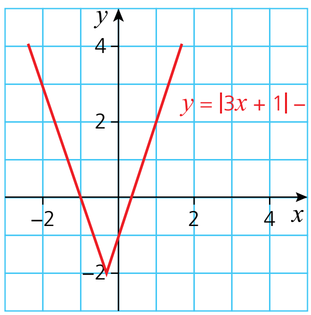
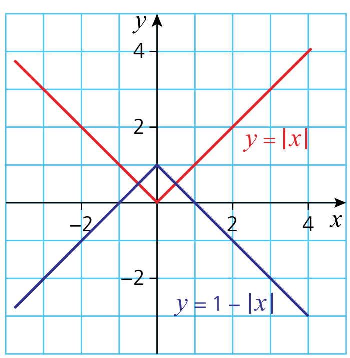


b **i** $\left(\pm\frac{1}{2}, \frac{1}{2}\right)$

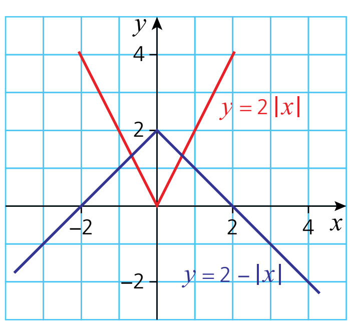
ii $\left(\pm\frac{2}{3}, \frac{4}{3}\right)$

iii $\left(\pm\frac{3}{4}, \frac{9}{4}\right)$

11 a **i**



ii



Past-paper questions Page 16

1 **(i)** $\frac{6}{4}$

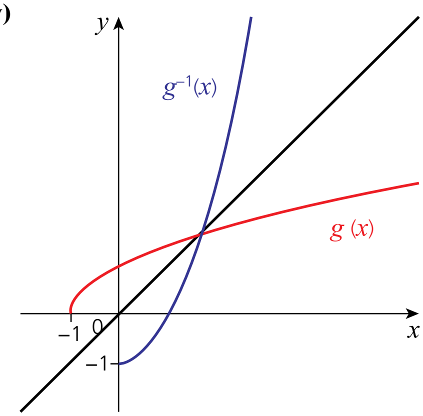
(ii) $\frac{4x}{3x+1}$

(iii) $g^{-1}(x) = x^2 - 1$

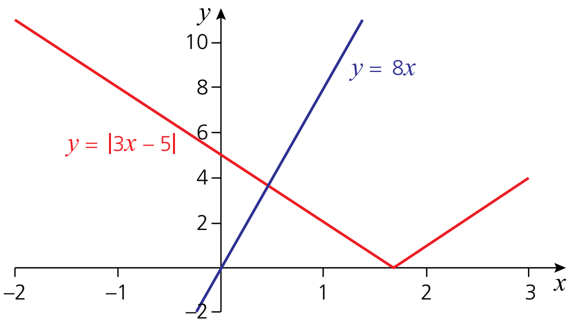
(Domain) $x > 0$

(Range) $g^{-1}(x) > -1$

(iv)



2 **i** and **ii**



iii $8x = \pm(3x - 5)$

leading to $x = \frac{5}{11}$ or 0.455

Chapter 2 Quadratic functions

Discussion point Page 19

3.22 and 0.78

Exercise 2.1 Page 27

1 **a** −5, 4

b 2, 3

c −4, 7

d −7, −6

2 **a** $\frac{1}{2}, 1$

b $-\frac{2}{3}, \frac{1}{3}$

c $-1, \frac{7}{2}$

d $-5, -\frac{2}{3}$

3 **a** ±13

b $\pm\frac{11}{2}$

c $\pm\frac{5}{4}$

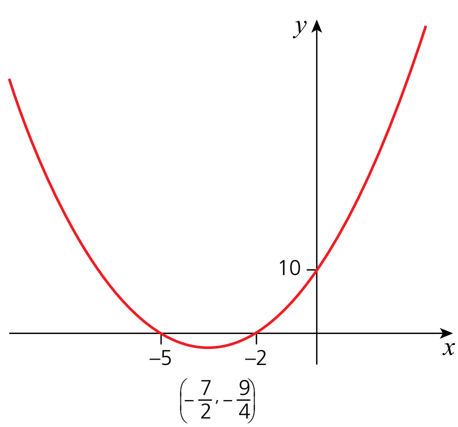
d $\pm\frac{3}{2}$

4 **a** **i** $(x + 5)(x + 2)$

ii $\left(-\frac{7}{2}, -\frac{9}{4}\right)$

iii minimum

iv

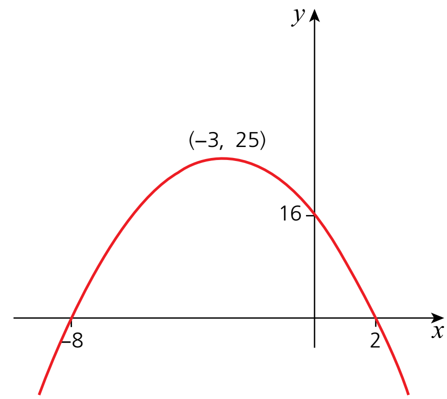


b **i** $(8 + x)(2 - x)$

ii $(-3, 25)$

iii maximum

iv

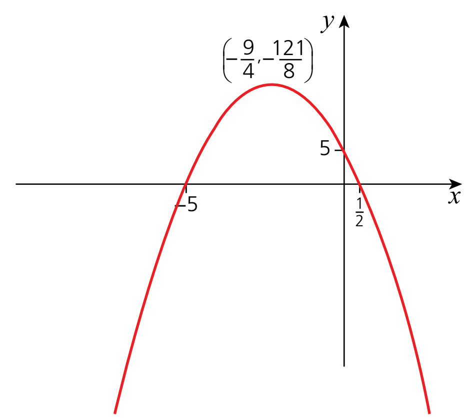


c **i** $(5 + x)(1 - 2x)$

ii $\left(-\frac{9}{4}, \frac{121}{8}\right)$

iii maximum

iv

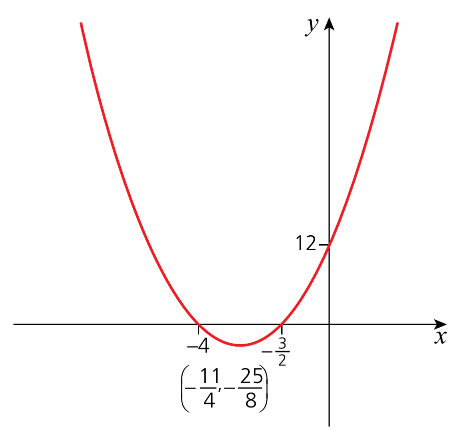


d **i** $(2x + 3)(x + 4)$

ii $\left(-\frac{11}{4}, -\frac{25}{8}\right)$

iii minimum

iv



5 **a** $(x+2)^2+5$

b $(x-5)^2-29$

c $\left(x+\frac{5}{2}\right)^2-\frac{53}{4}$

d $\left(x-\frac{9}{2}\right)^2-\frac{89}{4}$

6 **a** $2(x-3)^2-13$

b $3(x+2)^2+8$

c $4(x-1)^2+1$

d $2\left(x+\frac{9}{4}\right)^2-\frac{33}{8}$

7 **a** $-2\pm\sqrt{13}$

b $3.5\pm\sqrt{14.25}$

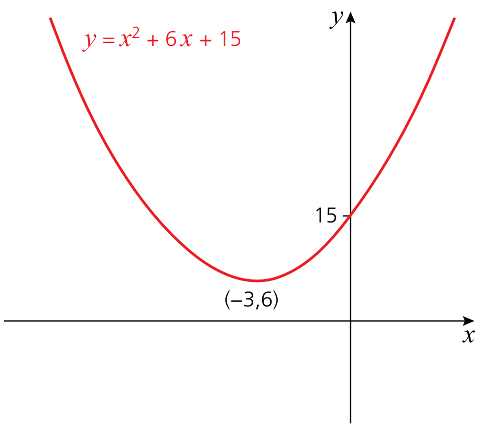
c $\frac{-3\pm\sqrt{27}}{2}=-\frac{3}{2}\pm\sqrt{\frac{27}{4}}$

d $1.5\pm\sqrt{6.75}$

8 **a** **i** $(-3, 6)$

ii minimum

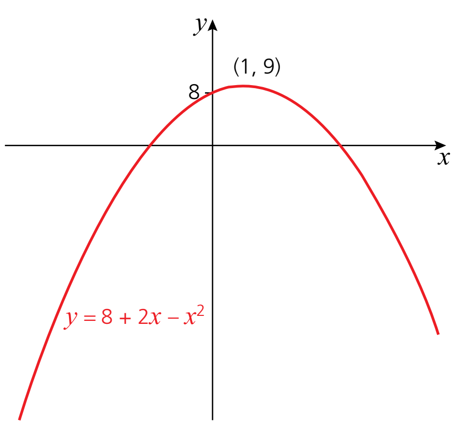
iii



8 **b** **i** $(1, 9)$

ii maximum

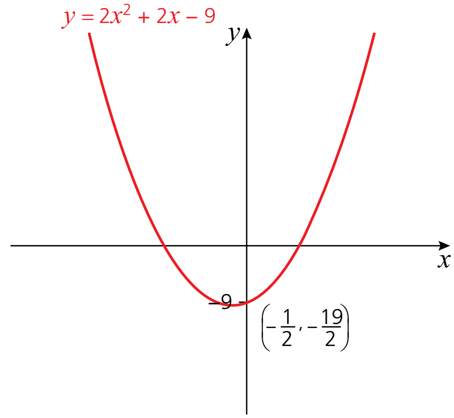
iii



8 **c** **i** $\left(-\frac{1}{2}, -\frac{19}{2}\right)$

ii minimum

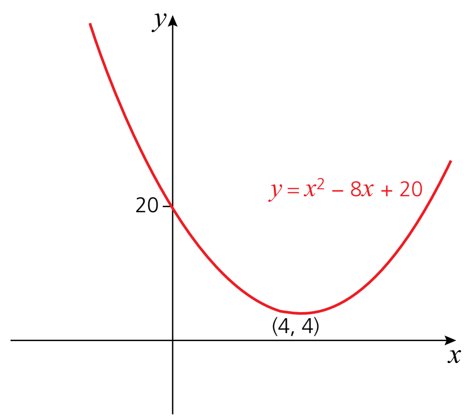
iii



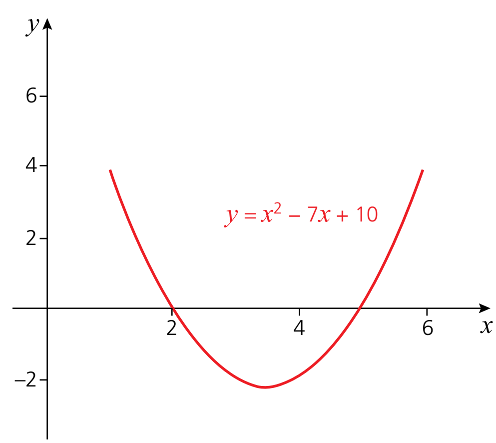
8 **d** **i** $(4, 4)$

ii minimum

iii

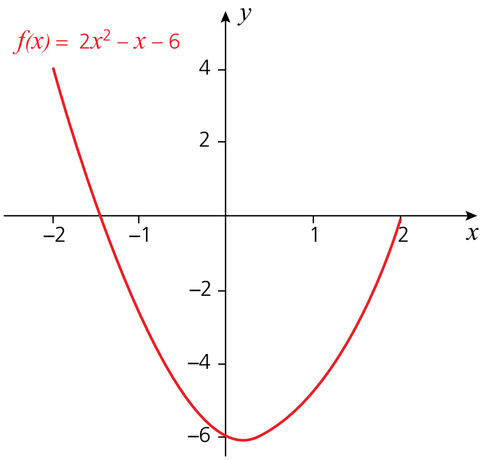


9 **a**



Range $\left[-\frac{9}{4}, 4\right]$

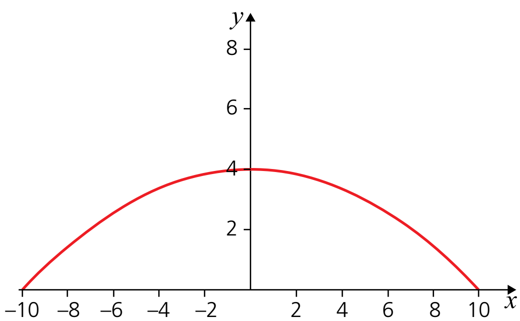
b



Range $\left[-\frac{49}{8}, 4\right]$

Real-world activity Page 27

1



2 Height 4 m, span 20 m

3 $80y = 400 - x^2$

Discussion point Page 32

Lines parallel to the y–axis are straight vertical lines and as such a quadratic curve will always intersect them rather than touch them.

Exercise 2.2 Page 36

1 **a** –2, –1
b ±3
c none
d $0, \frac{5}{2}$
e –6, 3
f –5 (repeated)
g $-\frac{1}{3}, \frac{1}{5}$
h $-3, \frac{1}{3}$

2 **a** i; ii –2.32, 4.32
b i; ii –1, 0
c i; ii –2.68, 1.68
d i; ii –2.27, 1.77

3 **a** –2.43, 0.93
b no real roots
c –0.18, –2.82
d 9, 5

4 **a** One real solution.
b No roots.
c No real roots.
d Distinct roots.
e No real roots.
f Equal roots.

5 **a** The line is a tangent to the curve at (–3, –27).
b Does not meet.
c $\left(\frac{2}{3}, \frac{2}{3}\right)\left(\frac{3}{2}, \frac{3}{2}\right)$
d (1.84, 6.68) (8.16, 19.32)
e (–4, 12) (1, 2)
f Tangent touches at (1.5, 18).
g Does not meet.
h Does not meet.

6 **a** $x < 1$ or $x > 5$
b $-4 \le a \le 1$
c $-2 < y < 2$
d $x \ne 2$
e $-4 < a < 2$
f $y < -1$ or $y > \frac{1}{3}$

Real world activity page 37

1 $y = 4 - \frac{4x^2}{49}$

$y = -4 + \frac{4x^2}{49}$

$y = 3 - \frac{x^2}{12}$

$y = -3 + \frac{x^2}{12}$

2 $y = 2 - \frac{8x^2}{49}$

$y = -2 + \frac{8x^2}{49}$

$y = \frac{3}{2} - \frac{x^2}{6}$

$y = -\frac{3}{2} + \frac{x^2}{6}$

Past-paper questions Page 38

1 **(i)** $2(x - \frac{1}{4})^2 + \frac{47}{8}$ 5.875

(ii) $\frac{47}{8}$ is minimum value when $x = \frac{1}{4}$

2 $4 < k < 12$ or $k > 4$ and $k < 12$

3 $mx + 2 = x^2 + 12x + 18$
$x^2 + (12 - m)x + 16 = 0$
$(12 - m)^2 = 4 \times 16$
leading to $m = 4, 20$

Chapter 3 Equations, inequalities and graphs

Discussion point Page 41

You can describe it in words, as in 'both x and y must be between –1 and 1' or you can do it in symbols, for example $-1 < x < 1$ and $-1 < y < 1$. Another way with symbols is to use the modulus sign and write $|x < 1|$ and $|y < 1|$. You could even use set notation and write it as $|x < 1| \cup |y < 1|$.

Discussion point Page 42

A small family car such as a Volkswagen Golf has an average consumption of 5.3 litres per 100 kilometres, so the fuel economy is at worst 0.530 litres per kilometre which is remarkably similar to Vettel's Ferrari Formula 1 car.

$$E = \frac{f}{d}$$

$$E = \frac{5.3}{100}$$

$$E = 0.530$$

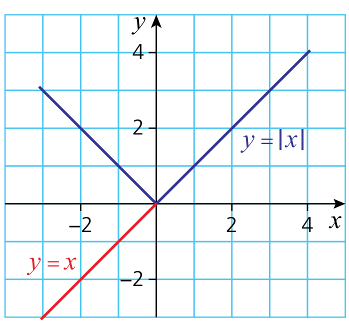
Find out the fuel consumption of other models of road vehicles and see how they compare.

Discussion point Page 44

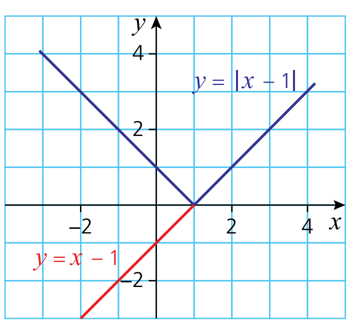
Using three points helps to eliminate errors. A mistake in the calculation using only two points would give a completely different line.

Exercise 3.1 Page 45

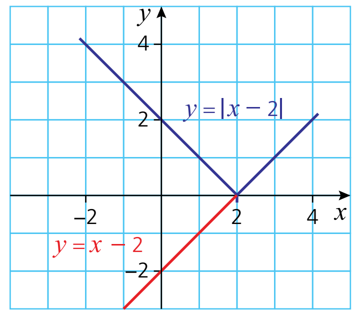
1 a



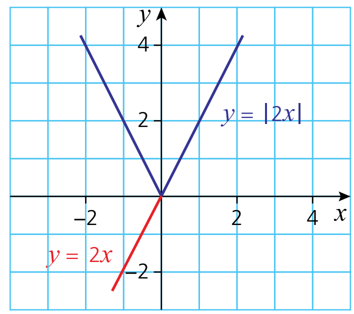
b



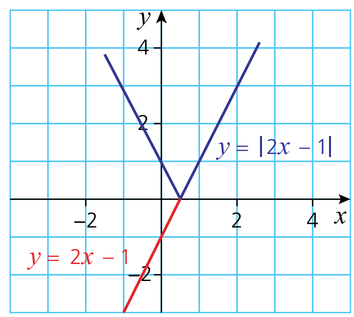
c



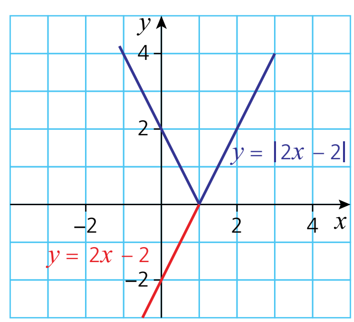
2 a



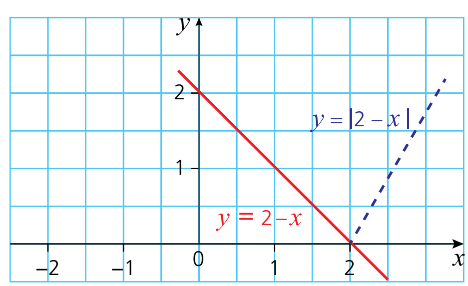
b



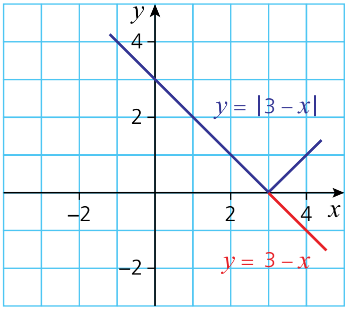
c



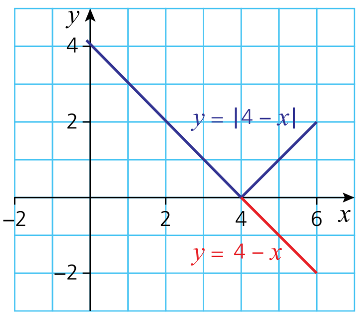
3 a



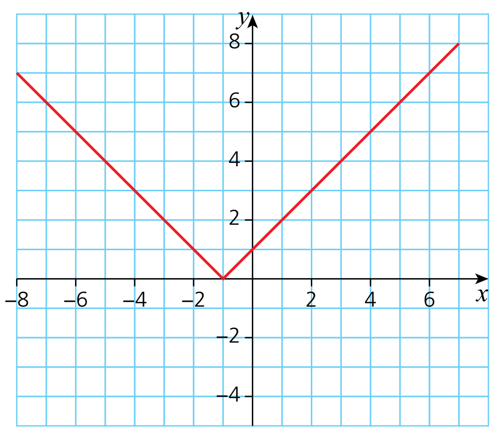
b



c

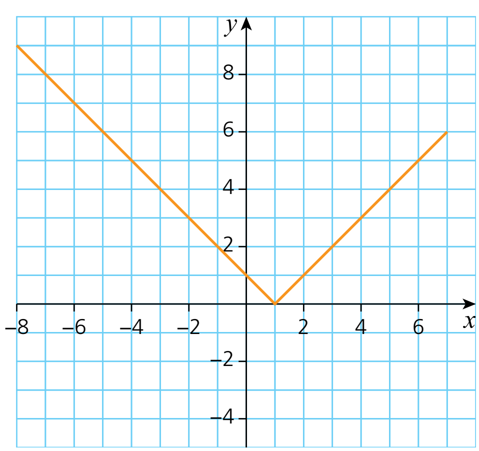


4 a



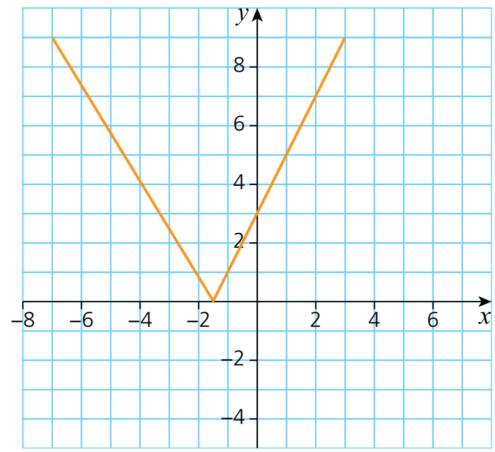
b −6, 4

5 a



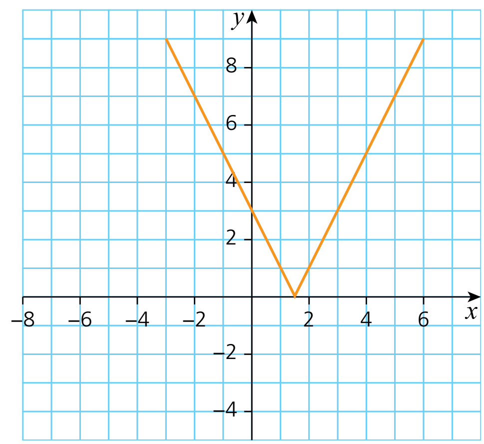
b −4, 6

6 a



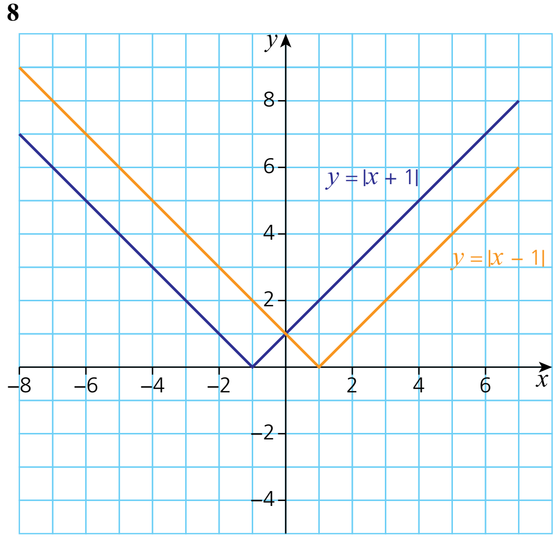
b −5, 2

7 a



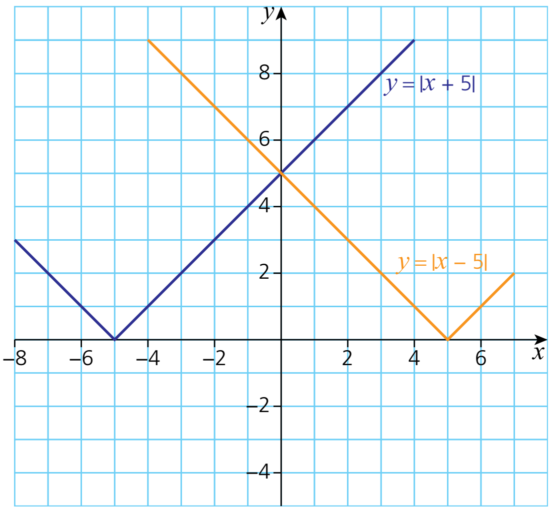
b −2, 58

8



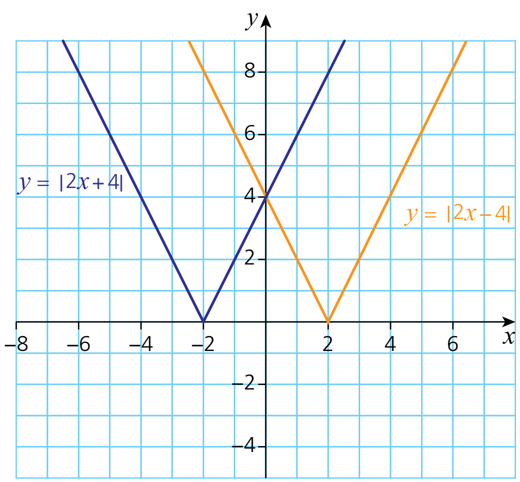
$x = 0$

9



$x = 0$

10



$x = 0$

Discussion point Page 48

$-(x + 7) = 4x$ is the same as $x + 7 = -4x$, which is already being considered.

Discussion point Page 49

It is easier to plot points with integer coordinates.

Exercise 3.2 Page 50

1 **a** $|x - 6| \leqslant 9$

b $|x - 6| \leqslant 10$

c $|x - 6| \leqslant 11$

2 **a** $-1 \leqslant x \leqslant 3$

b $-1 \leqslant x \leqslant 5$

c $-1 \leqslant x \leqslant 7$

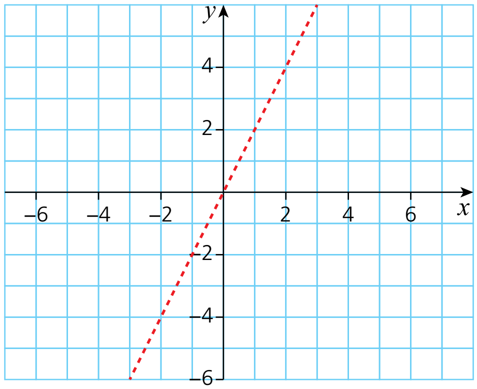
3 **a** $-3 < x < 5$

b $x < -3$ or $x > 5$

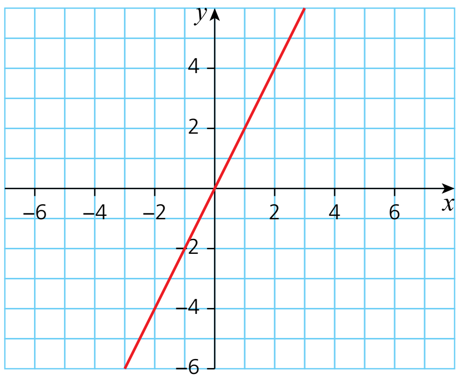
c $-4 < x < 1$

d $x < -4$ or $x > 1$

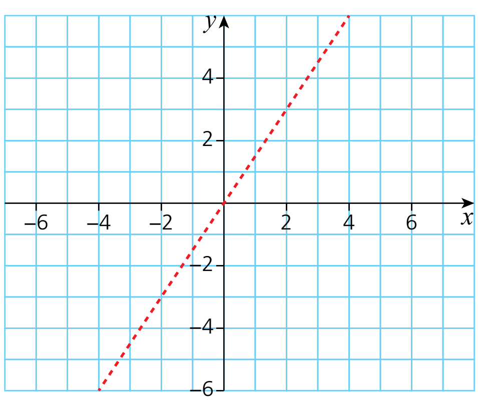
4 **a**



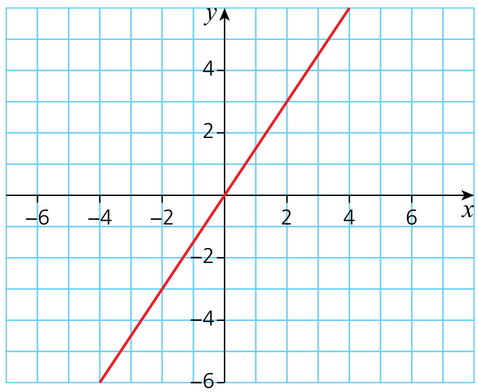
b



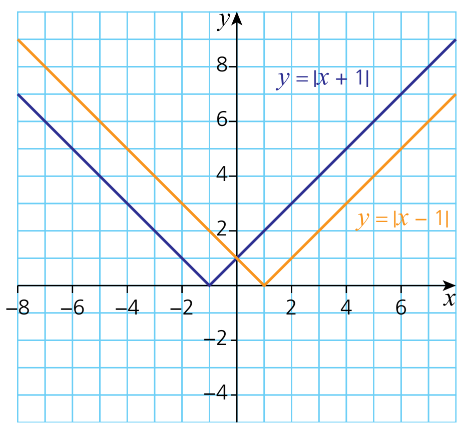
c



d

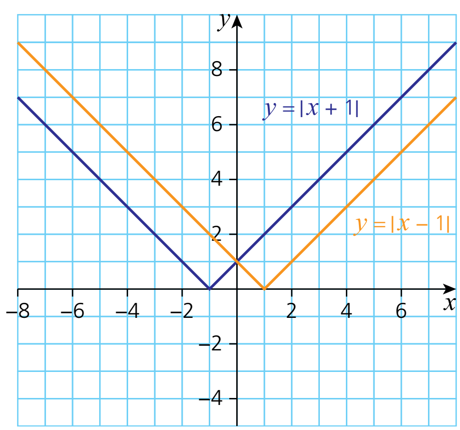


5 a i



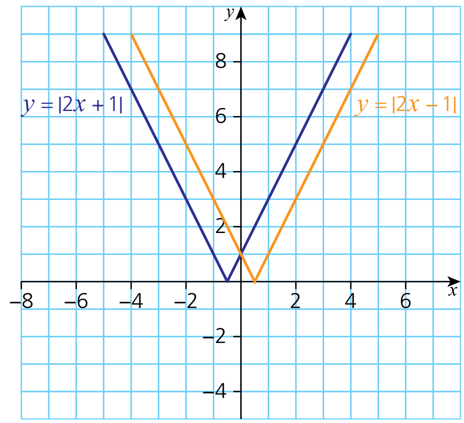
ii $x > 0$

b i



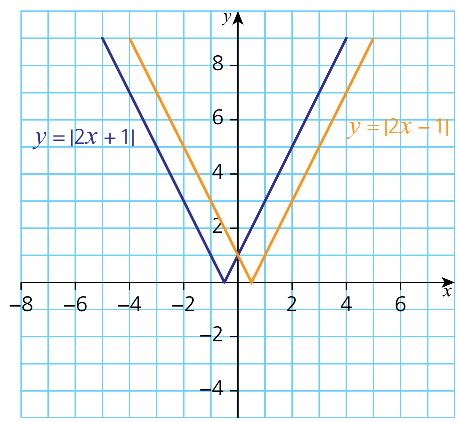
ii $x < 0$

c i



ii $x > 0$

d i



ii $x < 0$

6 a $y > x + 1$

b $y < = 2 - x$

c $y > = |x|$

Exercise 3.3 Page 53

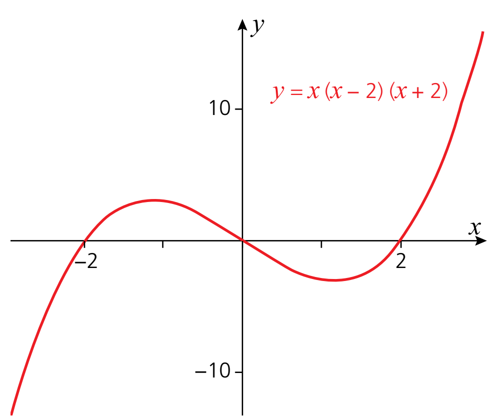
1 a 4

b 4

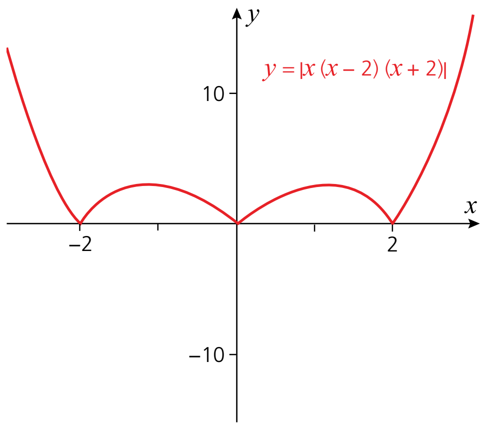
c 25

d no solutions

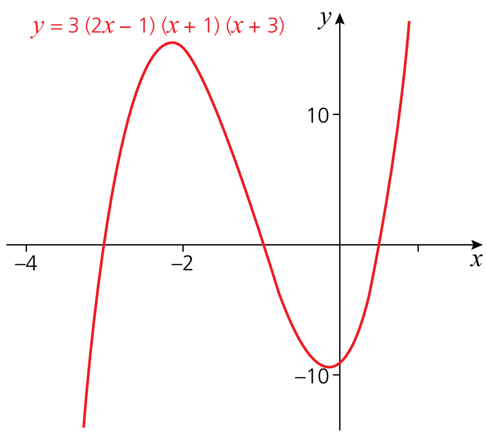
2 a



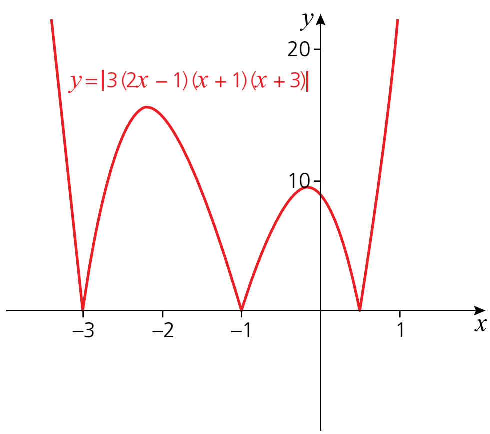
b



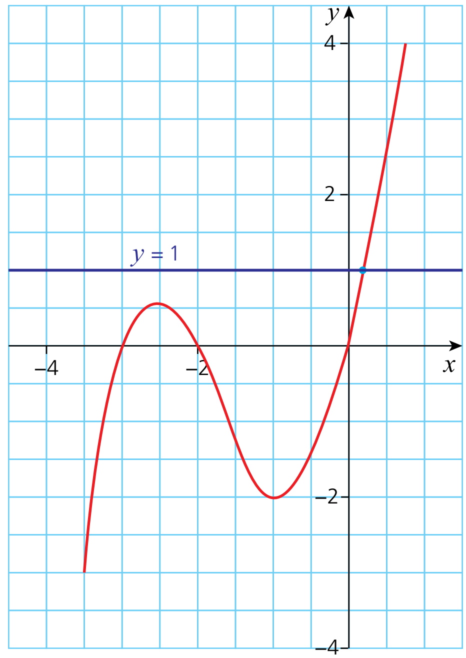
c



d

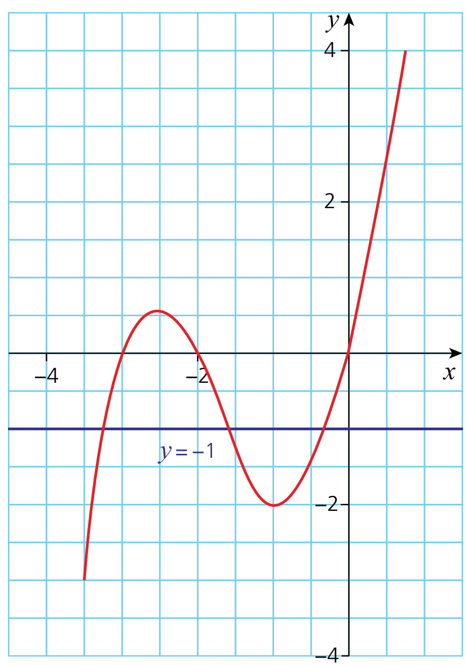


3 a



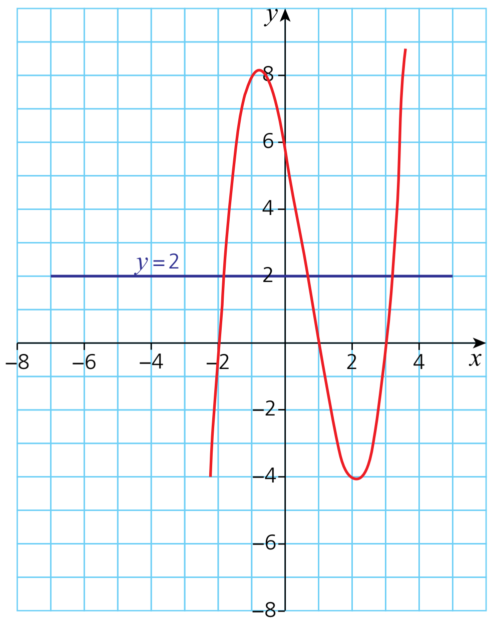
$x \geq 0.148$

b



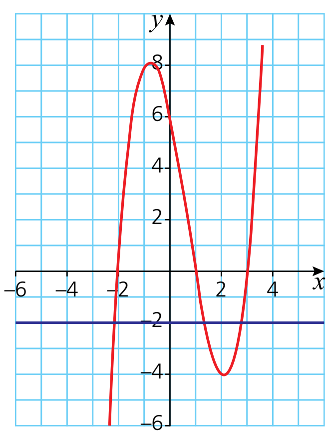
$x < -3.247$ or $-1.555 < x < -0.198$

c



$-1.856 < x < 0.678$ or $x > 3.177$

d



$x < -2.125$ or $1.363 < x < 2.762$

4 **a** $y = (x + 3)(x + 1)(x - 1)$

b $y = (2x + 3)(2x + 1)(2x - 1)$

c $y = (x + 2)(x - 2)^2$

5 **a** $y = |(x + 2)(x - 1)(x - 2)|$

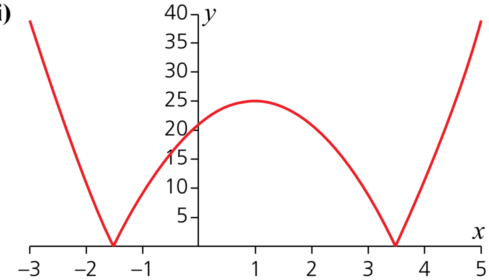
b $y = |(x + 3)(x + 2)(x - 1)|$

c $y = |(x + 2)(2x - 1)(x - 2)|$

6 There is no vertical scale.

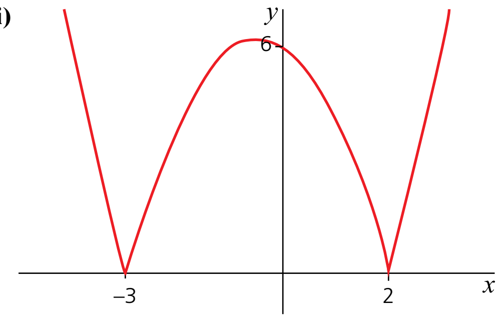
Past-paper questions Page 55

1 **(i)**



(ii) Two

2 **(i)**



$(-3, 0), (2, 0), (0, 6)$.

(ii) $\left(-\frac{1}{2}, \frac{25}{4}\right)$

(iii) $6.25 < x \leqslant 14$

3 $9x^2 + 2x - 1 < (x + 1)^2$

$8x^2 < 2$

$-\frac{1}{2} < x < \frac{1}{2}$

Chapter 4 Indices and surds

Discussion point Page 58

$7 + 7^2 + 7^3 + 7^4$. If you do the calculations the answer works out to be 2800.

Exercise 4.1 Page 60

1 **a** 2^{10}

b 5^1

c 3^3

d 6^9

e 4^6

f 5^{-4}

2 **a** 6×10^{14}

b 6×10^1

c 8×10^2

d 3×10^{12}

3 **a** $\left(\frac{1}{3}\right)^2$

b $\left(\frac{1}{5}\right)^4$

c $\left(\frac{3}{2}\right)^3$

d 3^6

4 **a** 1.5×10^5

b 2.4×10^3

c 5×10^8

d 5×10^{-4}

5 **a** 9

b 1

c 125

d 4096

e 81

f $\frac{1}{49}$

g 64

h $\frac{1}{32}$

i $\frac{16}{9}$

j 3

k 3

l 64

m 9

n $\frac{1}{4}$

o $\frac{1}{32}$

6 **a** $5^3, 3^5, 4^4$

b 2^7; 3^5 and 4^4

c 3^{-4}; 4^{-3} and 2^{-5}

7 **a** 7

b 14

c 1

d $-\frac{1}{2}$

e -3

f 3

g -3

h -2

8 **a** $6a^7$

b $8x^5 y^{-2}$

c $5b^3$

d $4p^{-6} q^{-5}$

e $64m^3$

f $64s^{12}t^6$

9 $x = 8$ and $y = 27$

Discussion point Page 61

$2^2 = 4$ and $\sqrt{4} = 2$

Therefore, the square root of 4 is a whole number and is not a surd.

Exercise 4.2 Page 65

1 **a** $2\sqrt{3}$

b $5\sqrt{3}$

c $10\sqrt{3}$

d $9\sqrt{5}$

e $7\sqrt{3}$

f $5\sqrt{5}$

2 **a** $\sqrt{54}$

b $\sqrt{125}$

c $\sqrt{432}$

d $\sqrt{1700}$

3 **a** $\frac{5}{7}$

b $\frac{2\sqrt{6}}{3}$

c $\frac{2\sqrt{5}}{5}$

d $\frac{\sqrt{6}}{11}$

4 **a** $23 + 17\sqrt{2}$

b $8\sqrt{3}$

5 **a** 5

b $5\sqrt{3} - 3$

c $18 + 8\sqrt{2}$

d $9 - 6\sqrt{2}$

6 **a** $\frac{\sqrt{6}}{6}$

b $4\sqrt{3}$

c $\frac{\sqrt{3}}{2}$

d $\frac{\sqrt{5} + \sqrt{2}}{3}$

e $\frac{2 + \sqrt{2}}{2}$

f $\frac{5 - 2\sqrt{5}}{5}$

7 **a** $\frac{5 + 4\sqrt{2}}{7}$

b $\frac{-15 + 9\sqrt{5}}{4}$

c $6 + 2\sqrt{6}$

8 $\sqrt{22}$

9 $x = 6\sqrt{2}$ cm. Area $= 72\,\text{cm}^2$

10 **a** h = 1.5 or $\frac{3}{2}$

b area $= \frac{3}{4}\sqrt{3}$ or $0.75\sqrt{3}$

Past-paper questions Page 66

1 $2+\sqrt{2}$

2 **(a)** Powers of 2: $4(3x-2)=3(2x)$

$x=\frac{4}{3}$

(b) $p=1, q=-\frac{4}{5}$

3 **(i)** $2^{5x}\times 2^{2y}=2^{-3}$

leads to $5x+2y=-3$

(ii) $7^{x}\times 49^{2y}=1$ can be written as $x+4y=0$

Solving $5x+2y=-3$ and $x+4y=0$

leads to $x=-\frac{2}{3}, y=\frac{1}{6}$

Chapter 5 Factors of polynomials

Discussion point Page 68

23, 128. Keep adding 105 (= 3 × 5 × 7) to find subsequent possible answers.

Exercise 5.1 Page 70

1 $x^4+3x^3-x^2-7x-4$

2 $x^4-3x^3+5x^2-x-2$

3 $4x^4-8x^3+3x^2+10x-5$

4 $x^4-4x^2+12x-9$

5 $4x^4-12x^3+9x^2-16$

6 $x^4-6x^3+13x^2-12x+4$

7 x^2-2x-1

8 x^2-x-1

9 x^3-x^2+x-1

10 $\left((x-2)-\frac{12}{(x+2)}\right)$

Exercise 5.2 Page 75

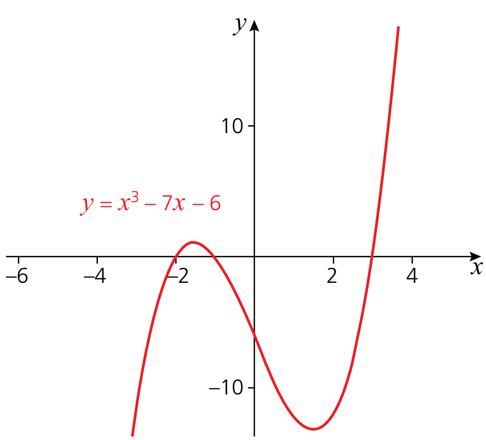
1 **a** f(1) = 0 factor

b f(−1) = −2 not a factor

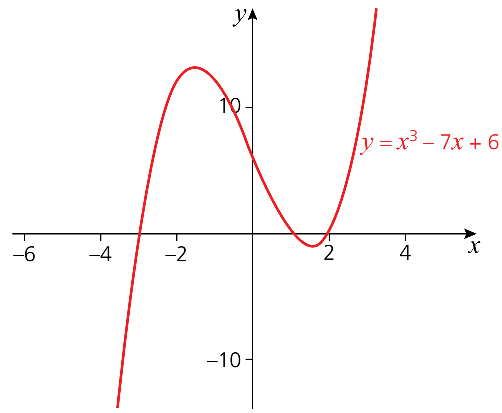
c f(1) = 0 factor

d f(−1) = −8 not a factor

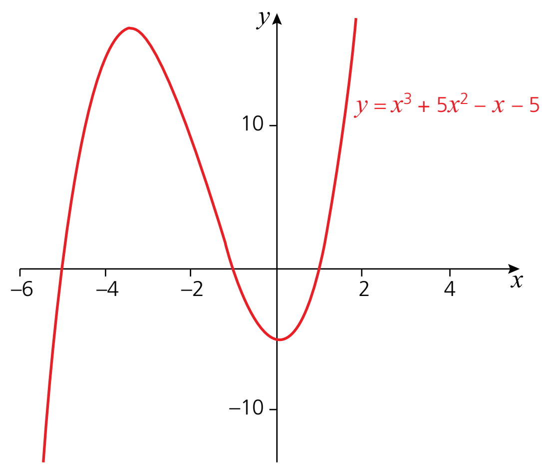
2 **a** $f(-2)=f(-1)=f(3)=0$ $(x+2)(x+1)(x-3)$



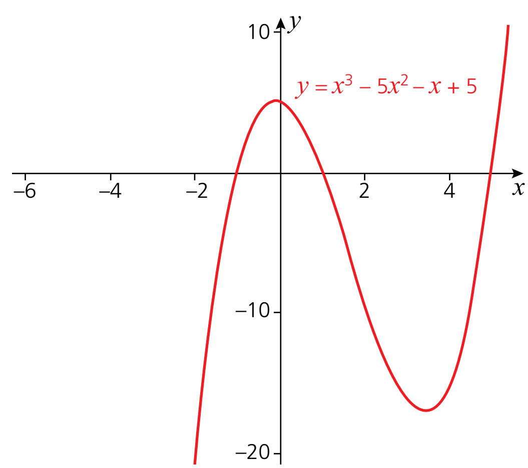
b $f(-3)=f(1)=f(2)=0$ $(x+3)(x-1)(x-2)$



c $f(-5)=f(-1)=f(1)=0$ $(x+5)(x+1)(x-1)$



d $f(-1)=f(1)=f(5)=0$ $(x+1)(x-1)(x-5)$



3 **a** $(x+1)(x^2+1)$
b $(x-1)(x^2+1)$
c $(x+2)(x^2+x+1)$
d $(x-2)(x^2-x-1)$

4 3

5 3

6 **a** $a+b=5\ 3a+b=27$
b $a=11, b=-6$

Exercise 5.3 Page 77

1 **a** $f(2)=6$
b $f(-2)=-10$
c $f(4)=136$
d $f(-4)=-144$

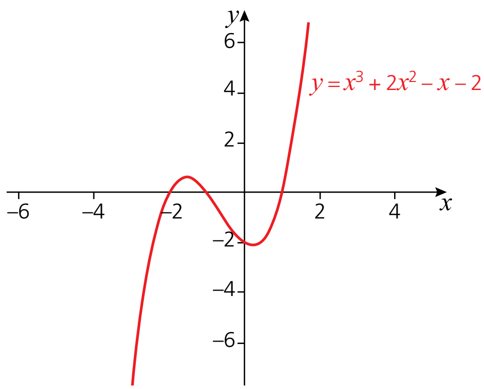
2 $a=-8, b=1$

3 $x=1,-2,-3$

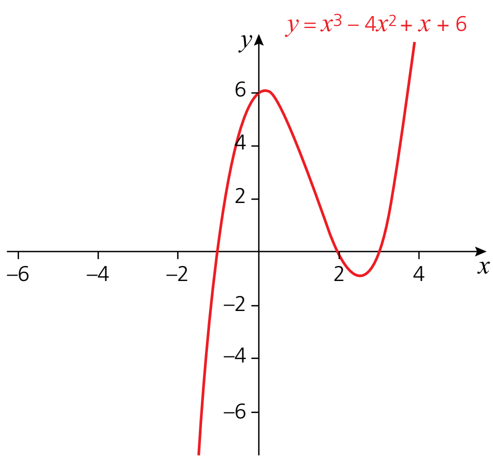
4 $a=1,-3$

5 $a=-7, b=-6$

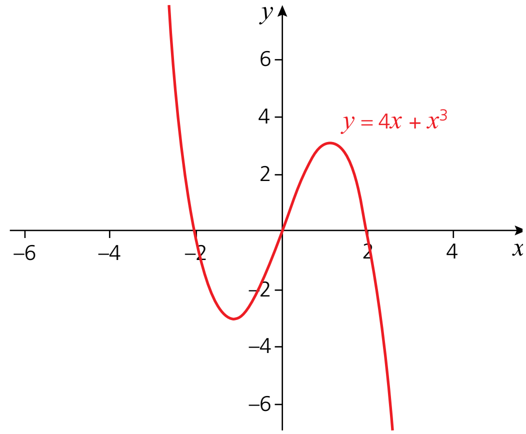
6 **a**



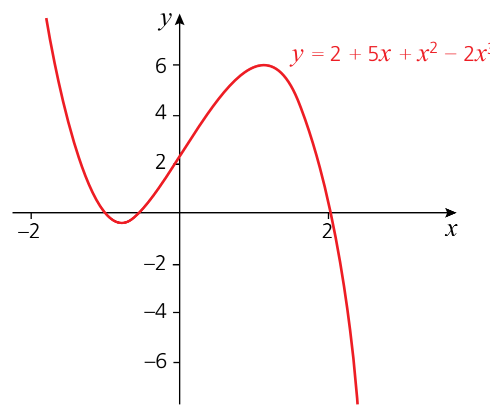
b



c



d



Past-paper questions Page 78

1 **(ii)** $(2x-1)(7x^2-4x+2)$

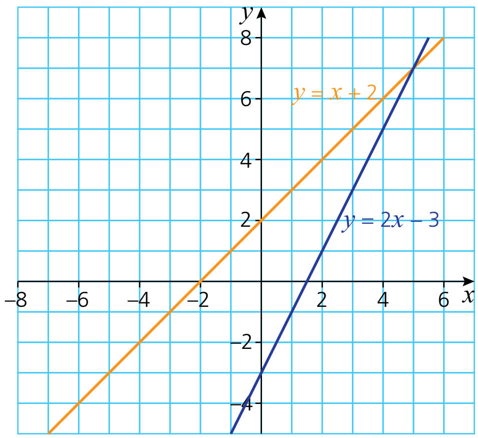
2 **(ii)** $f(-3)=-49$
(iii) $f(x)=(2x-1)(2x^2+3x-2)$
(iv) $f(x)=(2x-1)(2x-1)(x+2)$
Leading to $x=0.5, -2$

3 **(iii)** $6x^2-17x+20=0$ has no real roots
$x=-2$

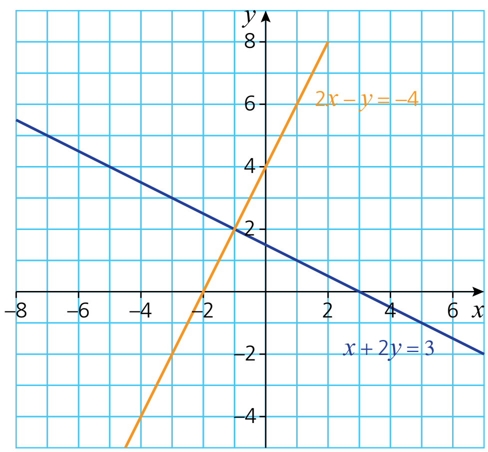
Chapter 6 Simultaneous equations

Exercise 6.1 Page 85

1 **a** $x = 5,\ y = 7$



b $x = -1,\ y = 2$



2 **a** $x = 3, y = 7$

b $x = 7, y = 3$

3 **a** $x = -2, y = 2$

b $x = 2, y = -2$

4 $x = 1, y = 1$

5 $x = -5, y = -2$

6 **a** $x = 3$ and $y = 1$

b $x = 3$ and $y = 0.5$

7 **a** $x = 2$ and $y = 3$

b $x = 2$ and $y = 1.5$

8 $x = 1$ and $y = -2$

9 $x = -3$ and $y = 4$

10 \$3.40

11 \$7

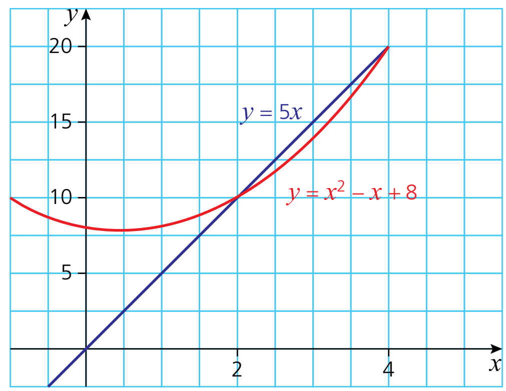
12 \$324

13 \$5

Exercise 6.2 Page 88

1 $x = 2,\ y = 3$ and $x = 2,\ y = -3$

2 $x = 2,\ y = 10$ and $x = 4,\ y = 20$



3 $x = -1,\ y = -4$ and $x = 4,\ y = 1$

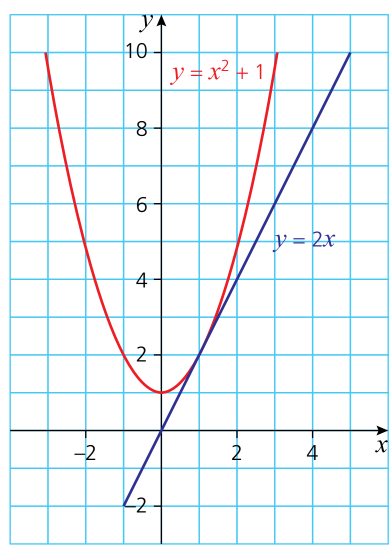
4 $x = -1.5,\ y = 11$ and $x = 1.25,\ y = 0$

5 A $(-4, 3)$ B $(3, 4)$

6 $r = 15,\ x = 4$

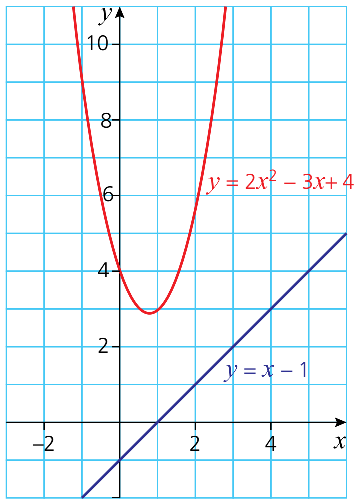
7 **a** $x = 1,\ y = 2$

b The line is a tangent to the curve.



8 **a** No solutions.

b The line does not intersect the curve.



Past–paper questions Page 89

1 $y = -0.5x + 3.75$

2 $3y + x - 2 = 0$

3 $k < -2, k > 8$

Chapter 7 Logarithmic and exponential functions

Discussion point Page 90

In 1700, about $2^{10} = 1024$. In 1000 about $2^{34} = 17\,000\,000\,000$ or 17 billion.

Discussion point Page 91

This is more than the present world population and much more than it was then. The calculation depends on the assumption that all your ancestors of any generation are different people and this is clearly not true; cousins, whether close or distant, must have married each other and so had some ancestors in common.

Discussion point Page 100

Answers will vary.

Exercise 7.1 Page 101

1 **a** 3
b 0
c 2
d –2

2 **a** 4
b 4
c 3
d –3

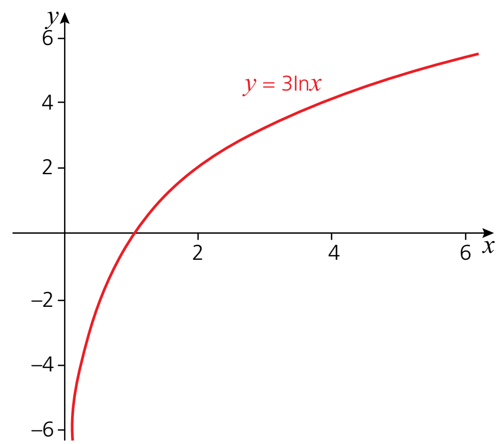
3 **a** 2
b 6
c –3
d –6

4 **a** $\log 15$
b $\log 64$
c $\log 4$
d $\log 5$
e $\log 72$
f $\log \frac{81}{64}$
g $\log \frac{1}{8}$

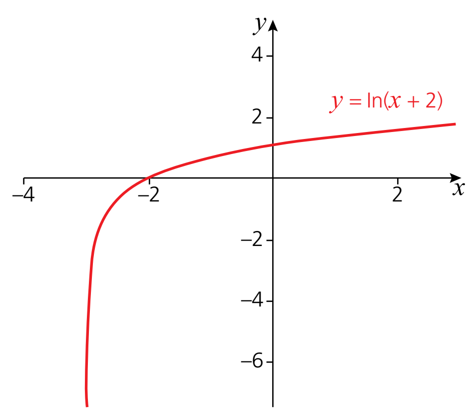
5 **a** $3\log x$
b $6\log x$
c $\frac{3}{2}\log x$

6 **a** 9.3 cm = 93 mm
b 517.06 cm^2 = 51706 mm^2

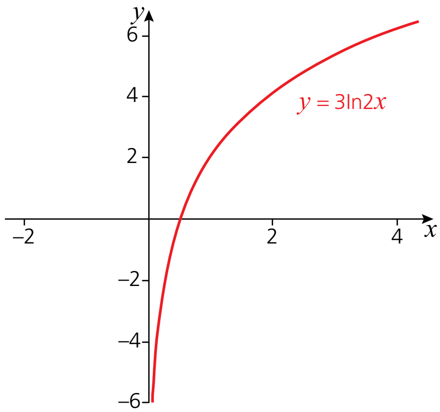
7 **a** Stretch in the y direction scale factor 3.



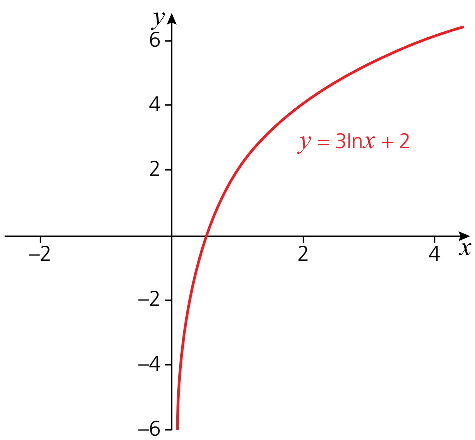
b Translation $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.



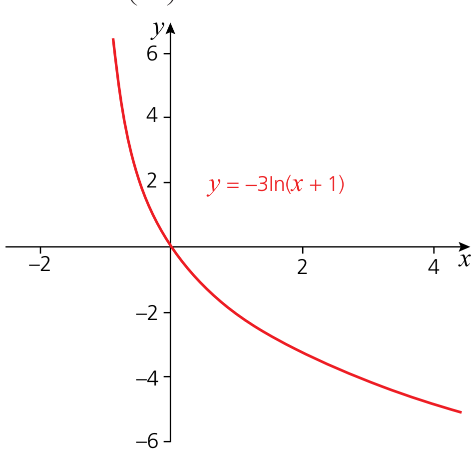
c Stretch in the y direction scale factor 3 and stretch in the x direction scale factor $\frac{1}{2}$.



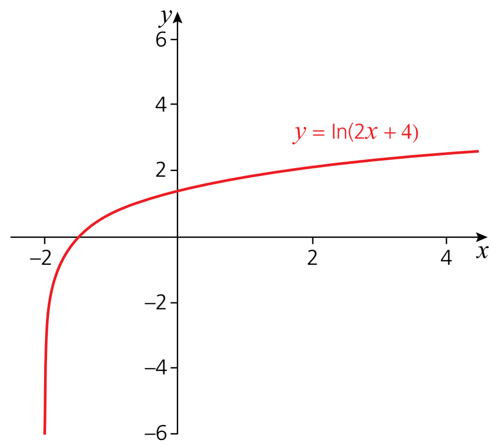
d Stretch in the y direction scale factor 3 and translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.



e Stretch in the y direction scale factor –3 and translation $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.



f Translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ followed by stretch in the x direction scale factor $\frac{1}{2}$.



8 a ii $y = \log(x-1)$
b iv $y = 3\ln x$
c v $y = \log(2-x)$
d vi $y = \ln(x+2)$
e i $y = \log(x+1)$
f iii $y = -\ln x$

9 a $\frac{1}{2}$
b $\frac{2}{9}$
c $\frac{1}{3}\ln 2, 0$

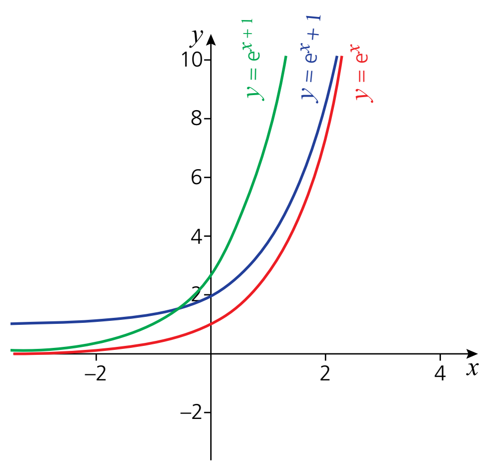
10 c = 0.993

11 $x = 3.15$

12 a 9.4 years (10 years)
b 0.197% per month
c i 113 months
ii 3435 days

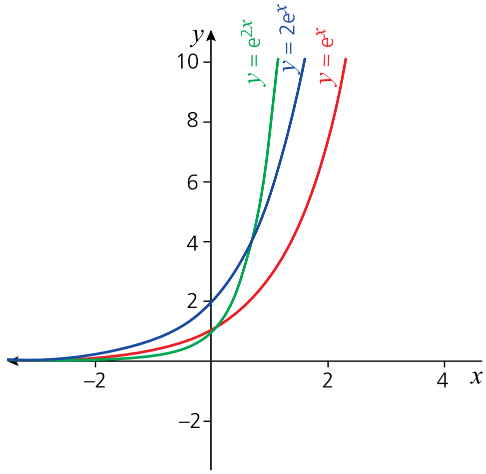
Exercise 7.2 Page 109

1 a



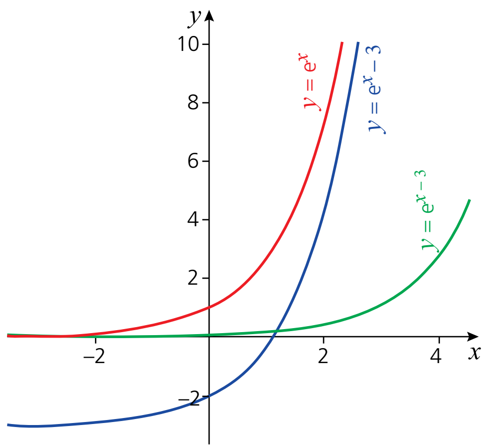
ii $y = e^x$ at $(0, 1)$ $y = e^x + 1$ at $(0, 2)$ and $y = e^{x+1}$ at $(0, e)$

b



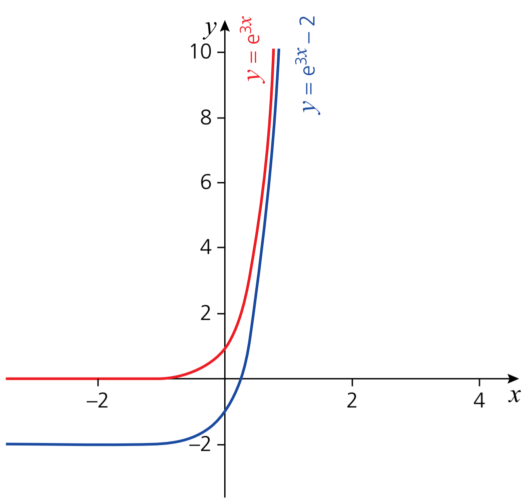
ii $y = e^x$ at $(0, 1)$ $y = e^{2x}$ at $(0, 1)$ and $y = 2e^x$ at $(0, 2)$

c

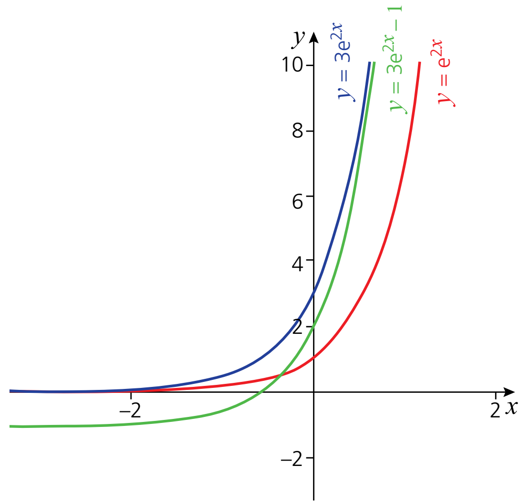


ii $y = e^x$ at $(0, 1)$ $y = e^x - 3$ at $(\ln 3, 0)$ and $(0, -2)$ and $y = e^{x-3}$ at $(0.\ e^{-3})$.

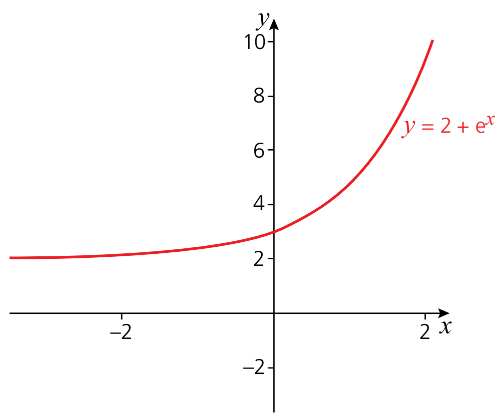
2



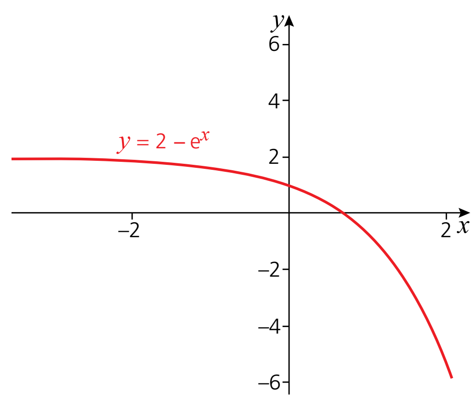
3



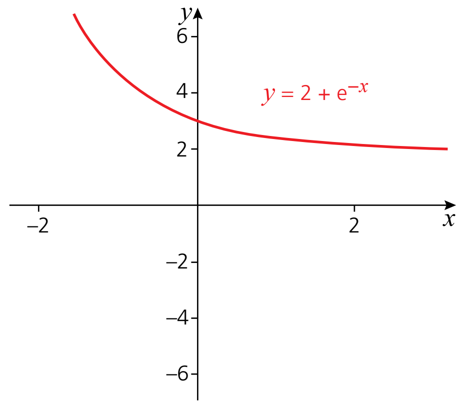
4 **a** $(0, 3)$



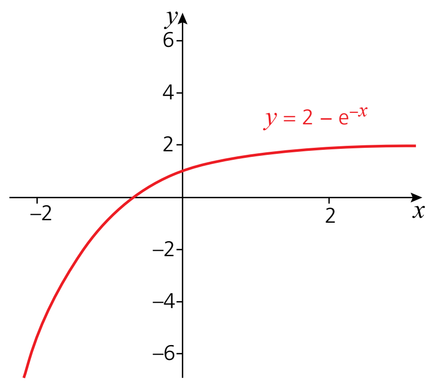
b $(0, 1)$



c (0, 3)



d (0, 1)



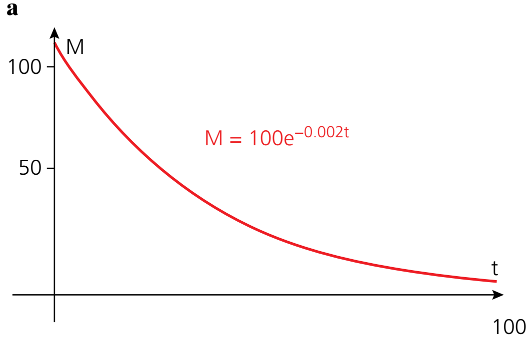
5 **a** 8.55
b 3.23
c 0.303
d 4.30

6 **a** \$4377
b 23 years

7 **a** 4999 m
b 42.59 seconds

8 **a** **ii** $y = e^x + 2$
b **iv** $y = 2 - e^{-x}$
c **i** $y = e^{2x}$
d **vi** $y = e^{-2x} - 1$
e **v** $y = 3e^{-x} - 5$
f **iii** $y = 2 - e^x$

9 **a**



b 347 days

10 **a** \$17.82
b 5 years

11 **a** 0.6309
b 0.65
c 2, 0

Past-paper questions Page 112

1 **(i)** $\log_a p + \log_a q = 9$

$2\log_a p + \log_a q = 15$

$\log_a p = 6$ and $\log_a q = 3$

(ii) $\log_p a + \log_q a = \dfrac{1}{\log_a p} + \dfrac{1}{\log_a q} = 0.5$

2 $a = b^2$, $2a - b = 3$

$2b^2 - b - 3 = 0$ or $4a^2 - 13a + 9 = 0$

leading to $a = \frac{9}{4}$, $b = \frac{3}{2}$

3 **i** $B = 900$

ii $B = 500 + 400e^2 = 3456$

iv $10000 = 500 + 400e^{0.2t}$

$e^{0.2t} = 23.75$

$0.2t = \ln 23.75$

$t = 16$ (days)

Chapter 8 Straight line graphs

Discussion point Page 114

The graph tells us that the natural length of the rope i.e. when there is no load attached is 15m. It shows a proportional relationship between the rope and the load; the heavier the load, the longer the rope becomes as it is stretched. It also tells you that for every extra 40 kg of load it stretches another 5 metres, so for every extra 1 kilogram it stretches 0.125 metres or 12.5 cm.

Activity Page 119

3 Gradient of AB $m_1 = \dfrac{q}{p}$

Gradient of BC $m_2 = -\dfrac{p}{q}$

Exercise 8.1 Page 120

1 **a** **i** 2 **ii** $-\frac{1}{2}$ **iii** $4\sqrt{5}$ **iv** (6, 7)

b **i** $-\frac{11}{5}$ **ii** $\frac{5}{11}$ **iii** $\sqrt{146}$ **iv** (7.5, –2.5)

c **i** $\frac{15}{2}$ **ii** $-\frac{2}{15}$ **iii** $\sqrt{229}$ **iv** (7, 7.5)

d **i** $-\frac{1}{5}$ **ii** 5 **iii** $\sqrt{26}$ **iv** (–0.5, –6.5)

3 **a** $6+3\left(\sqrt{10}\right)+3\left(\sqrt{2}\right)$

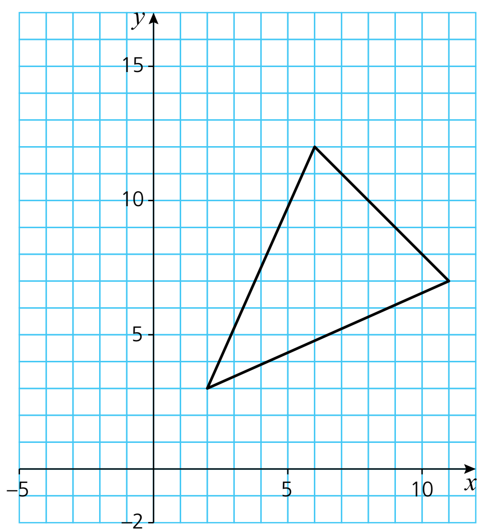
b Area = 9

4 **a** All sides $\sqrt{173}$

b Midpoints both $\left(\frac{7}{2}, \frac{1}{2}\right)$

c Gradient PQ $= -\frac{2}{13}$ gradient PS $= -\frac{13}{2}$ so not perpendicular. Rhombus.

5 **a**



c midpoint BC $\left(\frac{17}{2}, \frac{19}{2}\right)$

d $\frac{65}{2}$

6 **b** $(7, 4)$

7 **a** 2

b 1:4

8 **a** gradient BC = gradient AD $- -\frac{1}{5}$ so one pair of parallel sides so trapezium.

c $(8, 6)$

9 **a** **i** $y = 2x - 13$ **ii** $2y + x + 1 = 0$

b **i** $x + 3y = 17$ **ii** $y = 3x + 19$

c **i** $2x = 3y + 16$ **ii** $2y + 3x + 15 = 0$

10 **a** $2y = 3x - 13$

b $4y + 22x + 17 = 0$

c $9y + x = 18$

11 **a** $2y = x - 4$

b $(5, 0.5)$

c $y = -2x + 10.5$

d $(0, 10.5)$ $(5.25, 0)$

Exercise 8.2 Page 124

1 **a** iii **b** ii **c** i **d** iv

2 **a** $\ln y = \ln k + x \ln a$ Plot $\ln y$ against x gradient $\ln a$ and intercept $\ln k$

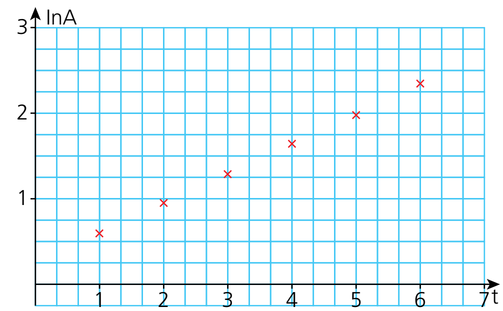
b $\ln y = \ln k + a \ln x$ Plot $\ln y$ against $\ln x$ gradient a and intercept $\ln k$

c $\ln y = \ln a + x \ln k$ Plot $\ln y$ against x gradient $\ln k$ and intercept $\ln a$

d $\ln y = \ln a + k \ln x$ Plot $\ln y$ against $\ln x$ gradient k and intercept $\ln a$

3 **b**

t	1	2	3	4	5	6
lnA	0.5878	0.9555	1.2809	1.6292	1.9741	2.3321



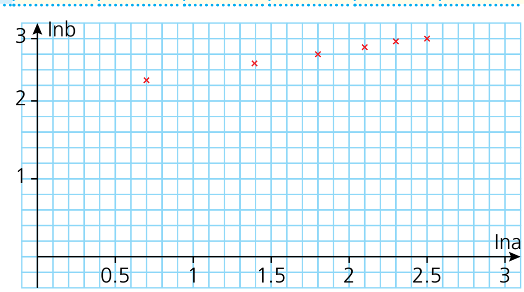
Points are close to a straight line.

c $k = 1.25$ $b = 1.44$

d **i** 4.3 days **ii** 6.45 cm^2

4 **b**

lna	0.6931	1.3863	1.7918	2.0794	2.3026	2.4849
lnb	2.2824	2.4932	2.6174	2.7014	2.7726	2.8273



Points are close to a straight line.

c) $n = 0.304$ $P = 7.938$

Answers

5 a

$\ln x$	0.2624	0.4700	0.6419	0.7885	0.9163	1.0296
$\ln y$	1.7579	2.1041	2.7014	3.0681	3.3878	3.6738

$a = 3$ $n = 2.5$

b The incorrect value is the second one. Using the answers given for part a, the correct value should be 9.7.

6 a $\ln P = \ln k + t \ln a$

b

t	1	2	3	4	5
$\ln p$	1.2809	1.4586	1.6487	1.8245	2.0149

$a = 1.2$ $k = 3.0$

c 115 thousand people. Not reliable.

Past–paper questions Page 125

1 (i) (3, 5)

(ii) $y = 2x - 1$

(iii) 15 square units

2 (i) D (3, 8) E (5.4, 9.2)

(ii) Area $= \frac{1}{2}(13 + 3) \times 4$

$= 32$

3 (i) $\sqrt{20}$ or 4.47

(ii) Gradient $AB = \frac{1}{2}, \perp$ gradient $= -2$

$\perp$ line $y - 4 = -2(x - 1)$

$y = -2x + 6$

(iii) (3,0) (−1, 8)

Chapter 9 Circular measure

Discussion point Page 128

26 centimetres per second.

Exercise 9.1 Page 133

1 a $\frac{2\pi}{3}$

b 3π

c $\frac{11\pi}{90}$

d $\frac{5\pi}{6}$

e $\frac{5\pi}{24}$

2 a 120°

b 100°

c 171°

d 25.7°

e 67.5°

3

Radius, r (cm)	Angle at centre, θ (degrees)	Angle at centre in radians	Arc length, s (cm)	Area, A (cm²)
8	120°	$\frac{2\pi}{3}$	$\frac{16\pi}{3}$	$\frac{64\pi}{3}$
10	28.6°	$\frac{7\pi}{44}$	5	25.0
5.73	60°	$\frac{\pi}{3}$	4	17.2
6	38.2°	$\frac{7\pi}{33}$	4	12
5.53	75°	$\frac{5\pi}{12}$	7.24	20

4

Radius, r (cm)	Angle at centre, θ (radians)	Arc length, s (cm)	Area, A (cm²)
10	$\frac{\pi}{3}$	$\frac{10\pi}{3}$	$\frac{50\pi}{3}$
12	2	24	144
20.37	$\frac{\pi}{4}$	16	162.97
5	2	10	25
6.51	$\frac{3\pi}{5}$	12.28	40

5 a $\frac{32\pi}{3}$ cm²

b $8 + \frac{16\pi}{3}$ cm

6 a 15π cm²

b 9 cm²

c $15\pi - 9$ cm²

7 a 33.82π

b 67.36 cm²

8 a $\frac{3\pi}{4}$

b 24π cm²

c $16\sqrt{2}$ cm²

d $24\pi - 16\sqrt{2} = 52.77$ cm²

Past–paper questions Page 135

1 (i) Area $= \frac{1}{2}\,18^2 \sin 1.5 - \frac{1}{2}10^2(1.5)$

$= 161.594 - 75$

$= 86.6$

(ii) $AC = 15$ or 10×1.5

$LBD = 36 \sin 0.75$

$BD = \sqrt{18^2 + 18^2 - (2 \times 18 \times 18 \cos 1.5)}$

$= 24.5$

Perimeter $= 15 + 24.5 + 16$

$= 55.5$

2 **(i)** $\sin\frac{\theta}{2} = \frac{6}{8}, \frac{\theta}{2} = 0.8481$

$\theta = 1.696$

(ii) Arc length $= (2\pi - 1.696) \times 8 = 36.697$

Perimeter $= 12 + (2\pi - 1.696) \times 8$

$= 48.7$

(iii) Area $= \frac{8^2}{2}(2\pi - 1.696) + \frac{8^2}{2}\sin 1.696$

$= 178.5$

3 **(i)** Area of sector $= \frac{1}{2} \times x^2 \times 0.8$

Area of triangle $=$

$\frac{1}{2} \times 5\cos 0.8 \times 5\sin 0.8 = 6.247\text{cm}^2$

$0.08x^2 = 6.247$

$x = 8.837$

(ii) Perimeter $= 19.85$ cm (2 d.p.)

(iii) Area $PQSR = 4 \times 6.247$

$= 25\,\text{cm}^2$

Chapter 10 Trigonometry

Discussion point Page 138

You can take measurements from the picture. Then use them to make a scale drawing of the cross–section of the pyramid that goes through its vertex, and measure the angle with a protractor.

This is not a very accurate method. The alternative is to use trigonometry to calculate the answer.

Exercise 10.1 Page 144

1 **a** $29^2 = 841 = 400 + 441 = 20^2 + 21^2$

b $\sin Q = \frac{20}{29}$

$\cos Q = \frac{21}{29}$

$\tan Q = \frac{20}{21}$

c **i** $\left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2 = \frac{400 + 441}{841} = 1$

ii $\tan Q = \frac{20}{29} \div \frac{21}{29} = \frac{20}{21}$

5 **a** 6 cm

6 **a** $2\sqrt{2}$ cm

7 **a** 4d

8 **a** 2 cm

Exercise 10.2 Page 147

1 **a** **i** $\frac{1}{2}$ **ii** $\frac{\sqrt{3}}{2}$ **iii** $\frac{1}{\sqrt{3}}$

b **i** 2 **ii** $\frac{2}{\sqrt{3}}$ **iii** $\sqrt{3}$

2 **a** **i** $\frac{1}{\sqrt{2}}$ **ii** $\frac{1}{\sqrt{2}}$ **iii** 1

b **i** $\sqrt{2}$ **ii** $\sqrt{2}$ **iii** 1

3 **a** **i** $\frac{\sqrt{3}}{2}$ **ii** $\frac{1}{2}$ **iii** $\sqrt{3}$

b **i** $\frac{2}{\sqrt{3}}$ **ii** 2 **iii** $\frac{1}{\sqrt{3}}$

4 **a** $B = 60°$ $C = 30°$

b $\sqrt{3}$

5 **a** $B = 30°$ $C = 60°$

b $AB = 2\sqrt{3}$ $BC = 4$

6 $\frac{4}{\sqrt{7}}, \frac{\sqrt{7}}{3}$

7 **a** 45°; 45°

b $\sec L = \sqrt{2}$ $\text{cosec} L = \sqrt{2}$ $\tan L = 1$

8 **b** 14.0°

9 **a** $0 < \alpha < 90$

$0 < \alpha < \frac{\pi}{2}$

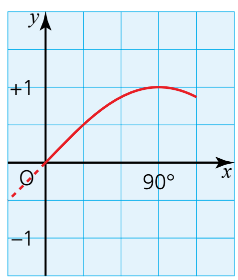
b No

c No

Discussion Point Page 151

1 The graphs continue the same wave patterns to the left.

2



Starting with $y = \sin x$ for 0° to 90°,

* reflect it in the line $x = 90°$ to obtain the part of the curve from 90° to 180°.

* rotate the curve from 0° to 180° through 180° about the point (180°, 0) to obtain the part of the curve between 180° and 360°. You now have the complete curve of $y = \sin x$ from 0° to 360°.

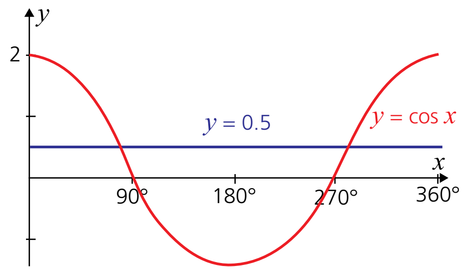
To obtain the curve of $y = \cos x$, translate the curve for $y = \sin x$ from 0° to 360° by –90° in the horizontal direction. Then translate the part of the curve between –90° and 0° by +360° in the horizontal direction.

Discussion point Page 154

The curves do not change.

Exercise 10.3 Page 155

1 a

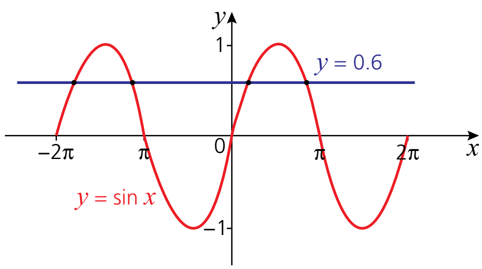


b 60° or 300°

c $(60 + 360n)°$ or $(300 + 360n)°$

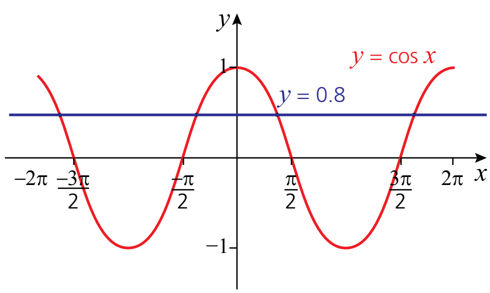
d $-\frac{1}{2}$

2 a



b $-1.2\pi, -1.8\pi, 0.2\pi, 0.8\pi$

c



d $-0.2\pi, -1.8\pi, 0.2\pi. 1.8\pi$

e For acute angles $\sin^{-1} 0.6 = \cos^{-1} 0.8$ and for other angles where both sin and cos are positive. For other angles, one or other is negative.

3 a 60°, 240°

b 30°, 150°

c 150°, 210°

d 30°, 210°

e 134.4°, 225.6°

f 72.5°, 287.5°

g 199.5°, 340.5°

h 180°

4 a $\frac{1}{\sqrt{2}}$

b $\frac{1}{2}$

c 1

d $\frac{\sqrt{3}}{2}$

e $-\frac{\sqrt{3}}{2}$

f 0

g $\frac{1}{\sqrt{2}}$

h $\frac{1}{\sqrt{2}}$

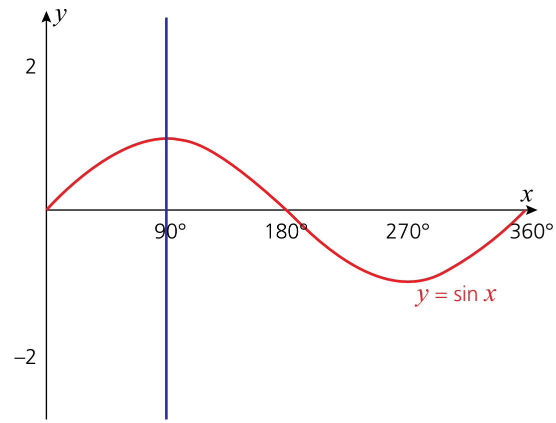
i 1

5 a 150°

b 139.7°

c 87.4°

6 a Graph has a line of symmetry at $x = 90$.



b i False **ii** True **iii** False **iv** True

7 a $0° < \alpha < 90°$

b No. If $\cos\alpha$ and $\sin\alpha$ are both positive, $\tan\alpha$ is positive.

c No. If $\cos\alpha = \tan\alpha$, $\tan\alpha = 1$.

If $\sin\alpha = 1$, $\cos\frac{\pi}{2} \neq 1$

8 a 80°, 320°

b 20°, 200°

c 55°, 325°

d 30°, 120°, 210°, 300°

e 60°, 300°

f 75°, 105°, 255°, 285°

g 10°, 50°, 130°, 170°, 250°, 290°

h 0°, 90°, 180°, 270°, 360°

i 15°, 75°, 135°, 195°, 255°, 315°

9 a 0.1 radians and 0.03π radians.

b $\frac{1}{3}\pi$

c –1.11 radians and -0.35π radians.

d –0.41 radians and -0.13π

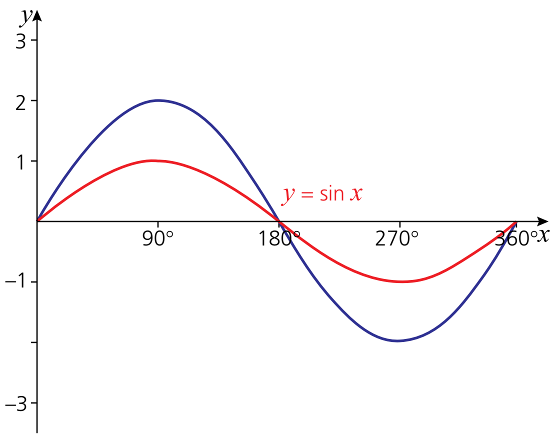
e 0 and $\frac{1}{2}\pi$

Discussion point Page 158

Stretching the point (90°, 1) by a scale factor of 2 parallel to the y–axis gives the point (90°, 2). Translating this through 3 units in the y–direction gives (90°, 5). If the translation is done before the stretch, the point (90°, 1) moves through (90°, 4) to (90°, 8), which is incorrect.

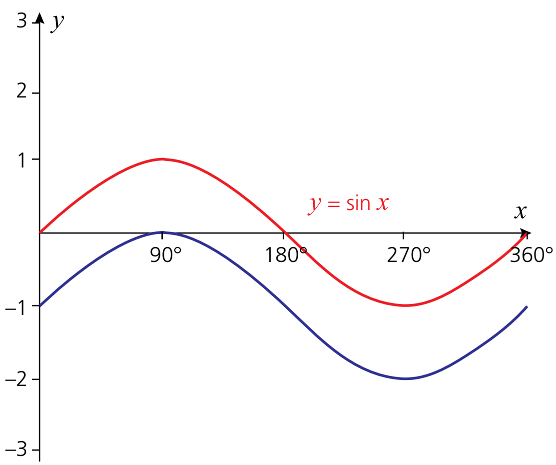
Exercise 10.4 Page 161

1 **i** **a**



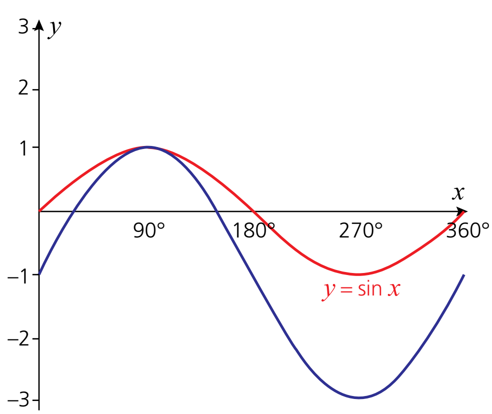
b Amplitude 2, period 360°.

ii **a**



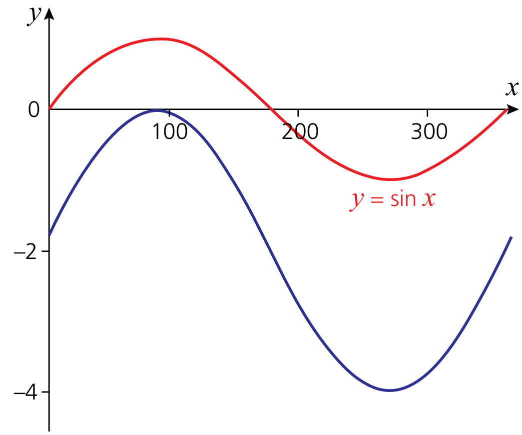
b Amplitude 1, period 360°.

iii **a**



b Amplitude 2, period 360°.

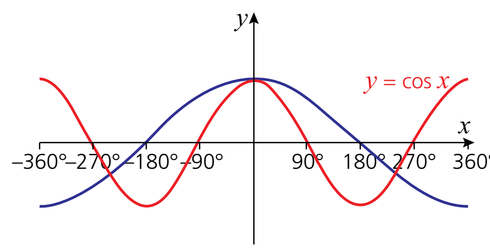
iv **a**



b Amplitude 2, period 360°.

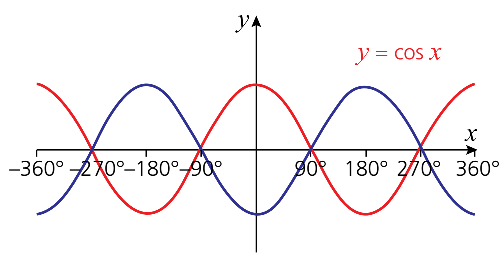
c The order of the transformation matters.

2 **i** **a** and **b**



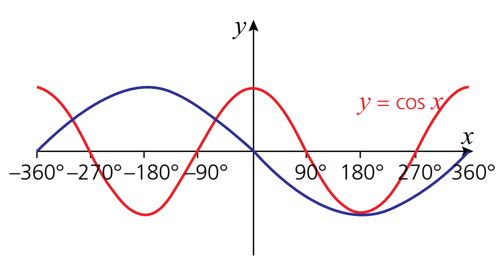
c Amplitude 1 period 720°.

ii **a** and **b**



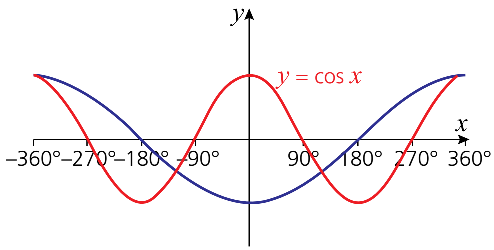
c Amplitude 1 period 360°.

iii **a** and **b**



c Amplitude 1 period 720°.

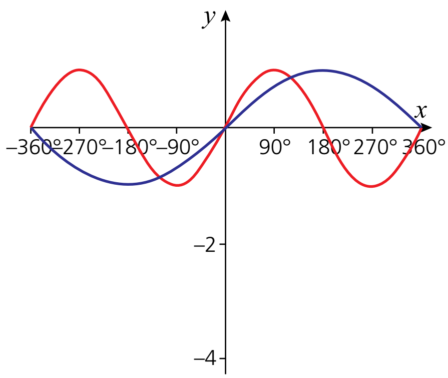
iv a and **b**



c Amplitude 1 period 720°.

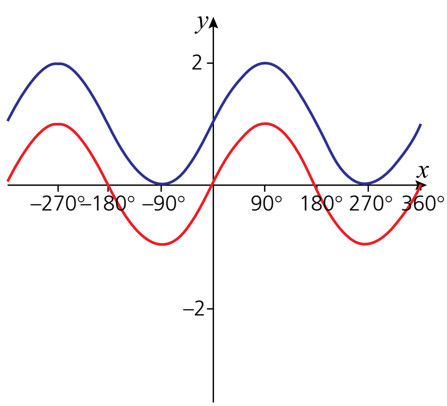
d The order of the transformation matters.

3 i a and **b**



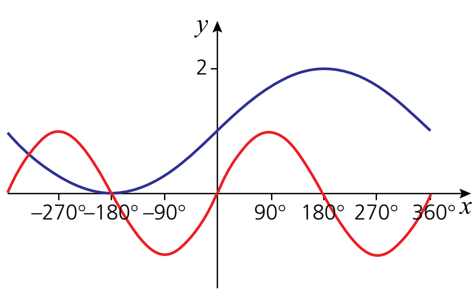
c Amplitude 1, period 720°.

ii a and **b**



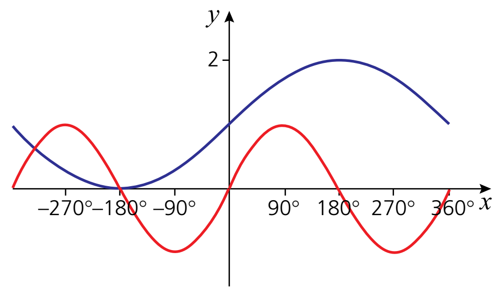
c Amplitude 1, period 360°.

iii a and **b**



c Amplitude 1, period 720°.

iv a and **b**



c Amplitude 1, period 720°.

d The order does not matter when the transformations are in different directions.

4 a Stretch in the y–direction scale factor 3 followed by a stretch in the x–direction scale factor $\frac{1}{2}$ (either order).

b Stretch in the y–direction scale factor 2 followed by a translation of one unit vertically upwards.

c Stretch in the y direction scale factor 2 followed by a translation to the right by 180 (either order).

d Translation to the left $\frac{\pi}{2}$ followed by a stretch in the y–direction scale factor 3 and a translation 3 units vertically upwards.

5 A stretch, scale factor 2, parallel to y–axis and a translation of 1 unit vertically upwards.

6 $y = \tan 2x$; a stretch, scale factor $\frac{1}{2}$, parallel to the x–axis.

7 Stretch in the x–direction scale factor $\frac{1}{2}$ followed by a stretch in the y–direction scale factor 3 and a translation vertically downwards by 2. (The stretch in the x–direction can be at any stage.)

Exercise 10.5 Page 168

5 50.74°; 125.26°

7 $\frac{\pi}{3}; \frac{2\pi}{3}; \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

9 a $-2\pi; 0; 2\pi$

b It is not possible to solve.

10 $\theta = -270°, -90°, 90°, 270°$

11 b $= 0°; 360°; 56.45°; 203.55°$

12 b $0; \pi; 2\pi$

Or $\frac{\pi}{6}; \frac{5\pi}{6}$

Past–paper questions Page 168

1 a $\sin^2 x = \frac{1}{4}$

$\sin x = (\pm)\frac{1}{2}$

$x = 30°, 150°, 210°, 330°$

b $(\sec^2 3y - 1) - 2\sec 3y - 2 = 0$

$\sec^2 3y - 2\sec 3y - 3 = 0$

$(\sec 3y + 1)(\sec 3y - 3) = 0$

leading to $\cos 3y = -1$, $\cos 3y = \frac{1}{3}$

$3y = 180°, 540°$ $3y = 70.5°, 289.5°, 430.5°$

$y = 60°, 180°, 23.5°, 96.5°, 143.5°$

c $z - \frac{\pi}{3} = \frac{\pi}{3}, \frac{4\pi}{3}$

$z = \frac{2\pi}{3}, \frac{5\pi}{3}$

2 a i $A = 3, B = 2$

ii $C = 4$

b 120 or $\frac{2\pi}{3}$

Amplitude = 5

3 $\text{LHS} = \frac{1+\cos\theta}{(1-\cos\theta)(1+\cos\theta)} + \frac{1-\cos\theta}{(1+\cos\theta)(1-\cos\theta)}$

$= \frac{2}{1-\cos^2\theta}$

$= \frac{2}{\sin^2\theta}$

$= 2\operatorname{cosec}^2\theta = \text{RHS}$

Chapter 11 Permutations and combinations

Discussion point Page 173

There are $6 \times 6 \times 6 \times 6 = 1296$ possibilities.

Suppose it takes 10 seconds to check each combination, so you check six combinations in 1 minute.

It will take $1296 \div (6 \times 60) = 3.6$ hours or 3 hours 36 minutes.

Exercise 11.1 Page 175

1 a 5040

b 72

c $\frac{12}{7}!$

2 a $\frac{1}{n+1}$

b $n-2$

3 a $(n+1)(n+2)$

b $n(n+1)$

4 a $\frac{9! \times 3!}{(6!)^2}$

b $\frac{15!}{3! \times 5!}$

c $\frac{(n+2)!}{n! \times 4!}$

5 a $8 \times 6!$

b $(n+1)(n-1)!$

6 $\frac{7!}{5!}$

7 24

8 5040

9 120

10 120

11 15!

12 a 6

b $8! = 40320$

c $9! = 362880$

d $6! = 720$

e $7! = 5040$

f $8! = 40320$

13 a 8!

b 7!

c 5040

d 35280

Discussion point (1) Page 178

No, it does not matter in which order the machine picked them since this is a combination, not a permutation.

Discussion point (3) Page 178

As the amount of numbers to choose from increases, the probability decreases because there are more possible combinations.

Discussion point (1) Page 179

$^nC_r = \frac{n!}{r!(n-r)!}, \quad {}^nP_r = \frac{n!}{(n-r)!}$

$^nC_r = \frac{1}{r!} \times \frac{n!}{(n-r)!} = \frac{1}{r!} \times {}^nP_r = \frac{^nP_r}{r!}$

Discussion point (2) Page 179

$^nC_r = \frac{n!}{r!(n-r)!}$

$\Rightarrow {}^nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1$

Similarly for ^nC_n.

Exercise 11.2 Page 179

1 a i 210 **ii** 3024 **iii** 1814400

b i 35 **ii** 126 **iii** 45

2 2730

3 1287

4 576

5 a 1716

b 63

6 a 210

b i $\frac{1}{42}$

ii $\frac{10}{21}$

7 a 210

b i $\frac{1}{14}$

ii $\frac{5}{42}$

8 **a** 8008
b 2940
c 7
d 84
9 **a** 1001
b 420
c 336
10 32
11 **a** **i** 15120
ii 1680
12 9080
13 **a** 495
b 360
14 **a** 21772800
b 126
15 **a** 56756700
b 3870720
c 192
16 **a** 79833600
b 3628800

Past-paper questions Page 181

1 **(i)** $\frac{14\times13\times12\times11\times10\times9}{6\times4\times3\times2\times1}$ or $\frac{14!}{8!\times6!}$
3003
(ii) Either 5 students + 1 teacher or 4 students + 2 teachers.
56 × 6 or 70 × 15
1386
(iii) 30
2 **(a) (i)** 360
(ii) 120
(b) (i) 924
(ii) 28
(iii) $924-({}^8C_3\times{}^4C_3)-({}^8C_2\times{}^4C_4)$
(i.e. 924 – 3M 3W – 2M 4W)
924 – 224 – 28
= 672
3 **(a) (i)** 15120
(ii) 210

Chapter 12 Series

Discussion point Page 184

50×2^{63} mg = 4.6×10^{14} tonnes. In 2017 the world production of wheat was about 7.5×10^8 tonnes. King Shirham made an agreement that he could not possibly fulfil!

Real world activity Page 188

1 $\frac{n-1}{2}$
2 Multiply the number in column 2 by $\frac{n-2}{3}$

Exercise 12.1 Page 191

1 **a** $1+4x+6x^2+x^4$
b $1+8x+24x^2+32x^3+16x^4$
c $1+12x+54x^2+108x^3+81x^4$
2 **a** $16+32x+24x^2+8x^3+x^4$
b $81+108x+54x^2+16x^3+x^4$
c $256+256x+96x^2+16x^3+x^4$
3 **a** $x^4+4x^3y+6x^2y^2+4xy^3+y^4$
b $x^4+8x^3y+24x^2y^2+32xy^3+16y^4$
c $x^4+12x^3y+54x^2y^2+108xy^3+81y^4$
4 **a** 10
b 21
c 35
d 21
e 1
f 286
5 **a** 15
b 21
c 28
6 $243+405kx+270k^2x^2$
7 $729x^6-4374x^4+10935x^2$
8 **a** $6t+2t^3$
9 **a** -5, -5
b 5, -5
10 **a** $1-12x+60x^2-160x^3$
b $64-576x+2160x^2-4320x^3$
c $729-5832x+19440x^2-34560x^3$
11 **a** $x^{10}+5x^7+10x^4+10x$
b $x^{10}-5x^7+10x^4-10x$
c $x^{15}+5x^{11}+10x^7+10x^3$
d $x^{15}-5x^{11}+10x^7-10x^3$
12 $n=5$ $a=3$

Exercise 12.2 Page 196

1 **a** yes: 2, 40
b no
c no
d yes: 4, 29
e yes: -4, -12
2 **a** 21
b 19
3 **a** d = 7.2
b 16 terms
4 **a** 5
b 2055
5 **a** 16, 20, 24
b 456
6 **a** 15
b 1140
7 **a** $a=8,\ d=6$
b 10
8 **a** 2
b 480
9 **a** 10000
b 10200
c 20200

d Answer c is the total of the other two (a and b) answers.

10 a 66000

b Sum negative for more than 30 terms.

11 a $7n-9$, 17th

b $\frac{7}{2}n^2-\frac{11}{2}n, 25$

12 a \$54000

b 10

13 a 47 days

b 126.9 km

14 a 16

b 35 cm

15 a $a=-25$ and $d=15$

b $n=16$

16 a 3

b 105

c $2n^2+n$

d $6n^2+n$

Discussion point Page 201

50×2^{63} mg $=4.6\times 10^{14}$ tonnes. In 2017 the world production of wheat was about 7.5×10^8 tonnes.

Discussion point Page 205

Something else

Exercise 12.3 Page 205

1 a yes 2, 192

b no

c yes: −1, 10

d yes: 1, 1

e no

f yes: $\frac{1}{2}, \frac{5}{32}$

g no

2 a 320

b 635

3 a 4

b 49152

4 a 16

b 8th

5 a 10

b 7161

6 a 9

b 199.609375

7 a 2

b 4.5

c 4603.58

8 a 0.5

b 16

9 a 0.1

b $\frac{8}{9}$

c $\frac{8}{11}$

10 a 0.7

b 14th

c $\frac{1000}{3}$

d 13

11 a $\frac{1}{3}$

b 5

12 a $\frac{1}{3}$

b $\frac{8}{3}^{(n-4)}$

c $324-\frac{4}{3}^{(n-4)}$

d 324

e 12 terms

13 a 10, 5, 2.5, 1.25, 0.625

b 0, 5, 7.5, 8.75, 9.375

c First is geometric with common ratio 0.5.

14 a 60

b 400°

15 a $15\times\left(\frac{5}{8}\right)^n$

b 79.27 m

18 a 4 and −4

b When $x=-4$; sum of the first ten terms = 629, 145.
When $x=4$; sum of the first ten terms = 1,048,575.

Past-paper questions Page 207

1 $120(p^7q^3)$ and $(45)(p^8q^2)$
$120p^7q^3=270p^8q^2$
$252p^5q^5=252$
$pq=1$ and $4q=9p$
leading to $p=\frac{2}{3}, q=\frac{3}{2}$

2 **(i)** -27.5 **(ii)** 38.5

3 **(i)** $64+192x+240x^2+160x^3$
(ii) 64

Chapter 13 Vectors in two dimensions

Discussion point Page 210

They tell you both the speed and the direction of the wind.

Exercise 13.1 Page 213

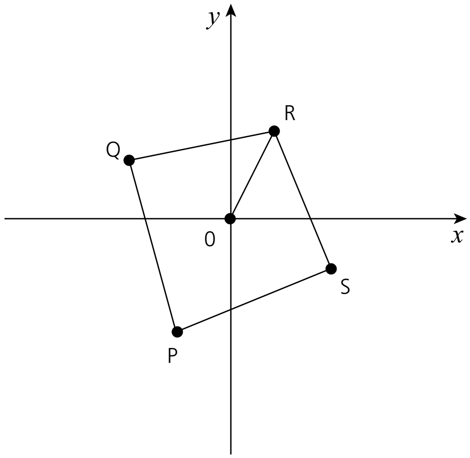
1 a $\begin{pmatrix}-4\\-2\end{pmatrix}$

b $\begin{pmatrix}6\\-4\end{pmatrix}$

c $\begin{pmatrix}-3\\0\end{pmatrix}$

d $\begin{pmatrix}-2\\-4\end{pmatrix}$

2 a



b i OR

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

ii RO

$\begin{pmatrix} -1 \\ -2 \end{pmatrix}$

c i PR

$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

ii $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$

d lengths of the vectors

i PQ

ii QR

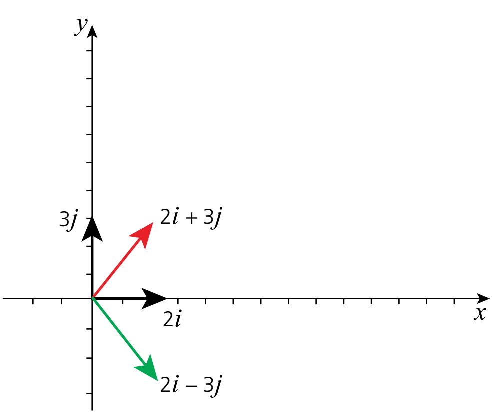
iii RS

iv SP

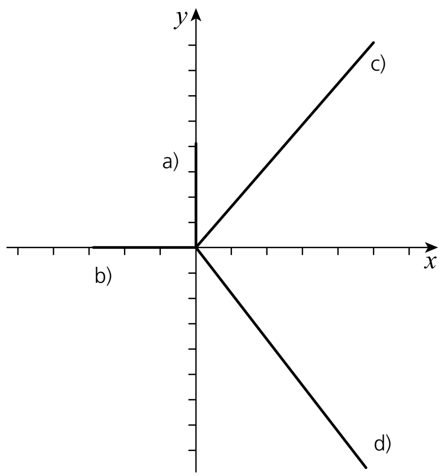
All the lengths of the vectors = $\sqrt{10}$.

e PQRS is a square because all angles are equal to 90°.

3 a b c d



4 i



ii

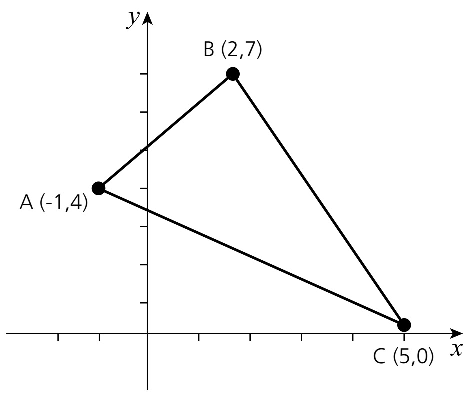
a 4

b 3

c $\sqrt{74}$

d $\sqrt{74}$

5 a



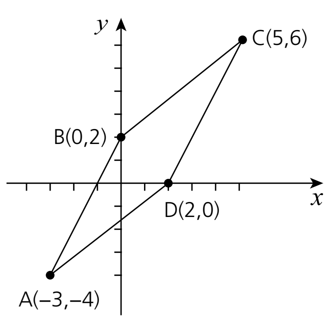
AB = 3i + 3j

BC = 3i – 7j

AC = 6i – 4j

b BC is the longest side of the triangle.

6 a



b OA = –3i – 4j

OB = 2j

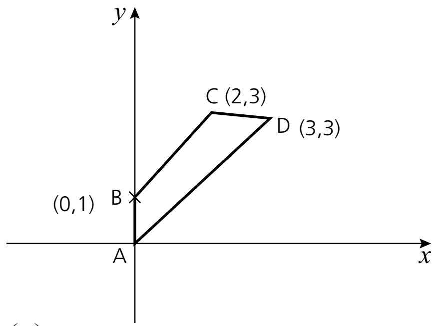
OC = 5i + 6j

OD = 2i

c AB = 3i + 6j
DC = 3i + 6j
BC = 5i + 4j
AD = 5i + 4j

d ABCD is a parallelogram.

7 **a**



b $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$

c Isosceles trapezoid.

Exercise 13.2 Page 220

1 **a** $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$

b $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

c $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$

2 **a** – i + 5j

b 13j

3 **a** 4i + 6j

b 6i – 4j

c 12j

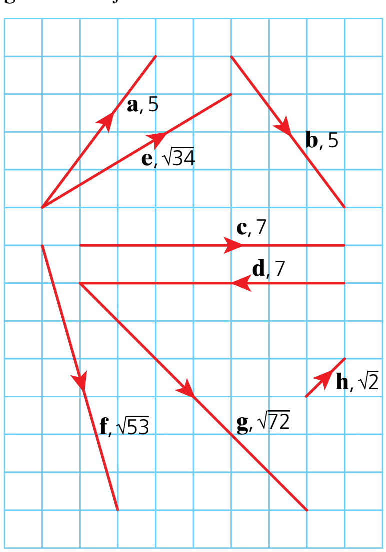
d 5i + 20j

e –4i + 25j

f i + 26j

g 11i – 15j

4



5 **a** **i** 2i + 3j

ii $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

b **i** i – 7j

ii $\begin{pmatrix} 1 \\ -7 \end{pmatrix}$

c **i** 3i + 5j

ii $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

d **i** 10i – 10j

ii $\begin{pmatrix} 10 \\ -10 \end{pmatrix}$

e **i** 4i – 2j

ii $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$

f **i** 4i + 2j

ii $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

6 **a** b – a

b a – b

c a

d – a

e b

f – b

g – a + b – a = b – 2a

h a – b + a = 2a – b

7 **a** AD = $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

DC = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

b AC = $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$

BD = $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c $(\frac{3}{2}; \frac{1}{2})$

d Gradient of AC = $\frac{5}{5} = 1$

Gradient of BD = $\frac{1}{-1} = -1$

They are perpendicular because $1 \times -1 = -1$.

8 **a** $\frac{3}{5}\text{i} - \frac{4}{5}\text{j}$

b $\left(\frac{5}{\sqrt{74}}\right)\text{i} + \left(\frac{7}{\sqrt{74}}\right)\text{j}$

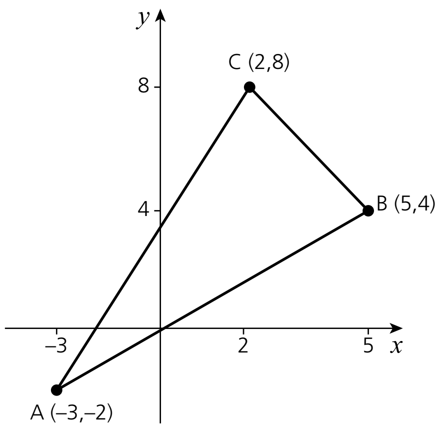
c $\begin{pmatrix} \frac{5}{13} \\ \frac{12}{13} \end{pmatrix}$

d $\begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} \end{pmatrix}$

e i

f $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

9 a



b $|AB| = 8i + 6j$
$|BC| = -3i + 4j$
$|CA| = -5i - 10j$

c $|AB| = 10$
$|BC| = 5$
$|CA| = 5\sqrt{5}$

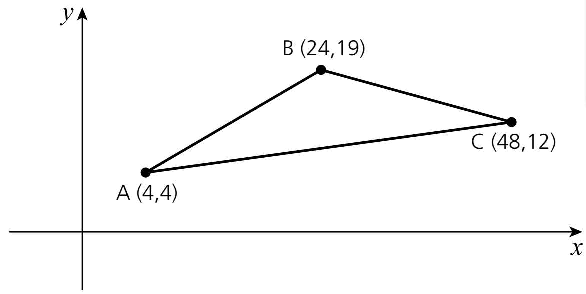
d Scalene triangle.

10 a **i** $OC = 4i$
ii $AB = 4i + 4j$
iii $BC = 4i - 4j$
iv $AD = 4i - 8j$
v $CD = -4i - 8j$
vi $AC = 8i$

b $|OC| = 4$
$|AB| = 16\sqrt{2}$
$|BC| = 16\sqrt{2}$
$|AD| = 4\sqrt{5}$
$|CD| = 4\sqrt{5}$
$|AC| = 8$

c AOB; BOC and ABC are both isosceles triangles and right triangles.

11 a



b $AB = 20i + 15j$
$BC = 24i - 7j$
$AC = 44i + 8j$

c $|AB| = 25$
$|BC| = 25$
$|AC| = 20\sqrt{(5)}$

d Isosceles triangle.

12 i Salman walks faster by 0.21 km per hour
ii 1.125 i + 20.444j
Distance = 20.47 kms

13 Direction = 1984.43°
Distance = 379.47
Time = 1 hour 54 minutes

Past-paper questions Page 222

1 **(ii)** 40**i** + 96**j**
(iii) $(40 + 10t)\mathbf{i} + (96 + 24t)\mathbf{j}$
(iv) $(120 - 22t)\mathbf{i} + (81 + 30t)\mathbf{j}$
(v) 1830 hours, 65**i** + 156**j**

2 **(i)** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
9**i** + 45**j**
(ii) $\overrightarrow{OC} = \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$
$\overrightarrow{OC} = 5\mathbf{i} + 12\mathbf{j}$
$|OC| - \sqrt{5^2 + 12^2}$
$= 13$
(iii) $\overrightarrow{OD} = \frac{4}{3}\mathbf{i} - 2\mathbf{j}$

3 $\overrightarrow{OC} = 5\mathbf{i} + 12\mathbf{j}$
$|\overrightarrow{OC}| = \sqrt{5^2 + 12^2} = 13$

Chapter 14 Differentiation

Discussion point Page 225

The connection is that gravity applies to an apple falling and to the planets in their orbits. Newton realised that the planets were held in their orbits round the Sun by the force of gravity but he needed calculus to work out their equations.

Exercise 14.1 Page 230

1 **a** $4x^3$
b $6x^2$
c 0
d 10

2 **a** $\frac{1}{2}x^{-\frac{1}{2}}$
b $\frac{5}{2}x^{-\frac{1}{2}}$
c $\frac{21}{2}t^{\frac{1}{2}}$
d $\frac{1}{2}x^{\frac{3}{2}}$

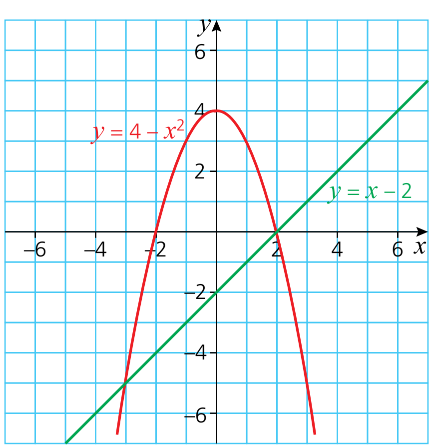
3 **a** $10x^4 + 8x$
b $12x^3 + 8$
c $3x^2$
d $1 - 15x^2$

4 **a** $-\frac{2}{x^3}$
b $-\frac{18}{x^4}$
c $\frac{2}{\sqrt{x}} + \frac{4}{x\sqrt{x}}$
d $\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$

5 **a** $2x - 1$
b $4x - 1$
c $1 - 5x^{-2}$
d $\frac{3}{2}x^{\frac{1}{2}}$

6 $(0, -9)$ 0, $(-3, 0)$ -6, $(3, 0)$ 6

7 **a**



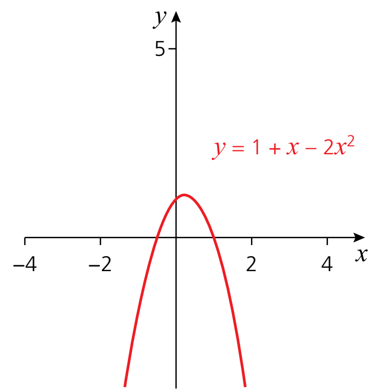
b $(-3, -5)$ $(2, 0)$
c $6, -4$

Discussion point Page 231

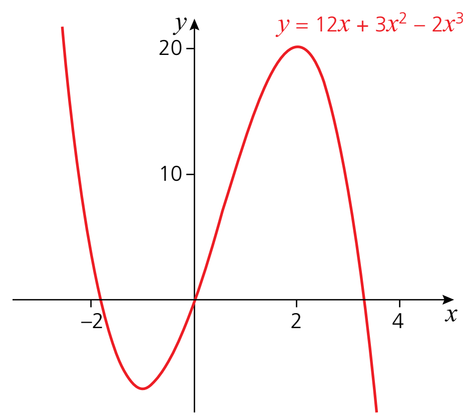
As the curve does not turn, the gradient is negative both to the left and to the right of D, D is a stationary point.

Exercise 14.2 Page 234

1 **i** $\frac{dy}{dx} = 1 - 4x$ $x - 0.25$
ii maximum
iii 1.125
iv



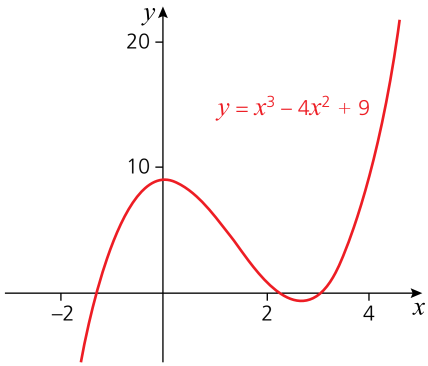
2 **i** $\frac{dy}{dx} = 12 + 6x - 6x^2$ $x = -1, 2$
ii minimum at $x = -1$, maximum at $x = 2$
iii $(-1, 7)$ $(2, 20)$
iv



3 **i** $\frac{dy}{dx} = 3x^2 - 8x$ $x = 0, \frac{8}{3}$
ii maximum at $x = 0$ minimum at $x = \frac{8}{3}$

iii $(0, 9)\ \left(\frac{8}{3}, -\frac{13}{27}\right)$

iv

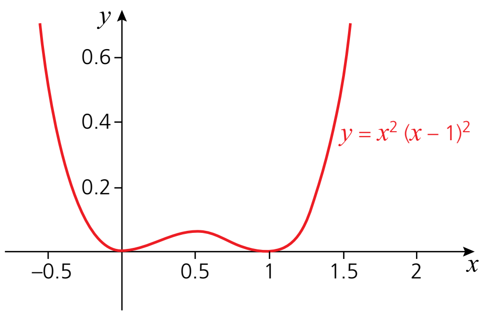


4 **i** $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 6x^2 + 2x \quad x = 0, 0.5, 1$

ii minimum at $x = 0$, maximum at $x = 0.5$, minimum at $x = 1$

iii $(0, 0)\ (0.5, 0.0625), (1, 0)$

iv

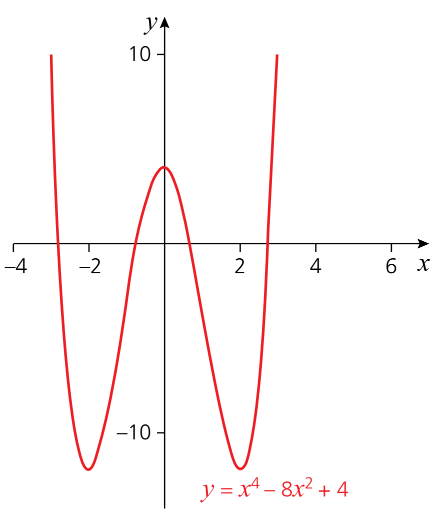


5 **i** $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 16x \quad x = -2, 0, 2$

ii minimum at $x = -2$, maximum at $x = 0$ minimum at $x = 2$

iii $(-2, -12), (0, 4), (2, -12)$

iv

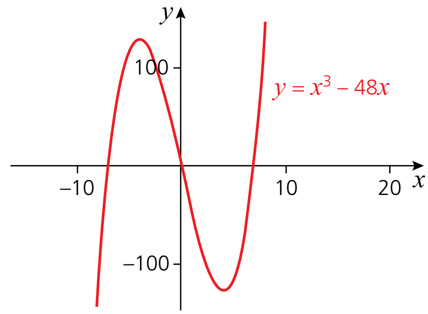


6 **i** $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 48 \quad x = -4, 4$

ii maximum at $x = -4$ minimum at $x = 4$

iii $(-4, 128), (4, -128)$

iv

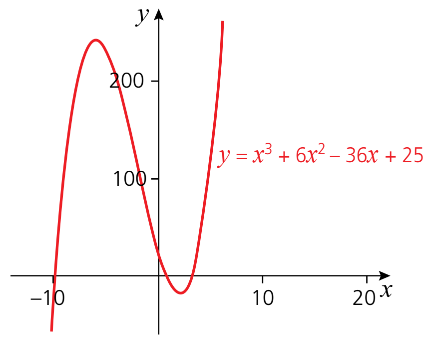


7 **i** $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 12x - 36 \quad x = -6, 2$

ii maximum at $x = -6$ minimum at $x = 2$

iii $(-6, 241), (2, -15)$

iv

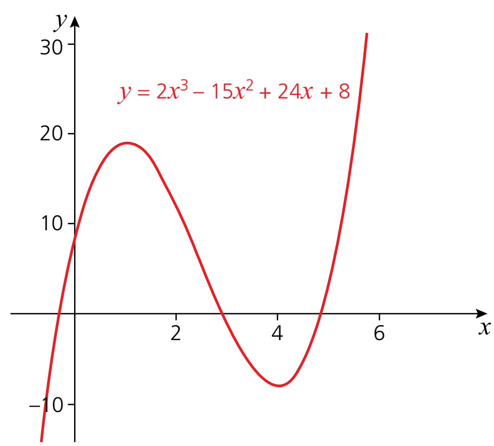


8 **i** $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 30x + 24 \quad x = 1, 4$

ii maximum at $x = 1$ minimum at $x = 4$

iii $(1, 19), (4, -8)$

iv



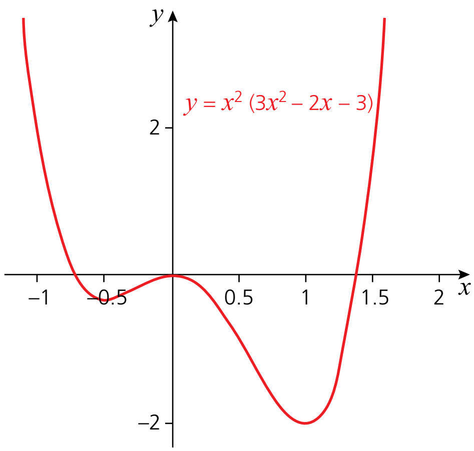
9 **a** $p = 4,\ q = -3$

b maximum value $\frac{4}{3}$ when $x = \frac{2}{3}$

10 a minimum at (–0.5, –0.3125)

maximum at (0, 0), minimum at (1, –2)

b



Exercise 14.3 Page 238

1 a $\frac{dy}{dx} = 3x^2 - 6x + 2, \frac{d^2y}{dx^2} = 6x - 6$

b $\frac{dy}{dx} = 12x^3 - 12x^2, \frac{d^2y}{dx^2} = 24x^2 - 24x$

c $\frac{dy}{dx} = 5x^4 - 5, \frac{d^2y}{dx^2} = 20x^3$

2 a maximum at (0.75, 2.875)

b maximum at $(\frac{1}{3}, 6.148)$, minimum at (1, 6)

c minimum at (–0.5, 0.75) maximum at (0, 1), minimum at (0.5, 0.75)

d maximum at (–1, 4) minimum at (1, –4)

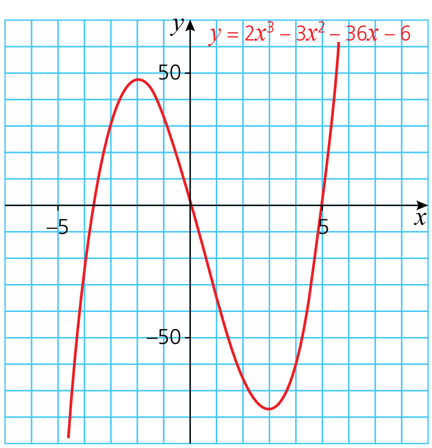
3 a $\frac{dy}{dx} = 6x^2 - 6x - 36,\ x = -2, 3$

b At $x = -2$, $\frac{d^2y}{dx^2} = -30$ maximum.

At $x = 3$, $\frac{d^2y}{dx^2} = 30$ maximum.

c (–2, 48), (3, –77)

d



4 The maximum area is equal to $3200\,m^2$

It occurs 40 m away from the wall; 80 m parallel to the wall.

5 a $V = \pi r^2(3 - r)$

b $V_{max} = 4\pi$ this occurs when $r = 2$ and $h = 1$

6 a $x = 6,\ y = 6,\ P = 24$ cm² and $\frac{d^2P}{dx^2} = \frac{2}{3}$ so minimum

b $x = 6,\ y = 6,\ A = 36$ cm² and $\frac{d^2A}{dx^2} = -2$ so maximum

Exercise 14.4 Page 241

1 a $\frac{dy}{dx} = 5 - 2x$

b –1

c $y = 9 - x$

d $y = x + 3$

2 a i $\frac{dy}{dx} = 6x - 3x^2$

ii 0

iii $y = 4$

iv $x = 2$

b i (3, 0)

ii –9

iii $y = 27 - 9x$

c $y = 0$

3 a (1, 0)

b $y = 2x - 2$

c $2y = 1 - x$ or $2y = -x + 1$

d Q (0, –2) R (0, 0.5)

The area of the triangle is equal to 3.125.

4 a $f'(x) = 3x^2 - 6x + 4$

b i 5

ii $x + 4y - 22 = 0$

iii $4y + x = 22$

c –1, 3

5 a $\frac{dy}{dx} = 3x^2 - 18x + 23$

b –1

c $y = -x + 5$

d Q (4, –3)

e $y = -x + 1$

6 a $2p - q = 16$

b $\frac{dy}{dx} = 3x^2 - p$

c 12

d (–2, 24)

e (0, 8)

f $12y = x + 96$

7 a $y = 3x - 5$

b $-\frac{1}{3}$

c $\left(\frac{1}{3}, -\frac{11}{9}\right)$

8 a $10 - 2x$

b $y = -2x + 15$

c $2y = x$

d $2y = x$ (The equation of the normal.)

9 b $(0, 0)$ tangent $y = 2x$ normal $2y + x = 0$

$(1, 0)$ tangent $2y + x = 1$ normal $y = 2x - 2$

c Opposite sides are parallel and adjacent sides are perpendicular so it is a rectangle.

Exercise 14.5 Page 253

1 a $3\cos x - 2\sec^2 x$

b $5\cos\theta$

c $-2\sin\theta - 2\cos\theta$

d $\frac{4}{x}$

e $\frac{1}{x}$

f $3e^x$

g $2e^x - \frac{1}{x}$

2 a $x\cos x + \sin x$

b $-x\sin x + \cos x$

c $x\sec^2 x + \tan x$

d $e^x\cos x + e^x\sin x$

e $-e^x\sin x + e^x\cos x$

f $e^x\sec^2 x + e^x\tan x$

3 a $\frac{x\cos x - \sin x}{x^2}$

b $\frac{\sin x - x\cos x}{\sin^2 x}$

c $-\frac{x\sin x + 2\cos x}{x^3}$

d $\frac{2x\cos x + x^2\sin x}{\cos^2 x}$

e $\frac{\tan x - x\sec^2 x}{\tan^2 x}$

f $\frac{x\sec^2 x - \tan x}{x^2}$

4 a $4(x+3)^3$

b $8(2x+3)^3$

c $8x(x^2+3)^3$

d $\frac{1}{2}\sqrt{(x+3)}$

e $\frac{1}{\sqrt{(2x+3)}}$

f $\frac{x}{\sqrt{(x^2+3)}}$

5 a $\frac{1}{(1+\cos x)}$

b $\frac{-\cos x - 1}{\sin^{2x} x}$

c $\cos 2x + \cos x$

d $\cos 2x - \sin x$

e $\cos 3x + \cos x + 2\cos 2x + \cos 3x$

f $2\cos 2x + 2\cos^2 x\sin x - \sin x - \sin^3 x$

6 a $e^x\ln x + \frac{e^x}{x}$

b $\frac{e^x\ln x - \frac{e^x}{x}}{(\ln x)^{(2)}}$

c $\frac{e^x - e^x\ln x}{7e^{2x}}$

7 a $e^x(\sin x + \cos x)$

b $-\frac{e^{-x}\sin x + e^{-x}\cos x}{\sin^2 x}$

c $\frac{\cos x + \sin x}{e^{-x}}$

8 b At $(0, -1)$ $\frac{dy}{dx} = \cos(0) + \sin(0)$

The equation of the tangent is equal to

$y = x - 1$

The equation of the normal is $y = -x - 1$

At $(\pi, 1)$ $\frac{dy}{dx} = \cos\pi + \sin\pi = -1$

The equation of the tangent is equal to

$y = -x + \pi + 1$

The equation of the normal is equal to

$y = x - \pi + 1$

9 b $\frac{dy}{dx} = 2\sec^2 x$

At $(0, -1)$

$\frac{dy}{dx} = 2\sec^{(2x0)} = 2$

The equation of tangent $-y = 2x - 1$

The equation of the normal: $y = -\left(\frac{1}{2}\right)x - 1$

At $\left(\frac{\pi}{4};1\right)\frac{dy}{dx} = 2\sec^2\left(\frac{\pi}{4}\right) = 2x\left(\sqrt{2}\right)^2 = 4$

The equation of the tangent; $y = 4x - \pi + 1$

The equation of the normal; $y = -\frac{1}{4}x + \frac{\pi}{16} + 1$

10 b The equation of the tangent is equal to $y = \frac{2}{e^{x-1}}$

The equation of the normal is equal to

$y = -\frac{e}{x} + \frac{e^2}{2} + 1$

11 c The equation of the tangent is equal to;

$y = ex - x + 1$

The equation of the normal is equal to:

$y = \left(\frac{-1}{e-1}\right)x + \frac{1}{(e-1)+e}$

Past-paper questions Page 254

1 (ii) $\frac{dA}{dx} = 8x - \frac{216}{x^2}$

When $\frac{dA}{dx} = 0$, $x = \sqrt[3]{27} = 3$

Dimensions are 3 by 6 by 4

(iii) Change in $A = -38p$, decrease

2 (i) $-2\sin 2x$ and $\frac{1}{3}\cos\left(\frac{x}{3}\right)$

$\frac{1}{3}\cos 2x \cos\left(\frac{x}{3}\right) - 2\sin 2x \sin\left(\frac{x}{3}\right)$

(ii) $\sec^2 x$ and $\frac{1}{x}$

$\frac{(\sec^2 x)(1 + 1\text{n}\, x) - \frac{1}{x}(\tan x)}{(1 + 1\text{n}\, x)^2}$

3 (a) $\frac{dy}{dx} = 12x^2 \ln(2x+1) + \frac{8x^3}{2x+1}$

Chapter 15 Integration

Discussion point Page 257

A marathon. She had clearly finished before 3 hours. You can estimate the distance she ran in each half hour, using the average speed. That suggests that by 2.5 hours she had run 41 760 metres and still had 435 metres to go. So an estimate is that she took a bit over 2 hours and 31 minutes.

Exercise 15.1 Page 261

1 a $y = 2x^2 + 2x + c$

b $y = 2x^3 - \frac{5}{2}x - x + c$

c $y = 3x - \frac{5}{4}x^4 + c$

d $y = x^3 - 2x^2 - 4x + c$

2 a $f(x) = \frac{5}{2}x^2 + 3x + c$

b $f(x) = \frac{1}{5}x^5 + \frac{1}{2}x^4 - \frac{1}{2}x^2 + 8x + c$

c $f(x) = \frac{1}{4}x^4 - \frac{4}{3}x^3 + x^2 - 8x + c$

d $f(x) = \frac{1}{3}x^3 - 7x^2 + 49x + c$

3 a $5x + c$

b $\frac{5}{4}x^4 + c$

c $x^2 - 3x + c$

d $\frac{3}{4}x^4 - 2x^2 + 3x + c$

4 a $9x - 3x^2 + \frac{1}{3}x^3 + c$

b $\frac{2}{3}x^3 - \frac{5}{2}x^2 - 3x + c$

c $\frac{1}{3}x^3 + x^2 + x + c$

d $\frac{4}{3}x^3 - 2x^2 + x + \text{c}$

5 a $y = x^2 - 3x + 6$

b $y = 4x + \frac{3}{4}x^4 - 208$

c $y = \frac{5}{2}x^2 - 6x - 18$

d $f(x) = \frac{1}{3}x^3 + x + 9$

e $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x - 44$

f $f(x) = \frac{4}{3}x^3 + 2x^2 + x - \frac{16}{3}$

6 a $y = \frac{4}{3}x^{\frac{3}{2}} - 1 + \frac{2}{3}$

b $y = \frac{1}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{3}$

7 a $y = x^2 + 3x + c$

b $y = x^2 + 3x + c$

c $y = x^2 + 3x - 11$

8 a $y = x^3 - 2x^2 + x - 12$

9 a $y = 2x^2 - x + c$

b $y = 2x^2 - x + c$

c $y = 2x^2 - x + 1$

d Curve passes above the point.

10 0

11 a $\frac{dy}{dx} = kx(x-2)$ for any $k \neq 0$

b $y = \frac{1}{3}x^3 - x^2 + 2$

Exercise 15.2 Page 264

1 7

2 255

3 4

4 16

5 $\frac{62}{3}$

6 30

7 591

8 $\frac{76}{3}$

9 $-\frac{4}{3}$

10 12

11 27

12 $\frac{16}{3}$

13 92

14 $-\frac{4}{3}$

15 $\frac{27}{2}$

16 0

17 28

18 $\frac{80}{3}$

Exercise 15.3 Page 265

1 9

2 36

3 2

4 $\frac{20}{3}$

5 0.25

6 $\frac{3125}{6}$

7 36

8 13.5

9 $\frac{64}{3}$

10 $\frac{64}{3}$

Exercise 15.4 Page 269

1 0.25

2 **a** $\frac{5}{12}, \frac{8}{3}$ below the axis

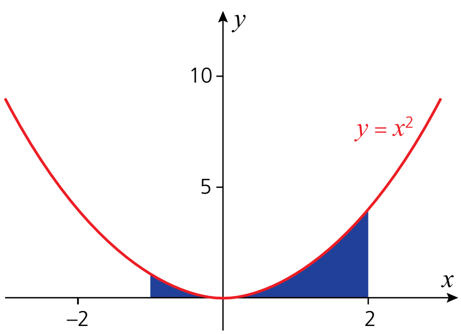
b $3\frac{1}{12}$

3 **a** $(3\sqrt{2}, 0)$

b $\frac{3}{5} \times 2^{\frac{2}{3}}$ (0.95244)

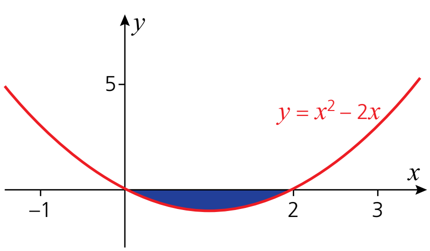
4 $\frac{253}{12}$

5 **a** and **b**



c 3

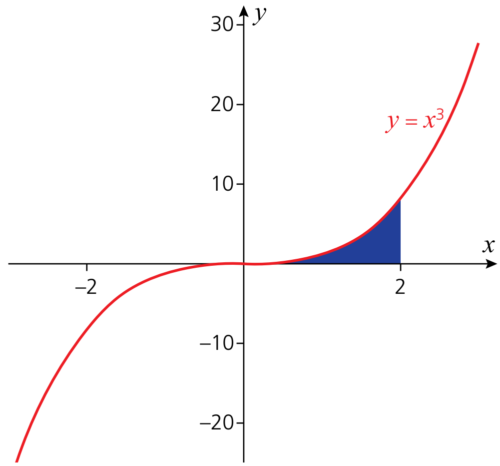
6 **a**



b $0 < x < 2$

c $\frac{4}{3}$ below the axis

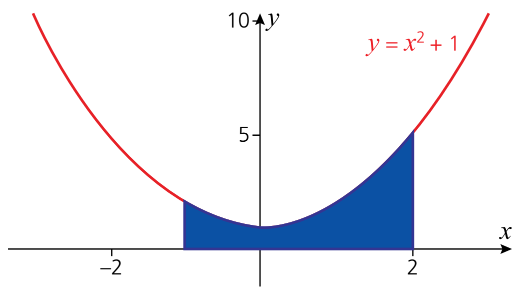
7 **a** and **b**



c 4

d 0

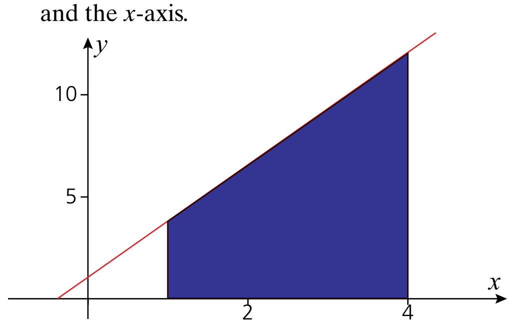
8 **a**



b 6

9 **a** 18

b Area between the lines $y = 2x + 1, x = 1, x = 4$ and the x-axis.



Exercise 15.5 Page 273

1 **a** $\frac{1}{3}\ln(3x+1) + c$

b $\frac{1}{15}(3x+1)^5 + c$

c $\frac{1}{3}e^{3x+1} + c$

d $(-\cos \frac{3x+1}{3}) + c$

e $(\sin\frac{3x+1}{3})+c$

f $3\ln(x-3)+c$

g $\frac{1}{8}(2x-1)^4+c$

h $2e^{2x-3}+c$

i $-\cos(3x)+c$

j $8\sin\left(\frac{x}{2}\right)+c$

k $\frac{2}{5}(x-2)^{\frac{5}{2}}+c$

l $\dfrac{(2x-1)^{\frac{5}{2}}}{5+c}$

2 a $\frac{1}{3}\ln\left(\frac{13}{7}\right)$

b 23632.4

c 147105.59 (2 d.p.)

d $\frac{1}{3}$

e $\frac{1}{\sqrt{3}}$

f $\ln6-\ln2=\ln3$

g 5904.8

h $5(1-e^{-4})$

i $\frac{1}{\sqrt{2}}$

j $\frac{1}{\sqrt{2}}$

Past-paper questions Page 273

1 (a) (i) $2=a-3,\ a=5$

(ii) $y=-5e^{1-x}-x^3+c$

$c=10$

$y=-5e^{1-x}-x^3+10$

(b) (i) $\frac{1}{7}\times\frac{3}{4}\times(7x+8)^{\frac{4}{3}}$

(ii) $\left[\frac{3}{28}(7x+8)^{\frac{4}{3}}\right]_0^8$

$=\frac{180}{7}$ or 25.7

2 (ii) When $\frac{dy}{dx}=0,\ (3x-1)(x-3)=0$

$x=\frac{1}{3},\ x=3$

(iii) Area

$=\frac{1}{2}(10+19)3-\left(\int_0^3 x^3-5x^2+3x+10\,dx\right)$

$=\frac{87}{2}-\left[\frac{x^4}{4}-\frac{5x^3}{3}+\frac{3x^2}{2}+10x\right]_0^3$

$\frac{87}{2}-\left(\frac{81}{4}-45+\frac{27}{2}+30\right)$

$=24.75$

3 (iii) $x=2,4$

(iv) $\int 3x-14+\frac{32}{x^2}dx=1.5x^2-14x-\frac{32}{x}(+c)$

Area $=\left[1.5x^2-14x-\frac{32}{x}\right]_2^4$

$=(-)2$

Chapter 16 Kinematics

Discussion point Page 277

No. One component of its motion will be in orbit round the Sun. Another component will be at right angles to it. Given the distance to Jupiter it is likely that it will complete several orbits of the Sun before it reaches Jupiter.

Discussion point Page 282

The particle is in motion, but has slowed down whilst still travelling towards O.

Exercise 16.1 Page 283

1 a i $v=10t-1,\ a=10$

ii 3, –1, 10

iii when $t=0.1,\ s=2.95$

b i $v=3-3t^2,\ a=-6t$

ii 0, 3, 0

iii when $t=1,\ s=2$

c i $v=4t^3-4,\ a=12t^2$

ii –6, –4, 0

iii when $t=1,\ s=-9$

d i $v=12t^2-3,\ a=24t$

ii 5, –3, 0

iii when $t=0.5,\ s=4$

e i $v=-4t+1,\ a=-4$

ii 5, 1, –4

iii when $t=0.25,\ s=5$

2 a $v=6t-3t^2\quad a=6-6t$

b $t=0$, 1 s

$t=2$; 4 s

c 4 m

d $-24\,ms^{-1}$ travelling in the negative direction

e $6\,ms^{-2}$

3 a 1 m

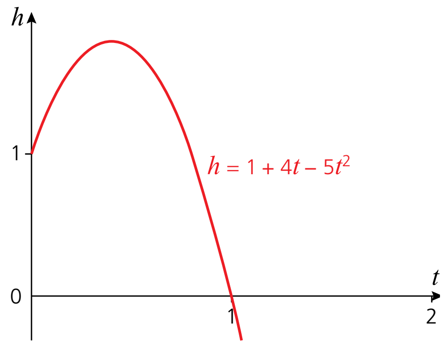
b $v=4-10t$

c 0.4 s

d 1.8 m

e 1 s

f



g -6

4 a $v = \frac{2}{3}t^3,\ a = 2t^2$

b 6 s

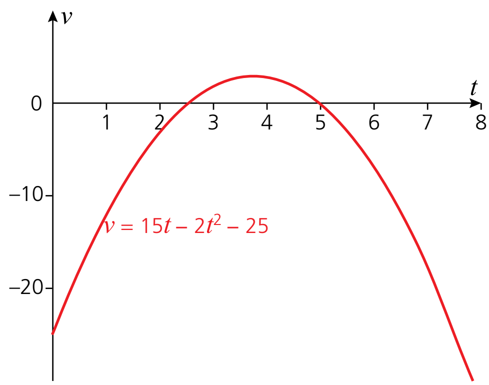
c 216 m, 144 ms^{-1}

5 a 2.5 s, 5 s

b 5 ms^{-2}, –5 ms^{-2}

c 3.125 ms^{-1}

d



Discussion Point page 286

If the velocity is $v = 3t^2 + 2$, the acceleration is given by $a = \frac{dv}{dt} = 6t$. Since this involves t it is not constant. So you cannot use the constant acceleration formula and must use calculus instead. You can also use calculus if the acceleration is constant.

Exercise 16.2 Page 287

1 a $v = 2t - 3t^2 + 1, s = t^2 - t^3 + t$

b $v = 2t^2 + 4,\ s = \frac{2}{3}t^3 + 4t + 3$

c $v = 4t^3 - 4t + 2,\ s = t^4 - 2t^2 + 2t + 1$

d $v = 2t + 2\ ,\ s = t^2 + 2t + 4$

e $v = 4t + \frac{1}{2}t^2 + 1,\ s = 2t^2 + \frac{1}{6}t^3 + t + 3$

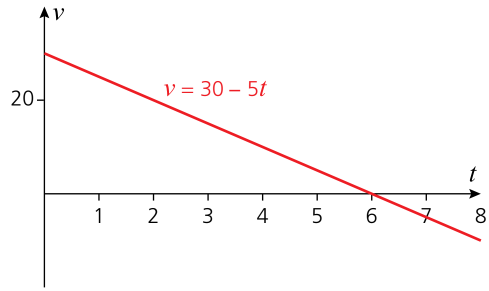
2 a $v = 3t^2 - 12t + 9,\ s = t^3 - 6t^2 + 9t$

b 3 s

3 a –3 ms^{-1}

b 2 s, –4 m

4 a



b 6 s

c 90 m

5 a $v = 4t + 6t^2$

b 48 m

6 a $a = 3t^2 - 8t + 4$

b $\frac{2}{3}$, 2 s Particle is slowing down

c 8.25 m

Past-paper questions Page 288

1 (i) $\frac{dy}{dt} = 36 - 6t$

When $\frac{dy}{dt} = 0, t = 6$

(ii) When $v = 0, t = 12$

(iii) $s = 18t^2 - t^3\ (+c)$

When $t = 12, s = 864$

(iv) When $s = 0, t = 18$

$v = -324$

So speed is 324

2 (i) Velocity = 9 m p/s

(ii) Acceleration = –78

(iii) Distance OP = 1.71

3 (i) $t = \sqrt{e^5 - 1}$ or $t^2 + 1 = e^5$

$t = 12.1$

(ii) distance = 1n 10 – 1n 5

= 1n 2 or 0.693

(iii) $v = \frac{2t}{t^2 + 1}$, $v = 0.8$

(iv) $a = \frac{(t^2 + 1)2 - 2t(2t)}{(t^2 + 1)^2}$

When $t = 2$, $a = -\frac{6}{25}$, or –0.24.

Index

U

V

Y

Z