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Mathematics
Core and Extended
Fifth edition

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The *Cambridge IGCSE™ Core and Extended Mathematics Student's Book* Fifth edition will help you to navigate the learning objectives confidently. It is supported by a Workbook as well as a print Teacher's Guide with digital teaching and learning access, and an interactive eBook on the Boost platform.

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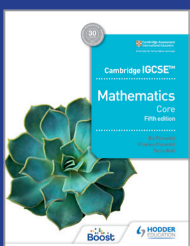

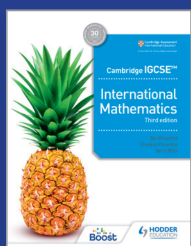

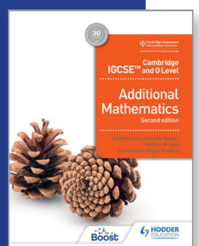

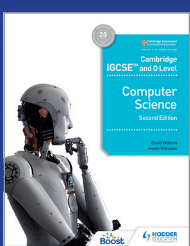

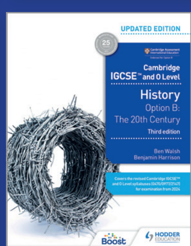

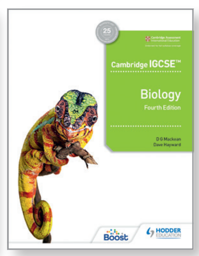

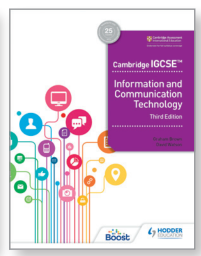

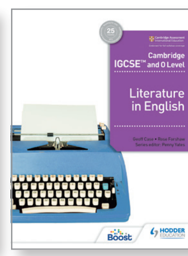

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Cambridge IGCSE™

Mathematics

Core and Extended

Fifth Edition

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Contents

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Introduction

This book has been written for all students of Cambridge IGCSE™ and IGCSE (9–1) Mathematics (0580/0980) for examination from 2025. It carefully and precisely follows the syllabus from Cambridge Assessment International Education. It provides the detail and guidance that are needed to support you throughout your course and help you to prepare for your examinations. The content is aimed particularly at students studying the Extended syllabus: our *Cambridge IGCSE™ Core Mathematics* Student's Book provides support for students focusing on the Core syllabus only.

How to use this book

To make your study of mathematics as rewarding and successful as possible, this book, endorsed by Cambridge International, offers the following important features:

Learning objectives

Each topic starts with an outline of the subject material and syllabus content to be covered. These opening pages show the learning objectives on the Extended syllabus, prefixed with an 'E'. All Core syllabus learning objectives are covered in the *Cambridge IGCSE Core Mathematics* Student's Book.

Organisation

Topics follow the order of the syllabus and are divided into chapters. In some cases, the order of the chapters is determined by continuity of the mathematics they cover, rather than the order of the syllabus. All instances where students should refer to other chapters are clearly explained in the text.

Within each chapter there is a blend of teaching, worked examples and exercises to help you build confidence and develop the skills and knowledge you need. In particular, there is an increased emphasis on non-calculator methods as well as suggestions for the use of scientific calculators.

At the end of each chapter there are comprehensive Student Assessment questions. You will also find sets of questions linked to the **Boost eBook** (boost-learning.com), which offer practice in topic areas that students often find difficult.

Although the authors have identified what material belongs exclusively to the Extended syllabus by use of a purple line, students studying the Extended syllabus are expected to be familiar with all the content of the Core syllabus as well. Material which is also part of the Core syllabus has been identified with a blue line.

ICT, mathematical modelling and problem-solving

Problem-solving is key to mathematical thinking and ICT can play a crucial role in this. Therefore, each topic ends with a section involving investigations and the use of ICT. The ICT investigations are, however, beyond the requirements of the syllabus and are identified with a yellow line as explained below.

Diagrams and working

Students are encouraged to draw diagrams when tackling questions where appropriate, and to show their full worked solutions. This is helpful for checking your own work, and also applies to any questions where use of a calculator is allowed.

Calculator and non-calculator questions

All exercise questions that should be attempted without a calculator are indicated by . Students should do as many calculations as possible without using a calculator. This will help to build understanding and confidence.

Some areas of mathematics, such as those using powers and roots, π, trigonometry and calculations with decimals, are more likely to require a calculator.

Core, Extended and Additional material

As this book covers the syllabus for both the Core and Extended content, we have used vertical coloured lines to distinguish between the two: a blue line and a purple line. Furthermore, there are a few instances where we have judged it to be appropriate to include some additional content that lies beyond the scope of the syllabus – where we consider it to be useful in supporting the syllabus content, and helpful in deepening understanding. This is indicated with a yellow line.

Key terms and glossary

It is important to understand and use mathematical terms; therefore, all key terms are highlighted in bold and explained in the glossary.

Answers and worked solutions

Answers to all questions are available free on hoddereducation.com/cambridgeextras

Worked solutions for the Student Assessment questions are available in *Cambridge IGCSE Core and Extended Mathematics Teacher's Guide with Boost Subscription*.

Callouts and Notes

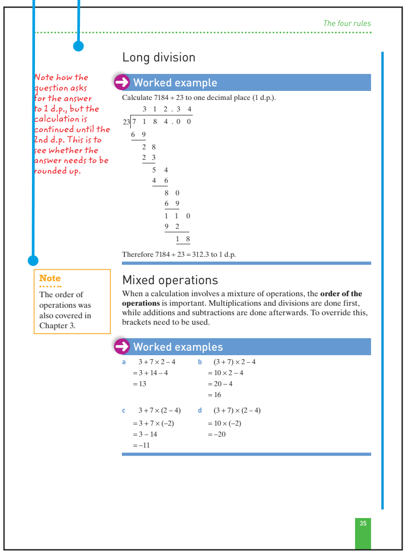
These commentaries provide additional explanations and encourage full understanding of mathematical principles. Notes give additional clarifications and tips.

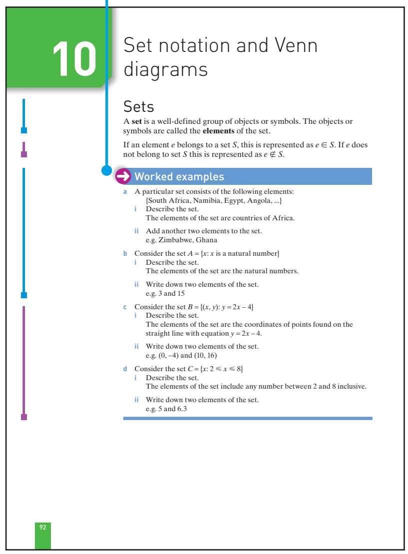
Worked examples

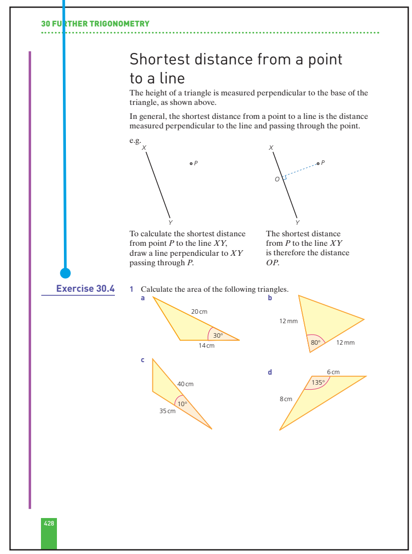
The worked examples cover important techniques and question styles. They are designed to reinforce the explanations, and give you step-by-step help for solving problems.

Exercises

These appear throughout the text and allow you to apply what you have learned. There are plenty of routine questions covering important mathematical techniques.







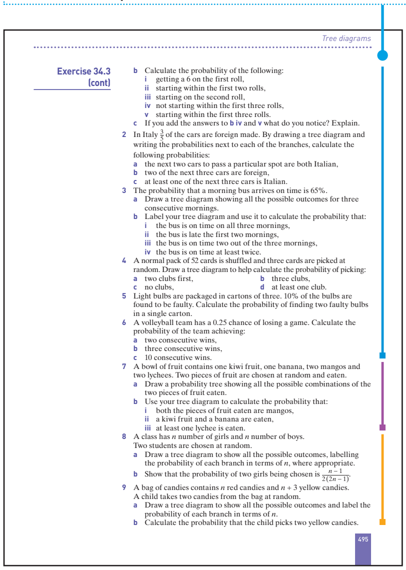
A blue line identifies content that is relevant for all students, regardless of whether they are studying the Core or Extended syllabus. A purple line identifies Extended content only. A yellow line identifies any material that lies beyond the scope of the syllabus (occasionally included because it can be helpful for students).

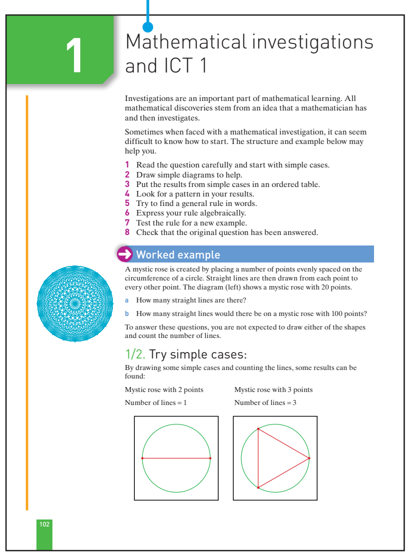
Mathematical investigations and ICT

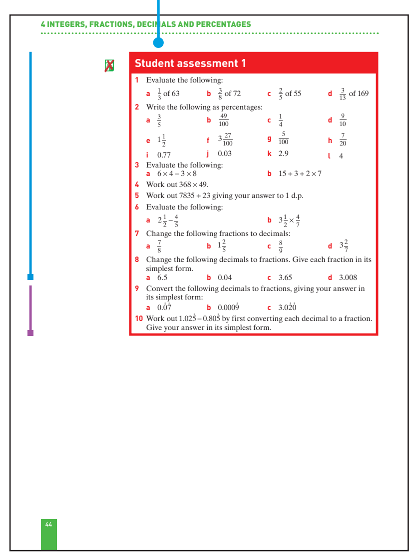
More problem-solving activities are provided at the end of each section to put what you've learned into practice.

Student assessments

End-of-chapter questions to test your understanding of the key topics and help to prepare you for your examinations.







Assessment

The information in this section is based on the Cambridge International syllabus. You should always refer to the appropriate syllabus document for the year of examination to confirm the details and for more information. The syllabus document is available on the Cambridge International website at www.cambridgeinternational.org

For Cambridge IGCSE™ Mathematics you will take two papers. If you are studying the Core syllabus, you will take Paper 1 and Paper 3. If you are studying the Extended syllabus, you will take Paper 2 and Paper 4. You may use a scientific calculator only for Papers 3 and 4, Paper 1 and Paper 2 are non-calculator papers.

Paper	Length	Type of questions
Paper 1 Non-calculator (Core)	1 hour 30 minutes	Structured and unstructured questions
Paper 2 Non-calculator (Extended)	2 hours	Structured and unstructured questions
Paper 3 Calculator (Core)	1 hour 30 minutes	Structured and unstructured questions
Paper 4 Calculator (Extended)	2 hours	Structured and unstructured questions

Command words

The command words that may appear in examinations are listed below. The command word used will relate to the context of the question.

Command word	What it means
Calculate	Work out from given facts, figures or information
Construct	Make an accurate drawing
Describe	State the points of a topic / give characteristics and main features
Determine	Establish with certainty
Explain	Set out purposes or reasons / make the relationships between things clear / say why and/or how and support with relevant evidence
Give	Produce an answer from a given source or recall/memory
Plot	Mark point(s) on a graph
Show (that)	Provide structured evidence that leads to a given result
Sketch	Make a simple freehand drawing showing the key features
State	Express in clear terms
Work out	Calculate from given facts, figures or information with or without the use of a calculator
Write	Give an answer in a specific form
Write down	Give an answer without significant working

Examination techniques

Make sure you check the instructions on the question paper, the length of the paper and the number of questions you have to answer.

Allocate your time sensibly between each question. Be sure not to spend too long on some questions because this might mean you don't have enough time to complete all of them. Make sure you show your working to show how you've reached your answer.

From the authors

Mathematics comes from the Greek word meaning *knowledge* or *learning*. Galileo Galilei (1564–1642) wrote 'the universe cannot be read until we learn the language in which it is written. It is written in mathematical language.' Mathematics is used in science, engineering, medicine, art, finance and so on, but mathematicians have always studied the subject for pleasure. They look for patterns in nature, for fun, as a game or a puzzle.

A mathematician may find that their puzzle solving helps to solve 'real life' problems. But trigonometry was developed without a 'real life' application in mind, before it was then applied to navigation and many other things. The algebra of curves was not 'invented' to send a rocket to Jupiter.

The study of mathematics is across all lands and cultures. A mathematician in Africa may be working with another in Japan to extend work done by a Brazilian in the USA.

People in all cultures have tried to understand the world around them, and mathematics has been a common way of furthering that understanding, even in cultures which have left no written records.

Each topic in this textbook has an introduction which tries to show how, over thousands of years, mathematical ideas have been passed from one culture to another. So when you are studying from this textbook, remember that you are following in the footsteps of earlier mathematicians who were excited by the discoveries they had made. These discoveries changed our world.

You may find some of the questions in this book difficult. It is easy when this happens to ask the teacher for help. Remember though that mathematics is intended to stretch the mind. If you are trying to get physically fit you do not stop as soon as things get hard. It is the same with mental fitness. Think logically. Try harder. In the end you are responsible for your own learning. Teachers and textbooks can only guide you. Be confident that you can solve that difficult problem.

Ric Pimentel

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TOPIC 1

Number

Contents

Learning objectives

E1.1 Types of number
Identify and use:
- natural numbers
- integers (positive, zero and negative)
- prime numbers
- square numbers
- cube numbers
- common factors
- common multiples
- rational and irrational numbers
- reciprocals.

E1.2 Sets
Understand and use set language, notation and Venn diagrams to describe sets and represent relationships between sets.

E1.3 Powers and roots
Calculate with the following:
- squares
- square roots
- cubes
- cube roots
- other powers and roots of numbers.

E1.4 Fractions, decimals and percentages
1. Use the language and notation of the following in appropriate contexts:
   - proper fractions
   - improper fractions
   - mixed numbers
   - decimals
   - percentages.
2. Recognise equivalence and convert between these forms.

E1.5 Ordering
Order quantities by magnitude and demonstrate familiarity with the symbols $=, \neq, >, <, \geqslant$ and $\leqslant$.

E1.6 The four operations
Use the four operations for calculations with integers, fractions and decimals, including correct ordering of operations and use of brackets.

E1.7 Indices I
1. Understand and use indices (positive, zero, negative and fractional).
2. Understand and use the rules of indices.

E1.8 Standard form

1. Use the standard form $A \times 10^n$ where n is a positive or negative integer, and $1 \leqslant A < 10$.
2. Convert numbers into and out of standard form.
3. Calculate with values in standard form.

E1.9 Estimation

1. Round values to a specified degree of accuracy.
2. Make estimates for calculations involving numbers, quantities and measurements.
3. Round answers to a reasonable degree of accuracy in the context of a given problem.

E1.10 Limits of accuracy

1. Give upper and lower bounds for data rounded to a specified accuracy.
2. Find upper and lower bounds of the results of calculations which have used data rounded to a specified accuracy.

E1.11 Ratio and proportion

Understand and use ratio and proportion, including:

- giving ratios in simplest form
- dividing a quantity in a given ratio
- using proportional reasoning and ratios in context.

E1.12 Rates

1. Use common measures of rate.
2. Apply other measures of rate.
3. Solve problems involving average speed.

E1.13 Percentages

1. Calculate a given percentage of a quantity.
2. Express one quantity as a percentage of another.
3. Calculate percentage increase or decrease.
4. Calculate with simple and compound interest.
5. Calculate using reverse percentages.

E1.14 Using a calculator

1. Use a calculator efficiently.
2. Enter values appropriately on a calculator.
3. Interpret the calculator display appropriately.

E1.15 Time

1. Calculations involving time: seconds (s), minutes (min), hours (h), days, weeks, months, years, including the relationship between units.
2. Calculate times in terms of the 24-hour and 12-hour clock.
3. Read clocks and timetables.

E1.16 Money

1. Calculate with money.
2. Convert from one currency to another.

E1.17 Exponential growth and decay

Use exponential growth and decay.

E1.18 Surds

1. Understand and use surds, including simplifying expressions.
2. Rationalise the denominator.

Hindu mathematicians

In 1300BCE a Hindu teacher named Laghada used geometry and trigonometry for his astronomical calculations. At around this time, other Indian mathematicians solved quadratic and simultaneous equations.

Much later in about 500CE, another Indian teacher, Aryabhata, worked on approximations for π (pi), and worked on the trigonometry of the sphere. He realised that not only did the planets go around the Sun but also that their paths were elliptical.

Brahmagupta, a Hindu, was the first to treat zero as a number in its own right. This helped to develop the decimal system of numbers.

One of the greatest mathematicians of all time was Bhascara, who, in the twelfth century, worked in algebra and trigonometry. He discovered that $\sin(A + B) = \sin A \cos B + \cos A \sin B$. His work was taken to Arabia and later to Europe.

A statue of Aryabhata (476–550)

Still alive today is the Indian woman mathematician Raman Parimala (born in 1948). Her work is famous in the fields of algebra and its connections with algebraic geometry and number theory.

1 Number and language

Vocabulary for sets of numbers

A **square** can be classified in many different ways. It is a quadrilateral but it is also a polygon and a two-dimensional shape. Just as shapes can be classified in many ways, so can numbers. Below is a description of some of the more common types of numbers.

Natural numbers

A child learns to count 'one, two, three, four, …'. These are sometimes called the counting numbers or whole numbers.

The child will say 'I am three', or 'I live at number 73'.

If we include the number 0, then we have the set of numbers called the **natural numbers**.

The set of natural numbers is {0, 1, 2, 3, 4, …}.

You should learn how to convert between values expressed in numbers and values expressed in words. For example, 12 014 is twelve thousand and fourteen; 1 745 233 is one million, seven hundred and forty-five thousand, two hundred and thirty-three.

Integers

On a cold day, the temperature may be 4 °C at 10 p.m. If the temperature drops by a further 6 °C, then the temperature is 'below zero'; it is –2 °C.

If you are overdrawn at the bank by $200, this might be shown on a bank statement as –$200.

The set of **integers** is {…, –3, –2, –1, 0, 1, 2, 3, …}.

Integers therefore are an extension of natural numbers. Every natural number is an integer.

Reciprocal

The **reciprocal** of a number is obtained when 1 is divided by that number. The reciprocal of 5 is $\frac{1}{5}$, the reciprocal of $\frac{2}{5}$ is $\frac{1}{\frac{2}{5}}$ which simplifies to $\frac{5}{2}$.

Exercise 1.1

1 Write the reciprocal of each of the following:

a $\frac{1}{8}$ **b** $\frac{7}{12}$ **c** $\frac{3}{5}$

d $1\frac{1}{2}$ **e** $3\frac{3}{4}$ **f** 6

Square numbers

The number 1 can be written as 1×1 or 1^2. This can be read as '1 squared' or '1 raised to the **power** of 2'.

The number 4 can be written as 2×2 or 2^2.

9 can be written as 3×3 or 3^2.

16 can be written as 4×4 or 4^2.

When an integer (whole number) is multiplied by itself, the result is a **square number**. In the examples above, 1, 4, 9 and 16 are all square numbers.

Cube numbers

The number 1 can be written as $1 \times 1 \times 1$ or 1^3. This can be read as '1 cubed' or '1 raised to the power of 3'.

The number 8 can be written as $2 \times 2 \times 2$ or 2^3.

27 can be written as $3 \times 3 \times 3$ or 3^3.

64 can be written as $4 \times 4 \times 4$ or 4^3.

When an integer is multiplied by itself and then by itself again, the result is a **cube number**. In the examples above 1, 8, 27 and 64 are all cube numbers.

Factors

The factors of 12 are all the numbers which will divide exactly into 12, i.e. 1, 2, 3, 4, 6 and 12.

Exercise 1.2

1 List all the factors of the following numbers:

a 6 **b** 9 **c** 7 **d** 15 **e** 24
f 36 **g** 35 **h** 25 **i** 42 **j** 100

Prime numbers

A **prime number** is one whose only **factors** are 1 and itself. (Note: 1 is not a prime number.)

Prime factors

The factors of 12 are 1, 2, 3, 4, 6 and 12.

Of these, 2 and 3 are prime numbers, so 2 and 3 are the **prime factors** of 12.

Exercise 1.3

1 In a 10 by 10 square, write the numbers 1 to 100.
Cross out number 1.
Cross out all the even numbers after 2 (these have 2 as a factor).
Cross out every third number after 3 (these have 3 as a factor).
Continue with 5, 7, 11 and 13, then list all the prime numbers less than 100.

2 List the prime factors of the following numbers:

a 15	**b** 18	**c** 24	**d** 16	**e** 20
f 13	**g** 33	**h** 35	**i** 70	**j** 56

An easy way to find prime factors is to divide by the prime numbers in order, smallest first.

Worked examples

a Find the prime factors of 18 and express it as a product of prime numbers:

	18
2	9
3	3
3	1

$18 = 2 \times 3 \times 3$ or 2×3^2

b Find the prime factors of 24 and express it as a product of prime numbers:

	24
2	12
2	6
2	3
3	1

$24 = 2 \times 2 \times 2 \times 3$ or $2^3 \times 3$

c Find the prime factors of 75 and express it as a product of prime numbers:

	75
3	25
5	5
5	1

$75 = 3 \times 5 \times 5$ or 3×5^2

Exercise 1.4

1 Find the prime factors of the following numbers and express them as a product of prime numbers:

a 12	b 32	c 36	d 40	e 44
f 56	g 45	h 39	i 231	j 63

Highest common factor

The factors of 12 can be listed as 1, 2, 3, 4, 6, 12.

The factors of 18 can be listed as 1, 2, 3, 6, 9, 18.

As can be seen, the factors 1, 2, 3 and 6 are common to both numbers. They are known as **common factors**. As 6 is the largest of the common factors, it is called the **highest common factor (HCF)** of 12 and 18.

The prime factors of 12 are $2 \times 2 \times 3$.

The prime factors of 18 are $2 \times 3 \times 3$.

So the highest common factor can be seen by inspection to be 2×3, i.e. 6.

Multiples

Multiples of 2 are 2, 4, 6, 8, 10, etc.
Multiples of 3 are 3, 6, 9, 12, 15, etc.
The numbers 6, 12, 18, 24, etc., are **common multiples** as these appear in both lists.
The **lowest common multiple (LCM)** of 2 and 3 is 6, since 6 is the smallest number divisible by 2 and 3.

The LCM of 3 and 5 is 15.

The LCM of 6 and 10 is 30.

Exercise 1.5

1 Find the HCF of the following numbers:

a 8, 12	b 10, 25	c 12, 18, 24
d 15, 21, 27	e 36, 63, 108	f 22, 110
g 32, 56, 72	h 39, 52	i 34, 51, 68
j 60, 144		

2 Find the LCM of the following:

a 6, 14	b 4, 15	c 2, 7, 10	d 3, 9, 10
e 6, 8, 20	f 3, 5, 7	g 4, 5, 10	h 3, 7, 11
i 6, 10, 16	j 25, 40, 100		

Rational and irrational numbers

A **rational number** is any number which can be expressed as a fraction. Examples of some rational numbers and how they can be expressed as a fraction are shown below:

$$0.2 = \tfrac{1}{5} \quad 0.3 = \tfrac{3}{10} \quad 7 = \tfrac{7}{1} \quad 1.53 = \tfrac{153}{100} \quad 0.\dot{2} = \tfrac{2}{9}$$

An **irrational number** cannot be expressed as a fraction. Examples of irrational numbers include:

$$\sqrt{2}, \quad \sqrt{5}, \quad 6 - \sqrt{3}, \quad \pi$$

In summary:

Rational numbers include:
- whole numbers,
- fractions,
- recurring decimals,
- terminating decimals.

A recurring decimal is one which repeats itself and has no end, e.g. 1.33333333...

A terminating decimal is one which has an end point, e.g. 5.2 or 0.45

Irrational numbers include:
- the **square root** of any number other than square numbers,
- a decimal which does not repeat or terminate (e.g. π).

Real numbers

The set of rational and irrational numbers together form the set of **real numbers**.

Exercise 1.6

1 State whether each number below is rational or irrational:

a 1.3 b $0.\dot{6}$ c $\sqrt{3}$

d $-2\frac{3}{5}$ e $\sqrt{25}$ f $\sqrt[3]{8}$

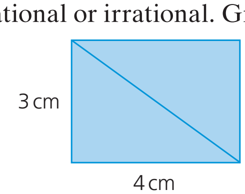
g $\sqrt{7}$ h 0.625 i $0.\dot{1}\dot{1}$

2 State whether each number below is rational or irrational:

a $\sqrt{2} \times \sqrt{3}$ b $\sqrt{2} + \sqrt{3}$ c $(\sqrt{2} \times \sqrt{3})^2$

d $\frac{\sqrt{8}}{\sqrt{2}}$ e $\frac{2\sqrt{5}}{2\sqrt{20}}$ f $4 + (\sqrt{9} - 4)$

3 In each of the following, decide whether the quantity required is rational or irrational. Give reasons for your answer.

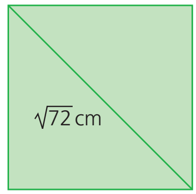
a


The length of the diagonal

b

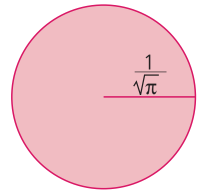

The circumference of the circle

c



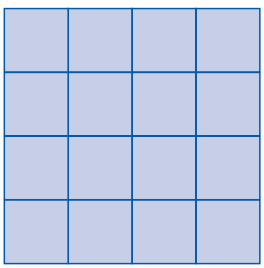
The side length of the square

d



The area of the circle

Square roots



The square shown contains 16 squares. It has sides of length 4 units.

So the square root of 16 is 4.

This can be written as $\sqrt{16} = 4$.

Note that 4×4 is 16 so 4 is the square root of 16.

However, -4×-4 is also 16 so -4 is also the square root of 16.

By convention, $\sqrt{16}$ means ‘the positive square root of 16’ so $\sqrt{16} = 4$ but the square root of 16 is ± 4, i.e. $+4$ or -4.

Note: -16 has no square root since any integer squared is positive.

Exercise 1.7

1 Find the following:

a $\sqrt{25}$ **b** $\sqrt{9}$ **c** $\sqrt{49}$ **d** $\sqrt{100}$
e $\sqrt{121}$ **f** $\sqrt{169}$ **g** $\sqrt{0.01}$ **h** $\sqrt{0.04}$
i $\sqrt{0.09}$ **j** $\sqrt{0.25}$

2 Use the $\sqrt{\ }$ key on your calculator to check your answers to Question 1.

3 Calculate the following:

a $\sqrt{\frac{1}{9}}$ **b** $\sqrt{\frac{1}{16}}$ **c** $\sqrt{\frac{1}{25}}$ **d** $\sqrt{\frac{1}{49}}$
e $\sqrt{\frac{1}{100}}$ **f** $\sqrt{\frac{4}{9}}$ **g** $\sqrt{\frac{9}{100}}$ **h** $\sqrt{\frac{49}{81}}$
i $\sqrt{2\frac{7}{9}}$ **j** $\sqrt{6\frac{1}{4}}$

Using a graph

Exercise 1.8

1 Copy and complete the table below for the equation $y = \sqrt{x}$.

x	0	1	4	9	16	25	36	49	64	81	100
y											

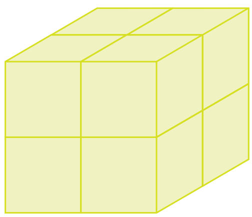
Plot the graph of $y = \sqrt{x}$.
Use your graph to find the approximate values of the following:

a $\sqrt{35}$ **b** $\sqrt{45}$ **c** $\sqrt{55}$ **d** $\sqrt{60}$ **e** $\sqrt{2}$

Exercise 1.8 (cont)

2 Check your answers to Question 1 above by using the $\sqrt{\ }$ key on your calculator.

Cube roots



The cube shown has sides of 2 units and occupies 8 cubic units of space. (That is, $2 \times 2 \times 2$.)

So the **cube root** of 8 is 2.

This can be written as $\sqrt[3]{8} = 2$.

$\sqrt[3]{\ }$ is read as 'the cube root of …'.

$\sqrt[3]{64}$ is 4, since $4 \times 4 \times 4 = 64$.

Note that $\sqrt[3]{64}$ is not -4

since $-4 \times -4 \times -4 = -64$

but $\sqrt[3]{-64}$ is -4.

Exercise 1.9

1 Find the following cube roots:

a $\sqrt[3]{8}$	**b** $\sqrt[3]{125}$	**c** $\sqrt[3]{27}$	**d** $\sqrt[3]{0.001}$
e $\sqrt[3]{0.027}$	**f** $\sqrt[3]{216}$	**g** $\sqrt[3]{1000}$	**h** $\sqrt[3]{1\,000\,000}$
i $\sqrt[3]{-8}$	**j** $\sqrt[3]{-27}$	**k** $\sqrt[3]{-1000}$	**l** $\sqrt[3]{-1}$

Further powers and roots

We have seen that the square of a number is the same as raising that number to the power of 2. For example, the square of 5 is written as 5^2 and means 5×5. Similarly, the cube of a number is the same as raising that number to the power of 3. For example, the cube of 5 is written as 5^3 and means $5 \times 5 \times 5$.

Numbers can be raised by other powers too. Therefore, 5 raised to the power of 6 can be written as 5^6 and means $5 \times 5 \times 5 \times 5 \times 5 \times 5$.

You will find a button on your calculator to help you to do this. On most calculators, it will look like y^x.

We have also seen that the square root of a number can be written using the $\sqrt{\ }$ symbol. Therefore, the square root of 16 is written as $\sqrt{16}$ and is 4, because $4 \times 4 = 16$.

The cube root of a number can be written using the $\sqrt[3]{\ }$ symbol. Therefore, the cube root of 27 is written as $\sqrt[3]{27}$ and is 3, because $3 \times 3 \times 3 = 27$.

Numbers can be rooted by other values as well. The fourth root of a number can be written using the symbol $\sqrt[4]{\ }$. Therefore the fourth root of 625 can be expressed as $\sqrt[4]{625}$ and is 5, because $5 \times 5 \times 5 \times 5 = 625$.

You will find a button on your calculator to help you to calculate with roots. On most calculators, it will look like $\sqrt[x]{y}$.

Exercise 1.10

1 Work out the following:

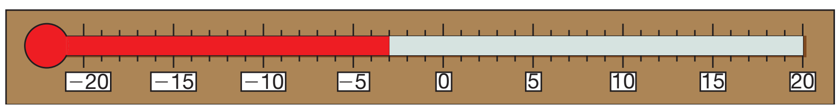
a 6^4 **b** $3^5 + 2^4$ **c** $(3^4)^2$

d $0.1^6 \div 0.01^4$ **e** $\sqrt[4]{2401}$ **f** $\sqrt[8]{256}$

g $\left(\sqrt[5]{243}\right)^3$ **h** $\left(\sqrt[9]{36}\right)^9$ **i** $2^7 \times \sqrt{\frac{1}{4}}$

j $\sqrt[6]{\frac{1}{64}} \times 2^7$ **k** $\sqrt[4]{5^4}$ **l** $\left(\sqrt[10]{59\,049}\right)^2$

Directed numbers



Worked example

The diagram above shows the scale of a thermometer. The temperature at 0400 was –3 °C. By 0900 the temperature had risen by 8 °C. What was the temperature at 0900?

$-3 + 8 = 5$, so the temperature is 5°C.

Exercise 1.11

1 The highest temperature ever recorded was in Libya. It was 58 °C. The lowest temperature ever recorded was –88 °C in Antarctica. What is the temperature difference?

2 Ms Okoro's bank account shows a positive amount of $105. Describe the amount in her account as a positive or negative number after each of these transactions is made in sequence:

a rent $140 **b** car insurance $283

c 1 week's salary $230 **d** food bill $72

e credit transfer $250

3 The roof of an apartment block is 130 m above ground level. The car park beneath the apartment is 35 m below ground level. How high is the roof above the floor of the car park?

4 A submarine is at a depth of 165 m. If the ocean floor is 860 m from the surface, how far is the submarine from the ocean floor?

Student assessment 1

1 State whether the following numbers are rational or irrational:

a 1.5 **b** $\sqrt{7}$ **c** $0.\dot{7}$

d $0.\dot{7}\dot{3}$ **e** $\sqrt{121}$ **f** π

2 Show, by expressing them as fractions or whole numbers, that the following numbers are rational:

a 0.625 **b** $\sqrt[3]{27}$ **c** 0.44

3 Find the value of:

a 9^2 **b** 15^2

c 0.2^2 **d** 0.7^2

4 Calculate:

a 3.5^2 **b** 4.1^2 **c** 0.15^2

5 Without using a calculator, find:

a $\sqrt{225}$ **b** $\sqrt{0.01}$ **c** $\sqrt{0.81}$

d $\sqrt{\frac{9}{25}}$ **e** $\sqrt{5\frac{4}{9}}$ **f** $\sqrt{2\frac{23}{49}}$

6 Without using a calculator, find:

a 4^3 **b** $(0.1)^3$ **c** $\left(\frac{2}{3}\right)^3$

7 Without using a calculator, find:

a $\sqrt[3]{27}$ **b** $\sqrt[3]{1000000}$ **c** $\sqrt[3]{\frac{64}{125}}$

8 Toby's bank statement for seven days in October is shown below:

Date	Payments ($)	Receipts ($)	Balance ($)
01/10			200
02/10	284		(a)
03/10		175	(b)
04/10	(c)		46
05/10		(d)	120
06/10	163		(e)
07/10		28	(f)

Copy and complete the statement by entering the amounts (a) to (f).

9 Using a calculator if necessary, work out:

a $2^6 \div 2^8$ **b** $4^5 \times \sqrt[6]{64}$ **c** $\sqrt[4]{81} \times 4^3$

2 Accuracy

Approximation

In many instances exact numbers are not necessary or even desirable. In those circumstances approximations are given. Approximations can take several forms; the most common forms are dealt with below.

Rounding

If 28 617 people attend a gymnastics competition, this figure can be reported to various levels of accuracy.

To the nearest 10 000 this figure would be **rounded** up to 30 000.

To the nearest 1000 the figure would be rounded up to 29 000.

To the nearest 100 the figure would be rounded down to 28 600.

In this type of situation, it is unlikely that the exact number would be reported.

Note: If a number falls exactly half-way, then it is rounded up.
For example, rounding 16 500 to the nearest thousand can be visualised as follows:

16 000 and 17 000 are the numbers in thousands either side of 16 500. As 16 500 falls exactly half-way, it gets rounded up to 17 000 if the answer is wanted to the nearest thousand.

Exercise 2.1

1 Round the following numbers to the nearest 1000:

a 68 786 **b** 74 245 **c** 89 000
d 4020 **e** 99 500 **f** 999 999

2 Round the following numbers to the nearest 100:

a 78 540 **b** 6858 **c** 14 099
d 8084 **e** 950 **f** 2984

3 Round the following numbers to the nearest 10:

a 485 **b** 692 **c** 8847
d 83 **e** 4 **f** 997

Decimal places

A number can also be approximated to a given number of **decimal places** (d.p.). This refers to the number of digits written after a decimal point.

Worked examples

a Write 7.864 to 1 d.p.

The answer needs to be written with one digit after the decimal point. However, to do this, the second digit after the decimal point also needs to be considered. If it is 5 or more, then the first digit is rounded up.

i.e. 7.864 is written as 7.9 to 1 d.p.

b Write 5.574 to 2 d.p.

The answer here is to be given with two digits after the decimal point. In this case, the third digit after the decimal point needs to be considered. As the third digit after the decimal point is less than 5, the second digit is not rounded up.

i.e. 5.574 is written as 5.57 to 2 d.p.

Exercise 2.2

1 Give the following to 1 d.p.

a 5.58	**b** 0.73	**c** 11.86
d 157.39	**e** 4.04	**f** 15.045
g 2.95	**h** 0.98	**i** 12.049

2 Give the following to 2 d.p.

a 6.473	**b** 9.587	**c** 16.476
d 0.088	**e** 0.014	**f** 9.3048
g 99.996	**h** 0.0048	**i** 3.0037

Significant figures

Numbers can also be approximated to a given number of **significant figures** (s.f.). In the number 43.25 the 4 is the most significant figure as it has a value of 40. In contrast, the 5 is the least significant as it only has a value of 5 hundredths.

Worked examples

a Write 43.25 to 3 s.f.

Only the three most significant digits are written in the answer; however, the fourth digit needs to be considered to see whether the third digit is to be rounded up or not.

i.e. 43.25 is written as 43.3 to 3 s.f.

b Write 0.0043 to 1 s.f.

In this example only two digits have any significance, the 4 and the 3. The 4 is the most significant and therefore is the only one of the two digits to be written in the answer.

i.e. 0.0043 is written as 0.004 to 1 s.f.

Exercise 2.3

1 Write the following to the number of significant figures written in brackets:

a 48 599 (1 s.f.) b 48 599 (3 s.f.) c 6841 (1 s.f.)
d 7538 (2 s.f.) e 483.7 (1 s.f.) f 2.5728 (3 s.f.)
g 990 (1 s.f.) h 2045 (2 s.f.) i 14.952 (3 s.f.)

2 Write the following to the number of significant figures written in brackets:

a 0.085 62 (1 s.f.) b 0.5932 (1 s.f.) c 0.942 (2 s.f.)
d 0.954 (1 s.f.) e 0.954 (2 s.f.) f 0.003 05 (1 s.f.)
g 0.003 05 (2 s.f.) h 0.009 73 (2 s.f.) i 0.009 73 (1 s.f.)

Appropriate accuracy

In many instances calculations carried out using a calculator produce answers which are not whole numbers. A calculator will give the answer to as many decimal places as will fit on its screen. In most cases this degree of accuracy is neither desirable nor necessary. Unless specifically asked for, answers should not be given to more than two decimal places. Indeed, one decimal place is usually sufficient. Alternatively, giving an answer correct to three significant figures is also considered an appropriate degree of accuracy.

Worked example

Calculate $4.64 \div 2.3$, giving your answer to an appropriate degree of accuracy.

The calculator will give the answer to $4.64 \div 2.3$ as 2.0173913.

However, the answer given to 1 d.p. is sufficient.

Therefore $4.64 \div 2.3 = 2.0$ (1 d.p.).

Estimating answers to calculations

Even though many calculations can be done quickly and effectively on a calculator, often an **estimate** for an answer, done without using a calculator, can be a useful check. Estimating an answer is done by rounding each of the numbers in such a way that the mental calculation becomes relatively straightforward.

Worked examples

a Estimate the answer to 57×246.

Here are two possibilities:
i $60 \times 200 = 12\,000$,
ii $50 \times 250 = 12\,500$.

b Estimate the answer to $6386 \div 27$.

$6000 \div 30 = 200$.

Note

≈ means approximately equal to.

c Estimate the answer to $\sqrt[3]{120} \times 48$.

As $\sqrt[3]{125} = 5$, $\sqrt[3]{120} \approx 5$

Therefore $\sqrt[3]{120} \times 48 \approx 5 \times 50$

$\approx 250.$

d Estimate the answer to $\frac{2^5 \times \sqrt[4]{600}}{8}$

An approximate answer can be calculated using the knowledge that $2^5 = 32$ and $\sqrt[4]{625} = 5$

Therefore $\frac{2^5 \times \sqrt[4]{600}}{8} \approx \frac{30 \times 5}{8} \approx \frac{150}{8}$

$\approx 20.$

Exercise 2.4

1 Calculate the following, giving your answer to an appropriate degree of accuracy:

a 23.456×17.89	**b** 0.4×12.62	**c** 18×9.24
d $76.24 \div 3.2$	**e** 7.6^2	**f** 16.42^3
g $\frac{2.3 \times 3.37}{4}$	**h** $\frac{8.31}{2.02}$	**i** $9.2 \div 4^2$

2 Without using a calculator, estimate the answers to the following:

a 62×19	**b** 270×12	**c** 55×60
d 4950×28	**e** 0.8×0.95	**f** 0.184×475

3 Without using a calculator, estimate the answers to the following:

a $3946 \div 18$	**b** $8287 \div 42$	**c** $906 \div 27$
d $5520 \div 13$	**e** $48 \div 0.12$	**f** $610 \div 0.22$

4 Without using a calculator, estimate the answers to the following:

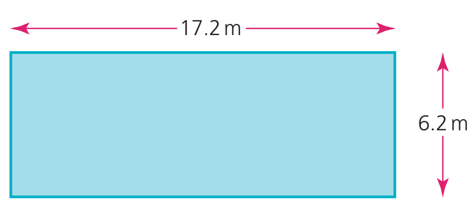
a $78.45 + 51.02$	**b** $168.3 - 87.09$	**c** 2.93×3.14
d $84.2 \div 19.5$	**e** $\frac{4.3 \times 752}{15.6}$	**f** $\frac{(9.8)^3}{(2.2)^2}$
g $\frac{\sqrt[3]{78} \times 6}{5^3}$	**h** $\frac{38 \times 6^3}{\sqrt[4]{9900}}$	**i** $\sqrt[4]{24} \times \sqrt[4]{26}$

5 Using estimation, identify which of the following are definitely incorrect. Explain your reasoning clearly.

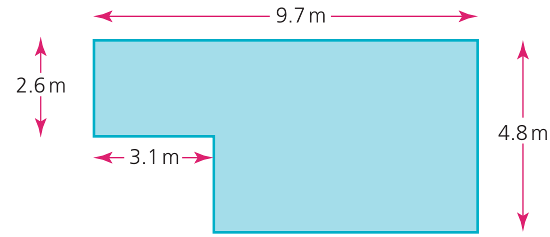
a $95 \times 212 = 20\,140$	**b** $44 \times 17 = 748$
c $689 \times 413 = 28\,457$	**d** $142\,656 \div 8 = 17\,832$
e $77.9 \times 22.6 = 2512.54$	**f** $\frac{8.42 \times 46}{0.2} = 19\,366$

6 Estimate the area of the shaded areas of the following shapes. Do *not* work out an exact answer.

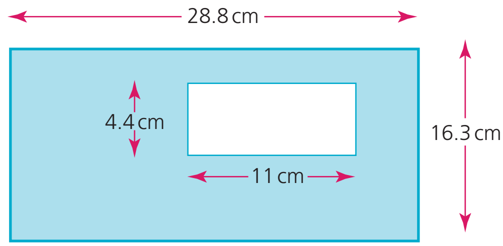
a



b

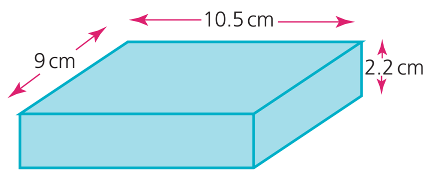


c

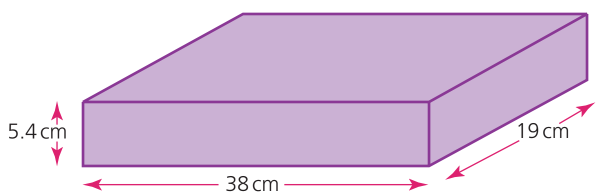


7 Estimate the volume of each of the solids below. Do *not* work out an exact answer.

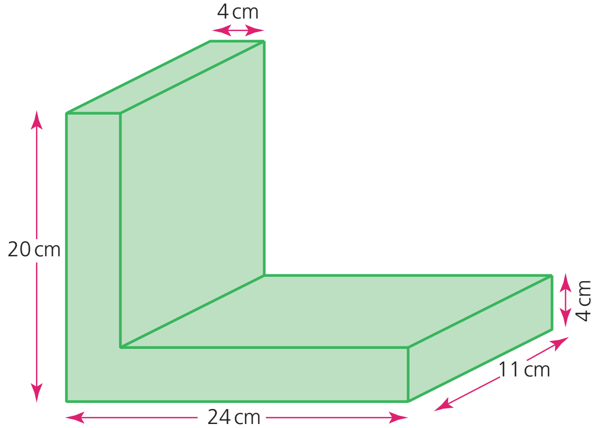
a



b



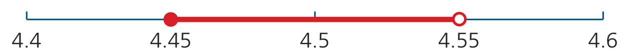
c



Upper and lower bounds

Numbers can be written to different degrees of accuracy. For example, although 4.5, 4.50 and 4.500 appear to represent the same number, they do not. This is because they are written to different degrees of accuracy.

4.5 is rounded to one decimal place and therefore any number from 4.45 up to but not including 4.55 would be rounded to 4.5. On a number line, this would be represented as:



As an inequality where x represents the number it would be expressed as:

$4.45 \leqslant x < 4.55$

4.45 is known as the **lower bound** of 4.5, while 4.55 is known as the **upper bound**.
Note that ○⟶ implies that the number is not included in the solution while ●⟶ implies that the number is included in the solution.

4.50 on the other hand is written to two decimal places and therefore only numbers from 4.495 up to but not including 4.505 would be rounded to 4.50. This therefore represents a much smaller range of numbers than when it is rounded to 4.5. Similarly, the range of numbers being rounded to 4.500 would be even smaller.

Worked example

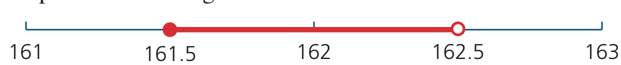
A girl's **height** is given as 162 cm to the nearest centimetre.

a Work out the lower and upper bounds within which her height can lie.

Lower bound = 161.5 cm

Upper bound = 162.5 cm

b Represent this range of numbers on a number line.



c If the girl's height is h cm, express this range as an inequality.

$161.5 \leqslant h < 162.5$

Exercise 2.5

1 Each of the following numbers is expressed to the nearest whole number.
  i Give the upper and lower bounds of each.
  ii Using x as the number, express the range in which the number lies as an inequality.
    a 6 b 83 c 152
    d 1000 e 100

2 Each of the following numbers is correct to one decimal place.
  i Give the upper and lower bounds of each.
  ii Using x as the number, express the range in which the number lies as an inequality.
    a 3.8 b 15.6 c 1.0
    d 10.0 e 0.3

3 Each of the following numbers is correct to two significant figures.
  i Give the upper and lower bounds of each.
  ii Using x as the number, express the range in which the number lies as an inequality.
    a 4.2 b 0.84 c 420
    d 5000 e 0.045 f 25 000

4 The mass of a sack of vegetables is given as 5.4 kg.
  a Illustrate the lower and upper bounds of the mass on a number line.
  b Using M kg for the mass, express the range of values in which M must lie as an inequality.

5 At a school sports day, the winning time for the 100 m race was given as 11.8 seconds.
  a Illustrate the lower and upper bounds of the winning time on a number line.
  b Using T seconds for the time, express the range of values in which T must lie as an inequality.

6 The capacity of a swimming pool is given as 620 m^3 correct to two significant figures.
  a Calculate the lower and upper bounds of the pool's capacity.
  b Using x cubic metres for the capacity, express the range of values in which x must lie as an inequality.

7 Hadiza is a surveyor. She measures the dimensions of a rectangular field to the nearest 10 m. The length is recorded as 630 m and the width is recorded as 400 m.
  a Calculate the lower and upper bounds of the length.
  b Using W metres for the width, express the range of values in which W must lie as an inequality.

Calculating with upper and lower bounds

When numbers are written to a specific degree of accuracy, calculations involving those numbers also give a range of possible answers.

Worked examples

a Calculate the upper and lower bounds for the following calculation, given that each number is given to the nearest whole number.

34×65

34 lies in the range $33.5 \leqslant x < 34.5$.

65 lies in the range $64.5 \leqslant x < 65.5$.

The lower bound of the calculation is obtained by multiplying together the two lower bounds. Therefore the minimum product is 33.5×64.5, i.e. 2160.75.

The upper bound of the calculation is obtained by multiplying together the two upper bounds. Therefore the maximum product is 34.5×65.5, i.e. 2259.75.

b Calculate the upper and lower bounds to $\frac{33.5}{22.0}$ given that each of the numbers is accurate to 1 d.p.

33.5 lies in the range $33.45 \leqslant x < 33.55$.

22.0 lies in the range $21.95 \leqslant x < 22.05$.

The lower bound of the calculation is obtained by dividing the lower bound of the numerator by the *upper* bound of the denominator. So the minimum value is $33.45 \div 22.05$, i.e. 1.52 (2 d.p.).

The upper bound of the calculation is obtained by dividing the upper bound of the numerator by the *lower* bound of the denominator. So the maximum value is $33.55 \div 21.95$, i.e. 1.53 (2 d.p.).

Exercise 2.6

1 Calculate the lower and upper bounds for the following calculations, if each of the numbers is given to the nearest whole number.

a 14×20 **b** 135×25 **c** 100×50

d $\frac{40}{10}$ **e** $\frac{33}{11}$ **f** $\frac{125}{15}$

g $\frac{12 \times 65}{16}$ **h** $\frac{101 \times 28}{69}$ **i** $\frac{250 \times 7}{100}$

j $\frac{44}{3^2}$ **k** $\frac{578}{17 \times 22}$ **l** $\frac{1000}{4 \times (3+8)}$

2 Calculate the lower and upper bounds for the following calculations, if each of the numbers is given to 1 d.p.

a $2.1 + 4.7$ **b** 6.3×4.8 **c** 10.0×14.9

d $17.6 - 4.2$ **e** $\frac{8.5 + 3.6}{6.8}$ **f** $\frac{7.7 - 6.2}{3.5}$

g $\frac{16.4^2}{(3.0 - 0.3)^2}$ **h** $\frac{100.0}{(50.0 - 40.0)^2}$ **i** $(0.1 - 0.2)^2$

3 Calculate the lower and upper bounds for the following calculations, if each of the numbers is given to 2 s.f.

a 64×320 **b** 1.7×0.65 **c** 4800×240

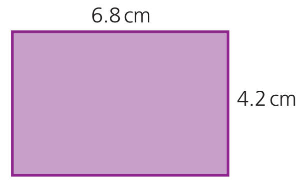
d $\frac{54\,000}{600}$ **e** $\frac{4.2}{0.031}$ **f** $\frac{200}{5.2}$

g $\frac{6.8 \times 42}{120}$ **h** $\frac{200}{(4.5 \times 6.0)}$ **i** $\frac{180}{(7.3 - 4.5)}$

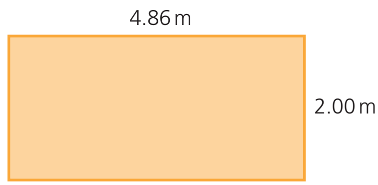
Exercise 2.7

1 The masses to the nearest 0.5 kg of two parcels are 1.5 kg and 2.5 kg. Calculate the lower and upper bounds of their combined mass.

2 Calculate the upper and lower bounds for the perimeter of the **rectangle** shown (below), if its dimensions are correct to 1 d.p.

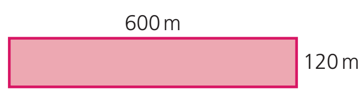


3 Calculate the upper and lower bounds for the perimeter of the rectangle shown (below), whose dimensions are accurate to 2 d.p.

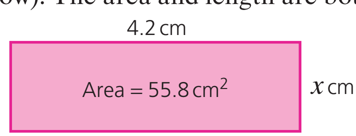


4 Calculate the upper and lower bounds for the **area** of the rectangle shown (below), if its dimensions are accurate to 1 d.p.

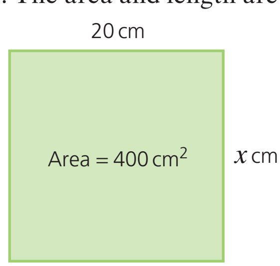
5 Calculate the upper and lower bounds for the area of the rectangle shown (below), whose dimensions are correct to 2 s.f.



6 Calculate the upper and lower bounds for the length marked x cm in the rectangle (below). The area and length are both given to 1 d.p.

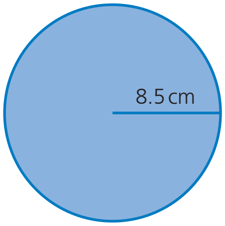


7 Calculate the upper and lower bounds for the length marked x cm in the rectangle (below). The area and length are both accurate to 2 s.f.



Exercise 2.7 (cont)

8 The radius of the circle shown (below) is given to 1 d.p. Calculate the upper and lower bounds of:
  a the **circumference**,
  b the area.



9 The area of the circle shown (below) is given to 2 s.f. Calculate the upper and lower bounds of:
  a the radius,
  b the circumference.

10 The mass of a cube of side 2 cm is given as 100 g. The side is accurate to the nearest millimetre and the mass accurate to the nearest gram. Calculate the maximum and minimum possible values for the density of the material (density = mass ÷ volume).

11 The distance to the nearest 100 000 km from Earth to the Moon is given as 400 000 km. The **average speed** to the nearest 500 km/h of a rocket to the Moon is given as 3500 km/h. Calculate the greatest and least time it could take the rocket to reach the Moon.

Student assessment 1

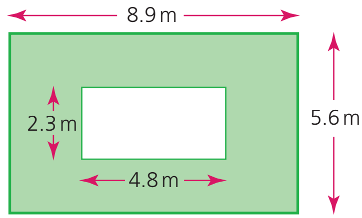
1 Round the following numbers to the degree of accuracy shown in brackets:
  a 2841 (nearest 100)
  b 7096 (nearest 10)
  c 48 756 (nearest 1000)
  d 951 (nearest 100)

2 Round the following numbers to the number of decimal places shown in brackets:
  a 3.84 (1 d.p.)
  b 6.792 (1 d.p.)
  c 0.8506 (2 d.p.)
  d 1.5849 (2 d.p.)
  e 9.954 (1 d.p.)
  f 0.0077 (3 d.p.)

3 Round the following numbers to the number of significant figures shown in brackets:
  a 3.84 (1 s.f.)
  b 6.792 (2 s.f.)
  c 0.7065 (1 s.f.)
  d 9.624 (1 s.f.)
  e 834.97 (2 s.f.)
  f 0.00451 (1 s.f.)
  g 62.4899 (5 s.f.)
  h 0.9997 (3 s.f.)

4 1 mile is 1760 yards. Estimate the number of yards in 11.5 miles.

5 Estimate the shaded area of the figure below:



6 Estimate the answers to the following.
Do *not* work out an exact answer.

a $\frac{5.3 \times 11.2}{2.1}$ **b** $\frac{(9.8)^2}{(4.7)^2}$ **c** $\frac{18.8 \times (7.1)^2}{(3.1)^2 \times (4.9)^2}$

7 A cuboid's dimensions are given as 12.32 cm by 1.8 cm by 4.16 cm. Calculate its volume, giving your answer to an appropriate degree of accuracy.

Student assessment 2

1 The following numbers are expressed to the nearest whole number. Illustrate on a number line the range in which each must lie.

a 7 **b** 40 **c** 300 **d** 2000

2 The following numbers are expressed correct to two significant figures. Representing each number by the letter x, express the range in which each must lie, using an inequality.

a 210 **b** 64 **c** 3.0 **d** 0.88

3 Some students measure the dimensions of their school's rectangular playing field to the nearest metre. The length was recorded as 350 m and the width as 200 m. Express the range in which the length and width lie using inequalities.

4 A boy's mass was measured to the nearest 0.1 kg. If his mass was recorded as 58.9 kg, illustrate on a number line the range within which it must lie.

5 An electronic clock is accurate to $\frac{1}{1000}$ of a second. The duration of a flash from a camera is timed at 0.004 seconds. Express the upper and lower bounds of the duration of the flash using inequalities.

6 The following numbers are rounded to the degree of accuracy shown in brackets. Express the lower and upper bounds of these numbers as an inequality.

a $x = 4.83$ (2 d.p.)
b $y = 5.05$ (2 d.p.)
c $z = 10.0$ (1 d.p.)
d $p = 100.00$ (2 d.p.)

Student assessment 3

1. Five animals have a mass, given to the nearest 10 kg, of: 40 kg, 50 kg, 50 kg, 60 kg and 80 kg. Calculate the least possible total mass.

2. A water tank measures 30 cm by 50 cm by 20 cm. If each of these measurements is given to the nearest centimetre, calculate the largest possible volume of the tank.

3. The volume of a cube is given as 125 cm^3 to the nearest whole number.
   - **a** Express as an inequality the upper and lower bounds of the cube's volume.
   - **b** Express as an inequality the upper and lower bounds of the length of each of the cube's edges.

4. The radius of a circle is given as 4.00 cm to 2 d.p. Express as an inequality the upper and lower bounds for:
   - **a** the circumference of the circle,
   - **b** the area of the circle.

5. A cylindrical water tank has a volume of 6000 cm^3 correct to 1 s.f. A full cup of water from the tank has a volume of 300 cm^3 correct to 2 s.f. Calculate the maximum number of full cups of water that can be drawn from the tank.

6. A match measures 5 cm to the nearest centimetre. 100 matches end to end measure 5.43 m correct to 3 s.f.
   - **a** Calculate the upper and lower limits of the length of one match.
   - **b** How can the limits of the length of a match be found to 2 d.p.?

3

Calculations and order

Ordering

The following symbols have a specific meaning in mathematics:

$=$ is equal to
$\neq$ is not equal to
$>$ is greater than
$\geqslant$ is greater than or equal to
$<$ is less than
$\leqslant$ is less than or equal to

$x \geqslant 3$ implies that x is greater than or equal to 3, i.e. x can be 3, 4, 4.2, 5, 5.6, etc.

$3 \leqslant x$ implies that 3 is less than or equal to x, i.e. x is still 3, 4, 4.2, 5, 5.6, etc.

Therefore:

$5 > x$ can be rewritten as $x < 5$, i.e. x can be 4, 3.2, 3, 2.8, 2, 1, etc.

$-7 \leqslant x$ can be rewritten as $x \geqslant -7$, i.e. x can be -7, -6, -5, etc.

These **inequalities** can also be represented on a number line:

$x < 5$

$x \geqslant -7$

Note that  implies that the number is not included in the solution while  implies that the number is included in the solution.

Worked examples

a The maximum number of players from one football team allowed on the pitch at any one time is eleven. Represent this information:

i as an inequality,
ii on a number line.

i Let the number of players be represented by the letter n. n must be less than or equal to 11. Therefore $n \leqslant 11$.

ii

b The maximum number of players from one football team allowed on the pitch at any one time is eleven. The minimum allowed is seven players. Represent this information:

i as an inequality,

ii on a number line.

i Let the number of players be represented by the letter n. n must be greater than or equal to 7, but less than or equal to 11. Therefore $7 \leqslant n \leqslant 11$.

ii

Exercise 3.1

1 Copy each of the following statements, and insert one of the symbols =, >, < into the space to make the statement correct:

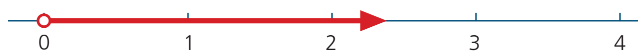
a 7×2 ... $8 + 7$
b 6^2 ... 9×4
c 5×10 ... 7^2
d 80 cm ... 1 m
e 1000 litres ... 1 m^3
f $48 \div 6$ ... $54 \div 9$

2 Represent each of the following inequalities on a number line, where x is a real number:

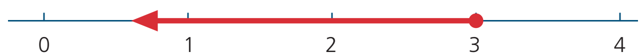
a $x < 2$
b $x \geqslant 3$
c $x \leqslant -4$
d $x \geqslant -2$
e $2 < x < 5$
f $-3 < x < 0$
g $-2 \leqslant x < 2$
h $2 \geqslant x \geqslant -1$

3 Write down the inequalities which correspond to the following number lines:

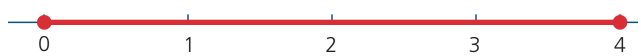
a



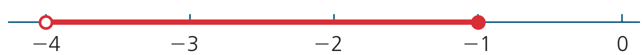
b



c



d



4 Write the following sentences using inequality signs:

a The maximum capacity of an athletics stadium is 20 000 people.
b In a class, the tallest student is 180 cm and the shortest is 135 cm.
c Five times a number plus 3 is less than 20.
d The maximum temperature in May was 25 °C.
e A farmer has between 350 and 400 apples on each tree in her orchard.
f In December, temperatures in Kenya were between 11 °C and 28 °C.

Exercise 3.2

1 Write the following decimals in order of magnitude, starting with the smallest:

6.0 0.6 0.66 0.606 0.06 6.6 6.606

2 Write the following fractions in order of magnitude, starting with the largest:

$\frac{1}{2}$ $\frac{1}{3}$ $\frac{6}{13}$ $\frac{4}{5}$ $\frac{7}{18}$ $\frac{2}{19}$

3 Write the following lengths in order of magnitude, starting with the smallest:
2 m 60 cm 800 mm 180 cm 0.75 m

4 Write the following masses in order of magnitude, starting with the largest:
4 kg 3500 g $\frac{3}{4}$ kg 700 g 1 kg

5 Write the following volumes in order of magnitude, starting with the smallest:
1 l 430 ml 800 cm^3 120 cl 150 cm^3

The order of operations

When carrying out calculations, care must be taken to ensure that they are carried out in the correct order.

Worked examples

a Use a scientific calculator to work out the answer to the following:

$2 + 3 \times 4 =$

[2] [+] [3] [×] [4] [=] 14

b Use a scientific calculator to work out the answer to the following:

$(2 + 3) \times 4 =$

[(] [2] [+] [3] [)] [×] [4] [=] 20

The reason why different answers are obtained is because, by convention, the operations have different priorities. These are as follows:

(1) brackets

(2) indices

(3) multiplication/division

(4) addition/subtraction.

Therefore in ***Worked example*** **a** 3×4 is **evaluated** first, and then the 2 is added, while in ***Worked example*** **b** $(2 + 3)$ is evaluated first, followed by multiplication by 4.

c Use a scientific calculator to work out why the answer to the following is –20:

$-4 \times (8 + -3) = -20$

The $(8 + -3)$ is evaluated first as it is in the brackets, the answer 5 is then multiplied by -4.

d Use a scientific calculator to work out why the answer to the following is –35:

$-4 \times 8 + -3 = -35$

The -4×8 is evaluated first as it is a multiplication, the answer -32 then has -3 added to it.

Exercise 3.3

In the following questions, evaluate the answers:

i in your head,

ii using a scientific calculator.

1 **a** $8 \times 3 + 2$ **b** $4 \div 2 + 8$
c $12 \times 4 - 6$ **d** $4 + 6 \times 2$
e $10 - 6 \div 3$ **f** $6 - 3 \times 4$

2 **a** $7 \times 2 + 3 \times 2$ **b** $12 \div 3 + 6 \times 5$
c $9 + 3 \times 8 - 1$ **d** $36 - 9 \div 3 - 2$
e $-14 \times 2 - 16 \div 2$ **f** $4 + 3 \times 7 - 6 \div 3$

3 **a** $(4 + 5) \times 3$ **b** $8 \times (12 - 4)$
c $3 \times (-8 + -3) - 3$ **d** $(4 + 11) \div (7 - 2)$
e $4 \times 3 \times (7 + 5)$ **f** $24 \div 3 \div (10 - 5)$

Exercise 3.4

In each of the following questions:

i Copy the calculation and put in any brackets which are needed to make it correct.

ii Check your answer using a scientific calculator.

1 **a** $6 \times 2 + 1 = 18$ **b** $1 + 3 \times 5 = 16$
c $8 + 6 \div 2 = 7$ **d** $9 + 2 \times 4 = 44$
e $9 \div 3 \times 4 + 1 = 13$ **f** $3 + 2 \times 4 - 1 = 15$

2 **a** $12 \div 4 - 2 + 6 = 7$ **b** $12 \div 4 - 2 + 6 = 12$
c $12 \div 4 - 2 + 6 = -5$ **d** $12 \div 4 - 2 + 6 = 1.5$
e $4 + 5 \times 6 - 1 = 33$ **f** $4 + 5 \times 6 - 1 = 29$
g $4 + 5 \times 6 - 1 = 53$ **h** $4 + 5 \times 6 - 1 = 45$

It is important to use brackets when dealing with more complex calculations.

Worked examples

Your calculator may have a fraction button. It may look like this:

a Evaluate the following using a scientific calculator:

$\dfrac{12+9}{10-3} =$

$\boxed{(}\boxed{1}\boxed{2}\boxed{+}\boxed{9}\boxed{)}\boxed{\div}\boxed{(}\boxed{1}\boxed{0}\boxed{-}\boxed{3}\boxed{)}\boxed{=}\ 3$

b Evaluate the following using a scientific calculator:

$\dfrac{20+12}{4^2} =$

$\boxed{(}\boxed{2}\boxed{0}\boxed{+}\boxed{1}\boxed{2}\boxed{)}\boxed{\div}\boxed{4}\boxed{x^2}\boxed{=}\ 2$

c Evaluate the following using a scientific calculator:

$\dfrac{90+38}{4^3} =$

$\boxed{(}\boxed{9}\boxed{0}\boxed{+}\boxed{3}\boxed{8}\boxed{)}\boxed{\div}\boxed{4}\boxed{x^y}\boxed{3}\boxed{=}\ 2$

Note: Different types of calculator have different ‘to the power of’ and ‘fraction’ buttons. It is therefore important that you get to know your calculator.

Exercise 3.5

Using a scientific calculator, evaluate the following:

1 **a** $\frac{9+3}{6}$ **b** $\frac{30-6}{5+3}$

c $\frac{40+9}{12-5}$ **d** $\frac{15\times2}{7+8}+2$

e $\frac{100+21}{11}+4\times3$ **f** $\frac{7+2\times4}{7-2}-3$

2 **a** $\frac{4^2-6}{2+8}$ **b** $\frac{3^2+4^2}{5}$

c $\frac{6^3-4^2}{4\times25}$ **d** $\frac{3^3\times4^4}{12^2}+2$

e $\frac{3+3^3}{5}+\frac{4^2-2^3}{8}$ **f** $\frac{(6+3)\times4}{2^3}-2\times3$

Exercise 3.6

In each of the following questions:

a Write the calculation represented by each problem.

b Work out the answer to each calculation.

1 The temperature of water in a beaker is initially 48°C. It is allowed to cool by 16°C before being heated up again. When heated up, the temperature of the water is trebled.
What is the temperature (T°C) of the water once it has been heated up?

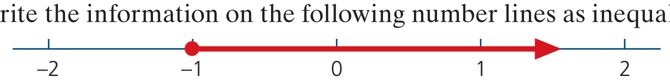
2 A submarine is initially at a depth of 400 m below the water's surface. It then dives a further distance so that it is at a depth double what it was initially. The submarine later climbs 620 m.
Calculate the depth D (m) the submarine climbs to.

3 Luis arranges five counters in a line. He squares the number of counters, then adds a further 11 counters. Finally, he divides the number of counters equally between himself and Dari, so they each receive N counters.
Calculate the number of counters (N) they each receive.

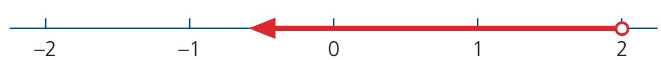
Student assessment 1

1 Write the information on the following number lines as inequalities:

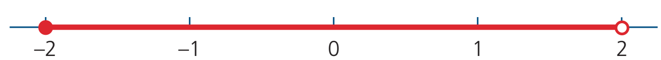
a



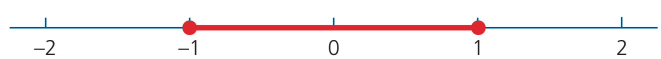
b



c



d



2 **a** Illustrate each of the following inequalities on a number line:

i $x \geqslant 3$ **ii** $x < 4$

iii $0 < x < 4$ **iv** $-3 \leqslant x < 1$

b Write down the smallest integer value which satisfies each inequality in part **a** above.

3 Write the following fractions in order of magnitude, starting with the smallest:

$\frac{4}{7}$ $\frac{3}{14}$ $\frac{9}{10}$ $\frac{1}{2}$ $\frac{2}{5}$

Student assessment 2

1 Evaluate the following:

a $6 \times 8 - 4$ b $3 + 5 \times 2$ c $3 \times 3 + 4 \times 4$

d $3 + 3 \times 4 + 4$ e $(5 + 2) \times 7$ f $18 \div 2 \div (5 - 2)$

2 Copy the following, if necessary inserting brackets to make the statement correct:

a $7 - 4 \times 2 = 6$ b $12 + 3 \times 3 + 4 = 33$

c $5 + 5 \times 6 - 4 = 20$ d $5 + 5 \times 6 - 4 = 56$

3 Evaluate the following using a calculator:

a $\frac{2^4 - 3^2}{2}$ b $\frac{(8 - 3) \times 3}{5} + 7$

Student assessment 3

1 Evaluate the following:

a $3 \times 9 - 7$ b $12 + 6 \div 2$

c $3 + 4 \div 2 \times 4$ d $6 + 3 \times 4 - 5$

e $(5 + 2) \div 7$ f $14 \times 2 \div (9 - 2)$

2 Copy the following, if necessary inserting brackets to make the statement correct:

a $7 - 5 \times 3 = 6$ b $16 + 4 \times 2 + 4 = 40$

c $4 + 5 \times 6 - 1 = 45$ d $1 + 5 \times 6 - 6 = 30$

3 Using a calculator, evaluate the following:

a $\frac{3^3 - 4^2}{2}$ b $\frac{(15 - 3) \div 3}{2} + 7$

4 Integers, fractions, decimals and percentages

Fractions

A single unit can be broken into equal parts called fractions, e.g. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$. If, for example, the unit is broken into ten equal parts and three parts are then taken, the fraction is written as $\frac{3}{10}$. That is, three parts out of ten parts.

In the fraction $\frac{3}{10}$:

The three is called the **numerator**.

The ten is called the **denominator**.

A **proper fraction** has its numerator less than its denominator, e.g. $\frac{3}{4}$.

An **improper fraction** has its numerator more than its denominator, e.g. $\frac{9}{2}$.

A **mixed number** is made up of a whole number and a proper fraction, e.g. $4\frac{1}{5}$.

Worked examples

a Find $\frac{1}{5}$ of 35.

This means 'divide 35 into 5 equal parts'.

$\frac{1}{5}$ of 35 is $35 \div 5 = 7$.

b Find $\frac{3}{5}$ of 35.

Since $\frac{1}{5}$ of 35 is 7, $\frac{3}{5}$ of 35 is 7×3.

That is, 21.

Exercise 4.1

1 Evaluate the following:

a $\frac{3}{4}$ of 20 **b** $\frac{4}{5}$ of 20 **c** $\frac{4}{9}$ of 45 **d** $\frac{5}{8}$ of 64

e $\frac{3}{11}$ of 66 **f** $\frac{9}{10}$ of 80 **g** $\frac{5}{7}$ of 42 **h** $\frac{8}{9}$ of 54

i $\frac{7}{8}$ of 240 **j** $\frac{4}{5}$ of 65

Changing a mixed number to an improper fraction

Worked examples

a Change $3\frac{5}{8}$ into an improper fraction.

$$3\frac{5}{8} = \frac{24}{8} + \frac{5}{8}$$
$$= \frac{24+5}{8}$$
$$= \frac{29}{8}$$

b Change the improper fraction $\frac{27}{4}$ into a mixed number.

$$\frac{27}{4} = \frac{24+3}{4}$$
$$= \frac{24}{4} + \frac{3}{4}$$
$$= 6\frac{3}{4}$$

Exercise 4.2

1 Change the following mixed numbers into improper fractions:

a $4\frac{2}{3}$ **b** $3\frac{3}{5}$ **c** $5\frac{7}{8}$ **d** $2\frac{5}{6}$

e $8\frac{1}{2}$ **f** $9\frac{5}{7}$ **g** $6\frac{4}{9}$ **h** $4\frac{1}{4}$

i $5\frac{4}{11}$ **j** $7\frac{6}{7}$ **k** $4\frac{3}{10}$ **l** $11\frac{3}{13}$

2 Change the following improper fractions into mixed numbers:

a $\frac{29}{4}$ **b** $\frac{33}{5}$ **c** $\frac{41}{6}$ **d** $\frac{53}{8}$

e $\frac{49}{9}$ **f** $\frac{17}{12}$ **g** $\frac{66}{7}$ **h** $\frac{33}{10}$

i $\frac{19}{2}$ **j** $\frac{73}{12}$

Decimals

H	T	U.	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
		3.	2	7	
		0.	0	3	8

3.27 is 3 units, 2 tenths and 7 hundredths.

i.e. $3.27 = 3 + \frac{2}{10} + \frac{7}{100}$

0.038 is 3 hundredths and 8 thousandths.

i.e. $0.038 = \frac{3}{100} + \frac{8}{1000}$

Note that 2 tenths and 7 hundredths is equivalent to 27 hundredths

i.e. $\frac{2}{10} + \frac{7}{100} = \frac{27}{100}$

and that 3 hundredths and 8 thousandths is equivalent to 38 thousandths.

i.e. $\frac{3}{100} + \frac{8}{1000} = \frac{38}{1000}$

Exercise 4.3

1 Write the following fractions as decimals:

a $4\frac{5}{10}$ b $6\frac{3}{10}$ c $17\frac{8}{10}$ d $3\frac{7}{100}$
e $9\frac{27}{100}$ f $11\frac{36}{100}$ g $4\frac{6}{1000}$ h $5\frac{27}{1000}$
i $4\frac{356}{1000}$ j $9\frac{204}{1000}$

2 Evaluate the following:

a $2.7 + 0.35 + 16.09$ b $1.44 + 0.072 + 82.3$
c $23.8 - 17.2$ d $16.9 - 5.74$
e $121.3 - 85.49$ f $6.03 + 0.5 - 1.21$
g $72.5 - 9.08 + 3.72$ h $100 - 32.74 - 61.2$
i $16.0 - 9.24 - 5.36$ j $1.1 - 0.92 - 0.005$

Percentages

A fraction whose denominator is 100 can be expressed as a **percentage**.

$\frac{29}{100}$ can be written as 29%

$\frac{45}{100}$ can be written as 45%

By using **equivalent fractions** to change the denominator to 100, other fractions can be written as percentages.

Worked example

Change $\frac{3}{5}$ to a percentage.

$\frac{3}{5} = \frac{3}{5} \times \frac{20}{20} = \frac{60}{100}$

$\frac{60}{100}$ can be written as 60%

Exercise 4.4

1 Express each of the following as a fraction with denominator 100, then write them as percentages:

a $\frac{29}{50}$ b $\frac{17}{25}$ c $\frac{11}{20}$ d $\frac{3}{10}$

e $\frac{23}{25}$ f $\frac{19}{50}$ g $\frac{3}{4}$ h $\frac{2}{5}$

2 Copy and complete the table of equivalents.

Fraction	$\frac{1}{10}$			$\frac{4}{10}$				$\frac{4}{5}$		$\frac{1}{4}$	
Decimal		0.2			0.5		0.7		0.9		
Percentage			30%			60%					75%

The four rules

Addition, subtraction, multiplication and division are mathematical operations.

Long multiplication

When carrying out long multiplication, it is important to remember place value.

Worked example

$184 \times 37 =$

$$\begin{array}{rrrrl} & 1 & 8 & 4 & \\ \times & & 3 & 7 & \\ \hline 1 & 2 & 8 & 8 & (184 \times 7) \\ 5 & 5 & 2 & 0 & (184 \times 30) \\ \hline 6 & 8 & 0 & 8 & (184 \times 37) \end{array}$$

Short division

Worked example

$453 \div 6 =$

$$\begin{array}{r} 7\ \ 5\ r3 \\ 6 \overline{)\,4\ 5\ {}^{3}3} \end{array}$$

It is usual, however, to give the final answer in decimal form rather than with a remainder. The division should therefore be continued:

$453 \div 6$

$$\begin{array}{r} 7\ \ 5\,.\ 5 \\ 6 \overline{)\,4\ 5\ {}^{3}3\,.\ {}^{3}0} \end{array}$$

Long division

Note how the question asks for the answer to 1 d.p., but the calculation is continued until the 2nd d.p. This is to see whether the answer needs to be rounded up.

Worked example

Calculate 7184 ÷ 23 to one decimal place (1 d.p.).

```
      3  1  2 . 3  4
   ________________
23 | 7  1  8  4 . 0  0
     6  9
     ----
        2  8
        2  3
        ----
           5  4
           4  6
           ----
              8  0
              6  9
              ----
              1  1  0
                 9  2
              -------
                 1  8
                 ----
```

Therefore 7184 ÷ 23 = 312.3 to 1 d.p.

Mixed operations

Note

The order of operations was also covered in Chapter 3.

When a calculation involves a mixture of operations, the **order of the operations** is important. Multiplications and divisions are done first, while additions and subtractions are done afterwards. To override this, brackets need to be used.

Worked examples

a $3 + 7 \times 2 - 4$
$= 3 + 14 - 4$
$= 13$

b $(3 + 7) \times 2 - 4$
$= 10 \times 2 - 4$
$= 20 - 4$
$= 16$

c $3 + 7 \times (2 - 4)$
$= 3 + 7 \times (-2)$
$= 3 - 14$
$= -11$

d $(3 + 7) \times (2 - 4)$
$= 10 \times (-2)$
$= -20$

Exercise 4.5

1 Evaluate the answer to each of the following:
   a $3 + 5 \times 2 - 4$ b $12 \div 8 + 6 \div 4$

2 Copy these equations and put brackets in the correct places to make them correct:
   a $6 \times 4 + 6 \div 3 = 20$ b $9 - 3 \times 7 + 2 = 54$

3 i Without using a calculator, work out the solutions to the following multiplications:
      a 785×38 b 164×253
   ii Use the answers to the above questions to deduce the answer to the following:
      a 7.85×3.8 b 1.64×2530

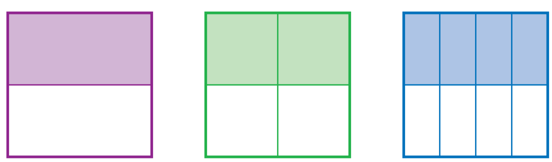
4 Work out the remainders in the following divisions:
   a $72 \div 7$ b $430 \div 9$

5 a A length of rail track is 9 m long. How many complete lengths will be needed to lay 1 km of track?
   b How many 35-cent stamps can be bought for 10 dollars?

6 Work out the following long divisions to 1 d.p.:
   a $7892 \div 7$ b $7892 \div 15$

Calculations with fractions

Equivalent fractions



$\frac{1}{2}$ $\frac{2}{4}$ $\frac{4}{8}$

It should be apparent that $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{4}{8}$ are equivalent fractions.

Similarly, $\frac{1}{3}$, $\frac{2}{6}$, $\frac{3}{9}$ and $\frac{4}{12}$ are equivalent, as are $\frac{1}{5}$, $\frac{10}{50}$ and $\frac{20}{100}$. Equivalent fractions are mathematically the same as each other. In the example diagrams, $\frac{1}{2}$ is mathematically the same as $\frac{4}{8}$. However, $\frac{1}{2}$ is a simplified form of $\frac{4}{8}$.

When carrying out calculations involving fractions it is usual to give your answer in its **simplest form**. Another way of saying 'simplest form' is '**lowest terms**'.

Writing a fraction in its 'simplest form' or in its 'lowest terms' means the same thing.

Worked examples

a Write $\frac{4}{22}$ in its simplest form.

Divide both the numerator and the denominator by their highest common factor.

The highest common factor of both 4 and 22 is 2.

Dividing both 4 and 22 by 2 gives $\frac{2}{11}$.

Therefore $\frac{2}{11}$ is $\frac{4}{22}$ written in its simplest form.

b Write $\frac{12}{40}$ in its lowest terms.

Divide both the numerator and the denominator by their highest common factor.

The highest common factor of both 12 and 40 is 4.

Dividing both 12 and 40 by 4 gives $\frac{3}{10}$.

Therefore $\frac{3}{10}$ is $\frac{12}{40}$ written in its lowest terms.

Exercise 4.6

1 Express the following fractions in their lowest terms. e.g. $\frac{12}{16} = \frac{3}{4}$

a $\frac{5}{10}$ **b** $\frac{7}{21}$ **c** $\frac{8}{12}$

d $\frac{16}{36}$ **e** $\frac{75}{100}$ **f** $\frac{81}{90}$

Addition and subtraction of fractions

For fractions to be either added or subtracted, the denominators need to be the same.

Worked examples

a $\frac{3}{11} + \frac{5}{11} = \frac{8}{11}$

b $\frac{7}{8} + \frac{5}{8} = \frac{12}{8} = 1\frac{1}{2}$

c $\frac{1}{2} + \frac{1}{3}$

$= \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

d $\frac{4}{5} - \frac{1}{3}$

$= \frac{12}{15} - \frac{5}{15} = \frac{7}{15}$

When dealing with calculations involving mixed numbers, it is sometimes easier to change them to improper fractions first.

Worked examples

a $5\frac{3}{4}-2\frac{5}{8}$

$=\frac{23}{4}-\frac{21}{8}$

$=\frac{46}{8}-\frac{21}{8}$

$=\frac{25}{8}=3\frac{1}{8}$

b $1\frac{4}{7}+3\frac{3}{4}$

$=\frac{11}{7}+\frac{15}{4}$

$=\frac{44}{28}+\frac{105}{28}$

$=\frac{149}{28}=5\frac{9}{28}$

Exercise 4.7

Evaluate each of the following and write the answer as a fraction in its simplest form:

1 a $\frac{3}{5}+\frac{4}{5}$ b $\frac{3}{11}+\frac{7}{11}$ c $\frac{2}{3}+\frac{1}{4}$

d $\frac{3}{5}+\frac{4}{9}$ e $\frac{8}{13}+\frac{2}{5}$ f $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$

2 a $\frac{1}{8}+\frac{3}{8}+\frac{5}{8}$ b $\frac{3}{7}+\frac{5}{7}+\frac{4}{7}$ c $\frac{1}{3}+\frac{1}{2}+\frac{1}{4}$

d $\frac{1}{5}+\frac{1}{3}+\frac{1}{4}$ e $\frac{3}{8}+\frac{3}{5}+\frac{3}{4}$ f $\frac{3}{13}+\frac{1}{4}+\frac{1}{2}$

3 a $\frac{3}{7}-\frac{2}{7}$ b $\frac{4}{5}-\frac{7}{10}$ c $\frac{8}{9}-\frac{1}{3}$

d $\frac{7}{12}-\frac{1}{2}$ e $\frac{5}{8}-\frac{2}{5}$ f $\frac{3}{4}-\frac{2}{5}+\frac{7}{10}$

4 a $\frac{3}{4}+\frac{1}{5}-\frac{2}{3}$ b $\frac{3}{8}+\frac{7}{11}-\frac{1}{2}$ c $\frac{4}{5}-\frac{3}{10}+\frac{7}{20}$

d $\frac{9}{13}+\frac{1}{3}-\frac{4}{5}$ e $\frac{9}{10}-\frac{1}{5}-\frac{1}{4}$ f $\frac{8}{9}-\frac{1}{3}-\frac{1}{2}$

5 a $2\frac{1}{2}+3\frac{1}{4}$ b $3\frac{3}{5}+1\frac{7}{10}$ c $6\frac{1}{2}-3\frac{2}{5}$

d $8\frac{5}{8}-2\frac{1}{3}$ e $5\frac{7}{8}-4\frac{3}{4}$ f $3\frac{1}{4}-2\frac{5}{9}$

6 a $2\frac{1}{2}+1\frac{1}{4}+1\frac{3}{8}$ b $2\frac{4}{5}+3\frac{1}{8}+1\frac{3}{10}$ c $4\frac{1}{2}-1\frac{1}{4}-3\frac{5}{8}$

d $6\frac{1}{2}-2\frac{3}{4}-3\frac{2}{5}$ e $2\frac{4}{7}-3\frac{1}{4}-1\frac{3}{5}$ f $4\frac{7}{20}-5\frac{1}{2}+2\frac{2}{5}$

Multiplication and division of fractions

Worked examples

a $\frac{3}{4}\times\frac{2}{3}$

$=\frac{6}{12}$

$=\frac{1}{2}$

b $3\frac{1}{2}\times 4\frac{4}{7}$

$=\frac{7}{2}\times\frac{32}{7}$

$=\frac{224}{14}$

$=16$

As already defined in Chapter 1, the reciprocal of a number is obtained when 1 is divided by that number. Therefore, the reciprocal of 5 is $\frac{1}{5}$ and the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$, and so on.

Dividing one fraction by another gives the same result as multiplying by the reciprocal.

Worked examples

a $\frac{3}{8} \div \frac{3}{4}$

$= \frac{3}{8} \times \frac{4}{3}$

$= \frac{12}{24}$

$= \frac{1}{2}$

b $5\frac{1}{2} \div 3\frac{2}{3}$

$= \frac{11}{2} \div \frac{11}{3}$

$= \frac{11}{2} \times \frac{3}{11}$

$= \frac{3}{2}$

Exercise 4.8

1 Write the reciprocal of each of the following:

a $\frac{1}{8}$ **b** $\frac{7}{12}$ **c** $\frac{3}{5}$

d $1\frac{1}{2}$ **e** $3\frac{3}{4}$ **f** 6

2 Evaluate the following:

a $\frac{3}{8} \times \frac{4}{9}$ **b** $\frac{2}{3} \times \frac{9}{10}$ **c** $\frac{5}{7} \times \frac{4}{15}$

d $\frac{3}{4}$ of $\frac{8}{9}$ **e** $\frac{5}{6}$ of $\frac{3}{10}$ **f** $\frac{7}{8}$ of $\frac{2}{5}$

3 Evaluate the following:

a $\frac{5}{8} \div \frac{3}{4}$ **b** $\frac{5}{6} \div \frac{1}{3}$ **c** $\frac{4}{5} \div \frac{7}{10}$

d $1\frac{2}{3} \div \frac{2}{5}$ **e** $\frac{3}{7} \div 2\frac{1}{7}$ **f** $1\frac{1}{4} \div 1\frac{7}{8}$

4 Evaluate the following:

a $\frac{3}{4} \times \frac{4}{5}$ **b** $\frac{7}{8} \times \frac{2}{3}$ **c** $\frac{3}{4} \times \frac{4}{7} \times \frac{3}{10}$

d $\frac{4}{5} \div \frac{2}{3} \times \frac{7}{10}$ **e** $\frac{1}{2}$ of $\frac{3}{4}$ **f** $4\frac{1}{2} \div 3\frac{1}{9}$

5 Evaluate the following:

a $\left(\frac{3}{8} \times \frac{4}{5}\right) + \left(\frac{1}{2} \text{ of } \frac{3}{5}\right)$ **b** $\left(1\frac{1}{2} \times 3\frac{3}{4}\right) - \left(2\frac{3}{5} \div 1\frac{1}{2}\right)$

c $\left(\frac{3}{5} \text{ of } \frac{4}{9}\right) + \left(\frac{4}{9} \text{ of } \frac{3}{5}\right)$ **d** $\left(1\frac{1}{3} \times 2\frac{5}{8}\right)^2$

6 Using the correct order of operations, evaluate the following:

a $\frac{1}{4} + \frac{2}{7} \times \frac{3}{4}$ **b** $\frac{1}{2} + \frac{3}{8} \div \frac{1}{4} - 2\frac{1}{2}$

c $\left(\frac{1}{6}\right)^2 - \frac{1}{8} \times \frac{5}{9}$ **d** $\left(5\frac{2}{3} - \frac{14}{3}\right)^2 \div \frac{2}{7} + 3\frac{1}{2}$

Changing a fraction to a decimal

To change a fraction to a decimal, divide the numerator by the denominator.

Worked examples

a Change $\frac{5}{8}$ to a decimal.

$$\begin{array}{r} 0.\ 6\ \ 2\ \ \ 5 \\ 8\overline{)5\,.\,0\ {}^{2}0\ {}^{4}0} \end{array}$$

b Change $2\frac{3}{5}$ to a decimal.

This can be represented as $2 + \frac{3}{5}$.

$$\begin{array}{r} 0.6 \\ 5\overline{)3.0} \end{array}$$

Therefore $2\frac{3}{5} = 2.6$

Exercise 4.9

1 Change the following fractions to decimals:

a $\frac{3}{4}$ **b** $\frac{4}{5}$ **c** $\frac{9}{20}$

d $\frac{17}{50}$ **e** $\frac{1}{3}$ **f** $\frac{3}{8}$

g $\frac{7}{16}$ **h** $\frac{2}{9}$ **i** $\frac{7}{11}$

2 Change the following mixed numbers to decimals:

a $2\frac{3}{4}$ **b** $3\frac{3}{5}$ **c** $4\frac{7}{20}$

d $6\frac{11}{50}$ **e** $5\frac{2}{3}$ **f** $6\frac{7}{8}$

g $5\frac{9}{16}$ **h** $4\frac{2}{9}$ **i** $5\frac{3}{7}$

Changing a decimal to a fraction

Changing a decimal to a fraction is done by knowing the 'value' of each of the digits in any decimal.

Worked examples

a Change 0.45 from a decimal to a fraction.

units	.	tenths	hundredths
0	.	4	5

0.45 is therefore equivalent to 4 tenths and 5 hundredths, which in turn is the same as 45 hundredths.

Therefore $0.45 = \frac{45}{100} = \frac{9}{20}$

b Change 2.325 from a decimal to a fraction.

units	.	tenths	hundredths	thousandths
2	.	3	2	5

Therefore $2.325 = 2\frac{325}{1000} = 2\frac{13}{40}$

Exercise 4.10

1 Change the following decimals to fractions. Give your fraction in its simplest form:

a 0.5	**b** 0.7	**c** 0.6
d 0.75	**e** 0.825	**f** 0.05
g 0.050	**h** 0.402	**i** 0.0002

2 Change the following decimals to mixed numbers in their simplest form:

a 2.4	**b** 6.5	**c** 8.2
d 3.75	**e** 10.55	**f** 9.204
g 15.455	**h** 30.001	**i** 1.0205

Recurring decimals

In Chapter 1, the definition of a rational number was given as any number that can be written as a fraction. These include integers, terminating decimals and recurring decimals. The examples given were:

$$0.2 = \frac{1}{5} \quad 0.3 = \frac{3}{10} \quad 7 = \frac{7}{1} \quad 1.53 = \frac{153}{100} \quad \text{and} \quad 0.\dot{2} = \frac{2}{9}$$

The first four examples are more straightforward to understand in terms of how the number can be expressed as a fraction. The fifth example shows a recurring decimal also written as a fraction. Any recurring decimal can be written as a fraction as any recurring decimal is also a rational number.

Changing a recurring decimal to a fraction

A recurring decimal is one in which the numbers after the decimal point repeat themselves infinitely. Which numbers are repeated is indicated by a dot above them.

$0.\dot{2}$ implies 0.222 222 222 222 222 …

$0.\dot{4}\dot{5}$ implies 0.454 545 454 545 …

$0.\dot{6}\dot{0}\dot{2}\dot{4}$ implies 0.602 460 246 024 …

Note: The last example is usually written in one of two other ways: $0.\dot{6}02\dot{4}$, where the dot only appears above the first and last numbers to be repeated, or as $0.\overline{6024}$, where a horizontal line is drawn above the numbers being repeated.

Entering $\frac{4}{9}$ into a calculator will produce 0.444 444 444…

Therefore $0.\dot{4} = \frac{4}{9}$.

The example below will prove this.

Worked examples

a Convert $0.\dot{4}$ to a fraction.

Let $x = 0.\dot{4}$. i.e. $x = 0.444\,444\,444\,444\ldots$

$10x = 4.\dot{4}$. i.e. $10x = 4.444\,444\,444\,444\ldots$

Subtracting x from $10x$ gives:

$$\begin{aligned} 10x &= 4.444\,444\,444\,444\ldots \\ -\quad x &= 0.444\,444\,444\,444\ldots \\ \hline 9x &= 4 \end{aligned}$$

Rearranging gives $x = \frac{4}{9}$

But $x = 0.\dot{4}$

Therefore $0.\dot{4} = \frac{4}{9}$

b Convert $0.\dot{6}\dot{8}$ to a fraction.

Let $x = 0.\dot{6}\dot{8}$ i.e. $x = 0.686\,868\,686\,868\,686\ldots$

$100x = 68.\dot{6}\dot{8}$ i.e. $100x = 68.686\,868\,686\,868\,686\ldots$

Subtracting x from $100x$ gives:

$$\begin{aligned} 100x &= 68.686\,868\,686\,868\,686\ldots \\ -\quad x &= 0.686\,868\,686\,868\,686\ldots \\ \hline 99x &= 68 \end{aligned}$$

Rearranging gives $x = \frac{68}{99}$

But $x = 0.\dot{6}\dot{8}$

Therefore $0.\dot{6}\dot{8} = \frac{68}{99}$

c Convert $0.0\dot{3}\dot{1}$ to a fraction.

Let $x = 0.0\dot{3}\dot{1}$ i.e. $x = 0.031\,313\,131\,313\,131\ldots$

$100x = 3.\dot{1}\dot{3}$ i.e. $100x = 3.131\,313\,131\,313\,131\ldots$

Subtracting x from $100x$ gives:

$$\begin{aligned} 100x &= 3.131\,313\,131\,313\,131\ldots \\ -\quad x &= 0.031\,313\,131\,313\,131\ldots \\ \hline 99x &= 3.1 \end{aligned}$$

Multiplying both sides of the equation by 10 eliminates the decimal to give:

$990x = 31$

Rearranging gives $x = \frac{31}{990}$

But $x = 0.0\dot{3}\dot{1}$

Therefore $0.0\dot{3}\dot{1} = \frac{31}{990}$

The method is therefore to let the recurring decimal equal x and then to multiply this by a multiple of 10 so that when one is subtracted from the other, either an integer (whole number) or terminating decimal (a decimal that has an end point) is left.

d Convert $2.0\overline{406}$ to a fraction.

Let $x = 2.0\overline{406}$ i.e. $x = 2.040\,640\,640\,640\ldots$

$1000x = 2040.\overline{640}$ i.e. $1000x = 2040.640\,640\,640\,640\ldots$

Subtracting x from $1000x$ gives:

$$\begin{aligned} 1000x &= 2040.640\,640\,640\,640\ldots \\ -\quad x &= 2.040\,640\,640\,640\ldots \\ \hline 999x &= 2038.6 \end{aligned}$$

Multiplying both sides of the equation by 10 eliminates the decimal to give:

$9990x = 20\,386$

Rearranging gives $x = \frac{20\,386}{9990} = 2\frac{406}{9990}$, which simplifies further to $2\frac{203}{4995}$.

But $x = 2.0\overline{406}$.

Therefore $2.0\overline{406} = 2\frac{203}{4995}$

Exercise 4.11

1 Convert each of the following recurring decimals to fractions in their simplest form:

a $0.\dot{3}$ **b** $0.\dot{7}$ **c** $0.\dot{4}\dot{2}$ **d** $0.\dot{6}\dot{5}$

2 Convert each of the following recurring decimals to fractions in their simplest form:

a $0.0\dot{5}$ **b** $0.0\dot{6}\dot{2}$ **c** $1.0\dot{2}$ **d** $4.00\dot{3}\dot{8}$

3 Work out the sum $0.1\dot{5} + 0.0\dot{4}$ by converting each decimal to a fraction first. Give your answer as a fraction in its simplest form.

4 Evaluate $0.\dot{2}\dot{7} - 0.\dot{1}0\dot{6}$ by converting each decimal to a fraction first. Give your answer as a fraction in its simplest form.

Student assessment 1

1 Evaluate the following:

a $\frac{1}{3}$ of 63 **b** $\frac{3}{8}$ of 72 **c** $\frac{2}{5}$ of 55 **d** $\frac{3}{13}$ of 169

2 Write the following as percentages:

a $\frac{3}{5}$ **b** $\frac{49}{100}$ **c** $\frac{1}{4}$ **d** $\frac{9}{10}$

e $1\frac{1}{2}$ **f** $3\frac{27}{100}$ **g** $\frac{5}{100}$ **h** $\frac{7}{20}$

i 0.77 **j** 0.03 **k** 2.9 **l** 4

3 Evaluate the following:

a $6 \times 4 - 3 \times 8$ **b** $15 \div 3 + 2 \times 7$

4 Work out 368×49.

5 Work out $7835 \div 23$ giving your answer to 1 d.p.

6 Evaluate the following:

a $2\frac{1}{2} - \frac{4}{5}$ **b** $3\frac{1}{2} \times \frac{4}{7}$

7 Change the following fractions to decimals:

a $\frac{7}{8}$ **b** $1\frac{2}{5}$ **c** $\frac{8}{9}$ **d** $3\frac{2}{7}$

8 Change the following decimals to fractions. Give each fraction in its simplest form.

a 6.5 **b** 0.04 **c** 3.65 **d** 3.008

9 Convert the following decimals to fractions, giving your answer in its simplest form:

a $0.\dot{0}\dot{7}$ **b** $0.000\dot{9}$ **c** $3.0\dot{2}\dot{0}$

10 Work out $1.02\dot{5} - 0.80\dot{5}$ by first converting each decimal to a fraction. Give your answer in its simplest form.

5 Further percentages

You should already be familiar with the percentage equivalents of simple fractions and decimals as outlined in the table below.

Fraction	Decimal	Percentage
$\frac{1}{2}$	0.5	50%
$\frac{1}{4}$	0.25	25%
$\frac{3}{4}$	0.75	75%
$\frac{1}{8}$	0.125	12.5%
$\frac{3}{8}$	0.375	37.5%
$\frac{5}{8}$	0.625	62.5%
$\frac{7}{8}$	0.875	87.5%
$\frac{1}{10}$	0.1	10%
$\frac{2}{10}$ or $\frac{1}{5}$	0.2	20%
$\frac{3}{10}$	0.3	30%
$\frac{4}{10}$ or $\frac{2}{5}$	0.4	40%
$\frac{6}{10}$ or $\frac{3}{5}$	0.6	60%
$\frac{7}{10}$	0.7	70%
$\frac{8}{10}$ or $\frac{4}{10}$	0.8	80%
$\frac{9}{10}$	0.9	90%

Simple percentages

Worked examples

a Of 100 sheep in a field, 88 are ewes.

i What percentage of the sheep are ewes?

88 out of 100 are ewes
= 88%

ii What percentage are not ewes?

12 out of 100
= 12%

b Convert the following percentages into fractions and decimals:

i 27%

$\frac{27}{100} = 0.27$

ii 5%

$\frac{5}{100} = \frac{1}{20} = 0.05$

c Convert $\frac{3}{16}$ to a percentage:

This example is more complicated as 16 is not a factor of 100.

Convert $\frac{3}{16}$ to a decimal first.

$3 \div 16 = 0.1875$

Convert the decimal to a percentage by multiplying by 100.

$0.1875 \times 100 = 18.75$

Therefore $\frac{3}{16} = 18.75\%$.

Exercise 5.1

1 There are 200 birds in a flock. 120 of them are female. What percentage of the flock are:

a female? **b** male?

2 Write these fractions as percentages:

a $\frac{7}{8}$ **b** $\frac{11}{15}$ **c** $\frac{7}{24}$ **d** $\frac{1}{7}$

3 Convert the following percentages to decimals:

a 39% **b** 47% **c** 83%
d 7% **e** 2% **f** 20%

4 Convert the following decimals to percentages:

a 0.31 **b** 0.67 **c** 0.09
d 0.05 **e** 0.2 **f** 0.75

Calculating a percentage of a quantity

Worked examples

a Find 25% of 300 m.

25% can be written as 0.25.

$0.25 \times 300\,\text{m} = 75\,\text{m}$.

b Find 35% of 280 m.

35% can be written as 0.35.

$0.35 \times 280\,\text{m} = 98\,\text{m}$.

Exercise 5.2

1 Write the percentage equivalent of the following fractions:

a $\frac{1}{4}$ b $\frac{2}{3}$ c $\frac{5}{8}$

d $1\frac{4}{5}$ e $4\frac{9}{10}$ f $3\frac{7}{8}$

2 Write the decimal equivalent of the following:

a $\frac{3}{4}$ b 80% c $\frac{1}{5}$

d 7% e $1\frac{7}{8}$ f $\frac{1}{6}$

3 Evaluate the following:

a 25% of 80 b 80% of 125 c 62.5% of 80

d 30% of 120 e 90% of 5 f 25% of 30

4 Evaluate the following:

a 17% of 50 b 50% of 17 c 65% of 80

d 80% of 65 e 7% of 250 f 250% of 7

5 In a class of 30 students, 20% travel to school by car, 10% walk and 70% travel by bus. Calculate the number of students who travel to school by:

a car b walking c bus

6 A survey conducted among 120 school children looked at which type of fruit they preferred. 55% said they preferred apple, 20% said they preferred mango, 15% preferred pineapple and 10% grapes. Calculate the number of children in each category.

7 A survey was carried out in a school to see what nationality its students were. Of the 220 students in the school, 65% were Australian, 20% were Pakistani, 5% were Greek and 10% belonged to other nationalities. Calculate the number of students of each nationality and how many were of other nationalities.

8 A shopkeeper keeps a record of the number of items she sells in one day. Of the 150 items she sold, 46% were newspapers, 24% were pens, 12% were books while the remaining 18% were other items. Calculate the number of each item sold and how many were other items.

Expressing one quantity as a percentage of another

To express one quantity as a percentage of another, first write the first quantity as a fraction of the second and then multiply by 100.

Worked example

In an examination, Yuji obtains 69 marks out of 75. Express this result as a percentage.

$\frac{69}{75} \times 100\% = 92\%$

Exercise 5.3

1 Express the first quantity as a percentage of the second.

a 24 out of 50
b 46 out of 125
c 7 out of 20
d 45 out of 90
e 9 out of 20
f 16 out of 40
g 13 out of 39
h 20 out of 35

2 A hockey team plays 42 matches. It wins 21, draws 14 and loses the rest. Express each of these results as a percentage of the total number of games played.

3 Four candidates stood in an election:
A received 24 500 votes
B received 18 200 votes
C received 16 300 votes
D received 12 000 votes

Express each of these as a percentage of the total votes cast.

4 A car manufacturer produces 155 000 cars a year. The cars are available for sale in six different colours. In one year, the numbers sold of each colour were:
Red 55 000
Blue 48 000
White 27 500
Silver 10 200
Green 9300
Black 5000

Express each of these as a percentage of the total number of cars produced. Give your answers to 1 d.p.

Percentage increases and decreases

Worked examples

a A shop assistant has a salary of \$16 000 per year. If his salary increases by 8%, calculate:

i the amount extra he receives each year,

ii his new annual salary.

i Increase = 8% of \$16 000
$= 0.08 \times \$16\,000 = \$1280$

ii New salary = old salary + increase
$= \$16\,000 + \$1280$ per year
$= \$17\,280$ per year

b A garage increases the price of a truck by 12%. If the original price was \$14 500, calculate its new price.

The original price represents 100%, therefore the increased price can be represented as 112%.

New price = 112% of \$14 500
$= 1.12 \times \$14\,500$
$= \$16\,240$

c A shop is having a sale. It sells a set of tools costing \$130 at a 15% discount. Calculate the sale price of the tools.

The old price represents 100%, therefore the new price can be represented as $(100 - 15)\% = 85\%$.

85% of $\$130 = 0.85 \times \$130$
$= \$110.50$

Exercise 5.4

1 Increase the following by the given percentage:

a 150 by 25% **b** 230 by 40% **c** 7000 by 2%
d 70 by 250% **e** 80 by 12.5% **f** 75 by 62%

2 Decrease the following by the given percentage:

a 120 by 25% **b** 40 by 5% **c** 90 by 90%
d 1000 by 10% **e** 80 by 37.5% **f** 75 by 42%

3 In the following questions the first number is increased to become the second number. Calculate the percentage increase in each case.

a $50 \rightarrow 60$ **b** $75 \rightarrow 135$ **c** $40 \rightarrow 84$
d $30 \rightarrow 31.5$ **e** $18 \rightarrow 33.3$ **f** $4 \rightarrow 13$

4 In the following questions the first number is decreased to become the second number. Calculate the percentage decrease in each case.

a $50 \rightarrow 25$ **b** $80 \rightarrow 56$ **c** $150 \rightarrow 142.5$
d $3 \rightarrow 0$ **e** $550 \rightarrow 352$ **f** $20 \rightarrow 19$

Exercise 5.4 (cont)

5 A farmer increases the yield on her farm by 15%. If her previous yield was 6500 tonnes, what is her present yield?

6 The cost of a computer in a store is reduced by 12.5% in a sale. If the computer was priced at $7800, what is its price in the sale?

7 A winter coat is priced at $100. In the sale, its price is reduced by 25%.
 a Calculate the sale price of the coat.
 b After the sale, its price is increased by 25%. Calculate the coat's price after the sale.

8 A farmer takes 250 chickens to be sold at a market. In the first hour, he sells 8% of his chickens. In the second hour, he sells 10% of those that were left.
 a How many chickens has he sold in total?
 b What percentage of the original number did he manage to sell in the two hours?

9 The number of fish on a fish farm increases by approximately 10% each month. If there were originally 350 fish, calculate to the nearest 100 how many fish there would be after 12 months.

Reverse percentages

Worked examples

a In a test, Ahmed answered 92% of the questions correctly. If he answered 23 questions correctly, how many did he get wrong?

92% of the marks is equivalent to 23 questions.

1% of the marks therefore is equivalent to $\frac{23}{92}$ questions.

So 100% is equivalent to $\frac{23}{92} \times 100 = 25$ questions.

Ahmed got 2 questions wrong.

b A boat is sold for $15 360. This represents a profit of 28% to the seller. What did the boat originally cost the seller?

The selling price is 128% of the original cost to the seller.

128% of the original cost is $15 360.

1% of the original cost is $\frac{\$15\,360}{128}$.

100% of the original cost is $\frac{\$15\,360}{128} \times 100$, i.e. $12 000.

Exercise 5.5

1 Calculate the value of X in each of the following:

a 40% of X is 240
b 24% of X is 84
c 85% of X is 765
d 4% of X is 10
e 15% of X is 18.75
f 7% of X is 0.105

2 Calculate the value of Y in each of the following:

a 125% of Y is 70
b 140% of Y is 91
c 210% of Y is 189
d 340% of Y is 68
e 150% of Y is 0.375
f 144% of Y is –54.72

3 In a geography textbook, 35% of the pages are coloured. If there are 98 coloured pages, how many pages are there in the whole book?

4 A town has 3500 families who own a car. If this represents 28% of the families in the town, how many families are there in total?

5 In a test, Isabel scored 88%. If she got three questions incorrect, how many did she get correct?

6 Water expands when it freezes. Ice is less dense than water so it floats. If the increase in volume is 4%, what volume of water will make an iceberg of 12 700 000 m^3? Give your answer to 3 s.f.

Student assessment 1

1 Find 40% of 1600 m.

2 A shop increases the price of a television by 8%. If the present price is \$320, what is the new price?

3 A car loses 55% of its value after four years. If it cost \$22 500 when new, what is its value after the four years?

4 Express the first quantity as a percentage of the second.

a 40 cm, 2 m
b 25 mins, 1 hour
c 450 g, 2 kg
d 3 m, 3.5 m
e 70 kg, 1 tonne
f 75 cl, 2.5 l

5 A house is bought for \$75 000 and then resold for \$87 000. Calculate the percentage profit.

6 A pair of shoes is priced at \$45. During a sale the price is reduced by 20%.

a Calculate the sale price of the shoes.
b What is the percentage increase in the price if after the sale it is restored to \$45?

Student assessment 2

1 Find 30% of 2500 m.

2 In a sale a shop reduces its prices by 12.5%. What is the sale price of a desk previously costing $600?

3 In the last six years the value of a house has increased by 35%. If it cost $72 000 six years ago, what is its value now?

4 Express the first quantity as a percentage of the second.
   a 35 mins, 2 hours
   b 650 g, 3 kg
   c 5 m, 4 m
   d 15 s, 3 mins
   e 600 kg, 3 tonnes
   f 35 cl, 3.5 l

5 Shares in a company are bought for $600. After a year, the same shares are sold for $550. Calculate the percentage loss.

6 In a sale, the price of a jacket originally costing $1700 is reduced by $400. Any item not sold by the last day of the sale is reduced by a further 50%. If the jacket is sold on the last day of the sale:
   a calculate the price it is finally sold for,
   b calculate the overall percentage reduction in price.

Student assessment 3

1 Calculate the original price for each of the following:

Selling price	Profit
$224	12%
$62.50	150%
$660.24	26%
$38.50	285%

2 Calculate the original price for each of the following:

Selling price	Loss
$392.70	15%
$2480	38%
$3937.50	12.5%
$4675	15%

3 In an examination, Aliya obtained 87.5% by gaining 105 marks. How many marks did she lose?

4 At the end of a year, a factory has produced 38 500 televisions. If this represents a 10% increase in productivity on last year, calculate the number of televisions that were made last year.

5 A computer manufacturer is expected to have produced 24 000 units by the end of this year. If this represents a 4% decrease on last year's output, calculate the number of units produced last year.

6 A company increased its productivity by 10% each year for the last two years. If it produced 56 265 units this year, how many units did it produce two years ago?

6 Ratio and proportion

Direct proportion

Workers in a pottery factory are paid according to how many plates they produce. The wage paid to them is said to be in **direct proportion** to the number of plates made. As the number of plates made increases so does their wage. Other workers are paid for the number of hours worked. For them the wage paid is in **direct proportion** to the number of hours worked. There are two main methods for solving problems involving direct proportion: the **ratio method** and the **unitary method**.

Worked example

A bottling machine fills 500 bottles in 15 minutes. How many bottles will it fill in $1\frac{1}{2}$ hours?

Note: The time units must be the same, so for either method the $1\frac{1}{2}$ hours must be changed to 90 minutes.

The ratio method

Let x be the number of bottles filled. Then:

$$\frac{x}{90} = \frac{500}{15}$$

so $x = \frac{500 \times 90}{15} = 3000$

3000 bottles are filled in $1\frac{1}{2}$ hours.

The unitary method

In 15 minutes, 500 bottles are filled.

Therefore in 1 minute, $\frac{500}{15}$ bottles are filled.

So in 90 minutes, $90 \times \frac{500}{15}$ bottles are filled.

In $1\frac{1}{2}$ hours, 3000 bottles are filled.

Exercise 6.1

Use either the ratio method or the unitary method to solve the problems below.

1 A machine prints four books in 10 minutes. How many will it print in 2 hours?

2 A gardener plants five apple trees in 25 minutes. If he continues to work at a constant rate, how long will it take him to plant 200 trees?

Exercise 6.1 (cont)

3 A television uses 3 units of electricity in 2 hours. How many units will it use in 7 hours? Give your answer to the nearest unit.

4 A bricklayer lays 1500 bricks in an 8-hour day. Assuming she continues to work at the same rate, calculate:
   a how many bricks she would expect to lay in a five-day week,
   b how long to the nearest hour it would take her to lay 10000 bricks.

5 A machine used to paint white lines on a road uses 250 litres of paint for each 8 km of road marked. Calculate:
   a how many litres of paint would be needed for 200 km of road,
   b what length of road could be marked with 4000 litres of paint.

6 An aircraft is cruising at 720 km/h and covers 1000 km. How far would it travel in the same period of time if the speed increased to 800 km/h?

7 A production line travelling at 2 m/s labels 150 tins. In the same period of time how many will it label at:
   a 6 m/s   b 1 m/s   c 1.6 m/s?

8 A car travels at an average speed of 80 km/h for 6 hours.
   a How far will it travel in the 6 hours?
   b What average speed will it need to travel at in order to cover the same distance in 5 hours?

If the information is given in the form of a ratio, the method of solution is the same.

Worked example

Tin and copper are mixed in the ratio 8 : 3. How much tin is needed to mix with 36 g of copper?

The ratio method

Let x grams be the mass of tin needed.

$$\frac{x}{36} = \frac{8}{3}$$

Therefore $x = \frac{8 \times 36}{3}$

$$= 96$$

So 96 g of tin is needed.

The unitary method

3 g of copper mixes with 8 g of tin.

1 g of copper mixes with $\frac{8}{3}$ g of tin.

So 36 g of copper mixes with $36 \times \frac{8}{3}$ g of tin.

Therefore 36 g of copper mixes with 96 g of tin.

Exercise 6.2

1 Sand and gravel are mixed in the ratio 5 : 3 to form ballast.
  a How much gravel is mixed with 750 kg of sand?
  b How much sand is mixed with 750 kg of gravel?

2 A recipe uses 150 g butter, 500 g flour, 50 g sugar and 100 g currants to make 18 small cakes.
  a Write the ratio of the amount of butter : flour : sugar : currants in its simplest form.
  b How much of each ingredient will be needed to make 72 cakes?
  c How many whole cakes could be made with 1 kg of butter?

3 A paint mix uses red and white paint in a ratio of 1 : 12.
  a How much white paint will be needed to mix with 1.4 litres of red paint?
  b If a total of 15.5 litres of paint is mixed, calculate the amount of white paint and the amount of red paint used. Give your answers to the nearest 0.1 litre.

4 Rebecca sells sacks of mixed bulbs to local people. The bulbs develop into two different colours of tulips, red and yellow. The colours are packaged in a ratio of 8 : 5 respectively.
  a If a sack contains 200 red bulbs, calculate the number of yellow bulbs.
  b If a sack contains 351 bulbs in total, how many of each colour would you expect to find?
  c One sack is packaged with a bulb mixture in the ratio 7 : 5 by mistake. If the sack contains 624 bulbs, how many more yellow bulbs would you expect to have compared with a normal sack of 624 bulbs?

5 A pure fruit juice is made by mixing the juices of oranges and mangos in the ratio of 9 : 2.
  a If 189 litres of orange juice are used, calculate the number of litres of mango juice needed.
  b If 605 litres of the juice are made, calculate the number of litres of orange juice and mango juice used.

Divide a quantity in a given ratio

Worked examples

a Divide 20 m in the ratio 3 : 2.

The ratio method

3 : 2 gives 5 parts.

$\frac{3}{5} \times 20\,\text{m} = 12\,\text{m}$

$\frac{2}{5} \times 20\,\text{m} = 8\,\text{m}$

20 m divided in the ratio 3 : 2 is 12 m : 8 m.

The unitary method

3 : 2 gives 5 parts.
5 parts is equivalent to 20 m.

1 part is equivalent to $\frac{20}{5}$ m.

Therefore 3 parts is $3 \times \frac{20}{5}$ m; that is 12 m.

Therefore 2 parts is $2 \times \frac{20}{5}$ m; that is 8 m.

b A factory produces cars in red, blue, white and green in the ratio 7 : 5 : 3 : 1. Out of a production of 48 000 cars how many are white?

7 + 5 + 3 + 1 gives a total of 16 parts.

Therefore, the total number of white cars $= \frac{3}{16} \times 48\,000 = 9000$.

Exercise 6.3

1 Divide 150 in the ratio 2 : 3.

2 Divide 72 in the ratio 2 : 3 : 4.

3 Divide 5 kg in the ratio 13 : 7.

4 Divide 45 minutes in the ratio 2 : 3.

5 Divide 1 hour in the ratio 1 : 5.

6 $\frac{7}{8}$ of a can of drink is water, the rest is syrup. What is the ratio of water to syrup?

7 $\frac{5}{9}$ of a litre carton of orange is pure orange juice, the rest is water. How many millilitres of each are in the carton?

8 55% of students in a school are boys.
- **a** What is the ratio of boys to girls?
- **b** How many boys and how many girls are there if the school has 800 students?

9 A piece of wood is cut in the ratio 2 : 3.
- **a** What fraction of the length is the longer piece?
- **b** If the piece of wood is 80 cm long, how long is the shorter piece?

10 A gas pipe is 7 km long. A valve is positioned in such a way that it divides the length of the pipe in the ratio 4 : 3. Calculate the distance of the valve from each end of the pipe.

11 The size of the angles of a quadrilateral are in the ratio 1 : 2 : 3 : 3. Calculate the size of each angle.

12 The angles of a triangle are in the ratio 3 : 5 : 4. Calculate the size of each angle.

13 A millionaire leaves 1.4 million dollars in her will to be shared between her three children in the ratio of their ages. If they are 24, 28 and 32 years old, calculate to the nearest dollar the amount they will each receive.

14 A small company makes a **profit** of $8000. This is divided between the directors in the ratio of their initial investments. If Malik put $20 000 into the firm, Zahra $35 000 and Ahmet $25 000, calculate the amount of the profit they will each receive.

Inverse proportion

Sometimes an increase in one quantity causes a decrease in another quantity. For example, if fruit is to be picked by hand, the more people there are picking the fruit, the less time it will take.

Worked examples

a If 8 people can pick the apples from the trees in 6 days, how long will it take 12 people?

8 people take 6 days.

1 person will take 6×8 days.

Therefore 12 people will take $\frac{6 \times 8}{12}$ days, i.e. 4 days.

b A cyclist averages a speed of 27 km/h for 4 hours. At what average speed would she need to cycle to cover the same distance in 3 hours?

Completing it in 1 hour would require cycling at 27×4 km/h.

Completing it in 3 hours requires cycling at $\frac{27 \times 4}{3}$ km/h; that is 36 km/h.

Exercise 6.4

1 A teacher shares sweets among 8 students so that they get 6 each. How many sweets would they each have had if there had been 12 students?

2 The table below represents the relationship between the speed and the time taken for a train to travel between two stations.

Speed (km/h)	60			120	90	50	10
Time (h)	2	3	4				

Copy and complete the table.

3 Six people can dig a trench in 8 hours.
- **a** How long would it take:
  - **i** 4 people **ii** 12 people **iii** 1 person?
- **b** How many people would it take to dig the trench in:
  - **i** 3 hours **ii** 16 hours **iii** 1 hour?

4 Chairs in a hall are arranged in 35 rows of 18.
- **a** How many rows would there be with 21 chairs to a row?
- **b** How many chairs would there be in each row if there were 15 rows?

5 A train travelling at 100 km/h takes 4 hours for a journey. How long would it take a train travelling at 60 km/h?

6 A worker in a sugar factory packs 24 cardboard boxes with 15 bags of sugar in each. If he had boxes which held 18 bags of sugar each, how many fewer boxes would be needed?

7 A swimming pool is filled in 30 hours by two identical pumps. How much quicker would it be filled if five similar pumps were used instead?

Compound measures

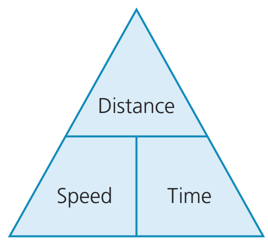
A **compound measure** is one made up of two or more other measures. The most common ones, which we will consider here, are speed, density and population density.

Speed is a compound measure as it is measured using distance and time.

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

Units of speed include metres per second (m/s) or kilometres per hour (km/h).

The relationship between speed, distance and time is often presented as shown in the diagram below:



i.e. $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$

$\text{Distance} = \text{Speed} \times \text{Time}$

$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$

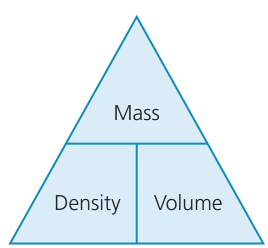
Similarly, $\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$

Density, which is a measure of the mass of a substance per unit of its volume, is calculated using the following formula:

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

Units of density include kilograms per cubic metre (kg/m^3) or grams per millilitre (g/ml).

The relationship between density, mass and volume, like speed, can also be presented in a helpful diagram as shown:



i.e. $\text{Density} = \frac{\text{Mass}}{\text{Volume}}$

$\text{Mass} = \text{Density} \times \text{Volume}$

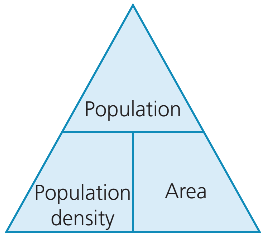
$\text{Volume} = \frac{\text{Mass}}{\text{Density}}$

Population density is also a compound measure as it is a measure of a population per unit of area.

$$\text{Population density} = \frac{\text{Population}}{\text{Area}}$$

An example of its units include the number of people per square kilometre (people/km^2).

Once again this can be represented in a triangular diagram as shown:



i.e. Population density $= \frac{\text{Population}}{\text{Area}}$

Population = Population density × Area

Area $= \frac{\text{Population}}{\text{Population density}}$

Worked examples

a A train travels a total distance of 140 km in $1\frac{1}{2}$ hours.

i Calculate the average speed of the train during the journey.

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

$$= \frac{140}{1\frac{1}{2}}$$

$$= 93\frac{1}{3}\text{ km/h}$$

ii During the journey, the train spent 15 minutes stopped at stations. Calculate the average speed of the train while it was moving.

Notice that the original time was given in hours, while the time spent stopped at stations is given in minutes. To proceed with the calculation, the units have to be consistent, i.e. either both in hours or both in minutes.

The time spent travelling is $1\frac{1}{2} - \frac{1}{4} = 1\frac{1}{4}$ hours.
Therefore:

$$\text{Average speed} = \frac{140}{1\frac{1}{4}}$$

$$= 112\text{ km/h}$$

iii If the average speed was 120 km/h, calculate how long the journey took.

$$\text{Total time} = \frac{\text{Total distance}}{\text{Average speed}}$$

$$= \frac{140}{120} = 1.1\dot{6}\text{ hours}$$

Note: It may be necessary to convert a decimal answer to hours and minutes.
To convert a decimal time to minutes, multiply by 60.

$0.1\dot{6} \times 60 = 10$

Therefore total time is 1 hr 10 mins or 70 mins.

b A village has a population of 540. Its total area is 8 km^2.

i Calculate the population density of the village.

$$\text{Population density} = \frac{\text{Population}}{\text{Area}}$$

$$= \frac{540}{8} = 67.5\text{ people/km}^2.$$

ii A building company wants to build some new houses in the existing area of the village. It is decided that the maximum desirable population density of the village should not exceed 110 people/km^2. Calculate the extra number of people the village can have.

Population = Population density × Area
= 110 × 8
= 880 people

Therefore the maximum number of extra people who will need housing is 880 – 540 = 340.

Exercise 6.5

1 Aluminium has a density of 2900 kg/m^3. A construction company needs four cubic metres of aluminium.
Calculate the mass of the aluminium needed.

2 A marathon race is 42 195 m in length. The world record in 2022 was 2 hrs, 1 min and 9 seconds held by Eliud Kipchoge of Kenya.
   a How many seconds in total did Eliud take to complete the race?
   b Calculate his average speed in m/s for the race, giving your answer to 2 decimal places.
   c What average speed would the runner need to maintain to complete the marathon in under two hours?

3 The approximate densities of four metals in g/cm^3 are given below:

Aluminium	2.9 g/cm^3
Brass	8.8 g/cm^3
Copper	9.3 g/cm^3
Steel	8.2 g/cm^3

A cube of an unknown metal has side lengths of 5 cm. The mass of the cube is 1.1 kg.
   a By calculating the cube's density, determine which metal the cube is likely to be made from.

Another cube made of steel has a mass of 4.0 kg.
   b Calculate the length of each of the sides of the steel cube, giving your answer to 1 d.p.

4 Singapore is the country with the highest population density in the world. Its population is 5 954 000 and it has a total area of 719 km^2.
   a Calculate Singapore's population density.

China is the country with the largest population.
   b Explain why China has not got the world's highest population density.
   c Find the area and population of your own country. Calculate your country's population density.

5 Kwabena has a rectangular field measuring 600 m × 800 m. He uses the field for grazing his sheep.
   a Calculate the area of the field in km^2.
   b 40 sheep graze in the field. Calculate the population density of sheep in the field, giving your answer in sheep/km^2.

Guidelines for keeping sheep state that the maximum population density for grazing sheep is 180/km^2.
   c Calculate the number of sheep Kwabena is allowed to graze in his field.

6 The formula linking pressure (P N/m^2), force (F N) and surface area (A m^2) is given as $P = \frac{F}{A}$. A square-based box exerts a force of 160 N on a floor. If the pressure on the floor is 1000 N/m^2, calculate the length, in cm, of each side of the base of the box.

Student assessment 1

1 A boat travels at an average speed of 15 km/h for 1 hour.
   a Calculate the distance it travels in one hour.
   b At what average speed will the boat have to travel to cover the same distance in $2\frac{1}{2}$ hours?

2 A ruler 30 cm long is broken into two parts in the ratio 8 : 7. How long are the two parts?

3 A recipe needs 400 g of flour to make 8 cakes. How much flour would be needed in order to make 24 cakes?

4 To make 6 jam tarts, 120 g of jam is needed. How much jam is needed to make 10 tarts?

5 The scale of a map is 1 : 25 000.
   a Two villages are 8 cm apart on the map. How far apart are they in real life? Give your answer in kilometres.
   b The distance from a village to the edge of a lake is 12 km in real life. How far apart would they be on the map? Give your answer in centimetres.

6 A motorbike uses petrol and oil mixed in the ratio 13 : 2.
   a How much of each is there in 30 litres of mixture?
   b How much petrol would be mixed with 500 ml of oil?

7 a A model car is a $\frac{1}{40}$ scale model. Express this as a ratio.
   b If the length of the real car is 5.5 m, what is the length of the model car?

8 An aunt gives a brother and sister $2000 to be divided in the ratio of their ages. If the girl is 13 years old and the boy 12 years old, how much will each get?

9 The angles of a triangle are in the ratio 2 : 5 : 8. Find the size of each of the angles.

10 A photocopying machine is capable of making 50 copies each minute.
   a If four identical copiers are used simultaneously, how long would it take to make a total of 50 copies?
   b How many copiers would be needed to make 6000 copies in 15 minutes?

11 It takes 16 hours for three bricklayers to build a wall. Calculate how long it would take for eight bricklayers to build a similar wall.

Student assessment 2

1 A cyclist travels at an average speed of 20 km/h for 1.5 hours.
  a Calculate the distance the cyclist travels in 1.5 hours.
  b What average speed will the cyclist need to travel in order to cover the same distance in 1 hour?

2 A piece of wood is cut in the ratio 3 : 7.
  a What fraction of the whole is the longer piece?
  b If the wood is 1.5 m long, how long is the shorter piece?

3 A recipe for two people requires $\frac{1}{4}$ kg of rice to 150 g of meat.
  a How much meat would be needed for five people?
  b How much rice would there be in 1 kg of the final dish?

4 The scale of a map is 1 : 10 000.
  a Two rivers are 4.5 cm apart on the map. How far apart are they in real life? Give your answer in metres.
  b Two towns are 8 km apart in real life. How far apart are they on the map? Give your answer in centimetres.

5 **a** A model train is a $\frac{1}{25}$ scale model. Express this as a ratio.
  b If the length of the model engine is 7 cm, what is the true length of the engine?

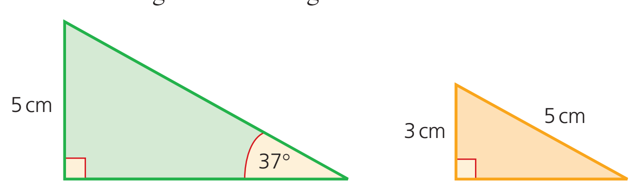
6 Divide 3 tonnes in the ratio 2 : 5 : 13.

7 The ratio of the angles of a quadrilateral is 2 : 3 : 3 : 4. Calculate the size of each of the angles.

8 The sum of the interior angles of a pentagon is 540°. The ratio of the interior angles is 2 : 3 : 4 : 4 : 5. Calculate the size of the largest angle.

9 A large swimming pool takes 36 hours to fill using three identical pumps.
  a How long would it take to fill using eight identical pumps?
  b If the pool needs to be filled in 9 hours, how many pumps will be needed?

10 The first triangle is an enlargement of the second. Calculate the size of the missing sides and angles.



11 A tap issuing water at a rate of 1.2 litres per minute fills a container in 4 minutes.
  a How long would it take to fill the same container if the rate was decreased to 1 litre per minute? Give your answer in minutes and seconds.
  b If the container is to be filled in 3 minutes, calculate the rate at which the water should flow.

7 Indices, standard form and surds

The index refers to the power to which a number is raised. In the example 5^3, the number 5 is raised to the power 3. The 3 is known as the **index**. Indices is the plural of index.

Worked examples

a $5^3 = 5 \times 5 \times 5$
$= 125$

b $7^4 = 7 \times 7 \times 7 \times 7$
$= 2401$

c $3^1 = 3$

Laws of indices

When working with numbers involving indices there are three basic laws which can be applied. These are:

1 $a^m \times a^n = a^{m+n}$
e.g. $2^3 \times 2^4 = 2^{3+4} = 2^7$

2 $a^m \div a^n$ or $\frac{a^m}{a^n} = a^{m-n}$
e.g. $5^6 \div 5^4 = 5^{6-4} = 5^2$

3 $(a^m)^n = a^{mn}$
e.g. $(4^5)^3 = 4^{5\times 3} = 4^{15}$

Positive indices

Worked examples

a Simplify $4^3 \times 4^2$.
$4^3 \times 4^2 = 4^{(3+2)}$
$= 4^5$

b Simplify $2^5 \div 2^3$.
$2^5 \div 2^3 = 2^{(5-3)}$
$= 2^2$

c Evaluate $3^3 \times 3^4$.
$3^3 \times 3^4 = 3^{(3+4)}$
$= 3^7$
$= 2187$

d Evaluate $(4^2)^3$.
$(4^2)^3 = 4^{(2\times 3)}$
$= 4^6$
$= 4096$

Exercise 7.1

1 Using indices, simplify the following expressions:

a $3 \times 3 \times 3$
b $2 \times 2 \times 2 \times 2 \times 2$
c 4×4
d $6 \times 6 \times 6 \times 6$
e $8 \times 8 \times 8 \times 8 \times 8 \times 8$
f 5

2 Simplify the following using indices:

a $2 \times 2 \times 2 \times 3 \times 3$
b $4 \times 4 \times 4 \times 4 \times 4 \times 5 \times 5$

Exercise 7.1 (cont)

c $3 \times 3 \times 4 \times 4 \times 4 \times 5 \times 5$
d $2 \times 7 \times 7 \times 7 \times 7$
e $1 \times 1 \times 6 \times 6$
f $3 \times 3 \times 3 \times 4 \times 4 \times 6 \times 6 \times 6 \times 6 \times 6$

3 Write out the following in full:
a 4^2 b 5^7 c 3^5
d $4^3 \times 6^3$ e $7^2 \times 2^7$ f $3^2 \times 4^3 \times 2^4$

4 Work out the value of the following:
a 2^5 b 3^4 c 8^2 d 6^3
e 10^6 f 4^4 g $2^3 \times 3^2$ h $10^3 \times 5^3$

Exercise 7.2

1 Simplify the following using indices:
a $3^2 \times 3^4$ b $8^5 \times 8^2$
c $5^2 \times 5^4 \times 5^3$ d $4^3 \times 4^5 \times 4^2$
e $2^1 \times 2^3$ f $6^2 \times 3^2 \times 3^3 \times 6^4$
g $4^5 \times 4^3 \times 5^5 \times 5^4 \times 6^2$ h $2^4 \times 5^7 \times 5^3 \times 6^2 \times 6^6$

2 Simplify the following:
a $4^6 \div 4^2$ b $5^7 \div 5^4$
c $2^5 \div 2^4$ d $6^5 \div 6^2$
e $\frac{6^5}{6^2}$ f $\frac{8^6}{8^5}$
g $\frac{4^8}{4^5}$ h $\frac{3^9}{3^2}$

3 Simplify the following:
a $(5^2)^2$ b $(4^3)^4$ c $(10^2)^5$
d $(3^3)^5$ e $(6^2)^4$ f $(8^2)^3$

4 Simplify the following:
a $\frac{2^2 \times 2^4}{2^3}$ b $\frac{3^4 \times 3^2}{3^5}$
c $\frac{5^6 \times 5^7}{5^2 \times 5^8}$ d $\frac{(4^2)^5 \times 4^2}{4^7}$
e $\frac{4^4 \times 2^5 \times 4^2}{4^3 \times 2^3}$ f $\frac{6^3 \times 6^3 \times 8^5 \times 8^6}{8^6 \times 6^2}$
g $\frac{(5^5)^2 \times (4^4)^3}{5^8 \times 4^9}$ h $\frac{(6^3)^4 \times 6^3 \times 4^9}{6^8 \times 4^8}$

The zero index

The zero index indicates that a number is raised to the power 0. A number raised to the power 0 is equal to 1. This can be explained by applying the laws of indices.

$\frac{a^m}{a^n} = a^{m-n}$ therefore $\frac{a^m}{a^m} = a^{m-m}$
$= a^0$

However, $\frac{a^m}{a^m} = 1$

therefore $a^0 = 1$

Negative indices

A negative index indicates that a number is being raised to a negative power: e.g. 4^{-3}.

Another **law of indices** states that $a^{-m} = \frac{1}{a^m}$. This can be proved as follows.

$$a^{-m} = a^{0-m}$$

$$= \frac{a^0}{a^m} \text{ (from the second law of indices)}$$

$$= \frac{1}{a^m}$$

therefore $a^{-m} = \frac{1}{a^m}$

Exercise 7.3

Evaluate the following:

1. a $2^3 \times 2^0$ b $5^2 \div 6^0$ c $5^2 \times 5^{-2}$
   d $6^3 \times 6^{-3}$ e $(4^0)^2$ f $4^0 \div 2^2$
2. a 4^{-1} b 3^{-2} c 6×10^{-2}
   d 5×10^{-3} e 100×10^{-2} f 10^{-3}
3. a 9×3^{-2} b 16×2^{-3} c 64×2^{-4}
   d 4×2^{-3} e 36×6^{-3} f 100×10^{-1}
4. a $\frac{3}{2^{-2}}$ b $\frac{4}{2^{-3}}$ c $\frac{9}{5^{-2}}$
   d $\frac{5}{4^{-2}}$ e $\frac{7^{-3}}{7^{-4}}$ f $\frac{8^{-6}}{8^{-8}}$

Exponential equations

Equations that involve indices as unknowns are known as **exponential equations**.

Worked examples

a Find the value of x if $2^x = 32$.

32 can be expressed as a power of 2,

$32 = 2^5$.

Therefore $2^x = 2^5$

$x = 5$

b Find the value of m if $3^{(m-1)} = 81$.

81 can be expressed as a power of 3,

$81 = 3^4$.

Therefore $3^{(m-1)} = 3^4$

$m - 1 = 4$

$m = 5$

Exercise 7.4

1 Find the value of x in each of the following:

a $2^x = 4$
b $2^x = 16$
c $4^x = 64$
d $10^x = 1000$
e $5^x = 625$
f $3^x = 1$

2 Find the value of z in each of the following:

a $2^{(z-1)} = 8$
b $3^{(z+2)} = 27$
c $4^{2z} = 64$
d $10^{(z+1)} = 1$
e $3^z = 9^{(z-1)}$
f $5^z = 125^z$

3 Find the value of n in each of the following:

a $\left(\frac{1}{2}\right)^n = 8$
b $\left(\frac{1}{3}\right)^n = 81$
c $\left(\frac{1}{2}\right)^n = 32$
d $\left(\frac{1}{2}\right)^n = 4^{(n+1)}$
e $\left(\frac{1}{2}\right)^{(n+1)} = 2$
f $\left(\frac{1}{16}\right)^n = 4$

4 Find the value of x in each of the following:

a $3^{-x} = 27$
b $2^{-x} = 128$
c $2^{(-x+3)} = 64$
d $4^{-x} = \frac{1}{16}$
e $2^{-x} = \frac{1}{256}$
f $3^{(-x+1)} = \frac{1}{81}$

Standard form

Standard form is also known as standard index form or sometimes as scientific notation. It involves writing large numbers or very small numbers in terms of powers of 10.

Positive indices and large numbers

$$100 = 1 \times 10^2$$
$$1000 = 1 \times 10^3$$
$$10\,000 = 1 \times 10^4$$
$$3000 = 3 \times 10^3$$

For a number to be in standard form it must take the form $A \times 10^n$ where the index n is a positive or negative integer and A must lie in the range $1 \leqslant A < 10$.

e.g. 3100 can be written in many different ways:

$3.1 \times 10^3 \quad 31 \times 10^2 \quad 0.31 \times 10^4$ etc.

However, only 3.1×10^3 satisfies the above conditions and therefore is the only one which is written in standard form.

Worked examples

a Write 72 000 in standard form.

7.2×10^4

b Write 4×10^4 as an ordinary number.

$4 \times 10^4 = 4 \times 10000$

$= 40000$

c Multiply the following and write your answer in standard form:

600×4000

$= 2400000$

$= 2.4 \times 10^6$

d Multiply the following and write your answer in standard form:

$(2.4 \times 10^4) \times (5 \times 10^7)$

$= 12 \times 10^{11}$

$= 1.2 \times 10^{12}$ when written in standard form

e Divide the following and write your answer in standard form:

$(6.4 \times 10^7) \div (1.6 \times 10^3)$

$= 4 \times 10^4$

f Add the following and write your answer in standard form:

$(3.8 \times 10^6) + (8.7 \times 10^4)$

Changing the indices to the same value gives the sum:

$(380 \times 10^4) + (8.7 \times 10^4)$

$= 388.7 \times 10^4$

$= 3.887 \times 10^6$ when written in standard form

g Subtract the following and write your answer in standard form:

$(6.5 \times 10^7) - (9.2 \times 10^5)$

Changing the indices to the same value gives

$(650 \times 10^5) - (9.2 \times 10^5)$

$= 640.8 \times 10^5$

$= 6.408 \times 10^7$ when written in standard form

Exercise 7.5

1 Which of the following are not in standard form?

a 6.2×10^5 **b** 7.834×10^{16}
c 8.0×10^5 **d** 0.46×10^7
e 82.3×10^6 **f** 6.75×10^1

2 Write the following numbers in standard form:

a 600 000 **b** 48 000 000
c 784 000 000 000 **d** 534 000
e 7 million **f** 8.5 million

Exercise 7.5 (cont)

3 Write the following in standard form:

a 68×10^5 b 720×10^6

c 8×10^5 d 0.75×10^8

e 0.4×10^{10} f 50×10^6

4 Write the following as ordinary numbers:

a 3.8×10^3 b 4.25×10^6

c 9.003×10^7 d 1.01×10^5

5 Multiply the following and write your answers in standard form:

a 200×3000 b 6000×4000

c 7 million × 20   d 500 × 6 million

e 3 million × 4 million   f 4500×4000

6 Light from the Sun takes approximately 8 minutes to reach Earth. If light travels at a speed of 3×10^8 m/s, calculate to three significant figures (s.f.) the distance from the Sun to the Earth.

Note

Core students may use a calculator for Questions 7–9.

7 Find the value of the following and write your answers in standard form:

a $(4.4 \times 10^3) \times (2 \times 10^5)$ b $(6.8 \times 10^7) \times (3 \times 10^3)$

c $(4 \times 10^5) \times (8.3 \times 10^5)$ d $(5 \times 10^9) \times (8.4 \times 10^{12})$

e $(8.5 \times 10^6) \times (6 \times 10^{15})$ f $(5.0 \times 10^{12})^2$

8 Find the value of the following and write your answers in standard form:

a $(3.8 \times 10^8) \div (1.9 \times 10^6)$ b $(6.75 \times 10^9) \div (2.25 \times 10^4)$

c $(9.6 \times 10^{11}) \div (2.4 \times 10^5)$ d $(1.8 \times 10^{12}) \div (9.0 \times 10^7)$

e $(2.3 \times 10^{11}) \div (9.2 \times 10^4)$ f $(2.4 \times 10^8) \div (6.0 \times 10^3)$

9 Find the value of the following and write your answers in standard form:

a $(3.8 \times 10^5) + (4.6 \times 10^4)$ b $(7.9 \times 10^9) + (5.8 \times 10^8)$

c $(6.3 \times 10^7) + (8.8 \times 10^5)$ d $(3.15 \times 10^9) + (7.0 \times 10^6)$

e $(5.3 \times 10^8) - (8.0 \times 10^7)$ f $(6.5 \times 10^7) - (4.9 \times 10^6)$

g $(8.93 \times 10^{10}) - (7.8 \times 10^9)$ h $(4.07 \times 10^7) - (5.1 \times 10^6)$

Negative indices and small numbers

A negative index is used when writing a number between 0 and 1 in standard form.

e.g.

$$
\begin{aligned}
100 &= 1 \times 10^2 \\
10 &= 1 \times 10^1 \\
1 &= 1 \times 10^0 \\
0.1 &= 1 \times 10^{-1} \\
0.01 &= 1 \times 10^{-2} \\
0.001 &= 1 \times 10^{-3} \\
0.0001 &= 1 \times 10^{-4}
\end{aligned}
$$

Note that A must still lie within the range $1 \leqslant A < 10$.

Worked examples

a Write 0.0032 in standard form.

3.2×10^{-3}

b Write 1.8×10^{-4} as an ordinary number.

$$\begin{aligned} 1.8 \times 10^{-4} &= 1.8 \div 10^4 \\ &= 1.8 \div 10000 \\ &= 0.00018 \end{aligned}$$

c Write the following numbers in order of magnitude, starting with the largest:

3.6×10^{-3} 5.2×10^{-5} 1×10^{-2} 8.35×10^{-2} 6.08×10^{-8}

8.35×10^{-2} 1×10^{-2} 3.6×10^{-3} 5.2×10^{-5} 6.08×10^{-8}

Exercise 7.6

1 Write the following numbers in standard form:
- a 0.0006
- b 0.000053
- c 0.000864
- d 0.000000088
- e 0.0000007
- f 0.0004145

2 Write the following numbers in standard form:
- a 68×10^{-5}
- b 750×10^{-9}
- c 42×10^{-11}
- d 0.08×10^{-7}
- e 0.057×10^{-9}
- f 0.4×10^{-10}

3 Write the following as ordinary numbers:
- a 8×10^{-3}
- b 4.2×10^{-4}
- c 9.03×10^{-2}
- d 1.01×10^{-5}

4 Deduce the value of n in each of the following cases:
- a $0.00025 = 2.5 \times 10^n$
- b $0.00357 = 3.57 \times 10^n$
- c $0.00000006 = 6 \times 10^n$
- d $0.004^2 = 1.6 \times 10^n$
- e $0.00065^2 = 4.225 \times 10^n$
- f $0.0002^n = 8 \times 10^{-12}$

5 Write these numbers in order of magnitude, starting with the largest:

3.2×10^{-4} 6.8×10^5 5.57×10^{-9} 6.2×10^3

5.8×10^{-7} 6.741×10^{-4} 8.414×10^2

Fractional indices

$16^{\frac{1}{2}}$ can be written as $(4^2)^{\frac{1}{2}}$.

$$\begin{aligned} (4^2)^{\frac{1}{2}} &= 4^{\left(2 \times \frac{1}{2}\right)} \\ &= 4^1 \\ &= 4 \end{aligned}$$

Therefore $16^{\frac{1}{2}} = 4$

but $\sqrt{16} = 4$

Therefore $16^{\frac{1}{2}} = \sqrt{16}$

Similarly:

$125^{\frac{1}{3}}$ can be written as $(5^3)^{\frac{1}{3}}$

$$(5^3)^{\frac{1}{3}} = 5^{\left(3\times\frac{1}{3}\right)}$$
$$= 5^1$$
$$= 5$$

Therefore $125^{\frac{1}{3}} = 5$

But $\sqrt[3]{125} = 5$

Therefore $125^{\frac{1}{3}} = \sqrt[3]{125}$

In general:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \sqrt[n]{(a^m)} \text{ or } \left(\sqrt[n]{a}\right)^m$$

Worked examples

a Evaluate $16^{\frac{1}{4}}$ without the use of a calculator.

$16^{\frac{1}{4}} = \sqrt[4]{16}$ Alternatively: $16^{\frac{1}{4}} = (2^4)^{\frac{1}{4}}$

$= \sqrt[4]{2^4}$ $\quad = 2^1$

$= 2$ $\quad = 2$

b Evaluate $25^{\frac{3}{2}}$ without the use of a calculator.

$25^{\frac{3}{2}} = (25^{\frac{1}{2}})^3$ Alternatively: $25^{\frac{3}{2}} = (5^2)^{\frac{3}{2}}$

$= (\sqrt{25})^3$ $\quad = 5^3$

$= 5^3$ $\quad = 125$

$= 125$

c Solve $32^x = 2$

$32^x = 2$

$(2^5)^x = 2^1$

$2^{5x} = 2^1$

$5x = 1$ Therefore $x = \frac{1}{5}$

d Solve $125^x = 5$

$125^x = 5$

$(5^3)^x = 5^1$

$5^{3x} = 5^1$

$3x = 1$ Therefore $x = \frac{1}{3}$

Exercise 7.7

Evaluate the following:

1 a $16^{\frac{1}{2}}$ b $25^{\frac{1}{2}}$ c $100^{\frac{1}{2}}$
 d $27^{\frac{1}{3}}$ e $81^{\frac{1}{2}}$ f $1000^{\frac{1}{3}}$

2 a $16^{\frac{1}{4}}$ b $81^{\frac{1}{4}}$ c $32^{\frac{1}{5}}$
 d $64^{\frac{1}{6}}$ e $216^{\frac{1}{3}}$ f $256^{\frac{1}{4}}$

3 a $4^{\frac{3}{2}}$ b $4^{\frac{5}{2}}$ c $9^{\frac{3}{2}}$
 d $16^{\frac{3}{2}}$ e $1^{\frac{5}{2}}$ f $27^{\frac{2}{3}}$

4 a $125^{\frac{2}{3}}$ b $32^{\frac{3}{5}}$ c $64^{\frac{5}{6}}$
 d $1000^{\frac{2}{3}}$ e $16^{\frac{5}{4}}$ f $81^{\frac{3}{4}}$

Solve the following:

5 a $16^x = 4$ b $8^x = 2$ c $9^x = 3$
 d $27^x = 3$ e $100^x = 10$ f $64^x = 2$

6 a $1000^x = 10$ b $49^x = 7$ c $81^x = 3$
 d $343^x = 7$ e $1\,000\,000^x = 10$ f $216^x = 6$

Exercise 7.8

Evaluate the following:

1 a $\frac{27^{\frac{2}{3}}}{3^2}$ b $\frac{7^{\frac{3}{2}}}{\sqrt{7}}$ c $\frac{4^{\frac{5}{2}}}{4^2}$
 d $\frac{16^{\frac{3}{2}}}{2^6}$ e $\frac{27^{\frac{5}{3}}}{\sqrt{9}}$ f $\frac{6^{\frac{4}{3}}}{6^{\frac{1}{3}}}$

2 a $5^{\frac{2}{3}} \times 5^{\frac{4}{3}}$ b $4^{\frac{1}{4}} \times 4^{\frac{1}{4}}$ c 8×2^{-2}
 d $3^{\frac{4}{3}} \times 3^{\frac{5}{3}}$ e $2^{-2} \times 16$ f $8^{\frac{5}{3}} \times 8^{-\frac{4}{3}}$

3 a $\frac{2^{\frac{1}{2}} \times 2^{\frac{5}{2}}}{2}$ b $\frac{4^{\frac{5}{6}} \times 4^{\frac{1}{6}}}{4^{\frac{1}{2}}}$ c $\frac{2^3 \times 8^{\frac{3}{2}}}{\sqrt{8}}$
 d $\frac{(3^2)^{\frac{3}{2}} \times 3^{-\frac{1}{2}}}{3^{\frac{1}{2}}}$ e $\frac{8^{\frac{1}{3}} + 7}{27^{\frac{1}{3}}}$ f $\frac{9^{\frac{1}{2}} \times 3^{\frac{5}{2}}}{3^{\frac{2}{3}} \times 3^{-\frac{1}{6}}}$

Surds

You have already encountered the various types of roots of numbers and have also worked with indices too.

So you will know that $\sqrt{9} = 3$, $\sqrt{\frac{4}{81}} = \frac{2}{9}$ and $\sqrt[4]{625} = 5$.

In each of these cases the answer was a rational number. If the root cannot be expressed as a rational number, then it is called a **surd**. Surds, therefore, are irrational.

$\sqrt{3}$, $\sqrt{7}$ and $\sqrt[3]{20}$ are examples of surds.

Mathematicians tend to prefer writing their answers using surds as surds are more accurate than providing a decimal approximation. For example, if an answer is given as $\sqrt{5}$ this is preferable to using a calculator to work out the answer as 2.236 067 977 (9 d.p.).

Rules of surds

As surds involve roots, they have similar rules to those of indices.

Multiplication rule:

e.g. $\sqrt{5} \times \sqrt{7} = \sqrt{35}$

Therefore, in general $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$

Division rule:

e.g. $\frac{\sqrt{5}}{\sqrt{8}} = \sqrt{\frac{5}{8}}$

Therefore, in general $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

Using the two rules above, calculations involving surds can often be simplified.

Worked examples

a Simplify the following calculation:

$\sqrt{3} \times \sqrt{12}$

$\sqrt{36} = 6$

Note that in this case, the result of multiplying the two numbers produced a square number, so it could be simplified further.

b Simplify $\frac{\sqrt{8}}{\sqrt{12}}$

$\sqrt{\frac{8}{12}} = \sqrt{\frac{2}{3}}$

A single surd can often be simplified, too. In general, a surd can be simplified if it has a square number as a factor.

To simplify $\sqrt{48}$, identify the largest square number factor of 48. In this case, 16 is the largest factor that is a square number. Therefore, using the multiplication rule, $\sqrt{48}$ can be written as $\sqrt{16} \times \sqrt{3}$. However, $\sqrt{16} = 4$; therefore, $\sqrt{48}$ can be simplified to $4\sqrt{3}$.

Worked examples

Simplify each of the following surds:

a $\sqrt{72}$

$\sqrt{72} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$

b $\sqrt{45}$

$\sqrt{45} = \sqrt{9} \times \sqrt{5} = 3\sqrt{5}$

Rationalising the denominator

By checking with your calculator, you will be able to see that $\frac{1}{\sqrt{3}}$ is the same as $\frac{\sqrt{3}}{3}$.

In mathematics, if a fraction involves a surd, it is more usual to write it without the surd in the denominator. In the two fractions above, it is more usual to write $\frac{\sqrt{3}}{3}$ rather than $\frac{1}{\sqrt{3}}$. The process of removing the surd from the denominator is known as **rationalising the denominator**.

Worked examples

Rationalise the denominator of the following fractions:

a $\frac{1}{\sqrt{3}}$

Multiplying both the numerator and denominator by $\sqrt{3}$ produces an equivalent fraction because $\frac{\sqrt{3}}{\sqrt{3}} = 1$

Therefore, $\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{\sqrt{3}}{3}$

b $\frac{2}{3\sqrt{5}}$

To eliminate the surd from the denominator, multiply both the numerator and denominator by $\sqrt{5}$. Once again this produces an equivalent fraction to the original one as $\frac{\sqrt{5}}{\sqrt{5}} = 1$

$$\frac{2}{3\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{3\sqrt{5} \times \sqrt{5}} = \frac{2\sqrt{5}}{3 \times 5} = \frac{2\sqrt{5}}{15}$$

More complicated expressions include ones where the denominator has more than one term, e.g. $\frac{1}{1+\sqrt{3}}$.

In this case, simply multiplying the numerator and denominator by $\sqrt{3}$ will not eliminate all the surds from the denominator, as it will produce $\frac{\sqrt{3}}{\sqrt{3}+3}$.

To rationalise $\frac{1}{1+\sqrt{3}}$, multiply both the numerator and the denominator by $1-\sqrt{3}$ (note that the sign has changed).

$$\frac{1}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{1-\sqrt{3}}{(1+\sqrt{3})(1-\sqrt{3})}$$

$$= \frac{1-\sqrt{3}}{1-\sqrt{3}+\sqrt{3}-3} = \frac{1-\sqrt{3}}{-2} = \frac{-1+\sqrt{3}}{2}$$

Note

To understand the maths behind the following method you may find it helpful to know what is meant by 'the difference of two squares', which is covered in Chapter 11.

Worked examples

a Rationalise the expression $\frac{4}{\sqrt{3}-2}$

Multiplying both the numerator and the denominator by $\sqrt{3}+2$ gives:

$$\frac{4}{\sqrt{3}-2}\times\frac{\sqrt{3}+2}{\sqrt{3}+2}=\frac{4(\sqrt{3}+2)}{(\sqrt{3}-2)(\sqrt{3}+2)}$$

$$=\frac{4\sqrt{3}+8}{3+2\sqrt{3}-2\sqrt{3}-4}=\frac{4\sqrt{3}+8}{-1}=-4\sqrt{3}-8$$

b Rationalise the expression $\frac{\sqrt{2}}{2\sqrt{3}+\sqrt{2}}$

Multiply both the numerator and the denominator by $2\sqrt{3}-\sqrt{2}$

$$\frac{\sqrt{2}}{2\sqrt{3}+\sqrt{2}}\times\frac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}-\sqrt{2}}=\frac{\sqrt{2}(2\sqrt{3}-\sqrt{2})}{(2\sqrt{3}+\sqrt{2})(2\sqrt{3}-\sqrt{2})}$$

$$=\frac{2\sqrt{6}-2}{12-2\sqrt{6}+2\sqrt{6}-2}=\frac{2\sqrt{6}-2}{10}=\frac{\sqrt{6}-1}{5}$$

Exercise 7.9

1 Simplify the following surds:

a $\sqrt{3}\times\sqrt{27}$ **b** $\sqrt{5}\times\sqrt{20}$ **c** $\sqrt{24}\times\sqrt{6}$

d $(2\sqrt{3})^2$ **e** $(5\sqrt{2})^2$ **f** $(3\sqrt{6})^2$

2 Simplify the following surds:

a $\frac{\sqrt{50}}{\sqrt{5}}$ **b** $\frac{\sqrt{18}}{\sqrt{2}}$ **c** $\left(\frac{\sqrt{10}}{\sqrt{5}}\right)^2$

d $\frac{\sqrt{2}}{\sqrt{5}\sqrt{6}}$ **e** $\left(\frac{3\sqrt{8}}{2\sqrt{7}}\right)^2$ **f** $\left(\frac{(6\sqrt{5})^2+(2\sqrt{5})^2}{20}\right)^2$

3 Express the following surds in their simplest form:

a $\sqrt{8}$ **b** $\sqrt{50}$ **c** $\sqrt{18}$ **d** $\sqrt{45}$

e $\sqrt{75}$ **f** $\sqrt{72}$ **g** $\sqrt{700}$ **h** $\sqrt{162}$

i $\sqrt{98}$ **j** $\sqrt{242}$ **k** $\sqrt{192}$ **l** $\sqrt{450}$

4 Rationalise the following fractions, simplifying where possible:

a $\frac{1}{\sqrt{6}}$ **b** $\frac{3}{\sqrt{5}}$ **c** $\frac{2}{\sqrt{8}}$ **d** $\frac{6}{\sqrt{6}}$

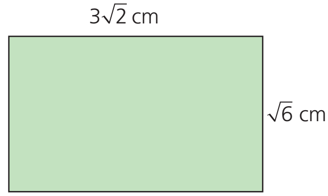
e $\frac{3}{5\sqrt{2}}$ **f** $-\frac{1}{2\sqrt{5}}$ **g** $\frac{5}{4\sqrt{10}}$ **h** $\frac{3}{8\sqrt{6}}$

5 Rationalise the denominators of the following fractions, simplifying where possible:

a $\frac{2}{5+\sqrt{3}}$ **b** $\frac{2}{\sqrt{2}-1}$ **c** $\frac{5}{\sqrt{2}+\sqrt{5}}$

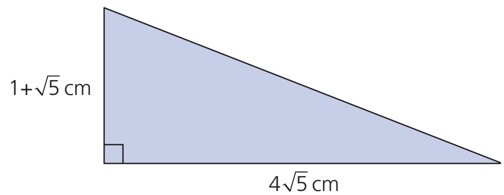
d $\frac{7\sqrt{3}}{2\sqrt{2}+5}$ **e** $\frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}-3\sqrt{3}}$ **f** $\frac{2+\sqrt{5}}{2-\sqrt{5}}$

6 A rectangle is shown below.



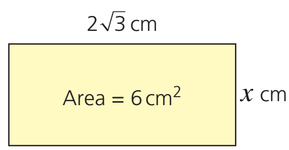
a Calculate its area.
b Calculate its perimeter.

7 A right-angled triangle has dimensions as shown below.



Calculate the area of the triangle.

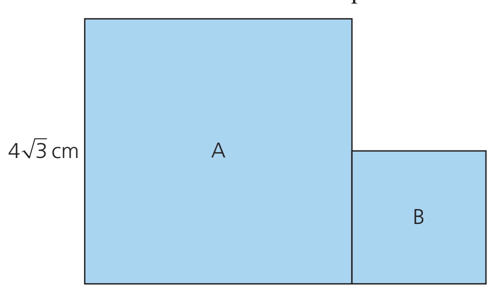
8 A rectangle of area $6\,\text{cm}^2$ is shown below.



Calculate the length of the side marked x, giving your answer in its simplest form.

9 A square has an area of $18\,\text{cm}^2$. Calculate its side length, giving your answer in its simplest form.

10 The shape below consists of two squares joined together.
The length of each side of square A is $4\sqrt{3}\,\text{cm}$.
The total area of the whole shape is $60\,\text{cm}^2$.



Calculate the following:
a the area of square A
b the area of square B
c the length of each side of square B
d the perimeter of the whole shape.

Student assessment 1

1 Using indices, simplify the following:
  a $3 \times 2 \times 2 \times 3 \times 27$ b $2 \times 2 \times 4 \times 4 \times 4 \times 2 \times 32$

2 Write the following out in full:
  a 6^5 b 2^{-5}

3 Work out the value of the following without using a calculator:
  a $3^3 \times 10^3$ b $1^{-4} \times 5^3$

4 Simplify the following using indices:
  a $2^4 \times 2^3$ b $7^5 \times 7^2 \times 3^4 \times 3^8$
  c $\frac{4^8}{2^{10}}$ d $\frac{(3^3)^4}{27^3}$
  e $\frac{7^6 \times 4^2}{4^3 \times 7^6}$ f $\frac{8^{-2} \times 2^6}{2^{-2}}$

5 Without using a calculator, evaluate the following:
  a $5^2 \times 5^{-1}$ b $\frac{4^5}{4^3}$
  c $\frac{7^{-5}}{7^{-7}}$ d $\frac{3^{-5} \times 4^2}{3^{-6}}$

6 Find the value of x in each of the following:
  a $2^{(2x+2)} = 128$ b $\frac{1}{4^{-x}} = \frac{1}{2}$
  c $3^{(-x+4)} = 81$ d $8^{-3x} = \frac{1}{4}$

Student assessment 2

1 Write the following numbers in standard form:
  a 6 million   b 0.0045   c 3 800 000 000
  d 0.000 000 361   e 460 million   f 3

2 Write the following as ordinary numbers:
  a 8.112×10^6 b 4.4×10^5 c 3.05×10^{-4}

3 Write the following numbers in order of magnitude, starting with the largest:

  3.6×10^2 2.1×10^{-3} 9×10^1 4.05×10^8 1.5×10^{-2} 7.2×10^{-3}

4 Write the following numbers:
  a in standard form,
  b in order of magnitude, starting with the smallest.

  15 million   430 000   0.000 435   4.8   0.0085

5 Deduce the value of n in each of the following:

- a $4750 = 4.75 \times 10^n$
- b $6\,440\,000\,000 = 6.44 \times 10^n$
- c $0.0040 = 4.0 \times 10^n$
- d $1000^2 = 1 \times 10^n$
- e $0.9^3 = 7.29 \times 10^n$
- f $800^3 = 5.12 \times 10^n$

6 Write the answers to the following calculations in standard form:

- a $50\,000 \times 2400$
- b $(3.7 \times 10^6) \times (4.0 \times 10^4)$
- c $(5.8 \times 10^7) + (9.3 \times 10^6)$
- d $(4.7 \times 10^6) - (8.2 \times 10^5)$

7 The speed of light is 3×10^8 m/s. Jupiter is 778 million km from the Sun. Calculate the number of minutes it takes for sunlight to reach Jupiter.

8 A star is 300 light years away from Earth. The speed of light is 3×10^5 km/s. Calculate the distance from the star to Earth. Give your answer in kilometres and written in standard form.

Student assessment 3

1 Evaluate the following:

- a $81^{\frac{1}{2}}$
- b $27^{\frac{1}{3}}$
- c $9^{\frac{1}{2}}$
- d $625^{\frac{3}{4}}$
- e $343^{\frac{2}{3}}$
- f $16^{-\frac{1}{4}}$
- g $\dfrac{1}{25^{-\frac{1}{2}}}$
- h $\dfrac{2}{16^{-\frac{3}{4}}}$

2 Evaluate the following:

- a $\dfrac{16^{\frac{1}{2}}}{2^2}$
- b $\dfrac{9^{\frac{5}{2}}}{3^3}$
- c $\dfrac{8^{\frac{4}{3}}}{8^{\frac{2}{3}}}$
- d $5^{\frac{6}{5}} \times 5^{\frac{4}{5}}$
- e $4^{\frac{3}{2}} \times 2^{-2}$
- f $\dfrac{27^{\frac{2}{3}} \times 3^{-2}}{4^{-\frac{3}{2}}}$
- g $\dfrac{(4^3)^{-\frac{1}{2}} \times 2^{\frac{3}{2}}}{2^{-\frac{3}{2}}}$
- h $\dfrac{(5^{\frac{2}{3}})^{\frac{1}{2}} \times 5^{\frac{2}{3}}}{3^{-2}}$

3 Draw a pair of axes with x from -4 to 4 and y from 0 to 10.

- a Plot a graph of $y = 3^{\frac{x}{2}}$.
- b Use your graph to estimate when $3^{\frac{x}{2}} = 5$.

4 Express the following surds in their simplest form.

- a $\sqrt{125}$
- b $\sqrt{80}$
- c $\sqrt{12} + \sqrt{48}$

5 Rationalise the following fractions, simplifying where possible.

- a $\dfrac{5}{2\sqrt{2}}$
- b $\dfrac{1}{2 - \sqrt{7}}$

6 The area (A) of a circle is calculated using the formula $A = \pi r^2$, where r is the radius of the circle.

Calculate the area of a circle with a radius of $3\sqrt{5}$ cm. Leave π in your answer.

Student assessment 4

1 Evaluate the following:

a $64^{\frac{1}{6}}$ b $27^{\frac{4}{3}}$ c $9^{-\frac{1}{2}}$ d $512^{\frac{2}{3}}$

e $\sqrt[3]{27}$ f $\sqrt[4]{16}$ g $\dfrac{1}{36^{-\frac{1}{2}}}$ h $\dfrac{2}{64^{-\frac{2}{3}}}$

2 Evaluate the following:

a $\dfrac{25^{\frac{1}{2}}}{9^{-\frac{1}{2}}}$ b $\dfrac{4^{\frac{5}{2}}}{2^3}$

c $\dfrac{27^{\frac{4}{3}}}{3^3}$ d $25^{\frac{3}{2}} \times 5^2$

e $4^{\frac{6}{4}} \times 4^{-\frac{1}{2}}$ f $\dfrac{27^{\frac{2}{3}} \times 3^{-3}}{9^{-\frac{1}{2}}}$

g $\dfrac{(4^2)^{-\frac{1}{4}} \times 9^{\frac{3}{2}}}{\left(\frac{1}{4}\right)^{\frac{1}{2}}}$ h $\dfrac{(5^{\frac{1}{3}})^{\frac{1}{2}} \times 5^{\frac{5}{6}}}{4^{-\frac{1}{2}}}$

3 Draw a pair of axes with x from −4 to 4 and y from 0 to 18.

a Plot a graph of $y = 4^{-\frac{x}{2}}$.

b Use your graph to estimate when $4^{-\frac{x}{2}} = 6$.

4 Simplify the following surds.

a $(6\sqrt{3})^2$ b $3\sqrt{2} \times 4\sqrt{6}$

5 Rationalise the following fractions, simplifying where possible.

a $\dfrac{7}{3\sqrt{3}}$ b $\dfrac{2}{\sqrt{7}+4}$

6 Calculate the area of a rectangle with a length of $(\sqrt{2} + 3\sqrt{6})$ cm and width $\sqrt{2}$ cm.

8 Money and finance

Currency conversions

In 2022, 1 euro (€) could be exchanged for 1.50 Australian dollars (A$).

Worked examples

a How many Australian dollars can be bought for €400?

€1 buys A$1.50.

€400 buys $1.50 \times 400 =$ A$600.

b How much does it cost, in euros, to buy A$940?

A$1.50 costs €1.

A$940 costs $\frac{1 \times 940}{1.5} =$ €626.67.

Exercise 8.1

The table shows the exchange rate for €1 into various currencies.

Australia	1.50 Australian dollars (A$)
India	75 rupees
Zimbabwe	412.8 Zimbabwe dollars (ZIM$)
South Africa	15 rand
Turkey	4.0 Turkish lira (L)
Japan	130 yen
Kuwait	0.35 dinar
USA	1.15 US dollars (US$)

1 Convert the following:
- a €25 into Australian dollars
- b €50 into rupees
- c €20 into Zimbabwean dollars
- d €300 into rand
- e €130 into Turkish lira
- f €40 into yen
- g €400 into dinar
- h €150 into US dollars

2 How many euros does it cost to buy the following:
- a A$500
- b 200 rupees
- c ZIM$10 000
- d 500 rand
- e 750 Turkish lira
- f 1200 yen
- g 50 dinar
- h US$150?

Earnings

Net pay is what is left after deductions such as tax, insurance and pension contributions are taken from **gross earnings**.

That is, Net pay = Gross pay – Deductions.

A **bonus** is an extra payment sometimes added to an employee's **basic pay**.

In many companies there is a fixed number of hours that an employee is expected to work. Any work done in excess of this **basic week** is paid at a higher rate, referred to as **overtime**. Overtime may be 1.5 times basic pay, called **time and a half**, or twice basic pay, called **double time**.

Piece work is another method of payment. Employees are paid for the number of articles made, not for the time taken.

Exercise 8.2

1 Kamal's gross pay is \$188.25. Deductions amount to \$33.43. What is his net pay?

2 Keiko's basic pay is \$128. She earns \$36 for overtime and receives a bonus of \$18. What is her gross pay?

3 Ella's gross pay is \$203. She pays \$54 in tax and \$18 towards her pension. What is her net pay?

4 Nameen works 35 hours for an hourly rate of \$8.30. What is his basic pay?

5 a Leeza works 38 hours for an hourly rate of \$4.15. In addition she works 6 hours of overtime at time and a half. What is her total gross pay?
b Deductions amount to 32% of her total gross pay. What is her net pay?

6 Pepe is paid \$5.50 for each basket of grapes he picks. One week he picks 25 baskets. How much is he paid?

7 Farah is paid \$5 for every 12 plates that she makes. This is her record for one week.

Mon	240
Tues	360
Wed	288
Thurs	192
Fri	180

How much is she paid?

8 Harry works at home, making clothes. The patterns and materials are provided by the company. The table shows the rates he is paid and the number of items he makes in one week:

Item	Rate	Number made
Jacket	\$25	3
Trousers	\$11	12
Shirt	\$13	7
Dress	\$12	0

a What are his gross earnings?
b Deductions amount to 15% of gross earnings. What is his net pay?

Profit and loss

Foodstuffs and manufactured goods are produced at a cost, known as the **cost price**, and sold at the **selling price**. If the selling price is greater than the cost price, a profit is made.

Worked example

A market trader buys oranges in boxes of 12 dozen for \$14.40 per box. He buys three boxes and sells all the oranges for 12c each. What is his profit or **loss**?

Cost price: $3 \times \$14.40 = \$43.20$

Selling price: $3 \times 144 \times 12\text{c} = \$51.84$

In this case he makes a profit of $\$51.84 - \$43.20$

His profit is \$8.64.

A second way of solving this problem would be:

\$14.40 for a box of 144 oranges is 10c each.

So cost price of each orange is 10c, and selling price of each orange is 12c. The profit is 2c per orange.

So 3 boxes would give a profit of $3 \times 144 \times 2\text{c}$.

That is, \$8.64.

Sometimes, particularly during sales or promotions, the selling price is reduced; this is known as a **discount**.

Worked example

In a sale, a skirt usually costing \$35 is sold at a 15% discount. What is the discount?

15% of $\$35 = 0.15 \times \$35 = \$5.25$

The discount is \$5.25.

Exercise 8.3

1. A market trader buys peaches in boxes of 120. She buys 4 boxes at a cost price of \$13.20 per box. She sells 425 peaches at 12c each. The rest are ruined. How much profit or loss does she make?
2. A shopkeeper buys 72 bars of chocolate for \$5.76. What is his profit if he sells them for 12c each?
3. A holiday company charters an aircraft to fly to Qatar at a cost of \$22 000. It then sells 150 seats at \$185 each and a further 35 seats at a 20% discount. Calculate the profit made per seat if the plane has 200 seats.
4. A car is priced at \$7200. The car dealer allows a customer to pay a one-third deposit and 12 payments of \$420 per month. How much extra does it cost the customer?
5. At an auction, a company sells 150 televisions for an **average** of \$65 each. The production cost was \$10 000. How much loss did the company make?

Percentage profit and loss

Most profits or losses are expressed as a percentage.

Profit or loss, divided by cost price, multiplied by 100 = % profit or loss.

Worked example

Abi buys a car for $7500 and sells it two years later for $4500. Calculate her loss over two years as a percentage of the cost price.

cost price = $7500 selling price = $4500 loss = $3000

$$\% \text{ Loss} = \frac{3000}{7500} \times 100 = 40$$

Her loss is 40%.

When something becomes worth less over a period of time, it is said to **depreciate**.

Exercise 8.4

1 Find the depreciation of the following cars as a percentage of the cost price. (C.P. = cost price, S.P. = selling price)

a Car A	C.P. $4500	S.P. $4005
b Car B	C.P. $9200	S.P. $6900

2 A company manufactures electrical items for the kitchen. Find the percentage profit on each of the following:

a Fridge	C.P. $50	S.P. $65
b Freezer	C.P. $80	S.P. $96

3 A developer builds a number of different types of house. Which type gives the developer the largest percentage profit?

Type A	C.P. $40 000	S.P. $52 000
Type B	C.P. $65 000	S.P. $75 000
Type C	C.P. $81 000	S.P. $108 000

4 Students in a school organise a disco. The disco company charges $350 hire charge. The students sell 280 tickets at $2.25. What is the percentage profit?

Interest

Interest can be defined as money added by a bank to sums deposited by customers. The money deposited is called the **principal**. The **percentage interest** is the given rate and the money is left for a fixed period of time.

A formula can be obtained for **simple interest**:

$$SI = \frac{Ptr}{100}$$

where SI = simple interest, i.e. the interest paid
P = the principal
t = time in years
r = rate percent

Worked examples

a Find the simple interest earned on $250 deposited for 6 years at 8% p.a.

$$SI = \frac{Ptr}{100}$$

$$SI = \frac{250 \times 6 \times 8}{100}$$

$$SI = 120$$

p.a. stands for per annum which means 'each year'

So the interest paid is $120.

b How long will it take for a sum of $250 invested at 8% to earn interest of $80?

$$SI = \frac{Ptr}{100}$$

$$80 = \frac{250 \times t \times 8}{100}$$

$$80 = 20t$$

$$4 = t$$

It will take 4 years.

c What rate per year must be paid for a principal of $750 to earn interest of $180 in 4 years?

$$SI = \frac{Ptr}{100}$$

$$180 = \frac{750 \times 4 \times r}{100}$$

$$180 = 30r$$

$$6 = r$$

The rate must be 6% per year.

d Find the principal which will earn interest of $120 in 6 years at 4%.

$$SI = \frac{Ptr}{100}$$

$$120 = \frac{P \times 6 \times 4}{100}$$

$$120 = \frac{24P}{100}$$

$$12\,000 = 24P$$

$$500 = P$$

So the principal is $500.

Exercise 8.5

All rates of interest given here are annual rates.

1 Find the simple interest paid in the following cases:

a Principal \$300 rate 6% time 4 years

b Principal \$750 rate 8% time 7 years

2 Calculate how long it will take for the following amounts of interest to be earned at the given rate.

a $P = \$500$ $r = 6\%$ $SI = \$150$

b $P = \$400$ $r = 9\%$ $SI = \$252$

3 Calculate the rate of interest per year which will earn the given amount of interest:

a Principal \$400 time 4 years interest \$112

b Principal \$800 time 7 years interest \$224

4 Calculate the principal which will earn the interest below in the given number of years at the given rate:

a $SI = \$36$ time = 3 years rate = 6%

b $SI = \$340$ time = 5 years rate = 8%

5 What rate of interest is paid on a deposit of \$2000 which earns \$400 interest in 5 years?

6 How long will it take a principal of \$350 to earn \$56 interest at 8% per year?

7 A principal of \$480 earns \$108 interest in 5 years. What rate of interest was being paid?

8 A principal of \$750 becomes a total of \$1320 in 8 years. What rate of interest was being paid?

9 \$1500 is invested for 6 years at 3.5% per year. What is the interest earned?

10 \$500 is invested for 11 years and becomes \$830 in total. What rate of interest was being paid?

Compound interest

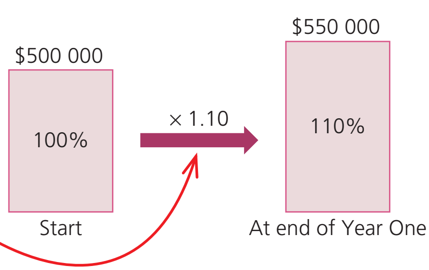
Note

A principal amount can be money deposited or an amount loaned.

Compound interest means that interest is paid not only on the principal amount, but also on the interest itself: it is compounded (or added to). This sounds complicated but the example below will make it clearer.

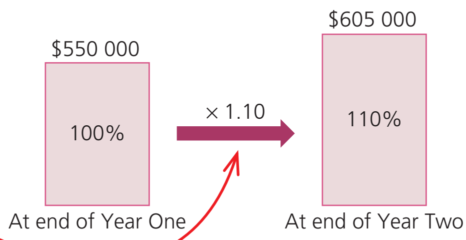
For example, a builder is going to build six houses on a plot of land. He borrows \$500 000 at 10% compound interest per annum and will pay the loan off in full after three years.

10% of \$500 000 is \$50 000, therefore at the end of the first year he will owe a total of \$550 000 as shown:



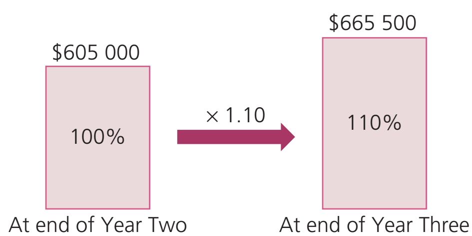
An increase of 10% is the same as multiplying by 1.10.

For the second year, the amount he owes increases again by 10%, but this is calculated by adding 10% to the amount he owed at the end of the first year, i.e. 10% of \$550 000. This can be represented using this diagram:



Once again an increase of 10% is the same as multiplying by 1.10.

For the third year, the amount he owes increases again by 10%. This is calculated by adding 10% to the amount he owed at the end of the second year, i.e. 10% of \$605 000 as shown:



Therefore, the compound interest he has to pay at the end of three years is \$665 500 – \$500 000 = \$165 500.

By looking at the diagrams above it can be seen that the principal amount has in effect been multiplied by 1.10 three times (this is the same as multiplying by 1.10^3), i.e. $\$500\,000 \times 1.10^3 = \$665\,500$.

The time taken for a debt to grow at compound interest can be calculated as shown in the next example.

Worked example

How long will it take for a debt to double at a compound interest rate of 27% per annum?

An interest rate of 27% implies a multiplier of 1.27.

The effect of applying this multiplier to a principal amount, P, is shown in the table:

Time (years)	0	1	2	3
Debt	P	$1.27P$	$1.27^2P = 1.61P$	$1.27^3P = 2.05P$

× 1.27 × 1.27 × 1.27

The debt will have more than doubled after 3 years.

Note

The opposite of exponential growth is known as **exponential decay**.

Compound interest is an example of a geometric sequence and therefore of **exponential** growth.

The interest is usually calculated annually, but there can be other time periods. Compound interest can be charged yearly, half-yearly, quarterly, monthly or daily. (In theory any time period can be chosen.)

Worked examples

a Find the compound interest paid on a loan of \$600 at a rate of 5% for 3 years.

An increase of 5% is equivalent to a multiplier of 1.05.

Therefore 3 years at 5% is calculated as $600 \times 1.05^3 = 694.58$ (2 d.p.).

The total payment is \$694.58, so the interest paid is \$694.58 – \$600 = \$94.58.

b Find the compound interest when \$3000 is invested for 18 months at a rate of 8.5% per year if the interest is calculated every 6 months.

Note: The interest for each time period of 6 months is $\frac{8.5}{2}\% = 4.25\%$.

There will be 3 time periods of 6 months each.

An increase of 4.25% is equivalent to a multiplier of 1.0425.

Therefore the total amount is $3000 \times 1.0425^3 = 3398.99$.

The interest earned is \$3398.99 – \$3000 = \$398.99.

Note

You should learn how to use the formula for calculating compound interest.

There is a formula for calculating the compound interest. It is written as:

$$I = P\left(1+\frac{r}{100}\right)^n - P$$

Where I = compound interest
P = the principal (amount originally borrowed)
r = interest rate
n = number of years

For the example of the builder earlier in this section, $P = 500000$ dollars, $r = 10\%$ and $n = 3$.

Therefore $I = 500\ 000\left(1+\frac{10}{100}\right)^3 - 500000 = 165500$ dollars.

Exercise 8.6

Using the formula for compound interest or otherwise, calculate the following:

1 A shipping company borrows \$70 million at 5% p.a. compound interest to build a new cruise ship. If it repays the debt after 3 years, how much interest will the company pay?

2 A woman borrows \$100 000 for home improvements. The compound interest rate is 15% p.a. and she repays the loan in full after 3 years. How much interest did she pay?

3 A man owes \$5000 on his credit cards. The rate of interest is 20% per year. If he doesn't repay any of the debt, how much will he owe after 4 years?

4 A school increases its intake by 10% each year. If it starts with 1000 students, how many will it have at the beginning of the fourth year of expansion?

5 8 million tonnes of fish were caught in the North Sea in 2019. If the catch is reduced by 20% each year for 4 years, what mass is caught at the end of this time?

6 How many years will it take for a debt to double at 42% p.a. compound interest?

7 How many years will it take for a debt to double at 15% p.a. compound interest?

8 A car loses value at a rate of 27% each year. How long will it take for its value to halve?

Student assessment 1

1 A visitor from Hong Kong receives 13.5 Pakistani rupees for each Hong Kong dollar.
  a How many Pakistani rupees would he get for HK$240?
  b How many Hong Kong dollars does it cost for 1 thousand rupees?

2 Below is a currency conversion table showing the amount of foreign currency received for 1 euro.

New Zealand	1.60 dollars (NZ$)
Brazil	3.70 reals

  a How many euros does it cost for NZ$1000?
  b How many euros does it cost for 500 Brazilian reals?

3 Laila works in a shop on Saturdays for 8.5 hours. She is paid $3.60 per hour. What is her gross pay for 4 weeks' work?

4 Razik makes cups and saucers in a factory. He is paid $1.44 per batch of cups and $1.20 per batch of saucers. What is his gross pay if he makes 9 batches of cups and 11 batches of saucers in one day?

5 Calculate the missing numbers from the simple interest table below:

Principal ($)	Rate (%)	Time (years)	Interest ($)
300	6	4	**a**
250	**b**	3	60
480	5	**c**	96
650	**d**	8	390
e	3.75	4	187.50

6 A family house was bought for $48 000 twelve years ago. It is now valued at $120 000. What is the average annual increase in the value of the house?

7 An electrician bought five broken washing machines for $550. She repaired them and sold them for $143 each. What was her percentage profit?

Student assessment 2

1 Find the simple interest paid on the following principal sums P, deposited in a savings account for t years at a fixed rate of interest of r%:

a	$P = \$550$	$t = 5$ years	$r = 3\%$
b	$P = \$8000$	$t = 10$ years	$r = 6\%$
c	$P = \$12\,500$	$t = 7$ years	$r = 2.5\%$

2 A sum of \$25 000 is deposited in a bank. After 8 years, the simple interest gained was \$7000. Calculate the annual rate of interest on the account assuming it remained constant over the 8 years.

3 A bank lends a business \$250 000. The annual rate of interest is 8.4%. When paying back the loan, the business pays an amount of \$105 000 in simple interest. Calculate the number of years the business took out the loan for.

4 A small business wishes to take out a \$10 000 loan from a bank. The bank has two loan options.
Option A: A loan with a **simple** interest rate of 4% per year.
Option B: A loan with a **compound** interest rate of 3% per year.

a If the loan is to be taken out for 10 years, calculate which loan is cheapest for the small business.

b How much will the business save over the 10 years by choosing the cheaper of the two loans?

5 Find the compound interest paid on the following principal sums P, deposited in a savings account for n years at a fixed rate of interest of r%:

a	$P = \$400$	$n = 2$ years	$r = 3\%$
b	$P = \$5000$	$n = 8$ years	$r = 6\%$
c	$P = \$18\,000$	$n = 10$ years	$r = 4.5\%$

6 A car is bought for \$12 500. Its value depreciates by 15% per year.

a Calculate its value after:
i 1 year **ii** 2 years

b After how many years will the car be worth less than \$1000?

9 Time

Times may be given in terms of the 12-hour clock. We tend to say, 'I get up at seven o'clock in the morning, play football at half past two in the afternoon, and go to bed before eleven o'clock'.

These times can be written as 7 a.m., 2.30 p.m. and 11 p.m.

In order to save confusion, most timetables are written using the 24-hour clock.

7 a.m. is written as 07 00

2.30 p.m. is written as 14 30

11.00 p.m. is written as 23 00

Worked example

A train covers the 480 km journey from Paris to Lyon at an average speed of 100 km/h. If the train leaves Paris at 08 35, when does it arrive in Lyon?

Time taken $= \frac{\text{distance}}{\text{speed}}$

Paris to Lyon $= \frac{480}{100}$ hours, that is, 4.8 hours.

4.8 hours is 4 hours and $(0.8 \times 60$ minutes$)$, that is, 4 hours and 48 minutes.

Departure 08 35; arrival 08 35 + 04 48

Arrival time is 13 23.

Note that 4.80 hrs does not represent 4 hrs and 80 minutes. This is because time is not a decimal system which has 10 as its base number. Time is a **sexagesimal** number system, with 60 as its base number, i.e. there are 60 minutes in an hour and 60 seconds in a minute.

As shown above, converting 0.8 hrs into minutes can be done by multiplying by 60.

i.e. 0.8 hrs is equivalent to $0.8 \times 60 = 48$ minutes.

Your scientific calculator will have a sexagesimal button and it will look similar to [°′and″].

To convert 4.80 hrs to hours, minutes and seconds, enter the number into your calculator followed by the [°′and″] button. The calculator will show an answer of [4] [°] [48] [′] [0] [″] which implies 4 hrs, 48 mins and 0 secs.

Exercise 9.1

1 Using your calculator, convert the following times written as decimals into time using hours, minutes and seconds.
  a 0.25 hrs b 3.765 hrs c 0.22 hrs

2 A train leaves a station at 06 24. The journey has 4 stops. Calculate the time the train arrives at each stop if the time taken from one to stop to the next is as follows:
Start to stop 1 takes 0.35 hrs.
Stop 1 to stop 2 takes 1.30 hrs.
Stop 2 to stop 3 takes 1.65 hrs.
Stop 3 to final stop takes 2.91 hrs.

3 a A journey to work takes Sangita three quarters of an hour. If she catches the bus at 07 55, when does she arrive?
  b Sangita catches a bus home each evening. The journey takes 55 minutes. If she catches the bus at 17 50, when does she arrive?

4 Jake cycles to school each day. His journey takes 70 minutes. When will he arrive if he leaves home at 07 15?

5 Find the time in hours and minutes for the following journeys of the given distance at the average speed stated:
  a 230 km at 100 km/h b 70 km at 50 km/h

6 Grand Prix racing cars cover a 120 km race at the following average speeds. How long do the first five cars take to complete the race? Answer in minutes and seconds.
First 240 km/h Second 220 km/h Third 210 km/h
Fourth 205 km/h Fifth 200 km/h

7 A train covers the 1500 km distance from Amsterdam to Barcelona at an average speed of 90 km/h. If the train leaves Amsterdam at 9.30 a.m. on Tuesday, when does it arrive in Barcelona?

8 A plane takes off at 16 25 for the 3200 km journey from Moscow to Athens. If the plane flies at an average speed of 600 km/h, when will it land in Athens?

9 A plane leaves London for Boston, a distance of 5200 km, at 09 45. The plane travels at an average speed of 800 km/h. If Boston time is five hours behind British time, what is the time in Boston when the aircraft lands?

Student assessment 1

1 A journey to school takes Monica 0.4 hours. What time does she arrive if she leaves home at 08 38?
2 A car travels 295 km at 50 km/h. How long does the journey take? Give your answer in hours and minutes.
3 A bus leaves Deltaville at 11 32. It travels at an average speed of 42 km/h. If it arrives in Eastwich at 12 42, what is the distance between the two towns?
4 A plane leaves Betatown at 17 58 and arrives at Fleckley at 05 03 the following morning. How long does the journey take? Give your answer in hours and minutes.

Student assessment 2

1 A journey to school takes Fouad 22 minutes. What is the latest time he can leave home if he must be at school at 08 40?

2 A plane travels 270 km at 120 km/h. How long does the journey take? Give your answer in hours and minutes.

3 A train leaves Alphaville at 13 27. It travels at an average speed of 56 km/h. If it arrives in Westwich at 16 12, what is the distance between the two towns?

4 A car leaves Gramton at 16 39. It travels a distance of 315 km and arrives at Halfield at 20 09.
   a How long does the journey take?
   b What is the car's average speed?

10 Set notation and Venn diagrams

Sets

A **set** is a well-defined group of objects or symbols. The objects or symbols are called the **elements** of the set.

If an element e belongs to a set S, this is represented as $e \in S$. If e does not belong to set S this is represented as $e \notin S$.

Worked examples

a A particular set consists of the following elements:

{South Africa, Namibia, Egypt, Angola, ...}

- **i** Describe the set.

  The elements of the set are countries of Africa.

- **ii** Add another two elements to the set.

  e.g. Zimbabwe, Ghana

b Consider the set $A = \{x: x \text{ is a natural number}\}$

- **i** Describe the set.

  The elements of the set are the natural numbers.

- **ii** Write down two elements of the set.

  e.g. 3 and 15

c Consider the set $B = \{(x, y): y = 2x - 4\}$

- **i** Describe the set.

  The elements of the set are the coordinates of points found on the straight line with equation $y = 2x - 4$.

- **ii** Write down two elements of the set.

  e.g. $(0, -4)$ and $(10, 16)$

d Consider the set $C = \{x: 2 \leqslant x \leqslant 8\}$

- **i** Describe the set.

  The elements of the set include any number between 2 and 8 inclusive.

- **ii** Write down two elements of the set.

  e.g. 5 and 6.3

Exercise 10.1

1 In the following questions:
i describe the set in words,
ii write down another two elements of the set.
- **a** {Asia, Africa, Europe, ...}
- **b** {2, 4, 6, 8, ...}
- **c** {Sunday, Monday, Tuesday, ...}
- **d** {January, March, July, ...}
- **e** {1, 3, 6, 10, ...}
- **f** {Mehmet, Michael, Mustapha, Matthew, ...}
- **g** {11, 13, 17, 19, ...}
- **h** {a, e, i, ...}
- **i** {Earth, Mars, Venus, ...}
- **j** $A = \{x: 3 \leqslant x \leqslant 12\}$
- **k** $S = \{y: -5 \leqslant y \leqslant 5\}$

2 The number of elements in a set A is written as n(A).
Give the value of n(A) for the finite sets in Questions 1 a–k above.

Subsets

If all the elements of one set X are also elements of another set Y, then X is said to be a **subset** of Y.
This is written as $X \subseteq Y$.

If a set A is empty (i.e. it has no elements in it), then this is called the **empty set** and it is represented by the symbol $\varnothing$. Therefore $A = \varnothing$. The empty set is a subset of all sets.

e.g. Three girls, Winnie, Natalie and Emma, form a set A
A = {Winnie, Natalie, Emma}
All the possible subsets of A are given below:
B = {Winnie, Natalie, Emma}
C = {Winnie, Natalie}
D = {Winnie, Emma}
E = {Natalie, Emma}
F = {Winnie}
G = {Natalie}
H = {Emma}
$I = \varnothing$

Note that the sets B and I above are considered as subsets of A.
i.e. $B \subseteq A$ and $I \subseteq A$

Similarly, $G \not\subseteq H$ implies that G is not a subset of H

Worked examples

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

a List subset B {even numbers}.

$B = \{2, 4, 6, 8, 10\}$

b List subset C {prime numbers}.

$C = \{2, 3, 5, 7\}$

Exercise 10.2

1 P = {whole numbers less than 30}
  a List the subset Q {even numbers}.
  b List the subset R {odd numbers}.
  c List the subset S {prime numbers}.
  d List the subset T {square numbers}.
  e List the subset U {triangle numbers}.

2 A = {whole numbers between 50 and 70}
  a List the subset B {multiples of 5}.
  b List the subset C {multiples of 3}.
  c List the subset D {square numbers}.

3 State whether each of the following statements is true or false:
  a {Algeria, Mozambique} $\subseteq$ {countries in Africa}
  b {mango, banana} $\subseteq$ {fruit}
  c $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$
  d {volleyball, basketball} $\not\subseteq$ {team sport}
  e {potatoes, carrots} $\subseteq$ {vegetables}

The universal set

The **universal set** ($\mathscr{E}$) for any particular problem is the set which contains all the possible elements for that problem.

The **complement** of a set A is the set of elements which are in $\mathscr{E}$ but not in A. The complement of A is identified as A'.

Notice that $\mathscr{E}' = \varnothing$ and $\varnothing' = \mathscr{E}$.

Worked examples

a If $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 2, 3, 4, 5\}$ what set is represented by A'?

A' consists of those elements in $\mathscr{E}$ which are not in A.

Therefore $A' = \{6, 7, 8, 9, 10\}$.

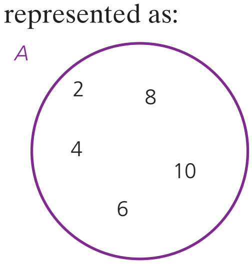
b If $\mathscr{E}$ = {all 3D shapes} and P = {**prisms**} what set is represented by P'?

P' = {all 3D shapes except prisms}.

Set notation and Venn diagrams

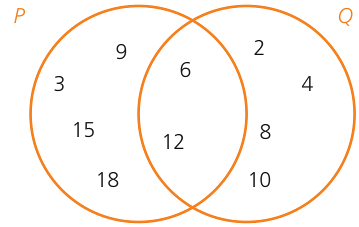
Venn diagrams are the principal way of showing sets diagrammatically. The method consists primarily of entering the elements of a set into a circle or circles.

Some examples of the uses of Venn diagrams are shown.
$A = \{2, 4, 6, 8, 10\}$ can be represented as:



Elements which are in more than one set can also be represented using a Venn diagram.

$P = \{3, 6, 9, 12, 15, 18\}$ and $Q = \{2, 4, 6, 8, 10, 12\}$ can be represented as:

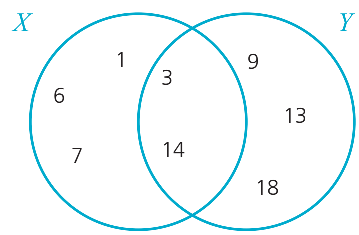


In the diagram above, it can be seen that those elements which belong to both sets are placed in the region of overlap of the two circles.

When two sets P and Q overlap as they do above, the notation $P \cap Q$ is used to denote the set of elements in the **intersection**, i.e. $P \cap Q = \{6, 12\}$.

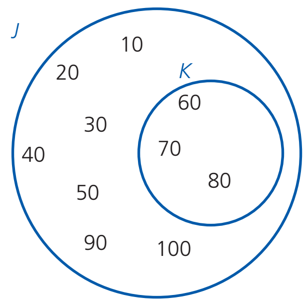
6 belongs to the set of $P \cap Q$; whilst 8 does not belong to the set of $P \cap Q$.

$X = \{1, 3, 6, 7, 14\}$ and $Y = \{3, 9, 13, 14, 18\}$ are represented as:



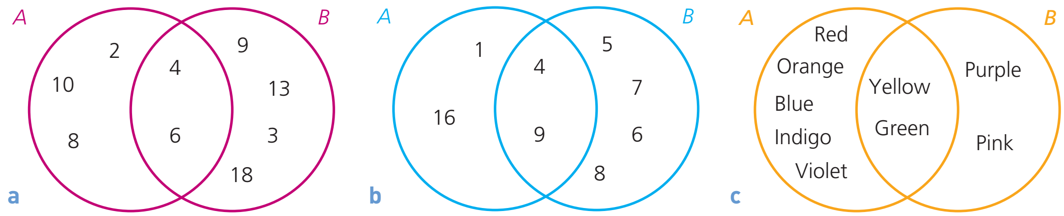
The **union** of two sets is everything which belongs to either or both sets and is represented by the symbol $\cup$.
Therefore in the example above $X \cup Y = \{1, 3, 6, 7, 9, 13, 14, 18\}$.

$J = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$ and $K = \{60, 70, 80\}$;
as discussed earlier, $K \subseteq J$ and can be represented as shown below:



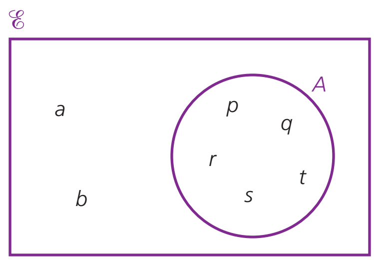
Exercise 10.3

1 Complete the statement $A \cap B = \{...\}$ for each of the Venn diagrams below.



2 Complete the statement $A \cup B = \{...\}$ for each of the Venn diagrams in Question 1 above.

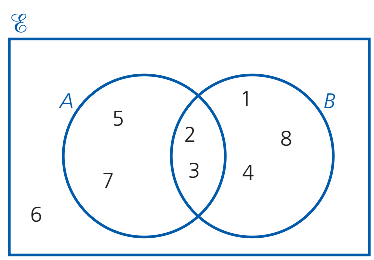
3



Copy and complete the following statements:

a $\mathscr{E} = \{...\}$ **b** $A' = \{...\}$

4

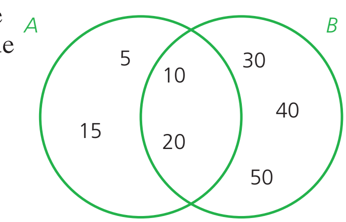


Copy and complete the following statements:

a $\mathscr{E} = \{...\}$ **b** $A' = \{...\}$ **c** $A \cap B = \{...\}$

d $A \cup B = \{...\}$ **e** $(A \cap B)' = \{...\}$ **f** $A \cap B' = \{...\}$

5 Using the Venn diagram (right), indicate whether the following statements are true or false. $\in$ means 'is an element of' and $\notin$ means 'is not an element of'.

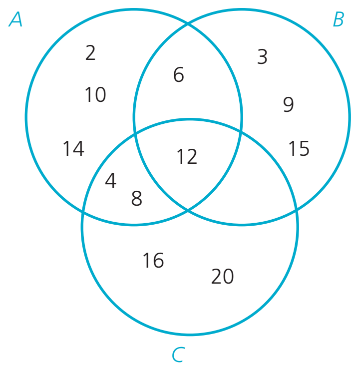


a $5 \in A$ **b** $20 \in B$

c $20 \notin A$ **d** $50 \in A$

e $50 \notin B$ **f** $A \cap B = \{10, 20\}$

6



a Describe in words the elements of:

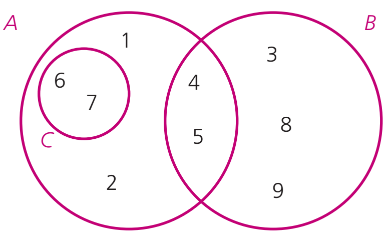
i set A **ii** set B **iii** set C

b Copy and complete the following statements:

i $A \cap B = \{...\}$ **ii** $A \cap C = \{...\}$ **iii** $B \cap C = \{...\}$

iv $A \cap B \cap C = \{...\}$ **v** $A \cup B = \{...\}$ **vi** $C \cup B = \{...\}$

7



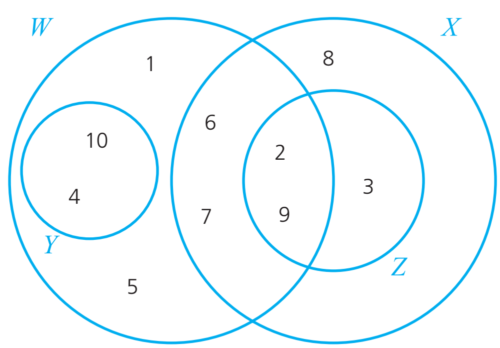
a Copy and complete the following statements:

i $A = \{...\}$ **ii** $B = \{...\}$ **iii** $C' = \{...\}$

iv $A \cap B = \{...\}$ **v** $A \cup B = \{...\}$ **vi** $(A \cap B)' = \{...\}$

b State, using set notation, the relationship between C and A.

8



a Copy and complete the following statements:

i $W = \{...\}$ **ii** $X = \{...\}$ **iii** $Z' = \{...\}$

iv $W \cap Z = \{...\}$ **v** $W \cap X = \{...\}$ **vi** $Y \cap Z = ...$

b Which of the named sets is a subset of X?

Exercise 10.4

1 $A = \{\text{Egypt, Libya, Morocco, Chad}\}$
$B = \{\text{Iran, Iraq, Turkey, Egypt}\}$
a Draw a Venn diagram to illustrate the above information.
b Copy and complete the following statements:
i $A \cap B = \{...\}$ ii $A \cup B = \{...\}$

2 $P = \{2, 3, 5, 7, 11, 13, 17\}$
$Q = \{11, 13, 15, 17, 19\}$
a Draw a Venn diagram to illustrate the above information.
b Copy and complete the following statements:
i $P \cap Q = \{...\}$ ii $P \cup Q = \{...\}$

3 $B = \{2, 4, 6, 8, 10\}$
$A \cup B = \{1, 2, 3, 4, 6, 8, 10\}$
$A \cap B = \{2, 4\}$
Represent the above information on a Venn diagram.

4 $X = \{a, c, d, e, f, g, l\}$
$Y = \{b, c, d, e, h, i, k, l, m\}$
$Z = \{c, f, i, j, m\}$
Represent the above information on a Venn diagram.

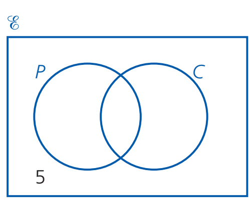
5 $P = \{1, 4, 7, 9, 11, 15\}$
$Q = \{5, 10, 15\}$
$R = \{1, 4, 9\}$
Represent the above information on a Venn diagram.

Problems involving sets

Worked example

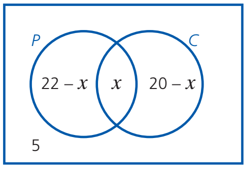
In a class of 31 students, some study Physics and some study Chemistry. If 22 study Physics, 20 study Chemistry and 5 study neither, calculate the number of students who take both subjects.

The information given above can be entered in a Venn diagram in stages.



The students taking neither Physics nor Chemistry can be put in first (as shown left).

This leaves 26 students to be entered into the set circles.



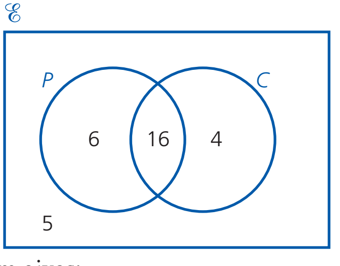
If x students take both subjects then

$$n(P) = 22 - x + x$$
$$n(C) = 20 - x + x$$
$$P \cup C = 31 - 5 = 26$$

Therefore $22 - x + x + 20 - x = 26$

$$42 - x = 26$$
$$x = 16$$

Substituting the value of x into the Venn diagram gives:



Therefore the number of students taking both Physics and Chemistry is 16.

Exercise 10.5

1 In a class of 35 students, 19 take Spanish, 18 take Korean and 3 take neither. Calculate how many take:
   a both Korean and Spanish,
   b just Spanish,
   c just Korean.

2 In a year group of 108 students, 60 liked football, 53 liked tennis and 10 liked neither. Calculate the number of students who liked football but not tennis.

3 In a year group of 113 students, 60 liked hockey, 45 liked rugby and 18 liked neither. Calculate the number of students who:
   a liked both hockey and rugby,
   b liked only hockey.

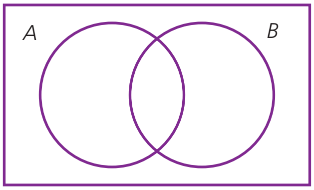
4 One year, 37 students sat an examination in Physics, 48 sat Chemistry and 45 sat Biology. 15 students sat Physics and Chemistry, 13 sat Chemistry and Biology, 7 sat Physics and Biology and 5 students sat all three.
   a Draw a Venn diagram to represent this information.
   b Calculate $n(P \cup C \cup B)$.

Student assessment 1

1 Describe the following sets in words:
   a {2, 4, 6, 8}
   b {2, 4, 6, 8, ...}
   c {1, 4, 9, 16, 25, ...}
   d {Arctic, Atlantic, Indian, Pacific}

2 Calculate the value of $n(A)$ for each of the sets shown below:
   a A = {days of the week}
   b A = {prime numbers between 50 and 60}
   c $A = \{x: x \text{ is an integer and } 5 \leqslant x \leqslant 10\}$
   d A = {days in a leap year}

3 Copy out the Venn diagram (below) twice.
   a On one copy, shade and label the region which represents $A \cap B$.
   b On the other copy, shade and label the region which represents $A \cup B$.



4 If $\mathscr{E}$ = {m, a, t, h, s} and A = {a, s}, what set is represented by A'?

5 If A = {a, b}, list all the subsets of A.

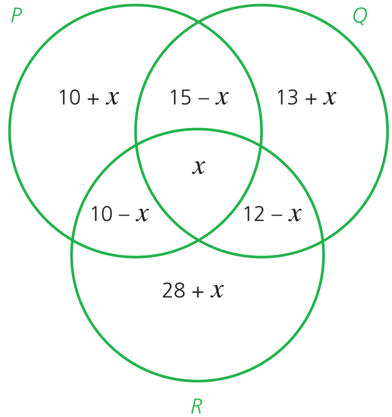
Student assessment 2

1 J = {London, Paris, Rome, Washington, Canberra, Ankara, Cairo}
K = {Cairo, Nairobi, Pretoria, Ankara}
  a Draw a Venn diagram to represent the above information.
  b Copy and complete the statement $J \cap K = \{...\}$.
  c Copy and complete the statement $J' \cap K = \{...\}$.
2 $M = \{x: x \text{ is an integer and } 2 \leqslant x \leqslant 20\}$
N = {prime numbers less than 30}
  a Draw a Venn diagram to illustrate the information above.
  b Copy and complete the statement $M \cap N = \{...\}$.
  c Copy and complete the statement $(M \cap N)' = \{...\}$.
3 $\mathscr{E}$ = {natural numbers}, M = {even numbers} and N = {multiples of 5}.
  a Draw a Venn diagram and place the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 in the appropriate places in it.
  b If $X = M \cap N$, describe set X in words.
4 If $A = \{2, 4, 6, 8\}$, write all the subsets of A with two or more elements.
5 In a region of mixed farming, farms keep goats, chickens or sheep. There are 77 farms altogether. 19 farms keep only goats, 8 keep only chickens and 13 keep only sheep. 13 keep both goats and chickens, 28 keep both chickens and sheep and 8 keep both goats and sheep.
  a Draw a Venn diagram to show the above information.
  b Calculate $n(G \cap C \cap S)$.

Student assessment 3

1 M = {a, e, i, o, u}
  a How many subsets are there of M?
  b List the subsets of M with four or more elements.
2 X = {lion, tiger, cheetah, leopard, puma, jaguar, cat}
Y = {elephant, lion, zebra, cheetah, gazelle}
Z = {anaconda, jaguar, tarantula, mosquito}
  a Draw a Venn diagram to represent the above information.
  b Copy and complete the statement $X \cap Y = \{...\}$.
  c Copy and complete the statement $Y \cap Z = \{...\}$.
  d Copy and complete the statement $X \cap Y \cap Z = \{...\}$.
3 A group of 40 people were asked whether they like cricket (C) and football (F). The number liking both cricket and football was three times the number liking only cricket. Adding 3 to the number liking only cricket and doubling the answer equals the number of people liking only football. Four said they did not like sport at all.
  a Draw a Venn diagram to represent this information.
  b Calculate $n(C \cap F)$.
  c Calculate $n(C \cap F')$.
  d Calculate $n(C' \cap F)$.

4 The Venn diagram below shows the number of elements in three sets P, Q and R.



If $n(P \cup Q \cup R) = 93$ calculate:

a x	**b** $n(P)$	**c** $n(Q)$
d $n(R)$	**e** $n(P \cap Q)$	**f** $n(Q \cap R)$
g $n(P \cap R)$	**h** $n(R \cup Q)$	**i** $n(P \cap Q)'$

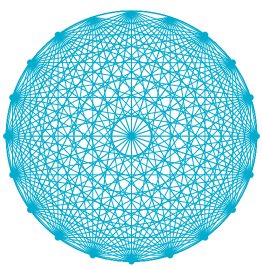
1 Mathematical investigations and ICT 1

Investigations are an important part of mathematical learning. All mathematical discoveries stem from an idea that a mathematician has and then investigates.

Sometimes when faced with a mathematical investigation, it can seem difficult to know how to start. The structure and example below may help you.

1 Read the question carefully and start with simple cases.
2 Draw simple diagrams to help.
3 Put the results from simple cases in an ordered table.
4 Look for a pattern in your results.
5 Try to find a general rule in words.
6 Express your rule algebraically.
7 Test the rule for a new example.
8 Check that the original question has been answered.

Worked example



A mystic rose is created by placing a number of points evenly spaced on the circumference of a circle. Straight lines are then drawn from each point to every other point. The diagram (left) shows a mystic rose with 20 points.

a How many straight lines are there?

b How many straight lines would there be on a mystic rose with 100 points?

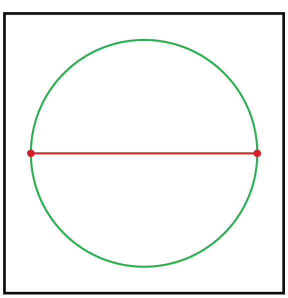
To answer these questions, you are not expected to draw either of the shapes and count the number of lines.

1/2. Try simple cases:

By drawing some simple cases and counting the lines, some results can be found:

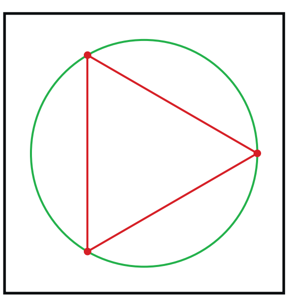
Mystic rose with 2 points

Number of lines = 1



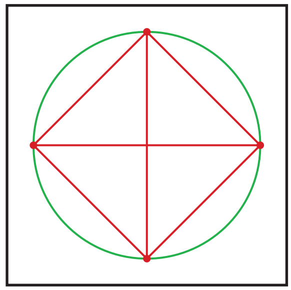
Mystic rose with 3 points

Number of lines = 3



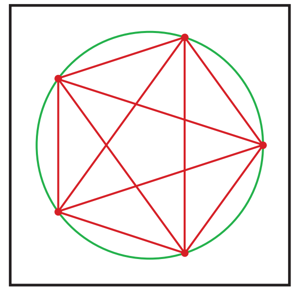
Mystic rose with 4 points

Number of lines = 6



Mystic rose with 5 points

Number of lines = 10



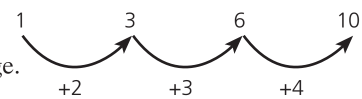
3. Enter the results in an ordered table:

Number of points	2	3	4	5
Number of lines	1	3	6	10

4/5. Look for a pattern in the results:

There are two patterns.

The first shows how the values change.



It can be seen that the difference between successive **terms** is increasing by one each time.

The problem with this pattern is that to find the 20th and 100th terms, it would be necessary to continue this pattern and find all the terms leading up to the 20th and 100th terms.

The second is the relationship between the number of points and the number of lines.

Number of points	2	3	4	5
Number of lines	1	3	6	10

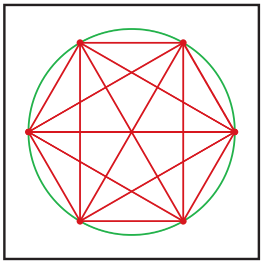
It is important to find a relationship that works for all values. For example, subtracting 1 from the number of points gives the number of lines in the first example only, so is not useful. However, halving the number of points and multiplying this by 1 less than the number of points works each time, i.e. number of lines = (half the number of points) × (one less than the number of points).

6. Express the rule algebraically:

The rule expressed in words above can be written more elegantly using algebra. Let the number of lines be l and the number of points be p.

$$l = \frac{1}{2}p(p - 1)$$

Note: Any letters can be used to represent the number of lines and the number of points, not just l and p.



7. Test the rule:

The rule was derived from the original results. It can be tested by generating a further result.

If the number of points $p = 6$, then the number of lines l is:

$$l = \frac{1}{2} \times 6(6 - 1)$$
$$= 3 \times 5$$
$$= 15$$

From the diagram to the left, the number of lines can also be counted as 15.

8. Check that the original questions have been answered

Using the formula, the number of lines in a mystic rose with 20 points is:

$$l = \frac{1}{2} \times 20(20 - 1)$$
$$= 10 \times 19$$
$$= 190$$

The number of lines in a mystic rose with 100 points is:

$$l = \frac{1}{2} \times 100(100 - 1)$$
$$= 50 \times 99$$
$$= 4950$$

Primes and squares

13, 41 and 73 are prime numbers.

Two different square numbers can be added together to make these prime numbers, e.g. $3^2 + 8^2 = 73$.

1 Find the two square numbers that can be added to make 13 and 41.
2 List the prime numbers less than 100.
3 Which of the prime numbers less than 100 can be shown to be the sum of two different square numbers?
4 Is there a rule to the numbers in Question 3?
5 Your rule is a predictive rule not a formula. Discuss the difference.

Football leagues

There are 18 teams in a football league.

1 If each team plays the other teams twice, once at home and once away, then how many matches are played in a season?
2 If there are t teams in a league, how many matches are played in a season?

ICT activity 1

In this activity, you will be using a spreadsheet to track the price of a company's shares over a period of time.

1 **a** Using the internet or a newspaper as a resource, find the value of a particular company's shares.

b Over a period of a month (or week), record the value of the company's shares. This should be carried out on a daily basis.

2 When you have collected all the values, enter them into a spreadsheet similar to the one shown on the left.

	A	B	C
1	Company Name		
2	Day	Share Price	Percentage Value
3	1	3.26	100
4	2	3.29	
5	3	4.11	
6	4		
7	5		
8			
9			
10			
11	etc	etc	

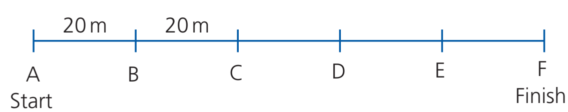
3 In column C, enter formulas that will calculate the value of the shares as a percentage of their value on day 1.

4 When the spreadsheet is complete, produce a graph showing how the percentage value of the share price changed over time.

5 Write a short report explaining the performance of the company's shares during that time.

ICT activity 2

The following activity requires the use of a graphing package.
The velocity of a student at different parts of a 100 m sprint will be analysed.
A running track is set out as shown below:



1 A student must stand at each of points A–F. The student at A runs the 100 m and is timed as they run past each of the points B–F by the students at these points, who each have a stopwatch.

2 Using the graphing package, plot a distance–time graph of the results by entering the data as pairs of coordinates, i.e. (time, distance).

3 Ensure that all the points are selected and draw a curve of best fit through them.

4 Select the curve and plot a coordinate of your choice on it. This point can now be moved along the curve using the cursor keys on the keyboard.

5 Draw a **tangent to the curve** through the point.

6 What does the gradient of the tangent represent?

7 At what point of the race was the student running fastest? How did you reach this answer?

8 Collect similar data for other students. Compare their graphs and running speeds.

9 Carefully analyse one of the graphs and write a brief report to the runner in which you should identify, giving reasons, the parts of the race they need to improve on.

TOPIC 2

Algebra and graphs

Contents

Learning objectives

E2.1 Introduction to algebra

1 Know that letters can be used to represent generalised numbers.
2 Substitute numbers into expressions and formulas.

E2.2 Algebraic manipulation

1 Simplify expressions by collecting like terms.
2 Expand products of algebraic expressions.
3 Factorise by extracting common factors.
4 Factorise expressions of the form:
- $ax + bx + kay + kby$
- $a^2x^2 - b^2y^2$
- $a^2 + 2ab + b^2$
- $ax^2 + bx + c$
- $ax^3 + bx^2 + cx$.

5 Complete the square for expressions in the form $ax^2 + bx + c$.

E2.3 Algebraic fractions

1 Manipulate algebraic fractions.
2 Factorise and simplify rational expressions.

E2.4 Indices II

1 Understand and use indices (positive, zero, negative and fractional).
2 Understand and use the rules of indices.

E2.5 Equations

1 Construct expressions, equations and formulas.
2 Solve linear equations in one unknown.
3 Solve fractional equations with numerical and linear algebraic denominators.
4 Solve simultaneous linear equations in two unknowns.
5 Solve simultaneous equations, involving one linear and one non-linear.

6 Solve quadratic equations by factorisation, completing the square and by use of the quadratic formula.
7 Change the subject of formulas.

E2.6 Inequalities

1 Represent and interpret inequalities, including on a number line.
2 Construct, solve and interpret linear inequalities.
3 Represent and interpret linear inequalities in two variables graphically.
4 List inequalities that define a given region.

E2.7 Sequences

1 Continue a given number sequence or pattern.
2 Recognise patterns in sequences, including the term-to-term rule, and relationships between different sequences.
3 Find and use the nth term of sequences.

E2.8 Proportion

Express direct and inverse proportion in algebraic terms and use this form of expression to find unknown quantities.

E2.9 Graphs in practical situations

1 Use and interpret graphs in practical situations, including travel graphs and conversion graphs.
2 Draw graphs from given data.
3 Apply the idea of rate of change to simple kinematics involving distance–time and speed–time graphs, acceleration and deceleration.
4 Calculate distance travelled as area under a speed–time graph.

E2.10 Graphs of functions

1 Construct tables of values, and draw, recognise and interpret graphs for functions of the following forms:
- ax^n (includes sums of no more than three of these)
- $ab^x + c$

where $n = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2, 3$; a and c are rational numbers; and b is a positive integer.
2 Solve associated equations graphically, including finding and interpreting roots by graphical methods.
3 Draw and interpret graphs representing exponential growth and decay problems.

E2.11 Sketching curves

Recognise, sketch and interpret graphs of the following functions:
a linear
b quadratic
c cubic
d reciprocal
e exponential.

E2.12 Differentiation

1 Estimate gradients of curves by drawing tangents.
2 Use the derivatives of functions of the form ax^n, where a is a rational constant and n is a positive integer or zero, and simple sums of not more than three of these.
3 Apply differentiation to gradients and stationary points (turning points).
4 Discriminate between maxima and minima by any method.

E2.13 Functions

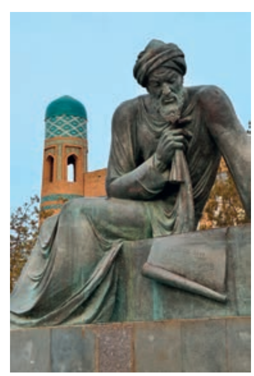
1 Understand functions, domain and range and use function notation.
2 Understand and find inverse functions $f^{-1}(x)$.
3 Form composite functions as defined by $gf(x) = g(f(x))$.

The founders of algebra

Abū Ja‘far Muḥammad ibn Mūsā al-Khwārizmī is called the ‘father of algebra’. He was born in Baghdad in 790CE. He wrote the book *Hisab al-jabr w’al-muqabala* in 830CE when Baghdad had the greatest university in the world and the greatest mathematicians studied there. He gave us the word ‘algebra’ and worked on quadratic equations. He also introduced the decimal system from India.

Muhammad al-Karaji was born in North Africa in what is now Morocco. He lived in the eleventh century and worked on the theory of indices. He also worked on an algebraic method of calculating square and cube roots. He may also have travelled to the University of Granada (then part of the Moorish Empire) where works of his can be found in the University library.

The poet Omar Khayyam is known for his long poem *The Rubaiyat*. He was also a fine mathematician, working on the binomial theorem. He introduced the symbol ‘shay’, which became our ‘x’.



Al-Khwārizmī (790–850)

11 Algebraic representation and manipulation

Algebra is a mathematical language and is at the heart of mathematics. As well as numbers, letters are also used. The letters are used to represent unknown quantities, or a variety of possible different values.

Using algebra may at first seem complicated, but as with any language, the more you use it and the more you understand its rules, the easier it becomes.

This topic deals with those rules.

Expanding a bracket

When removing brackets, every term inside the bracket must be multiplied by whatever is outside the bracket.

Worked examples

a $3(x+4)$
$= 3x + 12$

b $5x(2y+3)$
$= 10xy + 15x$

c $2a(3a+2b-3c)$
$= 6a^2 + 4ab - 6ac$

d $-4p(2p-q+r^2)$
$= -8p^2 + 4pq - 4pr^2$

e $-2x^2\left(1x+3y-\frac{1}{x}\right)$
$= -2x^3 - 6x^2y + 2x$

f $\frac{-2}{x}\left(-2x+4y+\frac{1}{x}\right)$
$= 4 - \frac{8y}{x} - \frac{2}{x^2}$

Exercise 11.1

Expand the following:

1 **a** $4(x-3)$ **b** $5(2p-4)$
c $-6(7x-4y)$ **d** $3(2a-3b-4c)$
e $-7(2m-3n)$ **f** $-2(8x-3y)$

2 **a** $3x(x-3y)$ **b** $a(a+b+c)$
c $4m(2m-n)$ **d** $-5a(3a-4b)$
e $-4x(-x+y)$ **f** $-8p(-3p+q)$

3 **a** $-(2x^2-3y^2)$ **b** $-(-a+b)$
c $-(-7p+2q)$ **d** $\frac{1}{2}(6x-8y+4z)$
e $\frac{3}{4}(4x-2y)$ **f** $\frac{1}{5}x(10x-15y)$

4 **a** $3r(4r^2-5s+2t)$ **b** $a^2(a+b+c)$
c $3a^2(2a-3b)$ **d** $pq(p+q-pq)$
e $m^2(m-n+nm)$ **f** $a^3(a^3+a^2b)$

Exercise 11.2

Expand and simplify the following:

1 a $3a - 2(2a + 4)$ b $8x - 4(x + 5)$
c $3(p - 4) - 4$ d $7(3m - 2n) + 8n$
e $6x - 3(2x - 1)$ f $5p - 3p(p + 2)$

2 a $7m(m + 4) + m^2 + 2$ b $3(x - 4) + 2(4 - x)$
c $6(p + 3) - 4(p - 1)$ d $5(m - 8) - 4(m - 7)$
e $3a(a + 2) - 2(a^2 - 1)$ f $7a(b - 2c) - c(2a - 3)$

3 a $\frac{1}{2}(6x + 4) + \frac{1}{3}(3x + 6)$ b $\frac{1}{4}(2x + 6y) + \frac{3}{4}(6x - 4y)$
c $\frac{1}{8}(6x - 12y) + \frac{1}{2}(3x - 2y)$ d $\frac{1}{5}(15x + 10y) + \frac{3}{10}(5x - 5y)$
e $\frac{2}{3}(6x - 9y) + \frac{1}{3}(9x + 6y)$ f $\frac{x}{7}(14x - 21y) - \frac{x}{2}(4x - 6y)$

Expanding a pair of brackets

When multiplying together expressions in brackets, it is necessary to multiply all the terms in one bracket by all the terms in the other bracket.

Worked examples

Expand the following:

a $(x + 3)(x + 5)$

	x	$+3$
x	x^2	$3x$
$+5$	$5x$	15

$= x^2 + 5x + 3x + 15$

$= x^2 + 8x + 15$

b $(x + 2)(2x - 1)$

	$2x$	-1
x	$2x^2$	$-x$
$+2$	$4x$	-2

$= 2x^2 - x + 4x - 2$

$= 2x^2 + 3x - 2$

Exercise 11.3

Expand the following and simplify your answer:

1 a $(x + 2)(x + 3)$ b $(x + 3)(x + 4)$
c $(x + 5)(x + 2)$ d $(x + 6)(x + 1)$
e $(x - 2)(x + 3)$ f $(x + 8)(x - 3)$

2 a $(x - 4)(x + 6)$ b $(x - 7)(x + 4)$
c $(x + 5)(x - 7)$ d $(x + 3)(x - 5)$
e $(x + 1)(x - 3)$ f $(x - 7)(x + 9)$

3 a $(2x - 3)(x - 3)$ b $(2x - 5)(x - 2)$
c $(x - 4)(3x - 8)$ d $(x + 3)(5x - 3)$
e $(2x - 3)^2$ f $(2x - 7)(3x - 5)$

4 a $(x + 3)(x - 3)$ b $(x + 7)(x - 7)$
c $(x - 8)(x + 8)$ d $(x + y)(x - y)$
e $(a + b)(a - b)$ f $(p - q)(p + q)$

Simple factorising

When factorising, the largest possible factor is removed from each of the terms and placed outside the brackets.

Worked examples

Factorise the following expressions:

a $10x + 15$
$= 5(2x + 3)$

b $8p - 6q + 10r$
$= 2(4p - 3q + 5r)$

c $-2q - 6p + 12$
$= 2(-q - 3p + 6)$

d $2a^2 + 3ab - 5ac$
$= a(2a + 3b - 5c)$

e $6ax - 12ay - 18a^2$
$= 6a(x - 2y - 3a)$

f $3b + 9ba - 6bd$
$= 3b(1 + 3a - 2d)$

Exercise 11.4

Factorise the following:

1 **a** $4x - 6$ **b** $18 - 12p$
c $6y - 3$ **d** $4a + 6b$
e $3p - 3q$ **f** $8m + 12n + 16r$

2 **a** $3ab + 4ac - 5ad$ **b** $8pq + 6pr - 4ps$
c $a^2 - ab$ **d** $4x^2 - 6xy$
e $abc + abd + fab$ **f** $3m^2 + 9m$

3 **a** $3pqr - 9pqs$ **b** $5m^2 - 10mn$
c $8x^2y - 4xy^2$ **d** $2a^2b^2 - 3b^2c^2$
e $12p - 36$ **f** $42x - 54$

4 **a** $18 + 12y$ **b** $14a - 21b$
c $11x + 11xy$ **d** $4s - 16t + 20r$
e $5pq - 10qr + 15qs$ **f** $4xy + 8y^2$

5 **a** $m^2 + mn$ **b** $3p^2 - 6pq$
c $pqr + qrs$ **d** $ab + a^2b + ab^2$
e $3p^3 - 4p^4$ **f** $7b^3c + b^2c^2$

6 **a** $m^3 - m^2n + mn^2$ **b** $4r^3 - 6r^2 + 8r^2s$
c $56x^2y - 28xy^2$ **d** $72m^2n + 36mn^2 - 18m^2n^2$

Substitution

Worked examples

Evaluate the expressions below if $a = 3$, $b = 4$, $c = -5$:

a $2a + 3b - c$
$= 6 + 12 + 5$
$= 23$

b $3a - 4b + 2c$
$= 9 - 16 - 10$
$= -17$

c $-2a + 2b - 3c$
$= -6 + 8 + 15$
$= 17$

d $a^2 + b^2 + c^2$
$= 9 + 16 + 25$
$= 50$

e $3a(2b - 3c)$
$= 9(8 + 15)$
$= 9 \times 23$
$= 207$

f $-2c(-a + 2b)$
$= 10(-3 + 8)$
$= 10 \times 5$
$= 50$

Exercise 11.5

Evaluate the following expressions if $p = 4$, $q = -2$, $r = 3$ and $s = -5$:

1 a $2p + 4q$ b $5r - 3s$
c $3q - 4s$ d $6p - 8q + 4s$
e $3r - 3p + 5q$ f $-p - q + r + s$

2 a $2p - 3q - 4r + s$ b $3s - 4p + r + q$
c $p^2 + q^2$ d $r^2 - s^2$
e $p(q - r + s)$ f $r(2p - 3q)$

3 a $2s(3p - 2q)$ b $pq + rs$
c $2pr - 3rq$ d $q^3 - r^2$
e $s^3 - p^3$ f $r^4 - q^5$

4 a $-2pqr$ b $-2p(q + r)$
c $-2rq + r$ d $(p + q)(r - s)$
e $(p + s)(r - q)$ f $(r + q)(p - s)$

5 a $(2p + 3q)(p - q)$ b $(q + r)(q - r)$
c $q^2 - r^2$ d $p^2 - r^2$
e $(p + r)(p - r)$ f $(-s + p)q^2$

Rearrangement of formulas

In the formula $a = 2b + c$, 'a' is the subject. In order to make either b or c the subject, the formula has to be rearranged.

Worked examples

Rearrange the following formulas to make the red letter the subject:

a $a = 2b + c$
$a - 2b = c$

b $2r + p = q$
$p = q - 2r$

c $ab = cd$
$\frac{ab}{d} = c$

d $\frac{a}{b} = \frac{c}{d}$
$ad = cb$
$d = \frac{cb}{a}$

Exercise 11.6

In the following questions, make the letter in red the subject of the formula:

1 **a** $m + n = r$ **b** $m + n = p$ **c** $2m + n = 3p$
d $3x = 2p + q$ **e** $ab = cd$ **f** $ab = cd$

2 **a** $3xy = 4m$ **b** $7pq = 5r$ **c** $3x = c$
d $3x + 7 = y$ **e** $5y - 9 = 3r$ **f** $5y - 9 = 3x$

3 **a** $6b = 2a - 5$ **b** $6b = 2a - 5$ **c** $3x - 7y = 4z$
d $3x - 7y = 4z$ **e** $3x - 7y = 4z$ **f** $2pr - q = 8$

4 **a** $\frac{p}{4} = r$ **b** $\frac{4}{p} = 3r$ **c** $\frac{1}{5}n = 2p$
d $\frac{1}{5}n = 2p$ **e** $p(q + r) = 2t$ **f** $p(q + r) = 2t$

5 **a** $3m - n = rt(p + q)$ **b** $3m - n = rt(p + q)$
c $3m - n = rt(p + q)$ **d** $3m - n = rt(p + q)$
e $3m - n = rt(p + q)$ **f** $3m - n = rt(p + q)$

6 **a** $\frac{ab}{c} = de$ **b** $\frac{ab}{c} = de$ **c** $\frac{ab}{c} = de$
d $\frac{a + b}{c} = d$ **e** $\frac{a}{c} + b = d$ **f** $\frac{a}{c} + b = d$

Further expansion

You will have seen earlier in this chapter how to expand a pair of brackets of the form $(x - 3)(x + 4)$. A similar method can be used to expand a pair of brackets of the form $(2x - 3)(3x - 6)$.

Worked examples

a Expand $(2x - 3)(3x - 6)$.

	$2x$	-3
$3x$	$6x^2$	$-9x$
-6	$-12x$	18

$= 6x^2 - 9x - 12x + 18$
$= 6x^2 - 21x + 18$

b Expand $(x-1)(x+2)(2x-5)$.

The expansion can be shown using diagrams as in example **a** above. To do so, it is easier to carry out the multiplication in steps.

Step 1: Expand $(x-1)(x+2)$

	x	-1
x	x^2	$-x$
$+2$	$2x$	-2

$= x^2 - x + 2x - 2$

$= x^2 + x - 2$

Step 2: Expand $(x^2 + x - 2)(2x - 5)$

	x^2	$+x$	-2
$2x$	$2x^3$	$2x^2$	$-4x$
-5	$-5x^2$	$-5x$	10

$= 2x^3 + 2x^2 - 4x - 5x^2 - 5x + 10$

$= 2x^3 - 3x^2 - 9x + 10$

Exercise 11.7

Expand the following brackets, giving your answer in its simplest form:

1 **a** $(y+2)(2y+3)$ **b** $(y+7)(3y+4)$
c $(2y+1)(y+8)$ **d** $(2y+1)(2y+2)$
e $(3y+4)(2y+5)$ **f** $(6y+3)(3y+1)$

2 **a** $(2p-3)(p+8)$ **b** $(4p-5)(p+7)$
c $(3p-4)(2p+3)$ **d** $(4p-5)(3p+7)$
e $(6p+2)(3p-1)$ **f** $(7p-3)(4p+8)$

3 **a** $(2x-1)(2x-1)$ **b** $(3x+1)^2$
c $(4x-2)^2$ **d** $(5x-4)^2$
e $(2x+6)^2$ **f** $(2x+3)(2x-3)$

4 **a** $(3+2x)(3-2x)$ **b** $(4x-3)(4x+3)$
c $(3+4x)(3-4x)$ **d** $(7-5y)(7+5y)$
e $(3+2y)(4y-6)$ **f** $(7-5y)^2$

5 **a** $(x+3)(3x+1)(x+2)$ **b** $(2x+4)(2x+1)(x-2)$
c $(-x+1)(3x-1)(4x+3)$ **d** $(-2x-3)(-x+1)(-x+5)$
e $(2x^2-3x+1)(-x+4)$ **f** $(4x-1)(-3x^2-3x-2)$

Further factorisation

Factorisation by grouping

Worked examples

Factorise the following expressions:

a $6x + 3 + 2xy + y$
$= 3(2x + 1) + y(2x + 1)$
$= (3 + y)(2x + 1)$
Note that $(2x + 1)$ was a common factor of both terms.

b $ax + ay - bx - by$
$= a(x + y) - b(x + y)$
$= (a - b)(x + y)$

c $2x^2 - 3x + 2xy - 3y$
$= x(2x - 3) + y(2x - 3)$
$= (x + y)(2x - 3)$

Exercise 11.8

Factorise the following by grouping:

1 **a** $ax + bx + ay + by$ **b** $ax + bx - ay - by$
c $3m + 3n + mx + nx$ **d** $4m + mx + 4n + nx$
e $3m + mx - 3n - nx$ **f** $6x + xy + 6z + zy$

2 **a** $pr - ps + qr - qs$ **b** $pq - 4p + 3q - 12$
c $pq + 3q - 4p - 12$ **d** $rs + rt + 2ts + 2t^2$
e $rs - 2ts + rt - 2t^2$ **f** $ab - 4cb + ac - 4c^2$

3 **a** $xy + 4y + x^2 + 4x$ **b** $x^2 - xy - 2x + 2y$
c $ab + 3a - 7b - 21$ **d** $ab - b - a + 1$
e $pq - 4p - 4q + 16$ **f** $mn - 5m - 5n + 25$

4 **a** $mn - 2m - 3n + 6$ **b** $mn - 2mr - 3rn + 6r^2$
c $pr - 4p - 4qr + 16q$ **d** $ab - a - bc + c$
e $x^2 - 2xz - 2xy + 4yz$ **f** $2a^2 + 2ab + b^2 + ab$

Difference of two squares

On expanding

$(x + y)(x - y)$
$= x^2 - xy + xy - y^2$
$= x^2 - y^2$

The reverse is that $x^2 - y^2$ factorises to $(x + y)(x - y)$. x^2 and y^2 are both square and therefore $x^2 - y^2$ is known as the **difference of two squares**.

Worked examples

a $p^2 - q^2$
$= (p + q)(p - q)$

b $4a^2 - 9b^2$
$= (2a)^2 - (3b)^2$
$= (2a + 3b)(2a - 3b)$

c $(mn)^2 - 25k^2$
$= (mn)^2 - (5k)^2$
$= (mn + 5k)(mn - 5k)$

d $4x^2 - (9y)^2$
$= (2x)^2 - (9y)^2$
$= (2x + 9y)(2x - 9y)$

Exercise 11.9

Factorise the following:

1 a $a^2 - b^2$ b $m^2 - n^2$ c $x^2 - 25$
d $m^2 - 49$ e $81 - x^2$ f $100 - y^2$

2 a $144 - y^2$ b $q^2 - 169$ c $m^2 - 1$
d $1 - t^2$ e $4x^2 - y^2$ f $25p^2 - 64q^2$

3 a $9x^2 - 4y^2$ b $16p^2 - 36q^2$ c $64x^2 - y^2$
d $x^2 - 100y^2$ e $(qr)^2 - 4p^2$ f $(ab)^2 - (cd)^2$

4 a $m^2n^2 - 9y^2$ b $\frac{1}{4}x^2 - \frac{1}{9}y^2$ c $(2x)^2 - (3y)^4$
d $p^4 - q^4$ e $4m^4 - 36y^4$ f $16x^4 - 81y^4$

Evaluation

Once factorised, numerical expressions can be evaluated.

Worked examples

Evaluate the following expressions:

a $13^2 - 7^2$
$= (13 + 7)(13 - 7)$
$= 20 \times 6$
$= 120$

b $6.25^2 - 3.75^2$
$= (6.25 + 3.75)(6.25 - 3.75)$
$= 10 \times 2.5$
$= 25$

Exercise 11.10

By factorising, evaluate the following:

1 a $8^2 - 2^2$ b $16^2 - 4^2$ c $49^2 - 1$
d $17^2 - 3^2$ e $88^2 - 12^2$ f $96^2 - 4^2$

2 a $45^2 - 25$ b $99^2 - 1$ c $27^2 - 23^2$
d $66^2 - 34^2$ e $999^2 - 1$ f $225 - 8^2$

Exercise 11.10 (cont)

3 a $8.4^2 - 1.6^2$ b $9.3^2 - 0.7^2$ c $42.8^2 - 7.2^2$
d $\left(8\frac{1}{2}\right)^2 - \left(1\frac{1}{2}\right)^2$ e $\left(7\frac{3}{4}\right)^2 - \left(2\frac{1}{4}\right)^2$ f $5.25^2 - 4.75^2$

4 a $8.62^2 - 1.38^2$ b $0.9^2 - 0.1^2$ c $3^4 - 2^4$
d $2^4 - 1$ e $1111^2 - 111^2$ f $2^8 - 25$

Factorising quadratic expressions

$x^2 + 5x + 6$ is known as a quadratic expression as the highest power of any of its terms is squared – in this case x^2.

It can be factorised by writing it as a product of two brackets.

Worked examples

a Factorise $x^2 + 5x + 6$.

	x	
x	x^2	
		$+6$

On setting up a 2×2 grid, some of the information can immediately be entered.

As there is only one term in x^2, this can be entered, as can the constant +6. The only two values which multiply to give x^2 are x and x. These too can be entered.

We now need to find two values which multiply to give +6 and which add to give $+5x$.

The only two values which satisfy both these conditions are +3 and +2.

Therefore $x^2 + 5x + 6 = (x + 3)(x + 2)$

	x	$+3$
x	x^2	$3x$
$+2$	$2x$	$+6$

b Factorise $x^2 + 2x - 24$.

	x	
x	x^2	
		-24

	x	$+6$
x	x^2	$+6x$
-4	$-4x$	-24

Therefore $x^2 + 2x - 24 = (x + 6)(x - 4)$

c Factorise $2x^2 + 11x + 12$.

	$2x$	
x	$2x^2$	
		12

	$2x$	$+3$
x	$2x^2$	$3x$
$+4$	$8x$	12

Therefore $2x^2 + 11x + 12 = (2x + 3)(x + 4)$

d Factorise $3x^2 + 7x - 6$.

	$3x$	
x	$3x^2$	
		-6

	$3x$	-2
x	$3x^2$	$-2x$
$+3$	$9x$	-6

Therefore $3x^2 + 7x - 6 = (3x - 2)(x + 3)$

Exercise 11.11

Factorise the following quadratic expressions:

1 **a** $x^2 + 7x + 12$ **b** $x^2 + 8x + 12$ **c** $x^2 + 13x + 12$
d $x^2 - 7x + 12$ **e** $x^2 - 8x + 12$ **f** $x^2 - 13x + 12$

2 **a** $x^2 + 6x + 5$ **b** $x^2 + 6x + 8$ **c** $x^2 + 6x + 9$
d $x^2 + 10x + 25$ **e** $x^2 + 22x + 121$ **f** $x^2 - 13x + 42$

3 **a** $x^2 + 14x + 24$ **b** $x^2 + 11x + 24$ **c** $x^2 - 10x + 24$
d $x^2 + 15x + 36$ **e** $x^2 + 20x + 36$ **f** $x^2 - 12x + 36$

4 **a** $x^2 + 2x - 15$ **b** $x^2 - 2x - 15$ **c** $x^2 + x - 12$
d $x^2 - x - 12$ **e** $x^2 + 4x - 12$ **f** $x^2 - 15x + 36$

5 **a** $x^2 - 2x - 8$ **b** $x^2 - x - 20$ **c** $x^2 + x - 30$
d $x^2 - x - 42$ **e** $x^2 - 2x - 63$ **f** $x^2 + 3x - 54$

6 **a** $2x^2 + 3x + 1$ **b** $2x^2 + 7x + 6$ **c** $2x^2 + x - 6$
d $2x^2 - 7x + 6$ **e** $3x^2 + 8x + 4$ **f** $3x^2 + 11x - 4$
g $4x^2 + 12x + 9$ **h** $9x^2 - 6x + 1$ **i** $6x^2 - x - 1$

Rearrangement of complex formulas

Worked examples

Make the letters in red the subject of each formula:

a

$$C = 2\pi r$$
$$\frac{C}{2\pi} = r$$

b

$$A = \frac{h}{2}(a+b)$$
$$2A = h(a+b)$$
$$\frac{2A}{h} = a+b$$
$$\frac{2A}{h} - a = b$$

c

$$x^2 + y^2 = h^2$$
$$y^2 = h^2 - x^2$$
$$y = \pm\sqrt{h^2 - x^2}$$

Note: not $y = h - x$

d

$$f = \sqrt{\frac{x}{k}}$$
$$f^2 = \frac{x}{k}$$
$$f^2 k = x$$

e

$$m = 3a\sqrt{\frac{p}{x}}$$

Square both sides

$$m^2 = \frac{9a^2 p}{x}$$
$$m^2 x = 9a^2 p$$
$$x = \frac{9a^2 p}{m^2}$$

f

$$A = \frac{y+x}{p+q^2}$$
$$A(p+q^2) = y+x$$
$$p+q^2 = \frac{y+x}{A}$$
$$q^2 = \frac{y+x}{A} - p$$
$$q = \pm\sqrt{\frac{y+x}{A} - p}$$

g

$$\frac{x}{4} = \frac{a-b}{3x}$$
$$3x^2 = 4(a-b)$$
$$x^2 = \frac{4(a-b)}{3}$$
$$x = \pm\sqrt{\frac{4(a-b)}{3}}$$

h

$$\frac{a}{bx+1} = \frac{b}{x}$$
$$ax = b(bx+1)$$
$$ax = b^2x + b$$
$$ax - b^2x = b$$
$$x(a-b^2) = b$$
$$x = \frac{b}{a-b^2}$$

Exercise 11.12

1 In the formulas below, make x the subject:

a $P = 2mx$ b $\frac{P}{Q} = rx$

2 In the following questions, make the letter in red the subject of the formula:

a $v = u + at$ (red: t) b $v^2 = u^2 + 2as$ (red: s)

c $s = ut + \frac{1}{2}at^2$ (red: u) d $s = ut + \frac{1}{2}at^2$ (red: a)

Exercise 11.13

In the formulas below, make x the subject:

1 a $T = 3x^2$ b $mx^2 = y^2$ c $x^2 + y^2 = p^2 - q^2$

d $m^2 + x^2 = y^2 - n^2$ e $p^2 - q^2 = 4x^2 - y^2$

2 a $\frac{P}{Q} = rx^2$ b $\frac{P}{Q} = \frac{x^2}{r}$ c $\frac{m}{n} = \frac{1}{x^2}$

d $\frac{r}{st} = \frac{w}{x^2}$ e $\frac{p+q}{r} = \frac{w}{x^2}$

3 a $\sqrt{x} = rp$ b $\frac{mn}{p} = \sqrt{x}$ c $g = \sqrt{\frac{k}{x}}$

d $r = 2\pi\sqrt{\frac{x}{g}}$ e $p^2 = \frac{4m^2r}{x}$ f $p = 2m\sqrt{\frac{r}{x}}$

In the following questions, make the letter in red the subject of the formula:

4 a $v^2 = u^2 + 2as$ (red: u) b $s = ut + \frac{1}{2}at^2$ (red: t)

5 a $A = \pi r\sqrt{s^2 + t^2}$ (red: r) b $A = \pi r\sqrt{h^2 + r^2}$ (red: h) c $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ (red: u)

d $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ (red: v) e $t = 2\pi\sqrt{\frac{l}{g}}$ (red: l) f $t = 2\pi\sqrt{\frac{l}{g}}$ (red: g)

6 a $\frac{xt}{7} = \frac{p+2}{3x}$ (red: x) b $\sqrt{a+2} = \frac{b-3}{\sqrt{a-2}}$ (red: a)

Exercise 11.14

1 The cost $\$x$ of printing n newspapers is given by the formula $x = 1.50 + 0.05n$.

a Calculate the cost of printing 5000 newspapers.

b Make n the subject of the formula.

c How many newspapers can be printed for \$25?

2 The formula $C = \frac{5}{9}(F - 32)$ can be used to convert temperatures in degrees Fahrenheit (°F) into degrees Celsius (°C).

a What temperature in °C is equivalent to 150 °F?

b What temperature in °C is equivalent to 12 °F?

c Make F the subject of the formula.

d Use your rearranged formula to find what temperature in °F is equivalent to 160 °C.

Exercise 11.14 (cont)

3 The height of Mount Kilimanjaro is given as 5900 m. The formula for the time taken, T hours, to climb to a height H metres is:

$$T = \frac{H}{1200} + k$$

where k is a constant.

a Calculate the time taken, to the nearest hour, to climb to the top of the mountain if $k = 9.8$.

b Make H the subject of the formula.

c How far up the mountain, to the nearest 100 m, could you expect to be after 14 hours?

4 The **volume of a cylinder** is given by the formula $V = \pi r^2 h$, where h is the height of the cylinder and r is the radius.

a Find the volume of a cylindrical post of length 7.5 m and a diameter of 30 cm.

b Make r the subject of the formula.

c A cylinder of height 75 cm has a volume of 6000 cm^3, find its radius correct to 3 s.f.

5 The formula for the **volume** V **of a sphere** is given as $V = \frac{4}{3}\pi r^3$.

a Find V if $r = 5$ cm.

b Make r the subject of the formula.

c Find the radius of a sphere of volume 2500 m^3.

Algebraic fractions

Simplifying algebraic fractions

The rules for fractions involving algebraic terms are the same as those for numeric fractions. However, the actual calculations are often easier when using algebra.

Worked examples

a $\frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$

b $\frac{a}{c} \times \frac{b}{d} = \frac{ab}{cd}$

c $\frac{\not{3}}{4} \times \frac{5}{\not{6}_2} = \frac{5}{8}$

d $\frac{\not{a}}{c} \times \frac{b}{2\not{a}} = \frac{b}{2c}$

e $\frac{\not{a}b}{e\not{c}} \times \frac{\not{c}d}{f\not{a}} = \frac{bd}{ef}$

f $\frac{x^5}{x^3} = \frac{\not{x} \times \not{x} \times \not{x} \times x \times x}{\not{x} \times \not{x} \times \not{x}} = x^2$

g $\frac{2b}{5} \div \frac{b}{7} = \frac{2b}{5} \times \frac{7}{b} = \frac{14\not{b}}{5\not{b}} = \frac{14}{5} = 2.8$

Exercise 11.15

Simplify the following **algebraic fractions**:

1 a $\frac{x}{y} \times \frac{p}{q}$ b $\frac{x}{y} \times \frac{q}{x}$ c $\frac{p}{q} \times \frac{q}{r}$

d $\frac{ab}{c} \times \frac{d}{ab}$ e $\frac{ab}{c} \times \frac{d}{ac}$ f $\frac{p^2}{q^2} \times \frac{q^2}{p}$

2 **a** $\frac{m^3}{m}$ **b** $\frac{r^7}{r^2}$ **c** $\frac{x^9}{x^3}$

d $\frac{x^2y^4}{xy^2}$ **e** $\frac{a^2b^3c^4}{ab^2c}$ **f** $\frac{pq^2r^4}{p^2q^3r}$

3 **a** $\frac{4ax}{2ay}$ **b** $\frac{12pq^2}{3p}$ **c** $\frac{15mn^2}{3mn}$

d $\frac{24x^5y^3}{8x^2y^2}$ **e** $\frac{36p^2qr}{12pqr}$ **f** $\frac{16m^2n}{24m^3n^2}$

4 **a** $\frac{2}{b} \times \frac{a}{3}$ **b** $\frac{4}{x} \times \frac{y}{2}$ **c** $\frac{8}{x} \times \frac{x}{4}$

d $\frac{9y}{2} \times \frac{2x}{3}$ **e** $\frac{12x}{7} \times \frac{7}{4x}$ **f** $\frac{4x^3}{3y} \times \frac{9y^2}{2x^2}$

5 **a** $\frac{2ax}{3bx} \times \frac{4by}{a}$ **b** $\frac{3p^2}{2q} \times \frac{5q}{3p}$

c $\frac{p^2q}{rs} \times \frac{pr}{q}$ **d** $\frac{a^2b}{fc^2} \times \frac{cd}{bd} \times \frac{ef^2}{ca^2}$

6 **a** $\frac{8x^2}{3} \div \frac{2x}{5}$ **b** $\frac{3b^3}{2} \div \frac{4b^2}{3}$

Addition and subtraction of fractions

In arithmetic it is easy to add or subtract fractions with the same denominator. It is the same process when dealing with algebraic fractions.

Worked examples

a $\frac{4}{11} + \frac{3}{11}$
$= \frac{7}{11}$

b $\frac{a}{11} + \frac{b}{11}$
$= \frac{a+b}{11}$

c $\frac{4}{x} + \frac{3}{x}$
$= \frac{7}{x}$

If the denominators are different, the fractions need to be changed to form fractions with the same denominator.

Worked examples

a $\frac{2}{9} + \frac{1}{3}$
$= \frac{2}{9} + \frac{3}{9}$
$= \frac{5}{9}$

b $\frac{a}{9} + \frac{b}{3}$
$= \frac{a}{9} + \frac{3b}{9}$
$= \frac{a+3b}{9}$

c $\frac{4}{5a} + \frac{7}{10a}$
$= \frac{8}{10a} + \frac{7}{10a}$
$= \frac{15}{10a}$
$= \frac{3}{2a}$

Similarly, with subtraction, the denominators need to be the same.

Worked examples

a $\frac{7}{a} - \frac{1}{2a}$

$= \frac{14}{2a} - \frac{1}{2a}$

$= \frac{13}{2a}$

b $\frac{p}{3} - \frac{q}{15}$

$= \frac{5p}{15} - \frac{q}{15}$

$= \frac{5p - q}{15}$

c $\frac{5}{3b} - \frac{8}{9b}$

$= \frac{15}{9b} - \frac{8}{9b}$

$= \frac{7}{9b}$

Exercise 11.16

Simplify the following fractions:

1 **a** $\frac{1}{7} + \frac{3}{7}$ **b** $\frac{a}{7} + \frac{b}{7}$ **c** $\frac{5}{13} + \frac{6}{13}$

d $\frac{c}{13} + \frac{d}{13}$ **e** $\frac{x}{3} + \frac{y}{3} + \frac{z}{3}$ **f** $\frac{p^2}{5} + \frac{q^2}{5}$

2 **a** $\frac{5}{11} - \frac{2}{11}$ **b** $\frac{c}{11} - \frac{d}{11}$ **c** $\frac{6}{a} - \frac{2}{a}$

d $\frac{2a}{3} - \frac{5b}{3}$ **e** $\frac{2x}{7} - \frac{3y}{7}$ **f** $\frac{3}{4x} - \frac{5}{4x}$

3 **a** $\frac{5}{6} - \frac{1}{3}$ **b** $\frac{5}{2a} - \frac{1}{a}$ **c** $\frac{2}{3c} + \frac{1}{c}$

d $\frac{2}{x} + \frac{3}{2x}$ **e** $\frac{5}{2p} - \frac{1}{p}$ **f** $\frac{1}{w} - \frac{3}{2w}$

4 **a** $\frac{p}{4} - \frac{q}{12}$ **b** $\frac{x}{4} - \frac{y}{2}$ **c** $\frac{m}{3} - \frac{n}{9}$

d $\frac{x}{12} - \frac{y}{6}$ **e** $\frac{r}{2} + \frac{m}{10}$ **f** $\frac{s}{3} - \frac{t}{15}$

5 **a** $\frac{3x}{4} - \frac{2x}{12}$ **b** $\frac{3x}{5} - \frac{2y}{15}$ **c** $\frac{3m}{7} + \frac{m}{14}$

d $\frac{4m}{3p} - \frac{3m}{9p}$ **e** $\frac{4x}{3y} - \frac{5x}{6y}$ **f** $\frac{3r}{7s} + \frac{2r}{14s}$

Often one denominator is not a multiple of the other. In these cases the **lowest common multiple** of both denominators has to be found.

Worked examples

a $\frac{1}{4} + \frac{1}{3}$

$= \frac{3}{12} + \frac{4}{12}$

$= \frac{7}{12}$

b $\frac{1}{5} + \frac{2}{3}$

$= \frac{3}{15} + \frac{10}{15}$

$= \frac{13}{15}$

c $\frac{a}{3} + \frac{b}{4}$

$= \frac{4a}{12} + \frac{3b}{12}$

$= \frac{4a + 3b}{12}$

d $\frac{2a}{3} + \frac{3b}{5}$

$= \frac{10a}{15} + \frac{9b}{15}$

$= \frac{10a + 9b}{15}$

Exercise 11.17

Simplify the following fractions:

1 **a** $\frac{a}{2}+\frac{b}{3}$ **b** $\frac{a}{3}+\frac{b}{5}$ **c** $\frac{p}{4}+\frac{q}{7}$

d $\frac{2a}{5}+\frac{b}{3}$ **e** $\frac{x}{4}+\frac{5y}{9}$ **f** $\frac{2x}{7}+\frac{2y}{5}$

2 **a** $\frac{a}{2}-\frac{a}{3}$ **b** $\frac{a}{3}-\frac{a}{5}$ **c** $\frac{p}{4}+\frac{p}{7}$

d $\frac{2a}{5}+\frac{a}{3}$ **e** $\frac{x}{4}+\frac{5x}{9}$ **f** $\frac{2x}{7}+\frac{2x}{5}$

3 **a** $\frac{3m}{5}-\frac{m}{2}$ **b** $\frac{3r}{5}-\frac{r}{2}$ **c** $\frac{5x}{4}-\frac{3x}{2}$

d $\frac{2x}{7}+\frac{3x}{4}$ **e** $\frac{11x}{2}-\frac{5x}{3}$ **f** $\frac{2p}{3}-\frac{p}{2}$

4 **a** $p-\frac{p}{2}$ **b** $c-\frac{c}{3}$ **c** $x-\frac{x}{5}$

d $m-\frac{2m}{3}$ **e** $q-\frac{4q}{5}$ **f** $w-\frac{3w}{4}$

5 **a** $2m-\frac{m}{2}$ **b** $3m-\frac{2m}{3}$ **c** $2m-\frac{5m}{2}$

d $4m-\frac{3m}{2}$ **e** $2p-\frac{5p}{3}$ **f** $6q-\frac{6q}{7}$

6 **a** $p-\frac{p}{r}$ **b** $\frac{x}{y}+x$ **c** $m+\frac{m}{n}$

d $\frac{a}{b}+a$ **e** $2x-\frac{x}{y}$ **f** $2p-\frac{3p}{q}$

7 **a** $\frac{a}{3}+\frac{a+4}{2}$ **b** $\frac{2b}{5}+\frac{b-4}{3}$

c $\frac{c+2}{4}-\frac{2-c}{2}$ **d** $\frac{2(d-3)}{7}-\frac{3(2-d)}{2}$

Simplifying complex algebraic fractions

With more complex algebraic fractions, the method of getting a common denominator is still required.

Worked examples

a $\frac{2}{x+1}+\frac{3}{x+2}$

$=\frac{2(x+2)}{(x+1)(x+2)}+\frac{3(x+1)}{(x+1)(x+2)}$

$=\frac{2(x+2)+3(x+1)}{(x+1)(x+2)}$

$=\frac{2x+4+3x+3}{(x+1)(x+2)}$

$=\frac{5x+7}{(x+1)(x+2)}$

b $\frac{5}{p+3}-\frac{3}{p-5}$

$=\frac{5(p-5)}{(p+3)(p-5)}-\frac{3(p+3)}{(p+3)(p-5)}$

$=\frac{5(p-5)-3(p+3)}{(p+3)(p-5)}$

$=\frac{5p-25-3p-9}{(p+3)(p-5)}$

$=\frac{2p-34}{(p+3)(p-5)}$

c $\frac{x^2 - 2x}{x^2 + x - 6}$

$= \frac{x(x-2)}{(x+3)(x-2)}$

$= \frac{x}{x+3}$

d $\frac{x^2 - 3x}{x^2 + 2x - 15}$

$= \frac{x(x-3)}{(x-3)(x+5)}$

$= \frac{x}{x+5}$

Exercise 11.18

Simplify the following algebraic fractions:

1 **a** $\frac{1}{x+1} + \frac{2}{x+2}$ **b** $\frac{3}{m+2} - \frac{2}{m-1}$ **c** $\frac{2}{p-3} + \frac{1}{p-2}$

d $\frac{3}{w-1} - \frac{2}{w+3}$ **e** $\frac{4}{y+4} - \frac{4}{y+1}$ **f** $\frac{2}{m-2} - \frac{3}{m+3}$

2 **a** $\frac{x(x-4)}{(x-4)(x+2)}$ **b** $\frac{y(y-3)}{(y+3)(y-3)}$ **c** $\frac{(m+2)(m-2)}{(m-2)(m-3)}$

d $\frac{p(p+5)}{(p-5)(p+5)}$ **e** $\frac{m(2m+3)}{(m+4)(2m+3)}$ **f** $\frac{(m+1)(m-1)}{(m+2)(m-1)}$

3 **a** $\frac{x^2 - 5x}{(x+3)(x-5)}$ **b** $\frac{x^2 - 3x}{(x+4)(x-3)}$ **c** $\frac{y^2 - 7y}{(y-7)(y-3)}$

d $\frac{x(x-1)}{x^2 + 2x - 3}$ **e** $\frac{x(x+2)}{x^2 + 4x + 4}$ **f** $\frac{x(x+4)}{x^2 + 5x + 4}$

4 **a** $\frac{x^2 - x}{x^2 - 1}$ **b** $\frac{x^2 + 2x}{x^2 + 5x + 6}$ **c** $\frac{x^2 + 4x}{x^2 + x - 12}$

d $\frac{x^2 - 5x}{x^2 - 3x - 10}$ **e** $\frac{x^2 + 3x}{x^2 - 9}$ **f** $\frac{x^2 - 7x}{x^2 - 49}$

Student assessment 1

1 Expand the following and simplify where possible:

a $5(2a - 6b + 3c)$ **b** $3x(5x - 9)$

c $-5y(3xy + y^2)$ **d** $3x^2(5xy + 3y^2 - x^3)$

e $5p - 3(2p - 4)$ **f** $4m(2m - 3) + 2(3m^2 - m)$

g $\frac{1}{3}(6x - 9) + \frac{1}{4}(8x + 24)$ **h** $\frac{m}{4}(6m - 8) + \frac{m}{2}(10m - 2)$

2 Factorise the following:

a $12a - 4b$ **b** $x^2 - 4xy$

c $8p^3 - 4p^2q$ **d** $24xy - 16x^2y + 8xy^2$

3 If $x = 2$, $y = -3$ and $z = 4$, evaluate the following:

a $2x + 3y - 4z$ **b** $10x + 2y^2 - 3z$

c $z^2 - y^3$ **d** $(x + y)(y - z)$

e $z^2 - x^2$ **f** $(z + x)(z - x)$

4 Rearrange the following formulas to make the green letter the subject:

a $x = 3p + q$ **b** $3m - 5n = 8r$

c $2m = \frac{3y}{t}$ **d** $x(w + y) = 2y$

e $\frac{xy}{2p} = \frac{rs}{t}$ **f** $\frac{x+y}{w} = m + n$

Student assessment 2

1 Expand the following and simplify where possible:

a $3(2x - 3y + 5z)$ **b** $4p(2m - 7)$

c $-4m(2mn - n^2)$ **d** $4p^2(5pq - 2q^2 - 2p)$

e $4x - 2(3x + 1)$ **f** $4x(3x - 2) + 2(5x^2 - 3x)$

g $\frac{1}{5}(15x - 10) - \frac{1}{3}(9x - 12)$ **h** $\frac{x}{2}(4x - 6) + \frac{x}{4}(2x + 8)$

2 Factorise the following:

a $16p - 8q$ **b** $p^2 - 6pq$

c $5p^2q - 10pq^2$ **d** $9pq - 6p^2q + 12q^2p$

3 If $a = 4$, $b = 3$ and $c = -2$, evaluate the following:

a $3a - 2b + 3c$ **b** $5a - 3b^2$

c $a^2 + b^2 + c^2$ **d** $(a + b)(a - b)$

e $a^2 - b^2$ **f** $b^3 - c^3$

4 Rearrange the following formulas to make the green letter the subject:

a $p = 4m + n$ **b** $4x - 3y = 5z$

c $2x = \frac{3y}{5p}$ **d** $m(x + y) = 3w$

e $\frac{pq}{4r} = \frac{mn}{t}$ **f** $\frac{p+q}{r} = m - n$

Student assessment 3

1 Expand the following and simplify where possible:

a $(x-4)(x+2)$ **b** $(x-8)^2$

c $(x+y)^2$ **d** $(x-11)(x+11)$

e $(3x-2)(2x-3)$ **f** $(5-3x)^2$

2 **a** Factorise the following fully:

i $pq-3rq+pr-3r^2$ **ii** $1-t^4$

b By factorising, evaluate the following:

i 875^2-125^2 **ii** $7.5^2-2.5^2$

3 Factorise the following:

a $x^2-4x-77$ **b** x^2-6x+9

c x^2-144 **d** $3x^2+3x-18$

e $2x^2+5x-12$ **f** $4x^2-20x+25$

4 Make the letter in green the subject of the formula:

a $mf^2=p$ **b** $m=5t^2$

c $A=\pi r\sqrt{p+q}$ **d** $\frac{1}{x}+\frac{1}{y}=\frac{1}{t}$

5 Simplify the following algebraic fractions:

a $\frac{x^7}{x^3}$ **b** $\frac{mn}{p}\times\frac{pq}{m}$

c $\frac{(y^3)^3}{(y^2)^3}$ **d** $\frac{28pq^2}{7pq^3}$

e $\frac{m^2n}{2}\div\frac{m^2}{n^2}$ **f** $\frac{7b^3}{c}\div\frac{4b^2}{3c^3}$

6 Simplify the following algebraic fractions:

a $\frac{m}{11}+\frac{3m}{11}-\frac{2m}{11}$ **b** $\frac{3p}{8}-\frac{9p}{16}$

c $\frac{4x}{3y}-\frac{7x}{12y}$ **d** $\frac{3m}{15p}+\frac{4n}{5p}-\frac{11n}{30p}$

e $\frac{2(y+4)}{3}-(y-2)$ **f** $3(y+2)-\frac{2y+3}{2}$

7 Simplify the following:

a $\frac{4}{(x-5)}+\frac{3}{(x-2)}$ **b** $\frac{a^2-b^2}{(a+b)^2}$ **c** $\frac{x-2}{x^2+x-6}$

Student assessment 4

1 The volume V of a cylinder is given by the formula $V = \pi r^2 h$, where h is the height of the cylinder and r is the radius.

a Find the volume of a cylindrical post 6.5 m long and with a diameter of 20 cm.

b Make r the subject of the formula.

c A cylinder of height 60 cm has a volume of 5500 cm^3, find its radius correct to 3 s.f.

2 The formula for the surface area of a closed cylinder is $A = 2\pi r(r + h)$, where r is the radius of the cylinder and h is its height.

a Find the surface area of a cylinder of radius 12 cm and height 20 cm, giving your answer to 3 s.f.

b Rearrange the formula to make h the subject.

c What is the height of a cylinder of surface area 500 cm^2 and radius 5 cm? Give your answer to 3 s.f.

3 The formula for finding the length d of the body diagonal of a cuboid whose dimensions are x, y and z is:

$$d = \sqrt{x^2 + y^2 + z^2}$$

a Find d when $x = 2$, $y = 3$ and $z = 4$.

b How long is the body diagonal of a block of concrete in the shape of a rectangular prism of dimensions 2 m, 3 m and 75 cm?

c Rearrange the formula to make x the subject.

d Find x when $d = 0.86$, $y = 0.25$ and $z = 0.41$.

4 A pendulum of length l metres takes T seconds to complete one full oscillation. The formula for T is:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where g m/s^2 is the acceleration due to gravity.

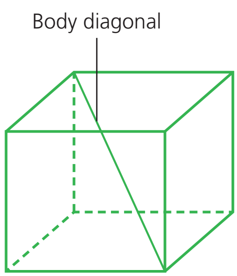
a Find T if $l = 5$ and $g = 10$.

b Rearrange the formula to make l the subject.

c How long is a pendulum which takes 3 seconds for one oscillation, if $g = 10$?

Note

The body diagonal of a cuboid is the straight line connecting any of its two non-adjacent vertices.



12 Algebraic indices

In Chapter 7, you saw how numbers can be expressed using indices. For example, $5 \times 5 \times 5 = 125$, therefore $125 = 5^3$. The 3 is called the **index**. **Indices** is the plural of index.

Three laws of indices were introduced:

1 $a^m \times a^n = a^{m+n}$

2 $a^m \div a^n$ or $\frac{a^m}{a^n} = a^{m-n}$

3 $(a^m)^n = a^{mn}$

Positive indices

Worked examples

a Simplify $d^3 \times d^4$

$$d^3 \times d^4 = d^{(3+4)} = d^7$$

b Simplify $\frac{(p^2)^4}{p^2 \times p^4}$

$$\frac{(p^2)^4}{p^2 \times p^4} = \frac{p^{2\times 4}}{p^{2+4}} = \frac{p^8}{p^6} = p^{8-6} = p^2$$

Exercise 12.1

1 Simplify the following:

a $c^5 \times c^3$

b $m^4 \div m^2$

c $(b^3)^5 \div b^6$

d $\frac{m^4n^9}{mn^3}$

e $\frac{6a^6b^4}{3a^2b^3}$

f $\frac{12x^5y^7}{4x^2y^5}$

g $\frac{4u^3v^6}{8u^2v^3}$

h $\frac{3x^6y^5z^3}{9x^4y^2z}$

2 Simplify the following:

a $4a^2 \times 3a^3$

b $2a^2b \times 4a^3b^2$

c $(2p^2)^3$

d $(4m^2n^3)^2$

e $(5p^2)^2 \times (2p^3)^3$

f $(4m^2n^2) \times (2mn^3)^3$

g $\frac{(6x^2y^4)^2 \times (2xy)^3}{12xy^6y^8}$

h $(ab)^d \times (ab)^e$

The zero index

As shown in Chapter 7, the zero index indicates that a number or algebraic term is raised to the power of zero. A term raised to the power of zero is always equal to 1. This is shown below.

$a^m \div a^n = a^{m-n}$ therefore $\dfrac{a^m}{a^m} = a^{m-m}$

$= a^0$

However, $\dfrac{a^m}{a^m} = 1$

therefore $a^0 = 1$

Negative indices

A negative index indicates that a number or an algebraic term is being raised to a negative power, e.g. a^{-4}.

As shown in Chapter 7, one law of indices states that:

$a^{-m} = \dfrac{1}{a^m}$. This is proved as follows.

$a^{-m} = a^{0-m}$

$= \dfrac{a^0}{a^m}$ (from the second law of indices)

$= \dfrac{1}{a^m}$

therefore $a^{-m} = \dfrac{1}{a^m}$

Exercise 12.2

1 Simplify the following:

a $c^3 \times c^0$

b $g^{-2} \times g^3 \div g^0$

c $(p^0)^3(q^2)^{-1}$

d $(m^3)^3(m^{-2})^5$

2 Simplify the following:

a $\dfrac{a^{-3} \times a^5}{(a^2)^0}$

b $\dfrac{(r^3)^{-2}}{(p^{-2})^3}$

c $(t^3 \div t^{-5})^2$

d $\dfrac{m^0 \div m^{-6}}{(m^{-1})^3}$

Fractional indices

It was shown in Chapter 7 that $16^{\frac{1}{2}} = \sqrt{16}$ and that $27^{\frac{1}{3}} = \sqrt[3]{27}$.

This can be applied to algebraic indices too.

In general:

$a^{\frac{1}{n}} = \sqrt[n]{a}$

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$ or $(\sqrt[n]{a})^m$

The last rule can be proved as shown below.

Using the laws of indices:

$a^{\frac{m}{n}}$ can be written as $(a^m)^{\frac{1}{n}}$ which in turn can be written as $\sqrt[n]{a^m}$.

Similarly:

$a^{\frac{m}{n}}$ can be written as $(a^{\frac{1}{n}})^m$ which in turn can be written as $(\sqrt[n]{a})^m$.

Worked examples

a Express $(\sqrt[3]{a})^4$ in the form $a^{\frac{m}{n}}$

$(\sqrt[3]{a}) = a^{\frac{1}{3}}$

Therefore $(\sqrt[3]{a})^4 = (a^{\frac{1}{3}})^4 = a^{\frac{4}{3}}$

b Express $b^{\frac{2}{5}}$ in the form $(\sqrt[n]{b})^m$

$b^{\frac{2}{5}}$ can be expressed as $(b^{\frac{1}{5}})^2$

$b^{\frac{1}{5}} = \sqrt[5]{b}$

Therefore $b^{\frac{2}{5}} = (b^{\frac{1}{5}})^2 = (\sqrt[5]{b})^2$

c Simplify $\dfrac{p^{\frac{1}{2}} \times p^{\frac{1}{3}}}{p}$

Using the laws of indices, the numerator $p^{\frac{1}{2}} \times p^{\frac{1}{3}}$ can be simplified to $p^{\left(\frac{1}{2}+\frac{1}{3}\right)} = p^{\frac{5}{6}}$.

Therefore $\dfrac{p^{\frac{5}{6}}}{p}$ can now be written as $p^{\frac{5}{6}} \times p^{-1}$

Using the laws of indices again, this can be simplified as $p^{\left(\frac{5}{6}-1\right)} = p^{-\frac{1}{6}}$

Therefore $\dfrac{p^{\frac{1}{2}} \times p^{\frac{1}{3}}}{p} = p^{-\frac{1}{6}}$

Other possible simplifications are $(\sqrt[6]{p})^{-1}$ or $\dfrac{1}{(\sqrt[6]{p})}$

Exercise 12.3

1 Rewrite the following in the form $a^{\frac{m}{n}}$:

a $(\sqrt[5]{a})^3$ **b** $(\sqrt[6]{a})^2$ **c** $(\sqrt[4]{a})^4$ **d** $(\sqrt[7]{a})^3$

2 Rewrite the following in the form $(\sqrt[n]{b})^m$:

a $b^{\frac{2}{7}}$ **b** $b^{\frac{8}{3}}$ **c** $b^{-\frac{2}{5}}$ **d** $b^{-\frac{4}{3}}$

3 Simplify the following algebraic expressions, giving your answer in the form $a^{\frac{m}{n}}$:

a $a^{\frac{1}{2}} \times a^{\frac{1}{4}}$ **b** $a^{\frac{2}{5}} \times a^{-\frac{1}{4}}$ **c** $\dfrac{\sqrt{a}}{a^{-2}}$ **d** $\dfrac{\sqrt[3]{a}}{a}$

4 Simplify the following algebraic expressions, giving your answer in the form $(\sqrt[n]{b})^m$:

a $\dfrac{\sqrt{b} \times b^{\frac{1}{4}}}{b^{-\frac{1}{5}}}$ **b** $\dfrac{b^{-\frac{1}{3}} \times \sqrt[3]{b}}{b^{\frac{2}{5}} \times b}$ **c** $\dfrac{b^3 \times b^{-\frac{1}{3}}}{b^{-2}}$ **d** $\dfrac{b^{-2} \times \sqrt[3]{b}}{\sqrt{b} \times (\sqrt[3]{b})^{-1}}$

5 Simplify the following:

a $\frac{1}{3}x^{\frac{1}{2}} \div 4x^{-2}$ **b** $\frac{2}{5}y^{\frac{1}{3}} \times 5y^{-\frac{2}{3}}$ **c** $\left(2p^{-\frac{1}{4}}\right)^2 \div \frac{1}{2}p^2$ **d** $3x^{-\frac{2}{3}} \div \frac{2}{3}x^{-\frac{1}{3}}$

Student assessment 1

1 Simplify the following using indices:

a $a \times a \times a \times b \times b$ **b** $d \times d \times e \times e \times e \times e \times e$

2 Write the following out in full:

a m^3 **b** r^4

3 Simplify the following using indices:

a $a^4 \times a^3$ **b** $p^3 \times p^2 \times q^4 \times q^5$

c $\dfrac{b^7}{b^4}$ **d** $\dfrac{(e^4)^5}{e^{14}}$

4 Simplify the following:

a $r^4 \times t^0$ **b** $\dfrac{(a^3)^0}{b^2}$ **c** $\dfrac{(m^0)^5}{n^{-3}}$

5 Simplify the following:

a $\dfrac{(p^2 \times p^{-5})^2}{p^3}$ **b** $\dfrac{(h^{-2} \times h^{-5})^{-1}}{h^0}$

Student assessment 2

1 Rewrite the following in the form $a^{\frac{m}{n}}$:

a $(\sqrt[8]{a})$ **b** $(\sqrt[5]{a})^{-2}$

2 Rewrite the following in the form $(\sqrt[n]{b^m})$:

a $b^{\frac{4}{9}}$ **b** $b^{-\frac{2}{3}}$

3 Simplify the following algebraic expressions, giving your answer in the form $a^{\frac{m}{n}}$:

a $a^{\frac{1}{3}} \times a^{\frac{3}{2}}$ **b** $\dfrac{\sqrt[3]{a}}{a^{-\frac{5}{6}}} \times a^2$

4 Simplify the following algebraic expressions, giving your answer in the form $(\sqrt[n]{t})^m$:

a $\dfrac{\sqrt{t} \times t^{\frac{2}{3}}}{t^{-\frac{1}{3}}}$ **b** $\dfrac{\sqrt[3]{t}}{t^2 \times t^{-\frac{2}{5}}}$

13 Equations and inequalities

An equation is formed when the value of an unknown quantity is needed.

Derive and solve linear equations with one unknown

Worked examples

Solve the following **linear equations**:

a $3x + 8 = 14$
$3x = 6$
$x = 2$

b $12 = 20 + 2x$
$-8 = 2x$
$-4 = x$

c $3(p + 4) = 21$
$3p + 12 = 21$
$3p = 9$
$p = 3$

d $4(x - 5) = 7(2x - 5)$
$4x - 20 = 14x - 35$
$4x + 15 = 14x$
$15 = 10x$
$1.5 = x$

e $6 = \frac{2x}{x - 4}$
$6(x - 4) = 2x$
$6x - 24 = 2x$
$4x - 24 = 0$
$4x = 24$
$x = 6$

f $\frac{-x}{2(x - 4)} = -3$
$\frac{-x}{2x - 8} = -3$
$-x = -3(2x - 8)$
$-x = -6x + 24$
$5x = 24$
$x = \frac{24}{5}$

Exercise 13.1

Solve the following linear equations:

1 a $3x = 2x - 4$ b $5y = 3y + 10$
c $2y - 5 = 3y$ d $p - 8 = 3p$
e $3y - 8 = 2y$ f $7x + 11 = 5x$

2 a $3x - 9 = 4$ b $4 = 3x - 11$
c $6x - 15 = 3x + 3$ d $4y + 5 = 3y - 3$
e $8y - 31 = 13 - 3y$ f $4m + 2 = 5m - 8$

3 a $7m - 1 = 5m + 1$ b $5p - 3 = 3 + 3p$
c $12 - 2k = 16 + 2k$ d $6x + 9 = 3x - 54$
e $8 - 3x = 18 - 8x$ f $2 - y = y - 4$

4 a $\frac{x}{2} = 3$ b $\frac{1}{2}y = 7$
c $\frac{x}{4} = 1$ d $\frac{1}{4}m = 3$
e $7 = \frac{x}{5}$ f $4 = \frac{1}{5}p$

5 **a** $\frac{x}{3} - 1 = 4$ **b** $\frac{x}{5} + 2 = 1$

c $\frac{2}{3}x = 5$ **d** $\frac{3}{4}x = 6$

e $\frac{1}{5}x = \frac{1}{2}$ **f** $\frac{2x}{5} = 4$

6 **a** $\frac{x+1}{2} = 3$ **b** $4 = \frac{x-2}{3}$

c $\frac{x-10}{3} = 4$ **d** $8 = \frac{5x-1}{3}$

e $\frac{2(x-5)}{3} = 2$ **f** $\frac{3(x-2)}{4} = 4x - 8$

7 **a** $6 = \frac{2(y-1)}{3}$ **b** $2(x+1) = 3(x-5)$

c $5(x-4) = 3(x+2)$ **d** $\frac{3+y}{2} = \frac{y+1}{4}$

e $\frac{7+2x}{3} = \frac{9x-1}{7}$ **f** $\frac{2x+3}{4} = \frac{4x-2}{6}$

8 **a** $\frac{14}{2x-3} = 2$ **b** $\frac{13}{3(x+5)} = \frac{1}{3}$

c $\frac{1}{x-1} - \frac{2}{x+4} = 0$ **d** $\frac{1}{x+2} - \frac{4}{x+5} = 0$

e $\frac{1}{x+3} - \frac{2}{3x} = 0$ **f** $\frac{5}{2x+6} = \frac{15}{44x-1}$

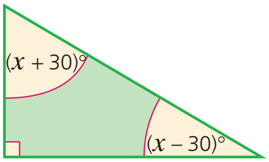
Constructing expressions and equations

In many cases, when dealing with the practical applications of mathematics, equations need to be constructed first before they can be solved. Often the information is either given within the context of a problem or in a diagram.

Worked examples

Note

All diagrams are not drawn to scale.



a **i** Write an expression for the sum of the angles in the triangle (left).

$(x + 30) + (x - 30) + 90$

ii Find the size of each of the angles in the triangle by constructing an equation and solving it to find the value of x.

The sum of the angles of a triangle is 180°.

$$\begin{aligned}(x + 30) + (x - 30) + 90 &= 180 \\ 2x + 90 &= 180 \\ 2x &= 90 \\ x &= 45\end{aligned}$$

The three angles are therefore: 90°, $x + 30 = 75°$, $x - 30 = 15°$.

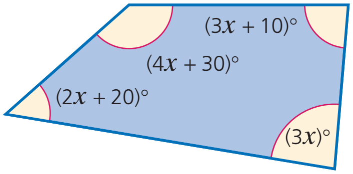
Check: 90° + 75° + 15° = 180°.

b **i** Write an expression for the sum of the angles in the quadrilateral (below).

$4x + 30 + 3x + 10 + 3x + 2x + 20$

ii Find the size of each of the angles in the quadrilateral by constructing an equation and solving it to find the value of x.
The sum of the angles of a quadrilateral is 360°.

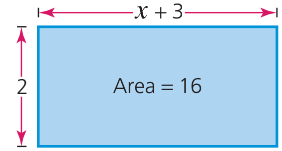
$$4x + 30 + 3x + 10 + 3x + 2x + 20 = 360$$
$$12x + 60 = 360$$
$$12x = 300$$
$$x = 25$$



The angles are:

$$4x + 30 = (4 \times 25) + 30 = 130°$$
$$3x + 10 = (3 \times 25) + 10 = 85°$$
$$3x = 3 \times 25 = 75°$$
$$2x + 20 = (2 \times 25) + 20 = 70°$$
$$\text{Total} = 360°$$

c Construct an equation and solve it to find the value of x in the diagram (right).



Area of rectangle = base × height

$$2(x + 3) = 16$$
$$2x + 6 = 16$$
$$2x = 10$$
$$x = 5$$

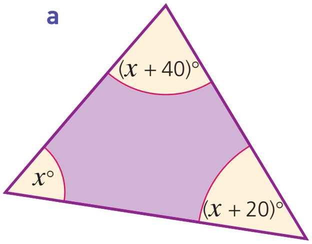
Exercise 13.2

In Questions 1–3:

i write an expression for the sum of the angles in each case, giving your answer in its simplest form,
ii construct an equation in terms of x,
iii solve the equation,
iv calculate the size of each of the angles,
v check your answers.

1

a



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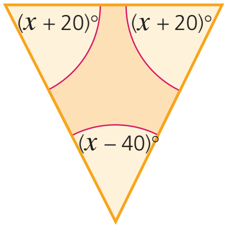

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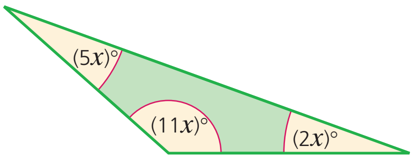

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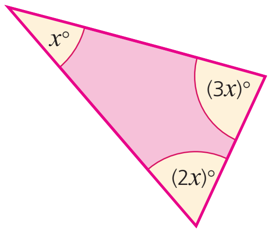

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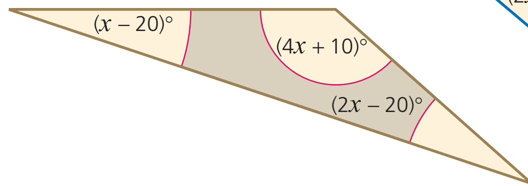

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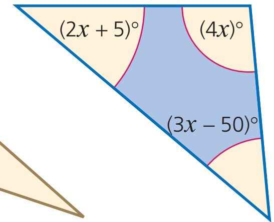
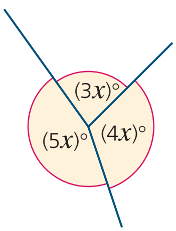


2 **a**



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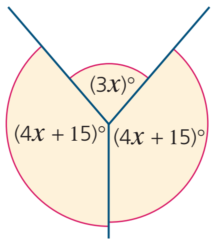

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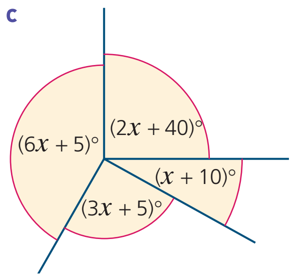

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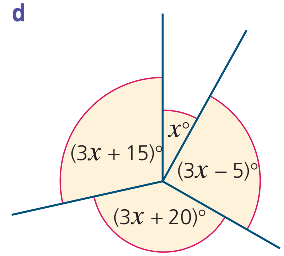
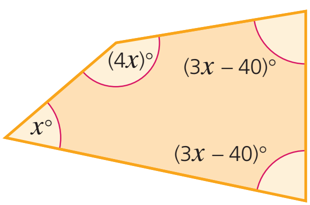


3 **a**



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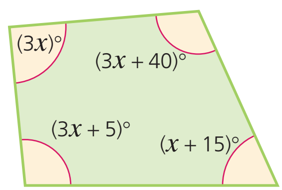

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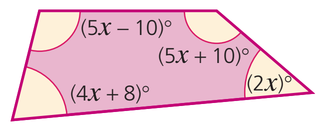

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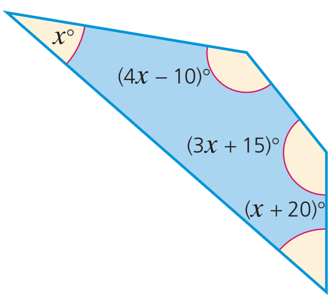

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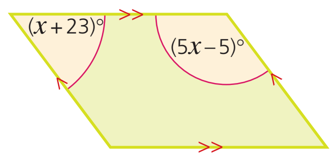
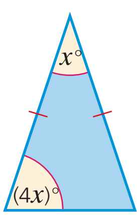
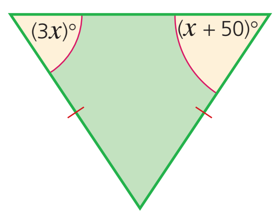


4 By constructing an equation and solving it, find the value of x in each of these isosceles triangles:

a



b



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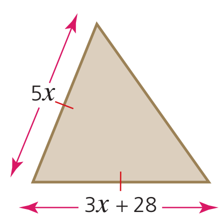
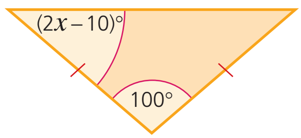


Exercise 13.2 (cont)

d



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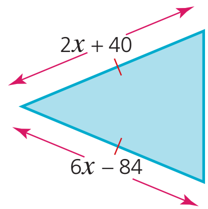

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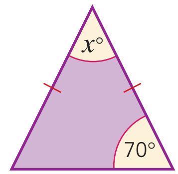
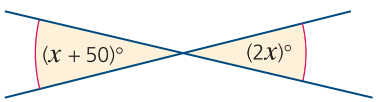


5 Using angle properties, calculate the value of x in each of these questions:

a



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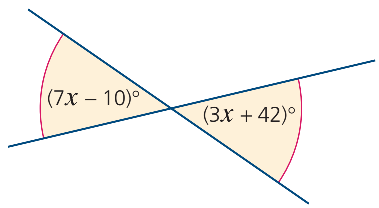

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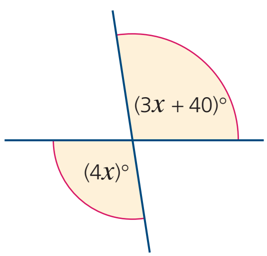

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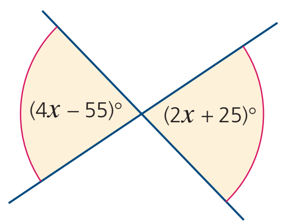
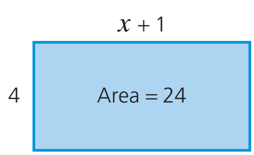
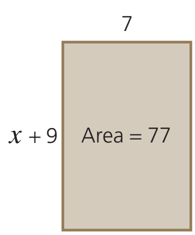


6 Calculate the value of x:

a



b



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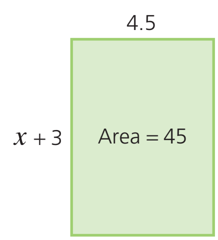

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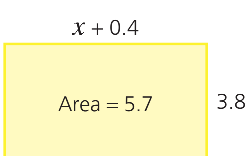
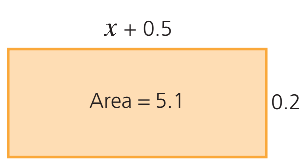


e



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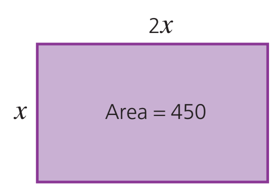


Simultaneous equations

When the values of two unknowns are needed, two equations need to be formed and solved. The process of solving two equations and finding a common solution is known as solving equations simultaneously.

The two most common ways of solving **simultaneous equations** algebraically are by **elimination** and by **substitution**.

By elimination

The aim of this method is to eliminate one of the unknowns by either adding or subtracting the two equations.

Worked examples

Solve the following simultaneous equations by finding the values of x and y which satisfy both equations.

a $3x + y = 9$ (1)
$5x - y = 7$ (2)

By adding equations (1) + (2) we eliminate the variable y:

$$8x = 16$$
$$x = 2$$

To find the value of y, we substitute $x = 2$ into either equation (1) or (2).

Substituting $x = 2$ into equation (1):

$$3x + y = 9$$
$$6 + y = 9$$
$$y = 3$$

To check that the solution is correct, the values of x and y are substituted into equation (2). If it is correct, then the left-hand side of the equation will equal the right-hand side.

$$5x - y = 7$$
$$\text{LHS} = 10 - 3 = 7$$
$$= \text{RHS} \checkmark$$

b $4x + y = 23$ (1)
$x + y = 8$ (2)

By subtracting the equations, i.e. (1) – (2), we eliminate the variable y:

$$3x = 15$$
$$x = 5$$

By substituting $x = 5$ into equation (2), y can be calculated:

$$x + y = 8$$
$$5 + y = 8$$
$$y = 3$$

Check by substituting both values into equation (1):

$$4x + y = 23$$
$$20 + 3 = 23$$
$$23 = 23$$

By substitution

The same equations can also be solved by the method known as **substitution**.

Worked examples

a $3x + y = 9$ (1)
$5x - y = 7$ (2)

Equation (2) can be rearranged to give: $y = 5x - 7$

This can now be substituted into equation (1):

$$3x + (5x - 7) = 9$$
$$3x + 5x - 7 = 9$$
$$8x - 7 = 9$$
$$8x = 16$$
$$x = 2$$

To find the value of y, $x = 2$ is substituted into either equation (1) or (2) as before, giving $y = 3$.

b $4x + y = 23$ (1)
$x + y = 8$ (2)

Equation (2) can be rearranged to give $y = 8 - x$.

This can be substituted into equation (1):

$$4x + (8 - x) = 23$$
$$4x + 8 - x = 23$$
$$3x + 8 = 23$$
$$3x = 15$$
$$x = 5$$

y can be found as before, giving a result of $y = 3$.

Exercise 13.3

Solve the following simultaneous equations either by elimination or by substitution:

1 **a** $x + y = 6$, $x - y = 2$ **b** $x + y = 11$, $x - y - 1 = 0$ **c** $x + y = 5$, $x - y = 7$

d $2x + y = 12$, $2x - y = 8$ **e** $3x + y = 17$, $3x - y = 13$ **f** $5x + y = 29$, $5x - y = 11$

2 **a** $3x + 2y = 13$, $4x = 2y + 8$ **b** $6x + 5y = 62$, $4x - 5y = 8$ **c** $x + 2y = 3$, $8x - 2y = 6$

d $9x + 3y = 24$, $x - 3y = -14$ **e** $7x - y = -3$, $4x + y = 14$ **f** $3x = 5y + 14$, $6x + 5y = 58$

3 **a** $2x + y = 14$, $x + y = 9$ **b** $5x + 3y = 29$, $x + 3y = 13$ **c** $4x + 2y = 50$, $x + 2y = 20$

d $x + y = 10$, $3x = -y + 22$ **e** $2x + 5y = 28$, $4x + 5y = 36$ **f** $x + 6y = -2$, $3x + 6y = 18$

4 **a** $x - y = 1$, $2x - y = 6$ **b** $3x - 2y = 8$, $2x - 2y = 4$ **c** $7x - 3y = 26$, $2x - 3y = 1$

d $x = y + 7$, $3x - y = 17$ **e** $8x - 2y = -2$, $3x - 2y = -7$ **f** $4x - y = -9$, $7x - y = -18$

5 **a** $x + y = -7$
$x - y = -3$

b $2x + 3y = -18$
$2x = 3y + 6$

c $5x - 3y = 9$
$2x + 3y = 19$

d $7x + 4y = 42$
$9x - 4y = -10$

e $4x - 4y = 0$
$8x + 4y = 12$

f $x - 3y = -25$
$5x - 3y = -17$

6 **a** $2x + 3y = 13$
$2x - 4y + 8 = 0$

b $2x + 4y = 50$
$2x + y = 20$

c $x + y = 10$
$3y = 22 - x$

d $5x + 2y = 28$
$5x + 4y = 36$

e $2x - 8y = 2$
$2x - 3y = 7$

f $x - 4y = 9$
$x - 7y = 18$

7 **a** $-4x = 4y$
$4x - 8y = 12$

b $3x = 19 + 2y$
$-3x + 5y = 5$

c $3x + 2y = 12$
$-3x + 9y = -12$

d $3x + 5y = 29$
$3x + y = 13$

e $-5x + 3y = 14$
$5x + 6y = 58$

f $-2x + 8y = 6$
$2x = 3 - y$

Further simultaneous equations

If neither x nor y can be eliminated by simply adding or subtracting the two equations then it is necessary to multiply one or both of the equations. The equations are multiplied by a number in order to make the coefficients of x (or y) numerically equal.

Worked examples

a $3x + 2y = 22$ (1)
$x + y = 9$ (2)

To eliminate y, equation (2) is multiplied by 2:

$3x + 2y = 22$ (1)
$2x + 2y = 18$ (3)

By subtracting (3) from (1), the variable y is eliminated:

$x = 4$

Substituting $x = 4$ into equation (2), we have:

$x + y = 9$
$4 + y = 9$
$y = 5$

Check by substituting both values into equation (1):

$3x + 2y = 22$
LHS = 12 + 10 = 22
= RHS ✓

b $5x - 3y = 1$ (1)
$3x + 4y = 18$ (2)

To eliminate the variable y, equation (1) is multiplied by 4, and equation (2) is multiplied by 3.

$20x - 12y = 4$ (3)
$9x + 12y = 54$ (4)

By adding equations (3) and (4) the variable y is eliminated:

$$29x = 58$$
$$x = 2$$

Substituting $x = 2$ into equation (2) gives:

$$3x + 4y = 18$$
$$6 + 4y = 18$$
$$4y = 12$$
$$y = 3$$

Check by substituting both values into equation (1):

$$5x - 3y = 1$$
$$\text{LHS} = 10 - 9 = 1$$
$$= \text{RHS} \checkmark$$

Exercise 13.4

Solve the following:

1 a $2x + y = 7$, $3x + 2y = 12$ b $5x + 4y = 21$, $x + 2y = 9$ c $x + y = 7$, $3x + 4y = 23$

d $2x - 3y = -3$, $3x + 2y = 15$ e $4x = 4y + 8$, $x + 3y = 10$ f $x + 5y = 11$, $2x - 2y = 10$

2 a $x + y = 5$, $3x - 2y + 5 = 0$ b $2x - 2y = 6$, $x - 5y = -5$ c $2x + 3y = 15$, $2y = 15 - 3x$

d $x - 6y = 0$, $3x - 3y = 15$ e $2x - 5y = -11$, $3x + 4y = 18$ f $x + y = 5$, $2x - 2y = -2$

3 a $3y = 9 + 2x$, $3x + 2y = 6$ b $x + 4y = 13$, $3x - 3y = 9$ c $2x = 3y - 19$, $3x + 2y = 17$

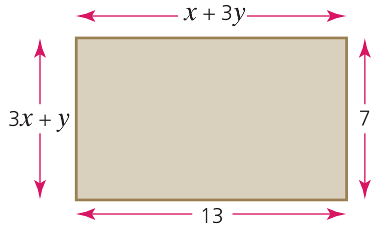
d $2x - 5y = -8$, $-3x - 2y = -26$ e $5x - 2y = 0$, $2x + 5y = 29$ f $8y = 3 - x$, $3x - 2y = 9$

4 a $4x + 2y = 5$, $3x + 6y = 6$ b $4x + y = 14$, $6x - 3y = 3$ c $10x - y = -2$, $-15x + 3y = 9$

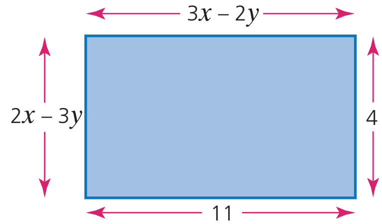
d $-2y = 0.5 - 2x$, $6x + 3y = 6$ e $x + 3y = 6$, $2x - 9y = 7$ f $5x - 3y = -0.5$, $3x + 2y = 3.5$

Exercise 13.5

1 The sum of two numbers is 17 and their difference is 3. Find the two numbers by forming two equations and solving them simultaneously.
2 The difference between two numbers is 7. If their sum is 25, find the two numbers by forming two equations and solving them simultaneously.
3 Find the values of x and y.



4 Find the values of x and y.



5 A man's age is three times his son's age. Ten years ago he was five times his son's age. By forming two equations and solving them simultaneously, find both of their ages.

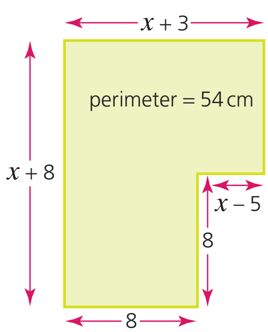
6 A grandfather is ten times older than his granddaughter. He is also 54 years older than her. How old is each of them?

Constructing further equations

Earlier in this chapter we looked at some simple examples of constructing and solving equations when we were given geometrical diagrams. This section extends this work with more complicated formulas and equations.

Worked examples

Construct and solve the equations below.



a Using the shape (left), construct an equation for the perimeter in terms of x. Find the value of x by solving the equation.

$$x + 3 + x + x - 5 + 8 + 8 + x + 8 = 54$$
$$4x + 22 = 54$$
$$4x = 32$$
$$x = 8$$

b A number is doubled, 5 is subtracted from it, and the total is 17. Find the number.

Let x be the unknown number.

$$2x - 5 = 17$$
$$2x = 22$$
$$x = 11$$

c 3 is added to a number. The result is multiplied by 8. If the answer is 64, calculate the value of the original number.

Let x be the unknown number.

$$8(x + 3) = 64$$
$$8x + 24 = 64$$
$$8x = 40$$
$$x = 5$$

or $8(x + 3) = 64$

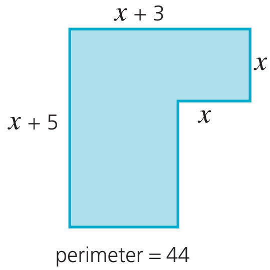
$$x + 3 = 8$$
$$x = 5$$

The original number = 5

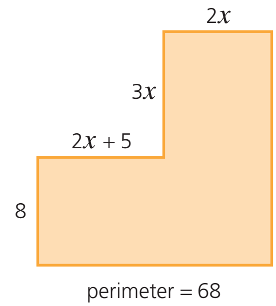
Exercise 13.6

1 Calculate the value of x:

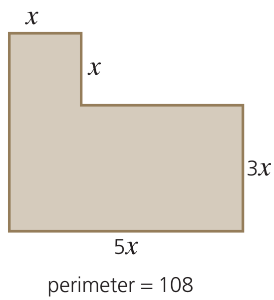
a



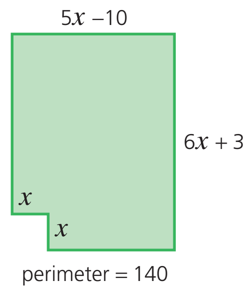
b



c



d



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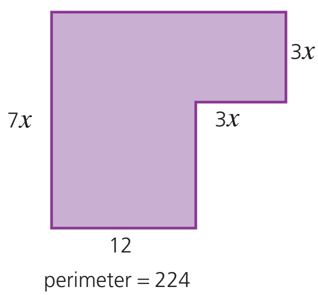

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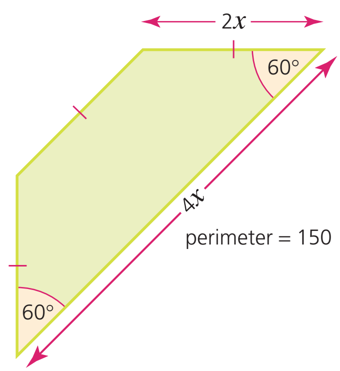


2 **a** A number is trebled and then 7 is added to it. If the total is 28, find the number.

b Multiply a number by 4 and then add 5 to it. If the total is 29, find the number.

c If 31 is the result of adding 1 to 5 times a number, find the number.

d Double a number and then subtract 9. If the answer is 11, what is the number?

e If 9 is the result of subtracting 12 from 7 times a number, find the number.

3 **a** Add 3 to a number and then double the result. If the total is 22, find the number.

b 27 is the answer when you add 4 to a number and then treble it. What is the number?

c Subtract 1 from a number and multiply the result by 5. If the answer is 35, what is the number?

d Add 3 to a number. If the result of multiplying this total by 7 is 63, find the number.
e Add 3 to a number. Quadruple the result. If the answer is 36, what is the number?

4 **a** Gabriella is x years old. Her brother is 8 years older and her sister is 12 years younger than she is. If their total age is 50 years, how old are they?
b A series of mathematics textbooks consists of four volumes. The first volume has x pages, the second has 54 pages more. The third and fourth volume each have 32 pages more than the second. If the total number of pages in all four volumes is 866, calculate the number of pages in each of the volumes.
c The five interior angles (in °) of a pentagon are x, $x + 30$, $2x$, $2x + 40$ and $3x + 20$. The sum of the interior angles of a pentagon is 540°. Calculate the size of each of the angles.
d A hexagon consists of three interior angles of equal size and a further three which are double this size. The sum of all six angles is 720°. Calculate the size of each of the angles.
e Four of the exterior angles of an octagon are the same size. The other four are twice as big. If the sum of the exterior angles is 360°, calculate the size of the interior angles.

Solving quadratic equations by factorising

You will need to be familiar with the work covered in Chapter 11 on the factorising of quadratics.

$x^2 - 3x - 10 = 0$ is a **quadratic equation**, which when factorised can be written as $(x - 5)(x + 2) = 0$.

Therefore either $x - 5 = 0$ or $x + 2 = 0$ since, if two things multiply to make zero, then one or both of them must be zero.

$x - 5 = 0$ or $x + 2 = 0$

$x = 5$ or $x = -2$

Worked examples

Solve the following equations to give two solutions for x:

a

$x^2 - x - 12 = 0$

$(x - 4)(x + 3) = 0$

so either $x - 4 = 0$ or $x + 3 = 0$

$x = 4$ or $x = -3$

b

$x^2 + 2x = 24$

This becomes $x^2 + 2x - 24 = 0$

$(x + 6)(x - 4) = 0$

so either $x + 6 = 0$ or $x - 4 = 0$

$x = -6$ or $x = 4$

c $x^2 - 6x = 0$

$x(x - 6) = 0$

so either $x = 0$ or $x - 6 = 0$

or $x = 6$

d $x^2 - 4 = 0$

$(x - 2)(x + 2) = 0$

so either $x - 2 = 0$ or $x + 2 = 0$

$x = 2$ or $x = -2$

Exercise 13.7

Solve the following quadratic equations by factorising:

1 a $x^2 + 7x + 12 = 0$ b $x^2 + 8x + 12 = 0$
c $x^2 + 13x + 12 = 0$ d $x^2 - 7x + 10 = 0$
e $x^2 - 5x + 6 = 0$ f $x^2 - 6x + 8 = 0$

2 a $x^2 + 3x - 10 = 0$ b $x^2 - 3x - 10 = 0$
c $x^2 + 5x - 14 = 0$ d $x^2 - 5x - 14 = 0$
e $x^2 + 2x - 15 = 0$ f $x^2 - 2x - 15 = 0$

3 a $x^2 + 5x = -6$ b $x^2 + 6x = -9$
c $x^2 + 11x = -24$ d $x^2 - 10x = -24$
e $x^2 + x = 12$ f $x^2 - 4x = 12$

4 a $x^2 - 2x = 8$ b $x^2 - x = 20$
c $x^2 + x = 30$ d $x^2 - x = 42$
e $x^2 - 2x = 63$ f $x^2 + 3x = 54$

Exercise 13.8

Solve the following quadratic equations:

1 a $x^2 - 9 = 0$ b $x^2 - 16 = 0$
c $x^2 = 25$ d $x^2 = 121$
e $x^2 - 144 = 0$ f $x^2 - 220 = 5$

2 a $4x^2 - 25 = 0$ b $9x^2 - 36 = 0$
c $25x^2 = 64$ d $x^2 = \frac{1}{4}$
e $x^2 - \frac{1}{9} = 0$ f $16x^2 - \frac{1}{25} = 0$

3 a $x^2 + 5x + 4 = 0$ b $x^2 + 7x + 10 = 0$
c $x^2 + 6x + 8 = 0$ d $x^2 - 6x + 8 = 0$
e $x^2 - 7x + 10 = 0$ f $x^2 + 2x - 8 = 0$

4 a $x^2 - 3x - 10 = 0$ b $x^2 + 3x - 10 = 0$
c $x^2 - 3x - 18 = 0$ d $x^2 + 3x - 18 = 0$
e $x^2 - 2x - 24 = 0$ f $x^2 - 2x - 48 = 0$

5 a $x^2 + x - 12 = 0$ b $x^2 + 8x = -12$
c $x^2 + 5x = 36$ d $x^2 + 2x = -1$
e $x^2 + 4x = -4$ f $x^2 + 17x = -72$

6 a $x^2 - 8x = 0$ b $x^2 - 7x = 0$
c $x^2 + 3x = 0$ d $x^2 + 4x = 0$
e $x^2 - 9x = 0$ f $4x^2 - 16x = 0$

7 a $2x^2 + 5x + 3 = 0$ b $2x^2 - 3x - 5 = 0$
c $3x^2 + 2x - 1 = 0$ d $2x^2 + 11x + 5 = 0$
e $2x^2 - 13x + 15 = 0$ f $12x^2 + 10x - 8 = 0$

8 **a** $x^2 + 12x = 0$ **b** $x^2 + 12x + 27 = 0$
c $x^2 + 4x = 32$ **d** $x^2 + 5x = 14$
e $2x^2 = 72$ **f** $3x^2 - 12 = 288$

Exercise 13.9

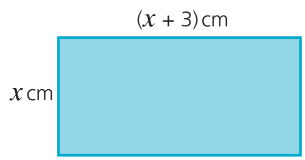
In the following questions, construct equations from the information given and then solve to find the unknown.

1 When a number x is added to its square, the total is 12. Find two possible values for x.

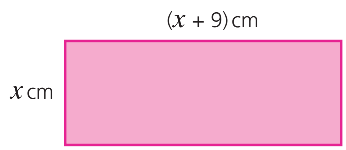
2 A number x is equal to its own square minus 42. Find two possible values for x.

Although a solution to a quadratic will give two solutions, sometimes one solution can be ignored due to the context of the question.

3 If the area of the rectangle (below) is 10 cm², calculate the only possible value for x.



4 If the area of the rectangle (below) is 52 cm², calculate the only possible value for x.

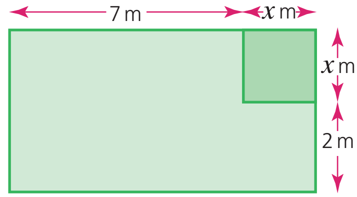


5 A triangle has a base length of $2x$ cm and a height of $(x - 3)$ cm. If its area is 18 cm², calculate its height and base length.

6 A triangle has a base length of $(x - 8)$ cm and a height of $2x$ cm. If its area is 20 cm², calculate its height and base length.

7 A right-angled triangle has a base length of x cm and a height of $(x - 1)$ cm. If its area is 15 cm², calculate the base length and height.

8 A rectangular garden has a square flower bed of side length x m in one of its corners. The remainder of the garden consists of lawn and has dimensions as shown (below). If the total area of the lawn is 50 m²:
a form an equation in terms of x,
b solve the equation,
c calculate the length and width of the whole garden.



The quadratic formula

In general a quadratic equation takes the form $ax^2 + bx + c = 0$ where a, b and c are integers. Quadratic equations can be solved by the use of the **quadratic formula** which states that:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Worked examples

a Solve the quadratic equation $x^2 + 7x + 3 = 0$.

$a = 1$, $b = 7$ and $c = 3$.

Substituting these values into the quadratic formula gives:

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times 3}}{2 \times 1}$$

$$x = \frac{-7 \pm \sqrt{49 - 12}}{2}$$

$$x = \frac{-7 \pm \sqrt{37}}{2}$$

When answers are left in this form, they are left in 'surd form', meaning that the answer is left with a surd and not written as an approximate decimal.

Therefore $x = \frac{-7 + 6.083}{2}$ or $x = \frac{-7 - 6.083}{2}$

$x = -0.459$ (3 s.f.) or $x = -6.54$ (3 s.f.)

This answer is given as a decimal. It is an approximate answer.

b Solve the quadratic equation $x^2 - 4x - 2 = 0$.

$a = 1$, $b = -4$ and $c = -2$.

Substituting these values into the quadratic formula gives:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - (4 \times 1 \times -2)}}{2 \times 1}$$

$$x = \frac{4 \pm \sqrt{16 + 8}}{2}$$

$$x = \frac{4 \pm \sqrt{24}}{2}$$

$$x = 2 \pm \sqrt{6}$$

This answer is in surd form.

Therefore $x = 2 + 2.449$ or $x = 2 - 2.449$

$x = 4.45$ (3 s.f.) or $x = -0.449$ (3 s.f.)

This answer is given as a decimal. It is an approximate answer.

Completing the square

Quadratics can also be solved by expressing them in terms of a **perfect square**. We look once again at the quadratic $x^2 - 4x - 2 = 0$.

The perfect square $(x - 2)^2$ can be expanded to give $x^2 - 4x + 4$. Notice that the x^2 and x terms are the same as those in the original quadratic.

Therefore $(x - 2)^2 - 6 = x^2 - 4x - 2$ and can be used to solve the quadratic.

$$(x - 2)^2 - 6 = 0$$

$$(x - 2)^2 = 6$$

$$x - 2 = \pm\sqrt{6}$$

$$x = 2 \pm \sqrt{6}$$

$x = 4.45$ (3 s.f.) or $x = -0.449$ (3 s.f.)

Exercise 13.10

Solve the following quadratic equations using either the quadratic formula or by completing the square. For each question:

i give your answers in surd form

ii and give your answers to 2 d.p.

1 a $x^2 - x - 13 = 0$ b $x^2 + 4x - 11 = 0$

c $x^2 + 5x - 7 = 0$ d $x^2 + 6x + 6 = 0$

e $x^2 + 5x - 13 = 0$ f $x^2 - 9x + 19 = 0$

2 a $x^2 + 7x + 9 = 0$ b $x^2 - 35 = 0$

c $x^2 + 3x - 3 = 0$ d $x^2 - 5x - 7 = 0$

e $x^2 + x - 18 = 0$ f $x^2 - 8 = 0$

3 a $x^2 - 2x - 2 = 0$ b $x^2 - 4x - 11 = 0$

c $x^2 - x - 5 = 0$ d $x^2 + 2x - 7 = 0$

e $x^2 - 3x + 1 = 0$ f $x^2 - 8x + 3 = 0$

4 a $2x^2 - 3x - 4 = 0$ b $4x^2 + 2x - 5 = 0$

c $5x^2 - 8x + 1 = 0$ d $-2x^2 - 5x - 2 = 0$

e $3x^2 - 4x - 2 = 0$ f $-7x^2 - x + 15 = 0$

Simultaneous equations involving one linear and one non-linear equation

So far we have dealt with the solution of linear simultaneous equations and also the solution of quadratic equations. However, solving equations simultaneously need not only deal with two linear equations.

Worked example

Solve the following linear and quadratic equations simultaneously.

$y = 2x + 3$ and $y = x^2 + x - 9$

Simultaneous equations involving one linear and one non-linear equation are solved using the method of substitution.

As both $2x + 3$ and $x^2 + x - 9$ are equal to y, then they must also be equal to each other.

$2x + 3 = x^2 + x - 9$

Rearranging the equation to collect all the terms on one side of the equation gives:

$x^2 - x - 12 = 0$

Note that equating a linear equation with a quadratic equation has produced a quadratic equation. This can therefore be solved in the normal way.

$x^2 - x - 12 = 0$

$(x - 4)(x + 3) = 0$

$x = 4$ and $x = -3$

Substituting these values of x into one of the original equations (the linear one is easier) will produce the corresponding y values.

When $x = 4$, $y = 2(4) + 3 = 11$

When $x = -3$, $y = 2(-3) + 3 = -3$

Exercise 13.11

Solve the following equations simultaneously:

1 $y = 4$ and $y = (x - 3)^2 + 4$

2 $y = -2x - 3$ and $y = x^2 + 5x + 3$

3 $y = 2x^2 + 4x + 1$ and $y = 3x + 11$

4 $y = 6x^2 - 5x + 2$ and $y = 5x + 6$

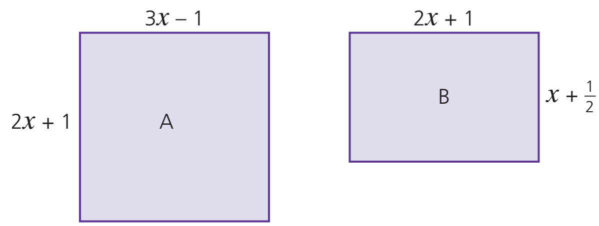
5 $y = -2x + 2$ and $y = 2(x - 3)^2 - 16$

6 $y = -5x^2 - 5x + 10$ and $y = 6x - 2$

7 $y = 11 - x$ and $y = \frac{30}{x}$

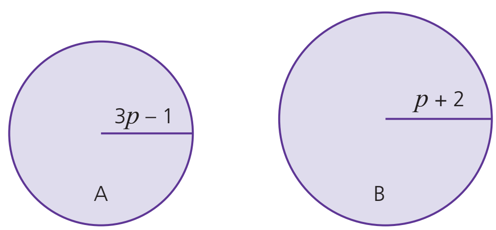
8 $y = 2x - 7 - \frac{3}{x}$ and $y = \frac{2}{x}$

9 Two rectangles are shown below.



Calculate the value of x if the area of rectangle A and the perimeter of rectangle B have the same numerical value.

10 Two circles are shown below:



Calculate the value of p if the area of circle A has the same numerical value as the perimeter of circle B. Give your answer correct to 2 d.p.

Linear inequalities

The statement

6 is less than 8

can be written as:

$6 < 8$

This inequality can be manipulated in the following ways:

adding 2 to each side:	$8 < 10$	this inequality is still true
subtracting 2 from each side:	$4 < 6$	this inequality is still true

multiplying both sides by 2:	$12 < 16$	this inequality is still true
dividing both sides by 2:	$3 < 4$	this inequality is still true
multiplying both sides by –2:	$-12 < -16$	this inequality is not true
dividing both sides by –2:	$-3 < -4$	this inequality is not true

As can be seen, when both sides of an inequality are either multiplied or divided by a negative number, the inequality is no longer true. For it to be true, the inequality sign needs to be changed around.

i.e. $-12 > -16$ and $-3 > -4$

When solving **linear inequalities**, the procedure is very similar to that for solving linear equations.

Worked examples

Remember:

implies that the number is not included in the solution. It is associated with $>$ and $<$.

implies that the number is included in the solution. It is associated with $\geqslant$ and $\leqslant$.

Solve the following inequalities and represent the solution on a number line:

a $15 + 3x < 6$

$3x < -9$

$x < -3$

b $17 \leqslant 7x + 3$

$14 \leqslant 7x$

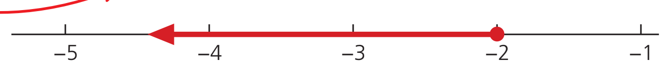
$2 \leqslant x$ that is $x \geqslant 2$

c $9 - 4x \geqslant 17$

$-4x \geqslant 8$

$x \leqslant -2$

Note the inequality sign has changed direction.



Exercise 13.12

Solve the following inequalities and illustrate your solution on a number line:

1. a $x + 3 < 7$ b $5 + x > 6$
   c $4 + 2x \leqslant 10$ d $8 \leqslant x + 1$
   e $5 > 3 + x$ f $7 < 3 + 2x$
2. a $x - 3 < 4$ b $x - 6 \geqslant -8$
   c $8 + 3x > -1$ d $5 \geqslant -x - 7$
   e $12 > -x - 12$ f $4 \leqslant 2x + 10$
3. a $\frac{x}{2} < 1$ b $4 \geqslant \frac{x}{3}$
   c $1 \leqslant \frac{x}{2}$ d $9x \geqslant -18$
   e $-4x + 1 < 3$ f $1 \geqslant -3x + 7$

Worked example

Find the range of values for which $7 < 3x + 1 \leqslant 13$ and illustrate the solutions on a number line.

This is in fact two inequalities, which can therefore be solved separately.

$7 < 3x + 1$ and $3x + 1 \leqslant 13$

$(-1) \rightarrow 6 < 3x$ $(-1) \rightarrow 3x \leqslant 12$

$(\div 3) \rightarrow 2 < x$ that is $x > 2$ $(\div 3) \rightarrow x \leqslant 4$

Exercise 13.13

Find the range of values for which the following inequalities are satisfied. Illustrate each solution on a number line:

1. a $4 < 2x \leqslant 8$ b $3 \leqslant 3x < 15$
   c $7 \leqslant 2x < 10$ d $10 \leqslant 5x < 21$
2. a $5 < 3x + 2 \leqslant 17$ b $3 \leqslant 2x + 5 < 7$
   c $12 < 8x - 4 < 20$ d $15 \leqslant 3(x - 2) < 9$

Student assessment 1

Solve the following equations:

1. a $y + 9 = 3$ b $3x - 5 = 13$
   c $12 - 5p = -8$ d $2.5y + 1.5 = 7.5$
2. a $5 - p = 4 + p$ b $8m - 9 = 5m + 3$
   c $11p - 4 = 9p + 15$ b $27 - 5r = r - 3$
3. a $\frac{p}{-2} = -3$ b $6 = \frac{2}{5}x$
   c $\frac{m - 7}{5} = 3$ d $\frac{4t - 3}{3} = 7$
4. a $\frac{2}{5}(t - 1) = 3$ b $5(3 - m) = 4(m - 6)$
   c $5 = \frac{2}{3}(x - 1)$ d $\frac{4}{5}(t - 2) = \frac{1}{4}(2t + 8)$
   e $\frac{4p}{9 - 3p} = \frac{20}{3}$ f $\frac{-5x}{3 - x} = \frac{5}{4}$

Solve the following simultaneous equations:

5 **a** $x + y = 11$
$x - y = 3$

b $5p - 3q = -1$
$-2p - 3q = -8$

c $3x + 5y = 26$
$x - y = 6$

d $2m - 3n = -9$
$3m + 2n = 19$

6 A straight line $y = -3x + 5$ is plotted on the same axes as the quadratic $y = 4x^2 + 8x + 2$.
Calculate the coordinates of the points of intersection.

Student assessment 2

1 The angles of a quadrilateral are x, $3x$, $(2x - 40)$ and $(3x - 50)$ degrees.
   a Construct an equation in terms of x.
   b Solve the equation.
   c Calculate the size of the four angles.

2 Three is subtracted from seven times a number. The result is multiplied by 5. If the answer is 55, calculate the value of the number by constructing an equation and solving it.

3 The interior angles of a pentagon are $9x$, $(5x + 10)$, $(6x + 5)$, $(8x - 25)$ and $(10x - 20)$ degrees. If the sum of the interior angles of a pentagon is 540°, find the size of each of the angles.

4 Solve the inequality below and illustrate your answer on a number line.

$$6 < 2x \leqslant 10$$

5 Solve the following quadratic equation by factorisation:

$$x^2 - x = 20$$

6 Solve the following quadratic equation by using the quadratic formula:

$$2x^2 - 7 = 3x$$

7 Solve the following equations simultaneously:

$$y = \frac{2}{x} + 1 \quad \text{and} \quad y = \frac{1}{2}x + 2$$

8 For what values of x is the following inequality true?

$$\frac{7}{x - 5} > 0$$

Student assessment 3

1 The angles of a triangle are $x°$, $y°$ and 40°. The difference between the two unknown angles is 30°.
   a Write down two equations from the information given above.
   b What is the size of the two unknown angles?

2 The interior angles of a pentagon increase by 10° as you progress clockwise.
   a Illustrate this information in a diagram.
   b Write an expression for the sum of the interior angles.
   c The sum of the interior angles of a pentagon is 540°. Use this to calculate the largest **exterior** angle of the pentagon.
   d Illustrate on your diagram the size of each of the five exterior angles.
   e Show that the sum of the exterior angles is 360°.

3 A flat sheet of card measures 12 cm by 10 cm. It is made into an open box by cutting a square of side x cm from each corner and then folding up the sides.
   a Illustrate the box and its dimensions on a simple 3D sketch.
   b Write an expression for the surface area of the outside of the box.
   c If the surface area is 56 cm^2, form and solve a quadratic equation to find the value of x.

4 a Show that $x - 2 = \frac{4}{x - 3}$ can be written as $x^2 - 5x + 2 = 0$.
   b Use the quadratic formula to solve $x - 2 = \frac{4}{x - 3}$

5 A right-angled triangle ABC has side lengths as follows: $AB = x$ cm, AC is 2 cm shorter than AB, and BC is 2 cm shorter than AC.
   a Illustrate this information on a diagram.
   b Using this information, show that $x^2 - 12x + 20 = 0$.
   c Solve the above quadratic and hence find the length of each of the three sides of the triangle.

14 Graphing inequalities and regions

Revision

An understanding of the following symbols is necessary:

$>$ means 'is greater than'

$\geqslant$ means 'is greater than or equal to'

$<$ means 'is less than'

$\leqslant$ means 'is less than or equal to'

Exercise 14.1

1 Solve each of the following inequalities:

a $15 + 3x < 21$ b $18 \leqslant 7y + 4$

c $19 - 4x \geqslant 27$ d $2 \geqslant \frac{y}{3}$

e $-4t + 1 < 1$ f $1 \geqslant 3p + 10$

2 Solve each of the following inequalities:

a $7 < 3y + 1 \leqslant 13$ b $3 \leqslant 3p < 15$

c $9 \leqslant 3(m - 2) < 15$ d $20 < 8x - 4 < 28$

Graphing an inequality

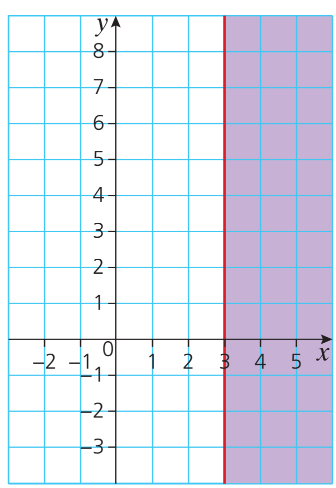
The solution to an inequality can also be illustrated on a graph.

Worked examples

a On a pair of axes, leave unshaded the region which satisfies the inequality $x \leqslant 3$.

To do this, the line $x = 3$ is drawn.

The region to the left of $x = 3$ represents the inequality $x \leqslant 3$ and therefore is unshaded as shown below.

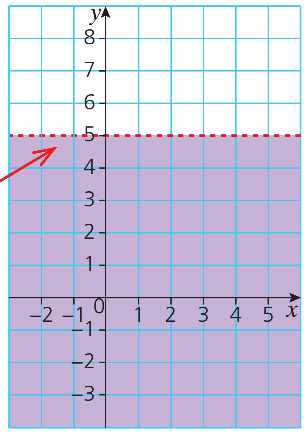


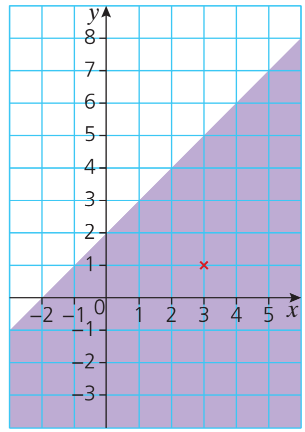
b On a pair of axes, leave unshaded the region which satisfies the inequality $y > 5$.

The line $y = 5$ is drawn first (in this case it is drawn as a broken line).

The region above the line $y = 5$ represents the inequality $y > 5$ and therefore is unshaded as shown (below left).

Note that a broken (dashed) line shows < or >, while a solid line shows ⩽ or ⩾.





c On a pair of axes, leave unshaded the region which satisfies the inequality $y \geqslant x + 2$.

The line $y = x + 2$ is drawn first (since it is included, this line is solid).

To know which region satisfies the inequality, and hence to know which side of the line to shade, the following steps are taken:

- Choose a point at random which does not lie on the line, e.g. (3, 1).
- Substitute those values of x and y into the inequality, i.e. $1 \geqslant 3 + 2$.
- If the inequality is false, then the region in which the point lies doesn't satisfy the inequality and can therefore be shaded as shown (above right).

Exercise 14.2

1 By drawing appropriate axes, leave unshaded the region which satisfies each of the following inequalities:

a $y \geqslant -x$ **b** $y \leqslant 2 - x$ **c** $x \geqslant y - 3$

d $x + y \geqslant 4$ **e** $2x - y \geqslant 3$ **f** $2y - x < 4$

Graphing more than one inequality

Several inequalities can be graphed on the same set of axes. If the regions which satisfy each inequality are left unshaded, then a solution can be found which satisfies all the inequalities, i.e. the region left unshaded by all the inequalities.

Worked example

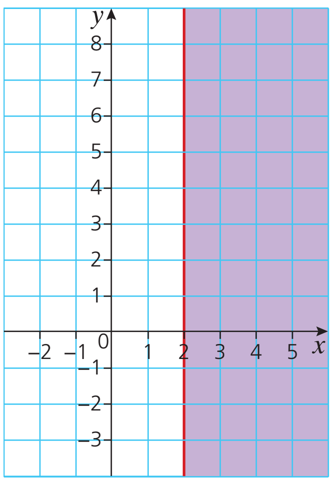
On the same pair of axes, leave unshaded the regions which satisfy the following inequalities simultaneously:

$x \leqslant 2 \qquad y > -1 \qquad y \leqslant 3 \qquad y \leqslant x + 2$

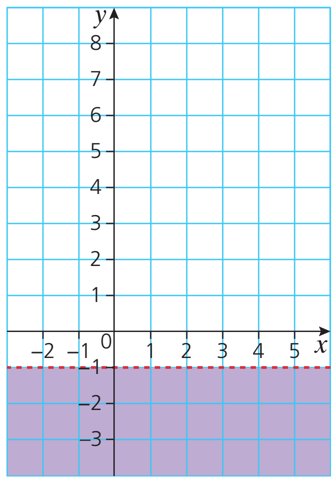
Hence find the region which satisfies all four inequalities.

If the four inequalities are graphed on separate axes, the solutions are as shown below:

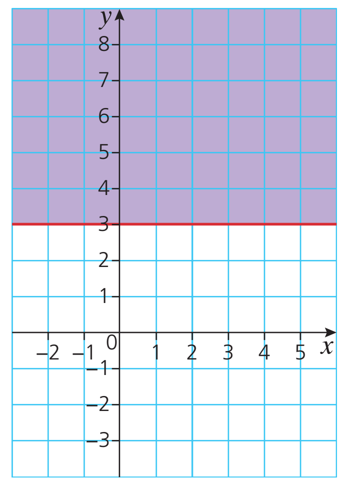
$x \leqslant 2$



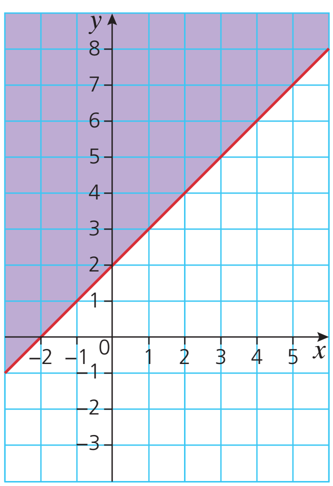
$y > -1$

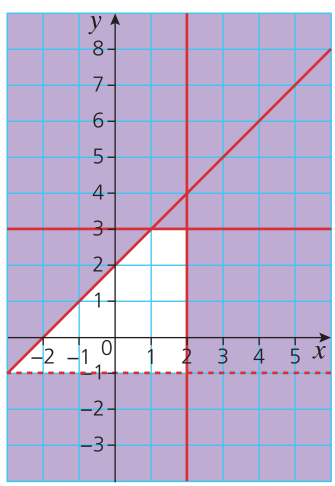


$y \leqslant 3$



$y \leqslant x + 2$





Combining all four on one pair of axes gives this diagram.

The unshaded region therefore gives a solution which satisfies all four inequalities.

Exercise 14.3

On the same pair of axes, plot the following inequalities and leave unshaded the region which satisfies all of them simultaneously.

1	$y \leqslant x$	$y > 1$	$x \leqslant 5$	
2	$x + y \leqslant 6$	$y < x$	$y \geqslant 1$	
3	$y \geqslant 3x$	$y \leqslant 5$	$x + y > 4$	
4	$2y \geqslant x + 4$	$y \leqslant 2x + 2$	$y < 4$	$x \leqslant 3$

Student assessment 1

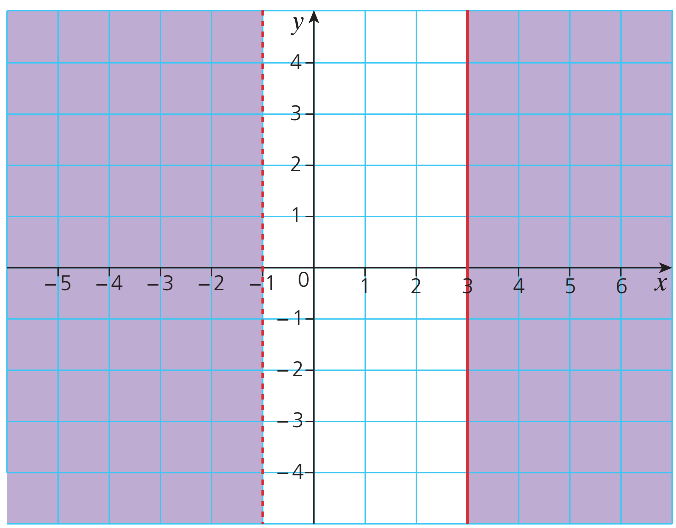
1 Solve the following inequalities:
   a $17 + 5x \leqslant 42$ **b** $3 \geqslant \frac{y}{3} + 2$

2 Solve the following inequalities:
   a $5 + 6x \leqslant 47$ **b** $4 \geqslant \frac{y+3}{3}$

3 Find the range of values for which:
   a $7 < 4y - 1 \leqslant 15$ **b** $18 < 3(p + 2) \leqslant 30$

4 Find the range of values for which:
   a $3 \leqslant 3p < 12$ **b** $24 < 8(x - 1) \leqslant 48$

5 Write the inequality which describes the unshaded region in the graph below.



6 **a** On the same axes, graph the following inequalities, leaving unshaded the region which satisfies all three.
   $x \geqslant -1$ $y \geqslant -2$ $10y + 11x \leqslant 24$
   b Calculate the area of the unshaded region.

7 **a** On a grid draw the following lines:
   $y = \frac{3}{2}x + 3$ $x + y = 3$ $y = -3$
   b **i** On the grid, leave unshaded the region satisfying all of the following inequalities:
   $y \leqslant \frac{3}{2}x + 3$ $x + y \leqslant 3$ $y \geqslant -3$
   ii Calculate the area of the unshaded region.

15 Sequences

A **sequence** is a collection of terms arranged in a specific order, where each term is obtained according to a rule. Examples of some simple sequences are given below:

2, 4, 6, 8, 10	1, 4, 9, 16, 25	1, 2, 4, 8, 16
1, 1, 2, 3, 5, 8	1, 8, 27, 64, 125	$10, 5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}$

You could discuss with another student the rules involved in producing the sequences above.

The terms of a sequence can be expressed as $T_1, T_2, T_3, \ldots, T_n$ where:

The use of subscripts is not on the Core syllabus and is Extended only, but it is a very useful way to describe terms in a sequence.

T_1 is the first term

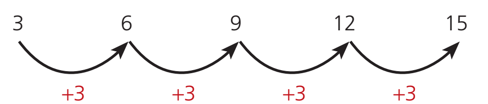
T_2 is the second term

T_n is the nth term

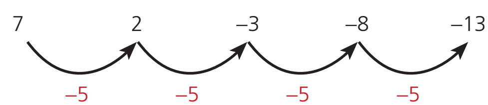
Therefore in the sequence 2, 4, 6, 8, 10, $T_1 = 2$, $T_2 = 4$, etc.

Linear sequences

In a **linear** sequence (also known as an arithmetic sequence) there is a **common difference** (d) between successive terms. Examples of some arithmetic sequences are given below:



$d = 3$



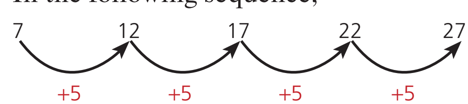
$d = -5$

Formulas for the terms of a linear sequence

There are two main ways of describing a sequence.

1 A **term-to-term rule**.

In the following sequence,



the term-to-term rule is +5 and the first term is 7.

This written using T_n notation is: $T_2 = T_1 + 5$, $T_3 = T_2 + 5$, etc. The general form is therefore written as $T_{n+1} = T_n + 5$, $T_1 = 7$, where T_n is the nth term and T_{n+1} the term after the nth term.

Note: It is important to give one of the terms, e.g. T_1, so that the exact sequence can be generated.

2 A formula for the *n*th term of a sequence.

This type of rule links each term to its position in the sequence, for example:

Position	1	2	3	4	5	n
Term	7	12	17	22	27	

We can deduce from the figures above that each term can be calculated by multiplying its position number by 5 and adding 2. Algebraically this can be written as the formula for the *n*th term:

nth term $= 5n + 2$ or $T_n = 5n + 2$

This textbook focuses on the generation and use of the rule for the *n*th term.

With a linear sequence, the rule for the *n*th term can be deduced by looking at the common difference, for example:

Position	1	2	3	4	5
Term	1	5	9	13	17

$+4$ $+4$ $+4$ $+4$

$t_n = 4n - 3$

nth term $= 4n - 3$

Position	1	2	3	4	5
Term	7	9	11	13	15

$+2$ $+2$ $+2$ $+2$

$t_n = 2n + 5$

nth term $= 2n + 5$

Position	1	2	3	4	5
Term	12	9	6	3	0

-3 -3 -3 -3

$t_n = -3n + 15$

nth term $= -3n + 15$

The common difference is the coefficient of n (i.e. the number by which n is multiplied). The constant is then worked out by calculating the number needed to make the term.

Exercise 15.1

1 For each of the sequences:
  i Write down the next two terms.
  ii Give an expression for the nth term.
    a 4, 7, 10, 13, 16, ...
    b 5, 9, 13, 17, 21, ...
    c 4, 9, 14, 19, 24, ...
    d 8, 10, 12, 14, 16, ...
    e 29, 22, 15, 8, 1, ...
    f 0, 4, 8, 12, 16, 20, ...
    g 1, 10, 19, 28, 37, ...
    h 15, 25, 35, 45, 55, ...
    i 9, 20, 31, 42, 53, ...
    j 1.5, 3.5, 5.5, 7.5, 9.5, 11.5, ...
    k 0.25, 1.25, 2.25, 3.25, 4.25, ...
    l 5, 4, 3, 2, 1, ...

2 The nth term for two different sequences are $2n + 8$ and $5n - 7$. Justifying your answers work out
  a which sequence if any, the term 22 belongs to
  b which sequence if any, the term 51 belongs to
  c a term which belongs to both sequences.

Worked examples

a A sequence is given by the term-to-term rule $T_{n+1} = T_n + 3$, where $T_1 = 0$. Generate the terms T_2, T_3 and T_4.

$T_2 = T_1 + 3$, therefore $T_2 = 0 + 3 = 3$

$T_3 = T_2 + 3$, therefore $T_3 = 3 + 3 = 6$

$T_4 = T_3 + 3$, therefore $T_4 = 6 + 3 = 9$

b Find the rule for the nth term of the sequence 12, 7, 2, –3, –8, …

Position	1	2	3	4	5	
Term	12	7	2	−3	−8	$t_n = -5n + 17$
	−5	−5	−5	−5		

Exercise 15.2

1 For each of the following sequences, the term-to-term rule and one of the terms are given.
Calculate the terms required and state whether the sequence generated is linear or not.
  a $T_{n+1} = T_n + 5$, where $T_1 = 2$. Calculate T_2, T_3 and T_4
  b $T_{n+1} = T_n - 3$, where $T_1 = 4$. Calculate T_2, T_3 and T_4
  c $T_{n+1} = T_n - 4$, where $T_3 = 2$. Calculate T_1, T_2 and T_4

Exercise 15.2 (cont)

d $T_{n+1} = 2T_n - 1$, where $T_2 = 0$. Calculate T_1, T_3 and T_4

e $T_{n+1} = \frac{T_n}{2} + 3$, where $T_1 = 2$. Calculate T_2, T_3 and T_4

f $T_{n+1} = \frac{T_n}{2} - 1$, where $T_3 = 10$. Calculate T_1, T_2 and T_4

2 For each of the following sequences:
i describe the term-to-term rule in words
ii write the term-to-term rule in its general form using the notation T_{n+1} and T_n
iii deduce the rule for the nth term
iv calculate the 10th term.
a 5, 8, 11, 14, 17
b 0, 4, 8, 12, 16
c $\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2}$
d 6, 3, 0, –3, –6
e –7, –4, –1, 2, 5
f –9, –13, –17, –21, –25

3 Copy and complete each of the following tables of linear sequences:

a

Position	1	2	5		50	n
Term				45		$4n - 3$

b

Position	1	2	5			n
Term				59	449	$6n - 1$

c

Position	1				100	n
Term		0	–5	–47		$-n + 3$

d

Position	1	2	3			n
Term	3	0	–3	–24	–294	

e

Position		5	7			n
Term	1	10	16	25	145	

f

Position	1	2	5		50	n
Term	–5.5	–7		–34		

4 For each of the following linear sequences:
i deduce the common difference d
ii write the term-to-term rule in its general form using the notation T_{n+1} and T_n
iii give the formula for the nth term
iv calculate the 50th term.

a 5, 9, 13, 17, 21
b 0, …, 2, …, 4
c −10, …, …, …, 2
d $T_1 = 6$, $T_9 = 10$
e $T_3 = -50$, $T_{20} = 18$
f $T_5 = 60$, $T_{12} = 39$

5 The first four terms of a linear sequence are 7, 19, 31, 43.
Decide whether each of the numbers below is in this linear sequence.
Justify your answers.
a −8
b 67
c 139
d 245

Sequences with quadratic and cubic rules

So far all the sequences we have looked at have been linear i.e. the term-to-term rule has a common difference and the rule for the *n*th term is linear and takes the form $T_n = an + b$. The rule for the *n*th term can be found algebraically using the method of differences and this method is particularly useful for more complex sequences.

Worked examples

a Deduce the rule for the *n*th term for the sequence 4, 7, 10, 13, 16, …

Firstly, produce a table of the terms and their positions in the sequence:

Position	1	2	3	4	5
Term	4	7	10	13	16

Extend the table to look at the differences:

Position	1		2		3		4		5
Term	4		7		10		13		16
1st difference		3		3		3		3	

As the row of 1st differences is constant, the rule for the *n*th term is linear and takes the form $T_n = an + b$.

By substituting the values of *n* into the rule, each term can be expressed in terms of *a* and *b*:

Position	1		2		3		4		5
Term	$a + b$		$2a + b$		$3a + b$		$4a + b$		$5a + b$
1st difference		a		a		a		a	

Compare the two tables in order to deduce the values of *a* and *b*:

$a = 3$

$a + b = 4$ therefore $b = 1$

The rule for the nth term $T_n = an + b$ can be written as $T_n = 3n + 1$.

For a linear rule, this method is perhaps overcomplicated. However, it is very efficient for quadratic and cubic rules.

b Deduce the rule for the nth term for the sequence 0, 7, 18, 33, 52, …

Entering the sequence in a table gives:

Position	1	2	3	4	5
Term	0	7	18	33	52

Extending the table to look at the differences gives:

Position	1	2	3	4	5
Term	0	7	18	33	52
1st difference	7	11	15	19	

The row of 1st differences is not constant, and so the rule for the nth term is not linear. Extend the table again to look at the row of 2nd differences:

Position	1	2	3	4	5
Term	0	7	18	33	52
1st difference	7	11	15	19	
2nd difference		4	4	4	

The row of 2nd differences is constant, and so the rule for the nth term is therefore a quadratic which takes the form $T_n = an^2 + bn + c$.

By substituting the values of n into the rule, each term can be expressed in terms of a, b and c as shown:

Position	1	2	3	4	5
Term	$a + b + c$	$4a + 2b + c$	$9a + 3b + c$	$16a + 4b + c$	$25a + 5b + c$
1st difference	$3a + b$	$5a + b$	$7a + b$	$9a + b$	
2nd difference		$2a$	$2a$	$2a$	

Comparing the two tables, the values of a, b and c can be deduced:

$2a = 4$ therefore $a = 2$

$3a + b = 7$ therefore $6 + b = 7$ giving $b = 1$

$a + b + c = 0$ therefore $2 + 1 + c = 0$ giving $c = -3$

The rule for the nth term $T_n = an^2 + bn + c$ can be written as $T_n = 2n^2 + n - 3$.

c Deduce the rule for the nth term for the sequence −6, −8, −6, 6, 34, …

Entering the sequence in a table gives:

Position	1	2	3	4	5
Term	−6	−8	−6	6	34

Extending the table to look at the differences:

Position	1	2	3	4	5
Term	−6	−8	−6	6	34
1st difference	−2	2	12	28	

The row of 1st differences is not constant, and so the rule for the nth term is not linear. Extend the table again to look at the row of 2nd differences:

Position	1	2	3	4	5
Term	−6	−8	−6	6	34
1st difference	−2	2	12	28	
2nd difference		4	10	16	

The row of 2nd differences is not constant either, and so the rule for the nth term is not quadratic. Extend the table by a further row to look at the row of 3rd differences:

Position	1	2	3	4	5
Term	−6	−8	−6	6	34
1st difference	−2	2	12	28	
2nd difference		4	10	16	
3rd difference		6	6		

The row of 3rd differences is constant, and so the rule for the nth term is therefore cubic which takes the form $T_n = an^3 + bn^2 + cn + d$.

By substituting the values of n into the rule, each term can be expressed in terms of a, b, c and d as shown:

Position	1	2	3	4	5
Term	$a + b + c + d$	$8a + 4b + 2c + d$	$27a + 9b + 3c + d$	$64a + 16b + 4c + d$	$125a + 25b + 5c + d$
1st difference	$7a + 3b + c$	$19a + 5b + c$	$37a + 7b + c$	$61a + 9b + c$	
2nd difference		$12a + 2b$	$18a + 2b$	$24a + 2b$	
3rd difference		$6a$	$6a$		

By comparing the two tables, equations can be formed and the values of a, b, c and d can be found:

$6a = 6$

therefore $a = 1$

$12a + 2b = 4$

therefore $12 + 2b = 4$ giving $b = -4$

$7a + 3b + c = -2$

therefore $7 - 12 + c = -2$ giving $c = 3$

$a + b + c + d = -6$

therefore $1 - 4 + 3 + d = -6$ giving $d = -6$

Therefore, the equation for the nth term is $T_n = n^3 - 4n^2 + 3n - 6$.

Exercise 15.3

By using a table if necessary, find the formula for the *n*th term of each of the following sequences:

1 2, 5, 10, 17, 26
2 0, 3, 8, 15, 24
3 6, 9, 14, 21, 30
4 9, 12, 17, 24, 33
5 –2, 1, 6, 13, 22
6 4, 10, 20, 34, 52
7 0, 6, 16, 30, 48
8 5, 14, 29, 50, 77
9 0, 12, 32, 60, 96
10 1, 16, 41, 76, 121

Exercise 15.4

Note

In Exercises 15.3 and 15.4, Core students can give their answers using *n*th terms whilst Extended students are expected to use T_n notation.

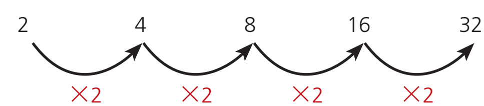
Use a table to find the formula for the *n*th term of the following sequences:

1 11, 18, 37, 74, 135
2 0, 6, 24, 60, 120
3 –4, 3, 22, 59, 120
4 2, 12, 36, 80, 150
5 7, 22, 51, 100, 175
6 7, 28, 67, 130, 223
7 1, 10, 33, 76, 145
8 13, 25, 49, 91, 157

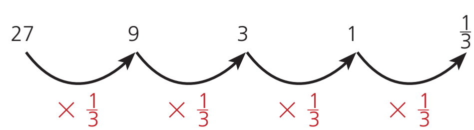
Exponential sequences

So far we have looked at sequences where there is a common difference between successive terms. There are, however, other types of sequences, e.g. 2, 4, 8, 16, 32. There is clearly a pattern to the way the numbers are generated as each term is double the previous term, but there is no common difference.

A sequence where there is a **common ratio** (r) between successive terms is known as an **exponential sequence** (or sometimes as a **geometric sequence**).



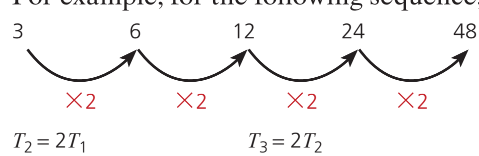
$r = 2$



$r = \frac{1}{3}$

As with an arithmetic sequence, there are two main ways of describing an exponential sequence.

1 The term-to-term rule.
For example, for the following sequence,



the general rule is $T_{n+1} = 2T_n$; $T_1 = 3$.

2 The formula for the nth term of an exponential sequence.
As with an arithmetic sequence, this rule links each term to its position in the sequence,

Position	1	2	3	4	5	n
Term	3	6	12	24	48	
	$\times 2$	$\times 2$	$\times 2$	$\times 2$		

to reach the second term the calculation is 3×2 or 3×2^1
to reach the third term, the calculation is $3 \times 2 \times 2$ or 3×2^2
to reach the fourth term, the calculation is $3 \times 2 \times 2 \times 2$ or 3×2^3
to reach the nth term, the calculation is $3 \times 2^{n-1}$
In general therefore

$$T_n = ar^{n-1}$$

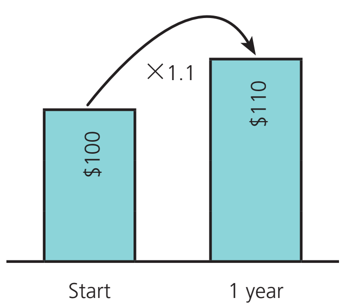
where a is the first term and r is the common ratio.

Applications of exponential sequences

In Chapter 8, simple and compound interest were shown as different ways that interest could be earned on money left in a bank account for a period of time. Here we look at compound interest as an example of an exponential sequence.

Compound interest

e.g. \$100 is deposited in a bank account and left untouched. After 1 year the amount has increased to \$110 as a result of interest payments. To work out the interest rate, calculate the multiplier from \$100 → \$110:



The multiplier is 1.1. This corresponds to a 10% increase. Therefore the simple interest rate is 10% in the first year.

Assume the money is left in the account and that the interest rate remains unchanged. Calculate the amount in the account after 5 years.

This is an example of an exponential sequence.

Number of years	0	1	2	3	4	5
Amount	100.00	110.00	121.00	133.10	146.41	161.05
	$\times 1.1$	$\times 1.1$	$\times 1.1$	$\times 1.1$	$\times 1.1$	

Alternatively, the amount after 5 years can be calculated using a variation of $T_n = ar^{n-1}$, i.e. $T_5 = 100 \times 1.1^5 = 161.05$. Note: As the number of years starts at 0, $\times 1.1$ is applied 5 times to get to the fifth year.

This is an example of compound interest as the previous year's interest is added to the total and included in the following year's calculation.

Worked examples

a Jivan deposits \$1500 in his savings account. The interest rate offered by the savings account is 6% each year for a 10-year period. Assuming Jivan leaves the money in the account, calculate how much interest he has gained after the 10 years.

An interest rate of 6% implies a common ratio of 1.06

Therefore $T_{10} = 1500 \times 1.06^{10} = 2686.27$

The amount of interest gained is $2686.27 - 1500 = \$1186.27$

b Adrienne deposits \$2000 in her savings account. The interest rate offered by the bank for this account is 8% compound interest per year. Calculate the number of years Adrienne needs to leave the money in her account for it to double in value.

An interest rate of 8% implies a common ratio of 1.08
The amount each year can be found using the term-to-term rule
$T_{n+1} = 1.08 \times T_n$

$T_1 = 2000 \times 1.08 = 2160$

$T_2 = 2160 \times 1.08 = 2332.80$

$T_3 = 2332.80 \times 1.08 = 2519.42$

…

$T_9 = 3998.01$

$T_{10} = 4317.85$

Adrienne needs to leave the money in the account for 10 years in order for it to double in value.
Although answers have been rounded to 2 d.p. in some places, the full answer has been stored in the calculator and used in the next calculation. This avoids rounding errors affecting the final solution.

Exercise 15.5

1 Identify which of the following are exponential sequences and which are not.

a 2, 6, 18, 54

b 25, 5, 1, $\frac{1}{5}$

c 1, 4, 9, 16

d –3, 9, –27, 81

e $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

f $\frac{1}{2}, \frac{2}{4}, \frac{4}{8}, \frac{8}{16}$

2 For the sequences in Question 1 that are exponential, calculate:

- **i** the common ratio r
- **ii** the next two terms
- **iii** a formula for the nth term.

3 The nth term of an exponential sequence is given by the formula $T_n = -6 \times 2^{n-1}$.
   a Calculate T_1, T_2 and T_3.
   b What is the value of n if $T_n = -768$?

4 Part of an exponential sequence is given below:
   $\ldots, -1, \ldots, \ldots, 64, \ldots$ where $T_2 = -1$ and $T_5 = 64$.
   Calculate:
   a the common ratio r
   b the value of T_1
   c the value of T_{10}.

5 A homebuyer takes out a loan with a mortgage company for \$200 000. The interest rate is 6% per year. If she is unable to repay any of the loan during the first 3 years, calculate the extra amount she will have to pay because of interest by the end of the third year.

6 A car is bought for \$10 000. It loses value at a rate of 20% each year.
   a Explain why the car is not worthless after 5 years.
   b Calculate its value after 5 years.
   c Explain why a depreciation of 20% per year means, in theory, that the car will never be worthless.

Combinations of sequences

So far we have looked at linear and exponential sequences and explored different ways of working out the term-to-term rule and the rule for the nth term. However, sometimes sequences are just variations of well-known ones. Being aware of these can often save a lot of time and effort.

Consider the sequence 2, 5, 10, 17, 26. By looking at the differences it could be established that the 2nd differences are constant and, therefore, the formula for the nth term will involve an n^2 term, but this takes quite a lot of time.

On closer inspection, when compared with the sequence 1, 4, 9, 16, 25, we can see that each of the terms in the sequence 2, 5, 10, 17, 26 has had 1 added to it.

1, 4, 9, 16, 25 is the well-known sequence of square numbers. The formula for the nth term for the sequence of square numbers is $T_n = n^2$, therefore the formula for the nth term for the sequence 2, 5, 10, 17, 26 is $T_n = n^2 + 1$.

Note

It is good practice to be aware of key sequences and to always check whether a sequence you are looking at is a variation of one of those. The key sequences include:

Square numbers 1, 4, 9, 16, 25….. where $T_n = n^2$
Cube numbers 1, 8, 27, 64, 125….. where $T_n = n^3$
Powers of two 2, 4, 8, 16, 32….. where $T_n = 2^n$
Triangle numbers 1, 3, 6, 10, 15….. where $T_n = \frac{1}{2}n(n+1)$

Worked examples

a Consider the sequence below:

2, 8, 18, 32

i By inspection, write the rule for the nth term of the sequence.

The terms of the sequence are double those of the sequence of square numbers, therefore $T_n = 2n^2$.

ii Write down the next two terms.

50, 72

b For the sequence below, by inspection, write down the rule for the nth term and the next two terms.

1, 3, 7, 15, 31

The terms of the sequence are one less than the terms of the sequence of powers of two. Therefore $T_n = 2^n - 1$

The next two terms are 63, 127.

Exercise 15.6

In each of the questions below:

i write down the rule for the nth term of the sequence by inspection

ii write down the next two terms of the sequence.

1 2, 5, 10, 17

2 3, 10, 29, 66

3 $\frac{1}{2}$, 2, $4\frac{1}{2}$, 8

4 1, 2, 4, 8

5 2, 6, 12, 20

6 2, 12, 36, 80

7 0, 4, 18, 48

8 6, 12, 24, 48

Student assessment 1

1 For each of the sequences given below:

i calculate the next two terms,

ii explain the pattern in words.

a 9, 18, 27, 36, ...

b 54, 48, 42, 36, ...

c 18, 9, 4.5, ...

d 12, 6, 0, −6, ...

e 216, 125, 64, ...

f 1, 3, 9, 27, ...

2 For each of the sequences shown below, give an expression for the nth term:

a 6, 10, 14, 18, 22, ...

b 13, 19, 25, 31, ...

c 3, 9, 15, 21, 27, ...

d 4, 7, 12, 19, 28, ...

e 0, 10, 20, 30, 40, ...

f 0, 7, 26, 63, 124, ...

3 For each of the following linear sequences:

i write down a formula for the nth term

ii calculate the 10th term.

a 1, 5, 9, 13, …

b 1, −2, −5, −8, …

4 Copy and complete both of the following tables of linear sequences:

a

Position	1	2	3	10		n
Term	17	14			–55	

b

Position	2	6	10		n
Term	–4	–2		35	

Student assessment 2

1 George deposits \$300 in a bank account. The bank offers 7% interest per year.
Assuming George does not take any money out of the account, calculate:
  a the amount of money in the account after 8 years
  b the minimum number of years the money must be left in the account, for the amount to be greater than \$350.

2 A computer loses 35% of its value each year. If the computer cost \$600 new, calculate:
  a its value after 2 years
  b its value after 10 years.

3 Part of an exponential sequence is given below:
…, …, 27, …, …, –1
where $T_3 = 27$ and $T_6 = -1$.
Calculate:
  a the common ratio r
  b the value T_1
  c the value of n if $T_n = -\frac{1}{81}$

4 Using a table of differences if necessary, calculate the rule for the nth term of the sequence 8, 24, 58, 116, 204 …

5 Using a table of differences, calculate the rule for the nth term of the sequence 10, 23, 50, 97, 170 …

6 For both of the following, calculate T_5 and T_{100}:
  a $T_n = 6n - 3$
  b $T_n = -\frac{1}{2}n + 4$

16 Proportion

Direct proportion

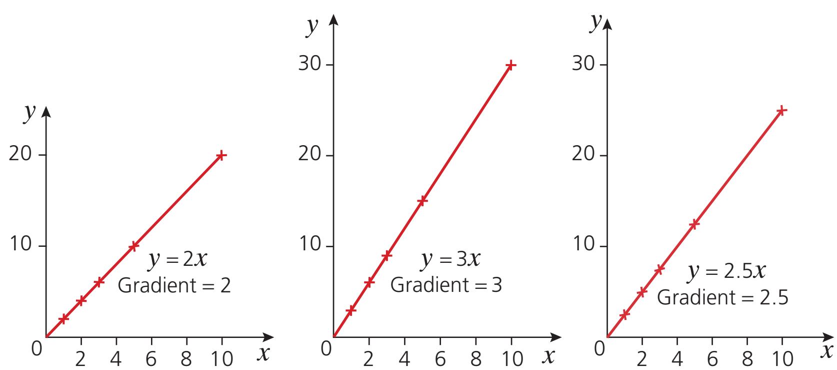
Consider the tables below:

x	0	1	2	3	5	10	$y = 2x$
y	0	2	4	6	10	20	

x	0	1	2	3	5	10	$y = 3x$
y	0	3	6	9	15	30	

x	0	1	2	3	5	10	$y = 2.5x$
y	0	2.5	5	7.5	12.5	25	

In each case y is **directly proportional** to x. This is written $y \propto x$. If any of these three tables is shown on a graph, the graph will be a straight line passing through the origin.



For any statement where $y \propto x$,

$$y = kx$$

where k is a constant equal to the gradient of the graph and is called the **constant of proportionality** or constant of variation.

Consider the tables below:

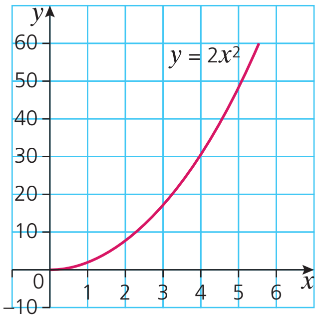
x	1	2	3	4	5	$y = 2x^2$
y	2	8	18	32	50	

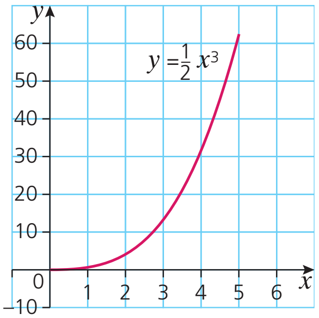
x	1	2	3	4	5	$y = \frac{1}{2}x^3$
y	$\frac{1}{2}$	4	$13\frac{1}{2}$	32	$62\frac{1}{2}$	

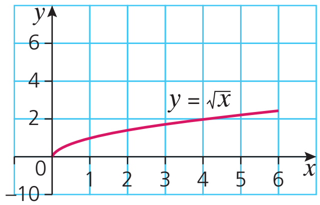
x	1	2	3	4	5	$y = \sqrt{x} = x^{\frac{1}{2}}$
y	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	

In the cases above, y is directly proportional to x^n, where $n > 0$. This can be written as $y \propto x^n$.

The graphs of each of the three equations are shown below:

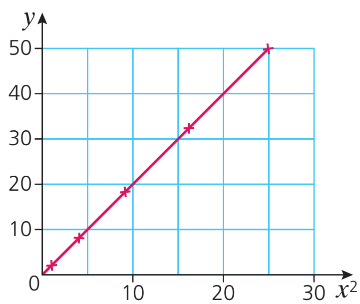






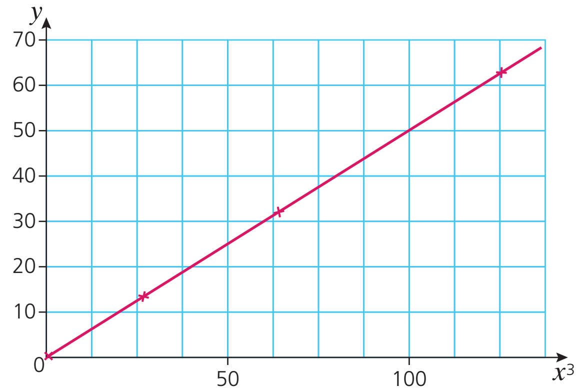
The graphs above, with (x, y) plotted, are not linear. However, if the graph of $y = 2x^2$ is plotted as (x^2, y), then the graph is linear and passes through the origin, demonstrating that $y \propto x^2$ as shown in the graph below.

x	1	2	3	4	5
x^2	1	4	9	16	25
y	2	8	18	32	50



Similarly, the graph of $y = \frac{1}{2}x^3$ is curved when plotted as (x, y), but is linear and passes through the origin if it is plotted as (x^3, y) as shown:

x	1	2	3	4	5
x^3	1	8	27	64	125
y	$\frac{1}{2}$	4	$13\frac{1}{2}$	32	$62\frac{1}{2}$



The graph of $y = \sqrt{x}$ is also linear if plotted as $(\sqrt{x}, y)$.

Inverse proportion

If y is **inversely proportional** to x, then $y \propto \frac{1}{x}$ and $y = \frac{k}{x}$.

If a graph of y against $\frac{1}{x}$ is plotted, this too will be a straight line passing through the origin.

Worked examples

a $y \propto x$. If $y = 7$ when $x = 2$, find y when $x = 5$.

$y = kx$

$7 = k \times 2$ so $k = 3.5$

When $x = 5$,

$y = 3.5 \times 5 = 17.5$

b $y \propto \frac{1}{x}$. If $y = 5$ when $x = 3$, find y when $x = 30$.

$y = \frac{k}{x}$

$5 = \frac{k}{3}$ so $k = 15$

When $x = 30$,

$y = \frac{15}{30} = 0.5$

Exercise 16.1

1 y is directly proportional to x. If $y = 6$ when $x = 2$, find:
- **a** the constant of proportionality
- **b** the value of y when $x = 7$
- **c** the value of y when $x = 9$
- **d** the value of x when $y = 9$
- **e** the value of x when $y = 30$.

2 y is directly proportional to x^2. If $y = 18$ when $x = 6$, find:
- **a** the constant of proportionality
- **b** the value of y when $x = 4$
- **c** the value of y when $x = 7$
- **d** the value of x when $y = 32$
- **e** the value of x when $y = 128$.

3 y is inversely proportional to x^3. If $y = 3$ when $x = 2$, find:
- **a** the constant of proportionality
- **b** the value of y when $x = 4$
- **c** the value of y when $x = 6$
- **d** the value of x when $y = 24$.

4 y is inversely proportional to x^2. If $y = 1$ when $x = 0.5$, find:
- **a** the constant of proportionality
- **b** the value of y when $x = 0.1$
- **c** the value of y when $x = 0.25$
- **d** the value of x when $y = 64$.

Exercise 16.2

1 Write each of the following in the form:
i $y \propto x$ **ii** $y = kx$.
- **a** y is directly proportional to x^3
- **b** y is inversely proportional to x^3
- **c** t is directly proportional to P
- **d** s is inversely proportional to t
- **e** A is directly proportional to r^2
- **f** T is inversely proportional to the square root of g.

2 If $y \propto x$ and $y = 6$ when $x = 2$, find y when $x = 3.5$.

3 If $y \propto \frac{1}{x}$ and $y = 4$ when $x = 2.5$ find:
- **a** y when $x = 20$
- **b** x when $y = 5$.

4 If $p \propto r^2$ and $p = 2$ when $r = 2$, find p when $r = 8$.

5 If $m \propto \frac{1}{r^3}$ and $m = 1$ when $r = 2$, find:
- **a** m when $r = 4$
- **b** r when $m = 125$.

6 If $y \propto x^2$ and $y = 12$ when $x = 2$, find y when $x = 5$.

Exercise 16.3

1 If a stone is dropped off the edge of a cliff, the height (h metres) of the cliff is proportional to the square of the time (t seconds) taken for the stone to reach the ground.
A stone takes 5 seconds to reach the ground when dropped off a cliff 125 m high.
- **a** Write down a relationship between h and t, using k as the constant of proportionality.
- **b** Calculate the constant of proportionality.
- **c** Find the height of a cliff if a stone takes 3 seconds to reach the ground.
- **d** Find the time taken for a stone to fall from a cliff 180 m high.

Exercise 16.3 (cont)

2 The velocity (v metres per second) of a body is known to be proportional to the square root of its kinetic energy (e joules). When the velocity of a body is 120 m/s, its kinetic energy is 1600 J.
   a Write down a relationship between v and e, using k as the constant of proportionality.
   b Calculate the value of k.
   c If $v = 21$, calculate the kinetic energy of the body in joules.

3 The length (l cm) of an edge of a cube is proportional to the cube root of its mass (m grams). It is known that if $l = 15$, then $m = 125$. Let k be the constant of proportionality.
   a Write down the relationship between l, m and k.
   b Calculate the value of k.
   c Calculate the value of l when $m = 8$.

4 The power (P) generated in an electrical circuit is proportional to the square of the current (I amps). When the power is 108 watts, the current is 6 amps.
   a Write down a relationship between P, I and the constant of proportionality, k.
   b Calculate the value of I when $P = 75$.

Student assessment 1

1 $y = kx$. When $y = 12$, $x = 8$.
   a Calculate the value of k.
   b Calculate y when $x = 10$.
   c Calculate y when $x = 2$.
   d Calculate x when $y = 18$.

2 $y = \frac{k}{x}$. When $y = 2$, $x = 5$.
   a Calculate the value of k.
   b Calculate y when $x = 4$.
   c Calculate x when $y = 10$.
   d Calculate x when $y = 0.5$.

3 $p = kq^3$. When $p = 9$, $q = 3$.
   a Calculate the value of k.
   b Calculate p when $q = 6$.
   c Calculate p when $q = 1$.
   d Calculate q when $p = 576$.

4 $m = \frac{k}{\sqrt{n}}$. When $m = 1$, $n = 25$.
   a Calculate the value of k.
   b Calculate m when $n = 16$.
   c Calculate m when $n = 100$.
   d Calculate n when $m = 5$.

5 $y = \frac{k}{x^2}$. When $y = 3$, $x = \frac{1}{3}$.
   a Calculate the value of k.
   b Calculate y when $x = 0.5$.
   c Calculate both values of x when $y = \frac{1}{12}$.
   d Calculate both values of x when $y = \frac{1}{3}$.

Student assessment 2

1 y is inversely proportional to x.

a Copy and complete the table below:

x	1	2	4	8	16	32
y				4		

b What is the value of x when $y = 20$?

2 Copy and complete the tables below:

a $y \propto x$

x	1	2	4	5	10
y		10			

b $y \propto \frac{1}{x}$

x	1	2	4	5	10
y	20				

c $y \propto \sqrt{x}$

x	4	16	25	36	64
y	4				

3 The pressure (P) of a given mass of gas is inversely proportional to its volume (V) at a constant temperature. If $P = 4$ when $V = 6$, calculate:

a P when $V = 30$

b V when $P = 30$.

4 The gravitational force (F) between two masses is inversely proportional to the square of the distance (d) between them. If $F = 4$ when $d = 5$, calculate:

a F when $d = 8$

b d when $F = 25$.

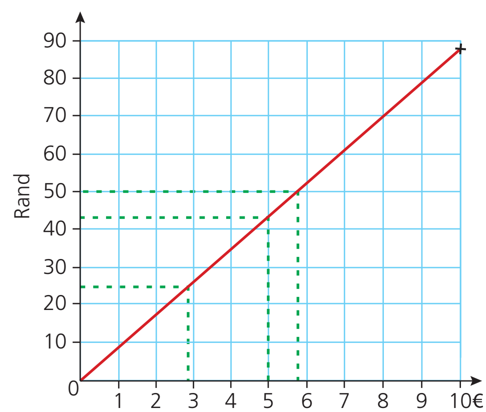
17 Graphs in practical situations

Conversion graphs

A straight-line graph can be used to convert one set of units to another. Examples include converting from one currency to another, converting distance in miles to kilometres and converting temperature from degrees Celsius to degrees Fahrenheit.

Worked example

The graph below converts South African rand into euros based on an exchange rate of €1 = 8.80 rand.



a Using the graph, estimate the number of rand equivalent to €5.

A line is drawn up from €5 until it reaches the plotted line, then across to the vertical axis.
From the graph it can be seen that €5 ≈ 44 rand.
(≈ is the symbol for 'is approximately equal to')

b Using the graph, find the cost in euros of a drink costing 25 rand.

A line is drawn across from 25 rand until it reaches the plotted line, then down to the horizontal axis.
From the graph it can be seen that the cost of the drink ≈ €2.80.

c If a meal costs 200 rand, use the graph to estimate its cost in euros.

The graph does not go up to 200 rand, therefore a factor of 200 needs to be used, e.g. 50 rand.
From the graph, 50 rand ≈ €5.70, therefore it can be deduced that 200 rand ≈ €22.80 (i.e. 4 × €5.70).

Exercise 17.1

1 Given that 80 km = 50 miles, draw a **conversion graph** up to 100 km.
Using your graph, estimate:
  a how many miles is 50 km,
  b how many kilometres is 80 miles,
  c the speed in miles per hour (mph) equivalent to 100 km/h,
  d the speed in km/h equivalent to 40 mph.

2 You can roughly convert temperature in degrees Celsius to degrees Fahrenheit by doubling the degrees Celsius and adding 30.
Draw a conversion graph up to 50 °C. Use your graph to estimate the following:
  a the temperature in °F equivalent to 25 °C,
  b the temperature in °C equivalent to 100 °F,
  c the temperature in °F equivalent to 0 °C,
  d the temperature in °C equivalent to 120 °F.

3 Given that 0 °C = 32 °F and 50 °C = 122 °F, on the same graph as in Question 2, draw a true conversion graph.
  i Use the true graph to calculate the conversions in Question 2.
  ii Where would you say the rough conversion is most useful?

4 Long-distance calls from New York to Harare are priced at 85 cents/min off peak and $1.20/min at peak times.
  a Draw, on the same axes, conversion graphs for the two different rates.
  b From your graph, estimate the cost of an 8 minute call made off peak.
  c Estimate the cost of the same call made at peak rate.
  d A caller has $4 of credit on his phone. Estimate how much more time he can talk for if he rings at off peak instead of at peak times.

5 A maths exam is marked out of 120. Draw a conversion graph to change the following marks to percentages.
  a 80 b 110 c 54 d 72

Speed, distance and time

You may already be aware of the following formula:

$$\text{distance} = \text{speed} \times \text{time}$$

Rearranging the formula gives:

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

Where the speed is not constant:

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}$$

Exercise 17.2

1 Find the average speed of an object moving:

a 30 m in 5s **b** 48 m in 12 s
c 78 km in 2 h **d** 50 km in 2.5 h
e 400 km in 2 h 30 min **f** 110 km in 2 h 12 min

2 How far will an object travel during:

a 10 s at 40 m/s **b** 7 s at 26 m/s
c 3 h at 70 km/h **d** 4 h 15 min at 60 km/h
e 10 min at 60 km/h **f** 1 h 6 min at 20 m/s?

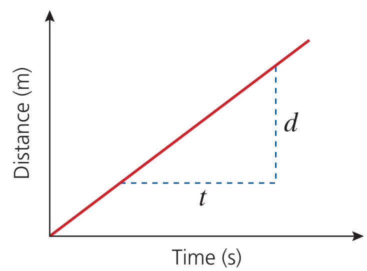
3 How long will it take to travel:

a 50 m at 10 m/s **b** 1 km at 20 m/s
c 2 km at 30 km/h **d** 5 km at 70 m/s
e 200 cm at 0.4 m/s **f** 1 km at 15 km/h?

Travel graphs

The graph of an object travelling at a constant speed is a straight line as shown.

Gradient = $\frac{d}{t}$



The units of the gradient are m/s, hence the gradient of a distance–time graph represents the speed at which the object is travelling.

Worked example

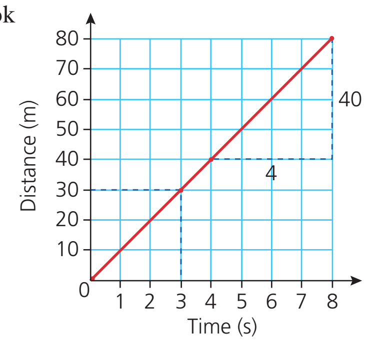
The graph (right) represents an object travelling at constant speed.

a From the graph, calculate how long it took to cover a distance of 30 m.

The time taken to travel 30 m is 3 seconds.

b Calculate the gradient of the graph.

Taking two points on the line, gradient $= \frac{40}{4} = 10$.



c Calculate the speed at which the object was travelling.

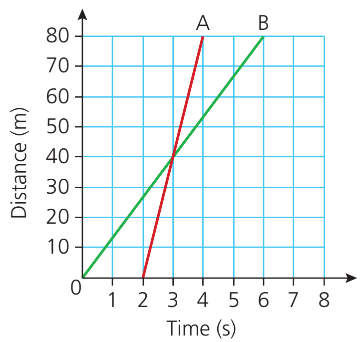
Gradient of a distance–time graph = speed.
Therefore the speed is 10 m/s.

Exercise 17.3

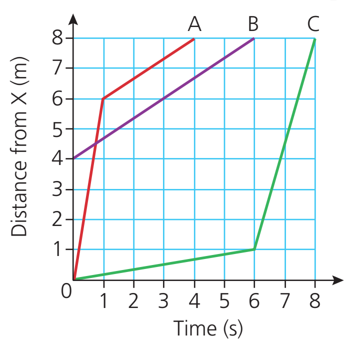
1 Draw a distance–time graph for the first 10 seconds of an object travelling at 6 m/s.

2 Draw a distance–time graph for the first 10 seconds of an object travelling at 5 m/s. Use your graph to estimate:
   a the time taken to travel 25 m,
   b how far the object travels in 3.5 seconds.

3 Two objects A and B set off from the same point and move in the same straight line. B sets off first, while A sets off 2 seconds later. Using the distance–time graph estimate:
   a the speed of each of the objects,
   b how far apart the objects would be 20 seconds after the start.



4 Three objects A, B and C move in the same straight line away from a point X. Both A and C change their speed during the journey, while B travels at the same constant speed throughout.

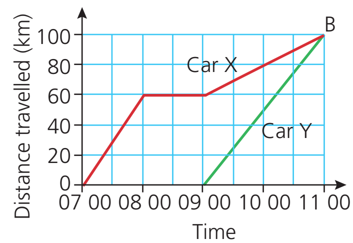


From the distance–time graph, estimate:
   a the speed of object B,
   b the two speeds of object A,
   c the average speed of object C,
   d how far object C is from X, 3 seconds from the start,
   e how far apart objects A and C are 4 seconds from the start.

The graphs of two or more journeys can be shown on the same axes. The shape of the graph gives a clear picture of the movement of each of the objects.

Worked example

The journeys of two cars, X and Y, travelling between A and B are represented on the distance–time graph (right). Car X and Car Y both reach point B 100 km from A at 11 00.



a Calculate the speed of Car X between 07 00 and 08 00.

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{60}{1} = \text{km/h} = 60 \text{ km/h}$$

b Calculate the speed of Car Y between 09 00 and 11 00.

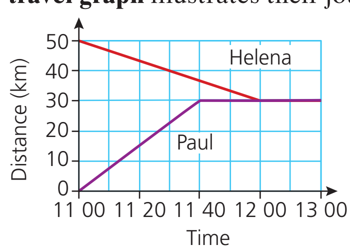
$$\text{speed} = \frac{100}{2} \text{ km/h} = 50 \text{ km/h}$$

c Explain what is happening to Car X between 08 00 and 09 00.

No distance has been travelled, therefore Car X is stationary.

Exercise 17.4

1 Two friends, Paul and Helena, arrange to meet for lunch at noon. They live 50 km apart and the restaurant is 30 km from Paul's home. The **travel graph** illustrates their journeys.



a What is Paul's average speed between 11 00 and 11 40?
b What is Helena's average speed between 11 00 and 12 00?
c What does the horizontal part of Paul's line represent?

2 A car travels at a speed of 60 km/h for 1 hour. It then stops for 30 minutes and then continues at a constant speed of 80 km/h for a further 1.5 hours. Draw a distance–time graph for this journey.

3 Fadi cycles for 1.5 hours at 10 km/h. He then stops for an hour and then travels for a further 15 km in 1 hour. Draw a distance–time graph of Fadi's journey.

4 Two friends leave their houses at 16 00. The houses are 4 km apart and the friends travel towards each other on the same road. Fyodor walks at 7 km/h and Yin walks at 5 km/h.
a On the same axes, draw a distance–time graph of their journeys.
b From your graph, estimate the time at which they meet.
c Estimate the distance from Fyodor's house to the point where they meet.

5 A train leaves a station P at 18 00 and travels to station Q 150 km away. It travels at a steady speed of 75 km/h. At 18 10 another train leaves Q for P at a steady speed of 100 km/h.

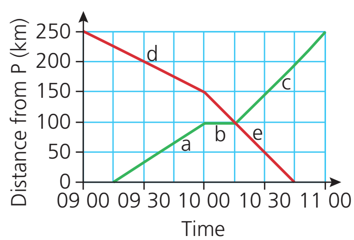
a On the same axes, draw a distance–time graph to show both journeys.

b From the graph, estimate the time at which both trains pass each other.

c At what distance from station Q do both trains pass each other?

d Which train arrives at its destination first?

6 A train sets off from town P at 09 15 and heads towards town Q 250 km away. Its journey is split into the three stages, a, b and c. At 09 00 a second train leaves town Q heading for town P. Its journey is split into the two stages, d and e.



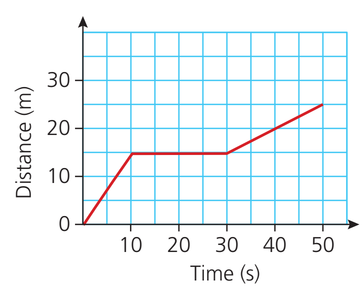
Using the graph, calculate the following:

a the speed of the first train during stages a, b and c,

b the speed of the second train during stages d and e.

Speed–time graphs, acceleration and deceleration

So far the graphs that have been dealt with have been like the one shown, i.e. distance–time graphs.



If the graph were of a girl walking, it would indicate that initially she was walking at a constant speed of 1.5 m/s for 10 seconds, then she stopped for 20 seconds and finally she walked at a constant speed of 0.5 m/s for 20 seconds.

Note

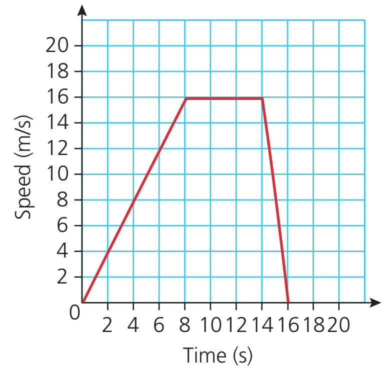
For a distance–time graph, the following is true:

- a straight line represents constant speed
- a horizontal line indicates no movement (i.e. speed is zero)
- the gradient of a line gives the speed.

This section also deals with the interpretation of travel graphs, but where the vertical axis represents the object's speed.

Worked example

The graph shows the speed of a car over a period of 16 seconds.



a Explain the shape of the graph.
For the first 8 seconds the speed of the car is increasing uniformly with time. This means it is **accelerating at a constant rate**. Between 8 and 14 seconds, the car is travelling at a constant speed of 16 m/s. Between 14 and 16 seconds, the speed of the car decreases uniformly. This means that it is **decelerating at a constant rate**.

b Calculate the rate of acceleration during the first 8 seconds.
From a speed–time graph, the acceleration is found by calculating the gradient of the line. Therefore:

$$\text{acceleration} = \frac{16}{8} = 2\,\text{m/s}^2$$

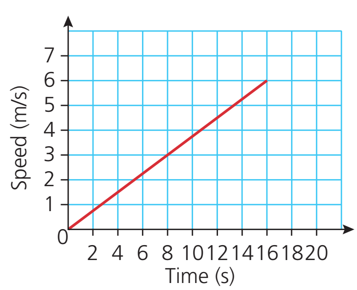
c Calculate the rate of deceleration between 14 and 16 seconds:

$$\text{deceleration} = \frac{16}{2} = 8\text{m/s}^2$$

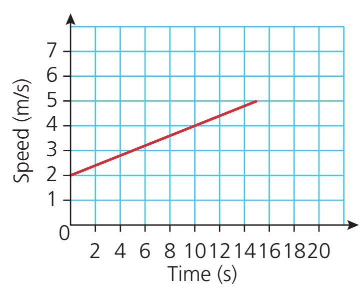
Exercise 17.5

Using the graphs below, calculate the acceleration/deceleration in each case.

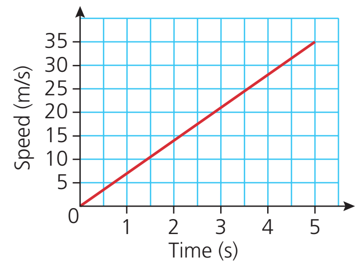
1



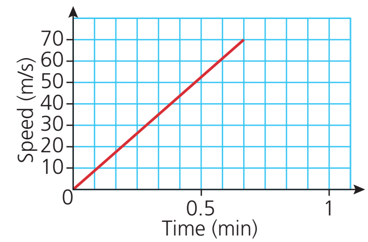
2



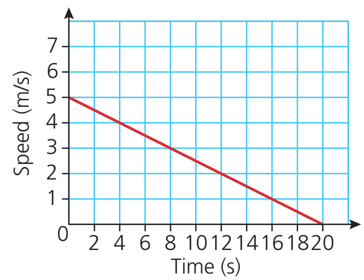
3



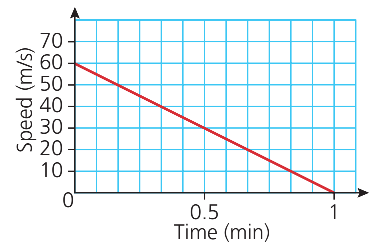
4



5



6

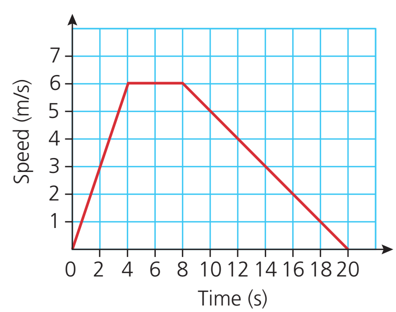


7 Sketch a graph to show an aeroplane accelerating from rest at a constant rate of 5 m/s^2 for 10 seconds.

8 A train travelling at 30 m/s starts to decelerate at a constant rate of 3 m/s^2. Sketch a speed–time graph showing the train's motion until it stops.

Exercise 17.6

1 The graph shows the speed–time graph of a boy running for 20 seconds.



Calculate:

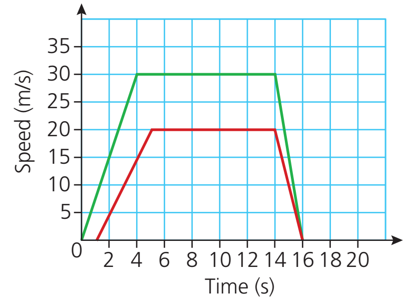
a the acceleration during the first four seconds,

b the acceleration during the second period of four seconds,

c the deceleration during the final twelve seconds.

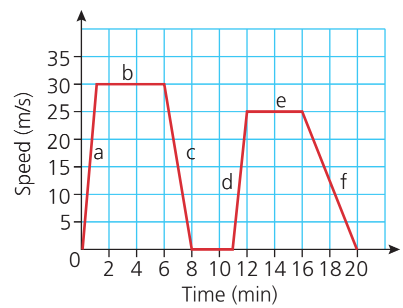
Exercise 17.6 (cont)

2 The speed–time graph represents a cheetah chasing a gazelle.



a Does the top graph represent the cheetah or the gazelle?
b Calculate the cheetah's acceleration in the initial stages of the chase.
c Calculate the gazelle's acceleration in the initial stages of the chase.
d Calculate the cheetah's deceleration at the end.

3 The speed–time graph represents a train travelling from one station to another.



a Calculate the acceleration during stage a.
b Calculate the deceleration during stage c.
c Calculate the deceleration during stage f.
d Describe the train's motion during stage b.
e Describe the train's motion 10 minutes from the start.

Area under a speed–time graph

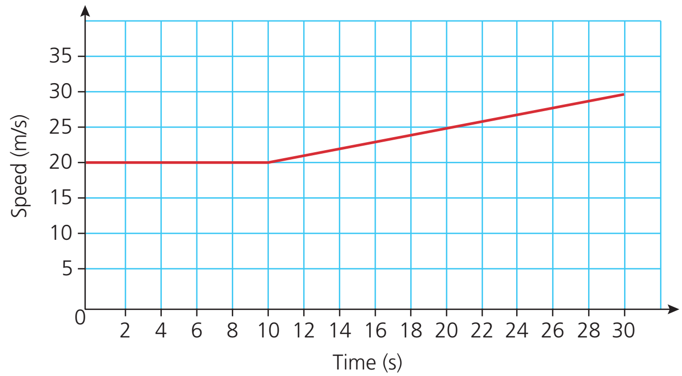
The area under a speed–time graph gives the distance travelled.

Worked example

The table below shows the speed of a train over a 30-second period.

Time (s)	0	5	10	15	20	25	30
Speed (m/s)	20	20	20	22.5	25	27.5	30

a Plot a speed–time graph for the first 30 seconds.

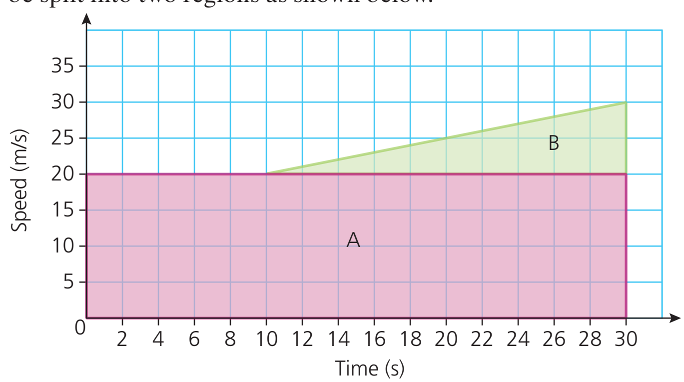


b Calculate the train's acceleration in the last 20 seconds.

Acceleration $= \frac{10}{20} = \frac{1}{2}\,\text{m/s}^2$

c Calculate the distance travelled during the 30 seconds.

This is calculated by working out the area under the graph. The graph can be split into two regions as shown below.



Distance represented by region A $= (20 \times 30)\,\text{m}$
$= 600\,\text{m}$

Distance represented by region B $= \left(\frac{1}{2} \times 20 \times 10\right)\text{m}$
$= 100\,\text{m}$

Total distance travelled $= (600 + 100)\,\text{m}$
$= 700\,\text{m}$

Exercise 17.7

1 The table below gives the speed of a boat over a 10-second period.

Time (s)	0	2	4	6	8	10
Speed (m/s)	5	6	7	8	9	10

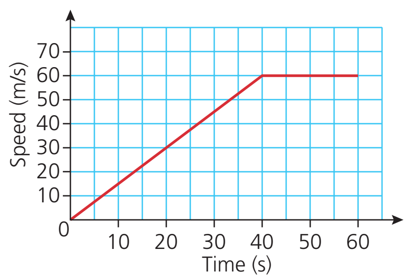
a Plot a speed–time graph for the 10-second period.
b Calculate the acceleration of the boat.
c Calculate the total distance travelled during the 10 seconds.

2 A cyclist travelling at 6 m/s applies the brakes and decelerates at a constant rate of 2 m/s^2.
a Copy and complete the table below.

Time (s)	0	0.5	1	1.5	2	2.5	3
Speed (m/s)	6						0

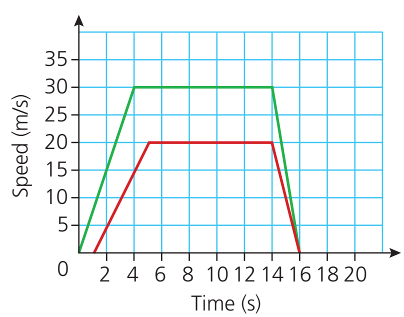
b Plot a speed–time graph for the 3 seconds shown in the table above.
c Calculate the distance travelled during the 3 seconds of deceleration.

3 A car accelerates as shown in the graph.



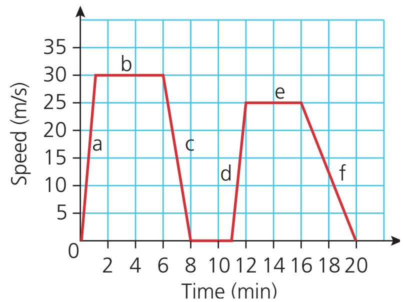
a Calculate the rate of acceleration in the first 40 seconds.
b Calculate the distance travelled over the 60 seconds shown.
c After what time had the motorist travelled half the distance?

4 The graph represents the cheetah and gazelle chase from Question 2 in Exercise 17.6.



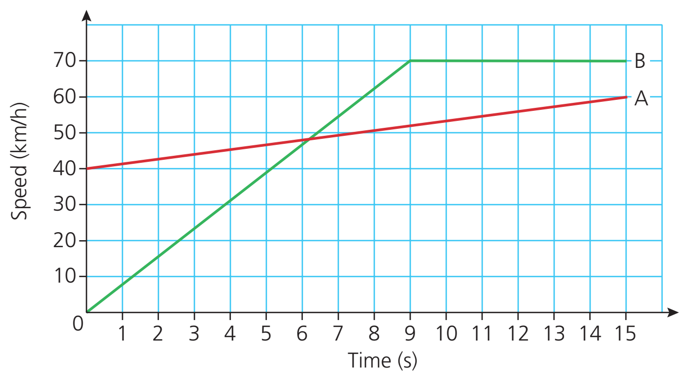
a Calculate the distance run by the cheetah during the chase.
b Calculate the distance run by the gazelle during the chase.

5 The graph represents the train journey from Question 3 in Exercise 17.6. Calculate, in km, the distance travelled during the 20 minutes shown.



6 An aircraft accelerates uniformly from rest at a rate of $10\,\text{m/s}^2$ for 12 seconds before it takes off. Calculate the distance it travels along the runway.

7 The speed–time graph below depicts the motion of two motorbikes A and B over a 15-second period.

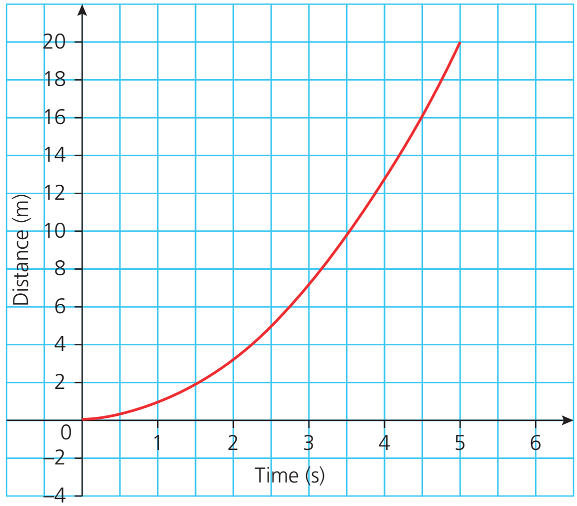


At the start of the graph, motorbike A overtakes a stationary motorbike B. Assume they then travel in the same direction.

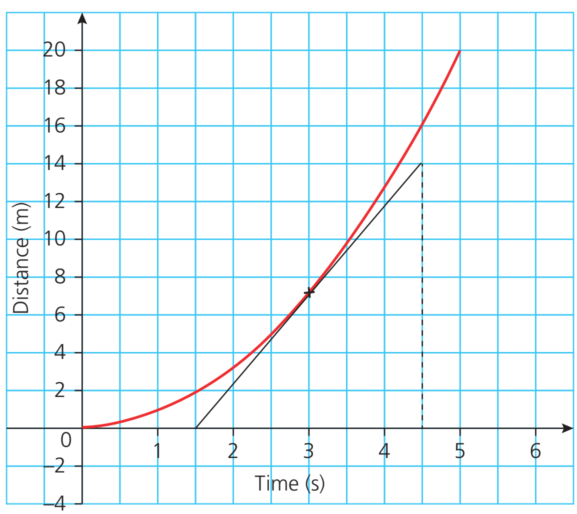
a Calculate motorbike A's acceleration over the 15 seconds in m/s^2.
b Calculate motorbike B's acceleration over the first 9 seconds in m/s^2.
c Calculate the distance travelled by A during the 15 seconds (give your answer to the nearest metre).
d Calculate the distance travelled by B during the 15 seconds (give your answer to the nearest metre).
e How far apart were the two motorbikes at the end of the 15-second period?

Non-linear travel graphs

So far, all the graphs investigated have involved straight lines. In real life, however, the motion of objects is unlikely to produce a straight line as changes in movement tend to be gradual rather than instantaneous.



The gradient of a curved distance–time graph is not constant (i.e. it is always changing). To find the speed of an object, for example, 3 seconds after the start, it is necessary to calculate the gradient of the curve at that particular point. This is done by drawing a tangent to the curve at that point. The gradient of the tangent is the same as the gradient of the curve at the same point.



The gradient of the tangent is calculated as

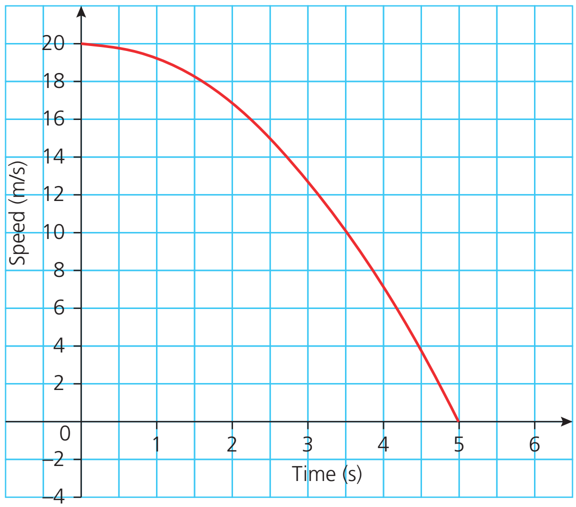
$$\frac{\text{vertical height}}{\text{horizontal distance}} = \frac{14 - 0}{4.5 - 1.5} = \frac{14}{3} = 4\frac{2}{3}.$$

Therefore the speed of the object after 3 seconds is approximately $4\frac{2}{3}$ m/s.

Note: The calculation is only approximate as the tangent is drawn by eye and therefore the readings are not exact.

Calculating the gradient of a curve is covered in more depth in Chapter 18: Graphs of functions.

The same method is used when calculating the acceleration of an object from a speed–time graph.



The graph above shows the motion of an object over a period of 6 seconds. To calculate the acceleration at a moment in time, for example, 4 seconds after the start, the gradient of the curve at that point needs to be calculated. This is done, once again, by drawing a tangent to the curve at that point and calculating its gradient.

The gradient of the tangent $= \frac{17-1}{2.5-5} = \frac{16}{-2.5} = -6.4$.

Therefore, the acceleration after 4 seconds is $-6.4\,\text{m/s}^2$ (i.e. the object is decelerating at $6.4\,\text{m/s}^2$).

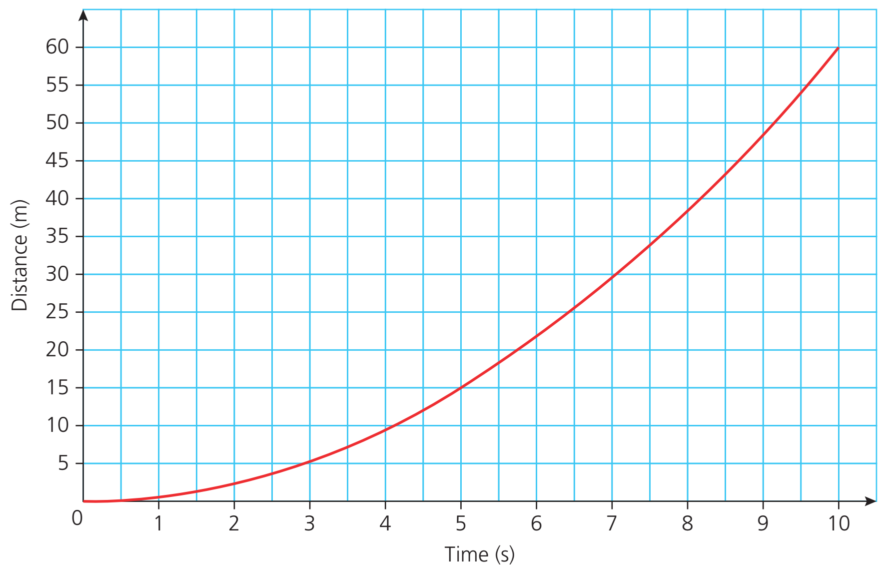
Worked example

A cyclist starts cycling from rest. For the first 10 seconds, her distance from the start is given by the equation $D = 0.6t^2$, where D is the distance travelled in metres and t is the time in seconds.

a Complete a distance–time table of results for the first 10 seconds of motion.

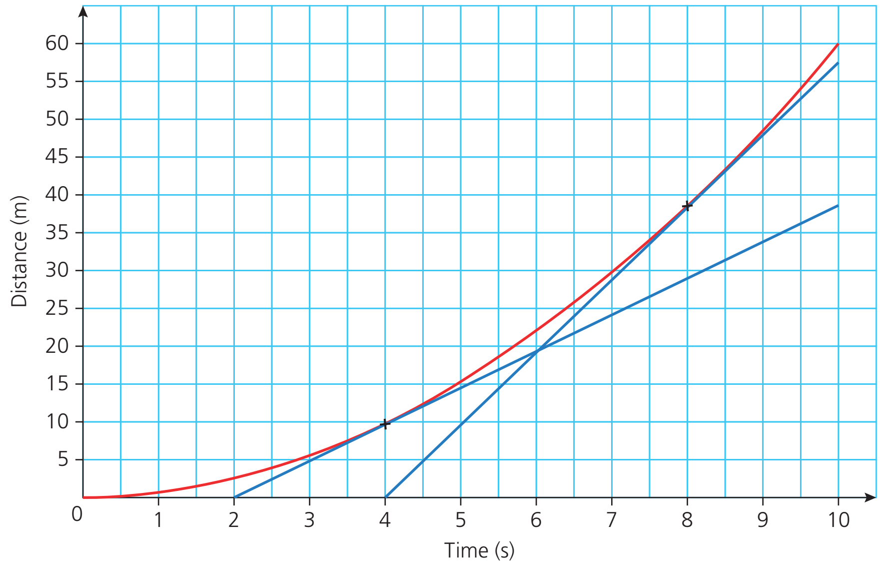
Time (s)	0	1	2	3	4	5	6	7	8	9	10
Distance (m)	0	0.6	2.4	5.4	9.6	15	21.6	29.4	38.4	48.6	60

b Plot a graph of the cyclist's motion on a distance–time graph.



c Calculate the speed of the cyclist after 4 s and after 8 s.

Tangents to the graph at 4 s and 8 s are drawn as shown:



The gradient of the tangent to the curve at 4 s = $\frac{36-0}{9.5-2} = \frac{36}{7.5} = 4.8$.

The speed after 4 s is 4.8 m/s.

The gradient of the tangent to the curve at 8 s = $\frac{48-0}{9-4} = \frac{48}{5} = 9.6$.

The speed after 8 s is 9.6 m/s.

Exercise 17.8

1 A stone is dropped off a tall cliff. The distance it falls is given by the equation $d = 4.9t^2$, where d is the distance fallen in metres and t is the time in seconds.

a Complete the table of results below for the first 10 seconds.

Time (s)	0	2	4	6	8	10
Distance (m)	0		78.4			

b Plot the results in a graph.

c From your graph, calculate the speed the stone was travelling at after 5 seconds.

2 A car travelling at 20 m/s applies the brakes. The speed of the car is given by the equation $s = 20 - 0.75t^2$, where s is the speed in m/s and t is the time in seconds.

a Plot a graph of the car's motion from the moment the brakes are applied until the moment it comes to rest.

b Estimate from your graph the time it takes for the car to come to rest.

c Calculate the car's deceleration after 2 seconds.

Exercise 17.8 (cont)

3 The distance of an object from its starting point is given by the equation $x = 2t^3 - t^2$, where x is the distance in metres from the start and t is the time in seconds.
   a Plot a distance–time graph for the first 6 seconds of motion.
   b Using your graph, estimate the speed of the object after 3.5 seconds.

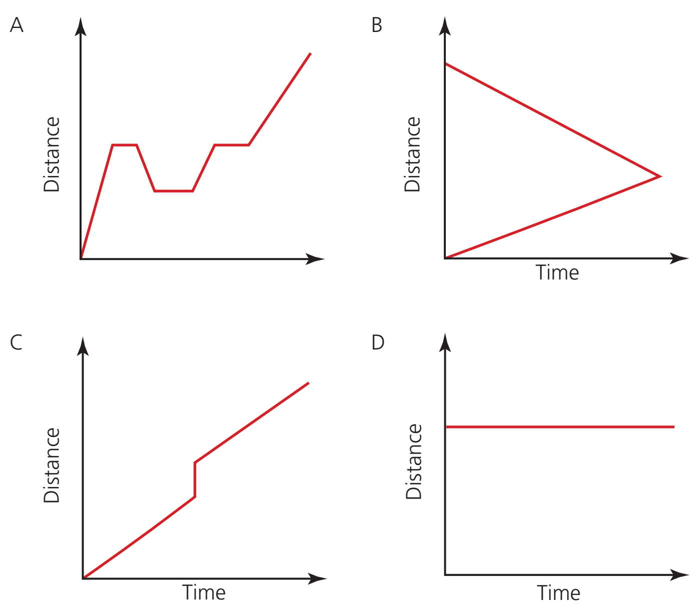
Student assessment 1

1 Absolute zero (0 K) is equivalent to –273 °C and 0 °C is equivalent to 273 K. Draw a conversion graph which will convert K into °C. Use your graph to estimate:
   a the temperature in K equivalent to –40 °C,
   b the temperature in °C equivalent to 100 K.

2 A plumber has a call-out charge of $70 dollars and then charges a rate of $50 per hour.
   a Draw a conversion graph and estimate the cost of the following:
      i a job lasting $4\frac{1}{2}$ hours, ii a job lasting $6\frac{3}{4}$ hours.
   b If a job cost $245, estimate from your graph how long it took to complete.

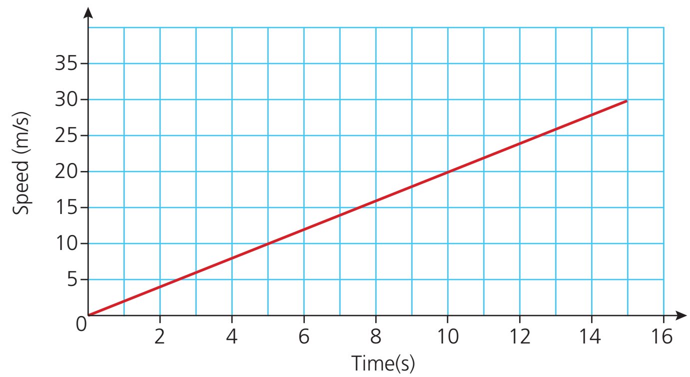
3 A boy lives 3.5 km from his school. He walks home at a constant speed of 9 km/h for the first 10 minutes. He then stops and talks to his friends for 5 minutes. He finally runs the rest of his journey home at a constant speed of 12 km/h.
   a Illustrate this information on a distance–time graph.
   b Use your graph to estimate the total time it took the boy to get home that day.

4 Below are four distance–time graphs A, B, C and D. Two of them are not possible.
   a Which two graphs are impossible?
   b Explain why the two you have chosen are not possible.



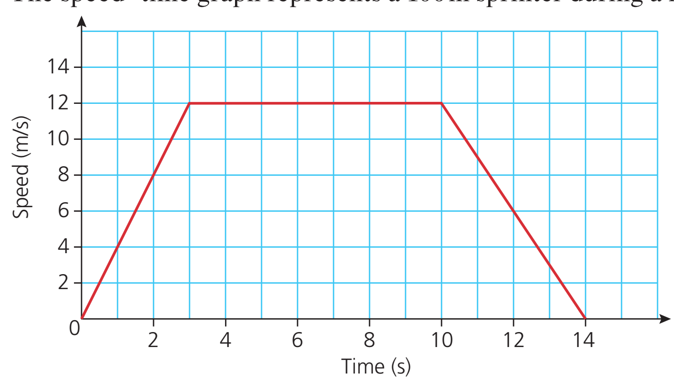
Student assessment 2

1 The graph below is a speed–time graph for a car accelerating from rest.



a Calculate the car's acceleration in m/s^2.
b Calculate, in metres, the distance the car travels in 15 seconds.
c How long did it take the car to travel half the distance?

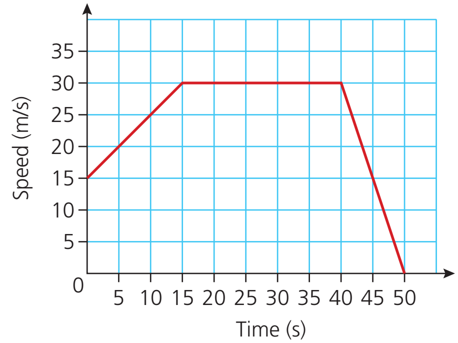
2 The speed–time graph represents a 100 m sprinter during a race.



a Calculate the sprinter's acceleration during the first two seconds of the race.
b Calculate the sprinter's deceleration at the end of the race.
c Calculate the distance the sprinter ran in the first 10 seconds.
d Calculate the sprinter's time for the 100 m race. Give your answer to 3 s.f.

3 A motorcyclist accelerates uniformly from rest to 50 km/h in 8 seconds. She then accelerates to 110 km/h in a further 6 seconds.
a Draw a speed–time graph for the first 14 seconds.
b Use your graph to find the total distance the motorcyclist travels. Give your answer in metres.

4 The graph shows the speed of a car over a period of 50 seconds.



a Calculate the car's acceleration in the first 15 seconds.
b Calculate the distance travelled while the car moved at constant speed.
c Calculate the total distance travelled.

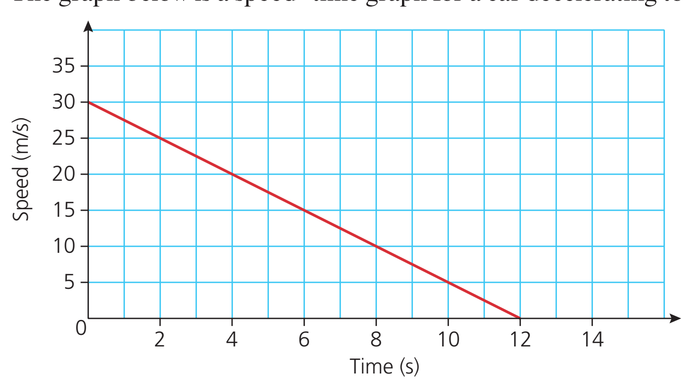
5 The distance of an object from its starting position is given by the equation:
$s = t^3 - 10t^2 + 21t + 25$, where s is the distance from the start in metres and t is the time in seconds.
a Copy and complete the table of results:

Time (secs)	0	1	2	3	4	5	6
Distance (m)			35			5	

b Plot a graph of distance from the start against time for the first 6 seconds of motion.
c Estimate from your graph when the object is stationary.
d Estimate the velocity of the object when $t = 5$ seconds.

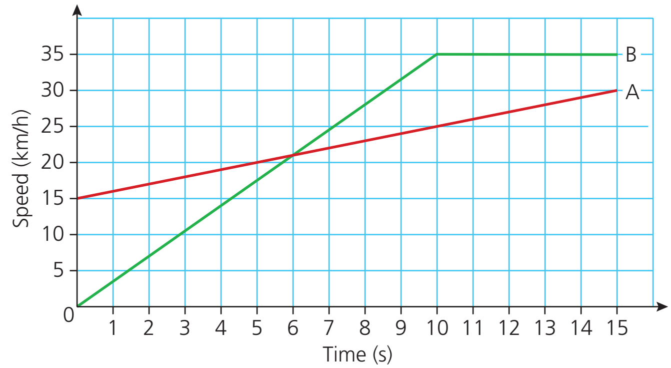
Student assessment 3

1 The graph below is a speed–time graph for a car decelerating to rest.



a Calculate the car's deceleration in m/s^2.
b Calculate, in metres, the distance the car travels in 12 seconds.
c How long did it take the car to travel half the distance?

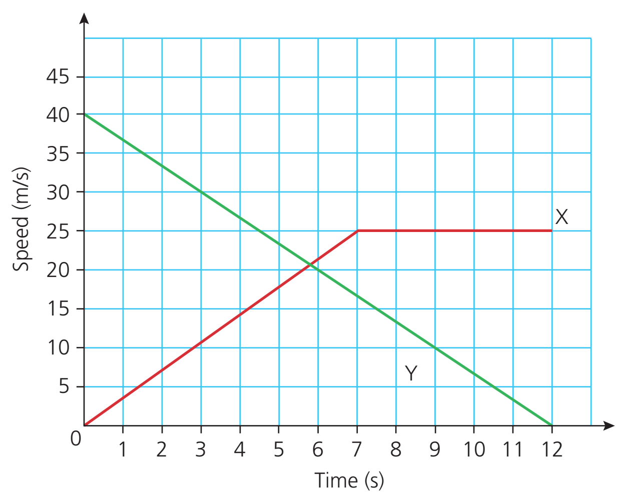
2 The graph below shows the speeds of two cars A and B over a 15-second period.



a Calculate the acceleration of car A in m/s^2.
b Calculate the distance travelled in metres during the 15 seconds by car A.
c Calculate the distance travelled in metres during the 15 seconds by car B.

3 A motor cycle accelerates uniformly from rest to 30 km/h in 3 seconds. It then accelerates to 150 km/h in a further 6 seconds.
a Draw a speed–time graph for the first 9 seconds.
b Use your graph to find the total distance the motor cycle travels. Give your answer in metres.

4 Two cars X and Y are travelling in the same direction. The speed–time graph (below) shows their speeds over 12 seconds.



a Calculate the deceleration of Y during the 12 seconds.
b Calculate the distance travelled by Y in the 12 seconds.
c Calculate the total distance travelled by X in the 12 seconds.

5 The speed of an object is recorded over a period of 5 seconds. Its speed is given by the equation $v = t^3 - 4t^2 - t + 14$, where v is the speed of the object in m/s and t is the time in seconds.

a Plot a speed–time graph for the object's motion in the first 5 seconds.

b Explain the acceleration of the object at the graph's lowest point. Justify your answer.

c Estimate the acceleration of the object when $t = 2$ seconds.

18 Graphs of functions

You should be familiar with the work covered in Chapter 21, Straight-line graphs, before working on this chapter.

Quadratic functions

The general expression for a **quadratic function** takes the form $ax^2 + bx + c$, where a, b and c are constants. Some examples of quadratic functions are:

$$y = 2x^2 + 3x + 12 \qquad y = x^2 - 5x + 6 \qquad y = 3x^2 + 2x - 3$$

A graph of a quadratic function produces a smooth curve called a **parabola**, for example:

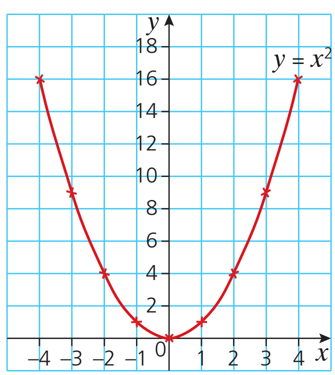
$y = x^2$

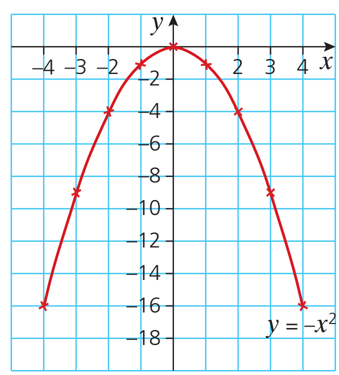
x	–4	–3	–2	–1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

$y = -x^2$

x	–4	–3	–2	–1	0	1	2	3	4
y	–16	–9	–4	–1	0	–1	–4	–9	–16

Notice how a quadratic function has a line of symmetry through its turning point.





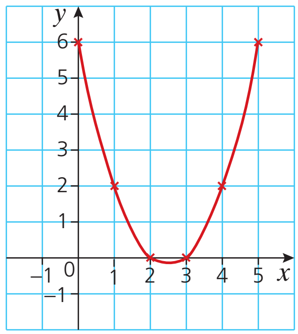
Worked examples

a Plot a graph of the function $y = x^2 - 5x + 6$ for $0 \leqslant x \leqslant 5$.

A table of values for x and y is given below:

x	0	1	2	3	4	5
y	6	2	0	0	2	6

These can then be plotted to give the graph:

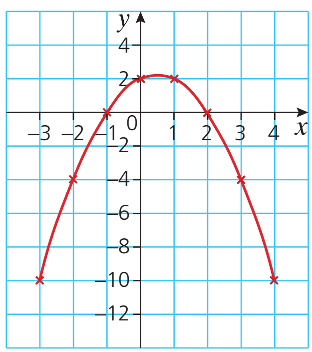


b Plot a graph of the function $y = -x^2 + x + 2$ for $-3 \leqslant x \leqslant 4$.

Drawing up a table of values gives:

x	–3	–2	–1	0	1	2	3	4
y	–10	–4	0	2	2	0	–4	–10

The graph of the function is given below:



Exercise 18.1

For each of the following quadratic functions, construct a table of values and then draw the graph.

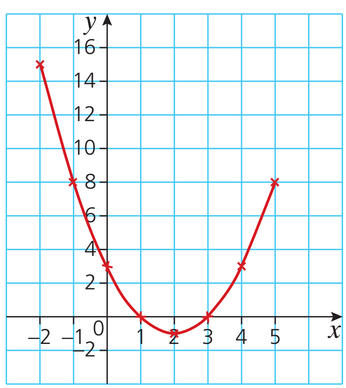
1. $y = x^2 + x - 2$, $\quad -4 \leqslant x \leqslant 3$
2. $y = -x^2 + 2x + 3$, $\quad -3 \leqslant x \leqslant 5$
3. $y = x^2 - 4x + 4$, $\quad -1 \leqslant x \leqslant 5$
4. $y = -x^2 - 2x - 1$, $\quad -4 \leqslant x \leqslant 2$
5. $y = x^2 - 2x - 15$, $\quad -4 \leqslant x \leqslant 6$
6. $y = 2x^2 - 2x - 3$, $\quad -2 \leqslant x \leqslant 3$
7. $y = -2x^2 + x + 6$, $\quad -3 \leqslant x \leqslant 3$
8. $y = 3x^2 - 3x - 6$, $\quad -2 \leqslant x \leqslant 3$
9. $y = 4x^2 - 7x - 4$, $\quad -1 \leqslant x \leqslant 3$
10. $y = -4x^2 + 4x - 1$, $\quad -2 \leqslant x \leqslant 3$

Graphical solution of a quadratic equation

Worked example

a Draw a graph of $y = x^2 - 4x + 3$ for $-2 \leqslant x \leqslant 5$.

x	–2	–1	0	1	2	3	4	5
y	15	8	3	0	–1	0	3	8



b Use the graph to solve the equation $x^2 - 4x + 3 = 0$.

To solve the equation it is necessary to find the values of x when $y = 0$, i.e. where the graph crosses the x-axis.

These points occur when $x = 1$ and $x = 3$ and are therefore the solutions.

Exercise 18.2

Solve each of the quadratic functions below by plotting a graph for the ranges of x stated.

1 $x^2 - x - 6 = 0$, $-4 \leqslant x \leqslant 4$

2 $-x^2 + 1 = 0$, $-4 \leqslant x \leqslant 4$

3 $x^2 - 6x + 9 = 0$, $0 \leqslant x \leqslant 6$

4 $-x^2 - x + 12 = 0$, $-5 \leqslant x \leqslant 4$

5 $x^2 - 4x + 4 = 0$, $-2 \leqslant x \leqslant 6$

6 $2x^2 - 7x + 3 = 0$, $-1 \leqslant x \leqslant 5$

7 $-2x^2 + 4x - 2 = 0$, $-2 \leqslant x \leqslant 4$

8 $3x^2 - 5x - 2 = 0$, $-1 \leqslant x \leqslant 3$

In the previous worked example, as $y = x^2 - 4x + 3$, a solution could be found to the equation $x^2 - 4x + 3 = 0$ by reading off where the graph crossed the x-axis. The graph can, however, also be used to solve other quadratic equations.

Worked example

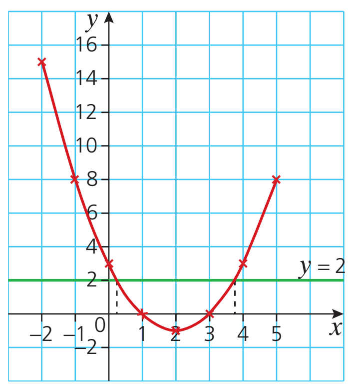
Use the graph of $y = x^2 - 4x + 3$ to solve the equation $x^2 - 4x + 1 = 0$.

$x^2 - 4x + 1 = 0$ can be rearranged to give:

$x^2 - 4x + 3 = 2$

2 has been added to both sides of the equation

Using the graph of $y = x^2 - 4x + 3$ and plotting the line $y = 2$ on the same graph, gives the graph shown below.



The point at which a graph of a function changes from a negative gradient to a positive gradient, or vice versa (that is, the point where the gradient is zero) is called a ***turning point****.* *See also stationary points on page 239.*

Where the curve and the line cross gives the solution to $x^2 - 4x + 3 = 2$ and hence also $x^2 - 4x + 1 = 0$.

Therefore the solutions to $x^2 - 4x + 1 = 0$ are

$x \approx 0.3$ and $x \approx 3.7$.

Exercise 18.3

Using the graphs that you drew in Exercise 18.2, solve the following quadratic equations. Show your method clearly.

1 $x^2 - x - 4 = 0$

2 $-x^2 - 1 = 0$

3 $x^2 - 6x + 8 = 0$

4 $-x^2 - x + 9 = 0$

5 $x^2 - 4x + 1 = 0$

6 $2x^2 - 7x = 0$

7 $-2x^2 + 4x = -1$

8 $3x^2 = 2 + 5x$

The completed square form and the graph of a quadratic equation

In Chapter 13, Exercise 13.10, we saw that a quadratic can be written in a form known as the completed square.

For example, $y = x^2 + 6x - 4$ can be written in completed square form as $y = (x + 3)^2 - 13$.

The general form of a quadratic equation written in completed square form is $y = a(x - b)^2 + c$, where a, b and c are constants.

To plot the graph of a quadratic written in completed square form, it is not necessary to expand the brackets first.

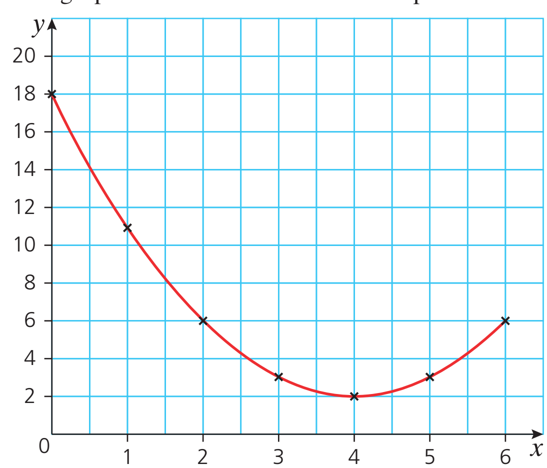
Worked examples

a Plot the graph of the function $y = (x - 4)^2 + 2$ for $0 \leqslant x \leqslant 6$.

Drawing up a table of values gives:

x	0	1	2	3	4	5	6
y	18	11	6	3	2	3	6

The graph of the function can now be plotted:



b Write the coordinates of the turning point of the graph.

The turning point in this case is the minimum point of the graph. It occurs at (4, 2).

In general, quadratic graphs take the parabola shape  or 

For the  shape the turning point is known as a minimum point, while for the  the turning point is known as the maximum point.

Exercise 18.4

For each of the quadratics in Questions 1–8:

- **a** Plot a graph of the function for the values of x given.
- **b** State whether the turning point is a maximum or a minimum point.
- **c** Write down the coordinates of the turning point.
- **d** In each case, write down the coordinates of the point on the curve which the curve's line of symmetry passes through.

1 $y = (x - 1)^2 + 4; \quad -2 \leqslant x \leqslant 4$

2 $y = (x + 3)^2 - 6; \quad -6 \leqslant x \leqslant 0$

3 $y = (x - 4)^2 - 1; \quad 0 \leqslant x \leqslant 7$

4 $y = -(x - 2)^2 + 3; \quad -1 \leqslant x \leqslant 5$

5 $y = -(x + 1)^2 - 5; \quad -4 \leqslant x \leqslant 3$

6 $y = 2(x - 1)^2 - 3; \quad -3 \leqslant x \leqslant 5$

7 $y = 3(x - 3)^2 - 20; \quad 0 \leqslant x \leqslant 6$

8 $y = -2(x - 5)^2 + 30; \quad 2 \leqslant x \leqslant 8$

9 **a** Describe any pattern you spot between the coordinates of the turning point and the equation of the quadratics in each of the graphs plotted in Questions 1–8 above.

b If the general equation of a quadratic in completed square form is given as $y = a(x - b)^2 + c$, write down the coordinates of the turning point.

10 Without plotting the graph, deduce the coordinates of the turning point in each of the following:

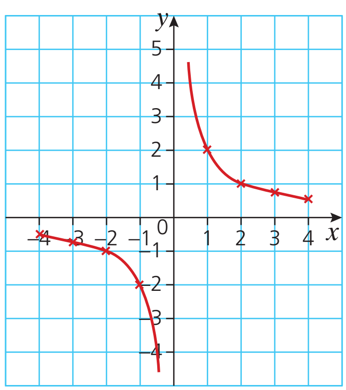
- **a** $y = (x - 6)^2 + 5$
- **b** $y = (x + 8)^2 - 6$
- **c** $y = -(x + 4)^2 + 12$
- **d** $y = -2(x - 12)^2 + 1$

The reciprocal function

Worked example

Draw the graph of $y = \frac{2}{x}$ for $-4 \leqslant x \leqslant 4$.

x	–4	–3	–2	–1	0	1	2	3	4
y	–0.5	–0.7	–1	–2	—	2	1	0.7	0.5



This is a **reciprocal function** giving a **hyperbola**.

Exercise 18.5

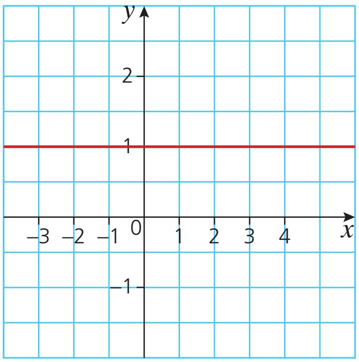
1 Plot the graph of the function $y = \frac{1}{x}$ for $-4 \leqslant x \leqslant 4$.

2 Plot the graph of the function $y = \frac{3}{x}$ for $-4 \leqslant x \leqslant 4$.

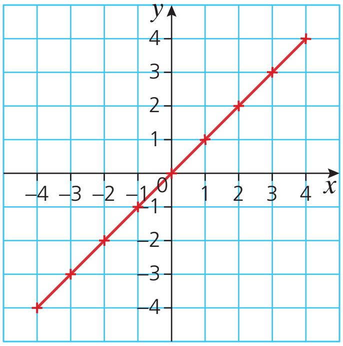
3 Plot the graph of the function $y = \frac{5}{2x}$ for $-4 \leqslant x \leqslant 4$.

Types of graph

Graphs of functions of the form ax^n take different forms depending on the values of a and n. The different types of line produced also have different names, as described below.



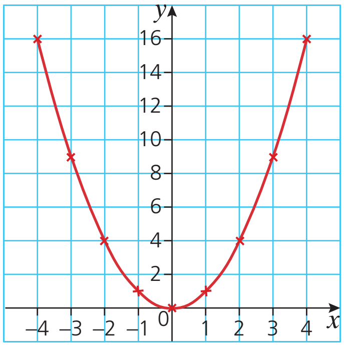
If $a = 1$ and $n = 0$, then $y = x^0$. This is a **linear** function giving a horizontal **straight line**.



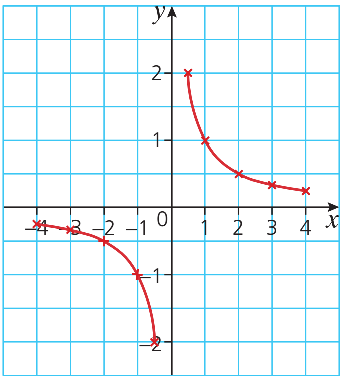
If $a = 1$ and $n = 1$, then $y = x^1$. This is a **linear** function giving a **straight line**.

Note

At Extended level, these four graphs can be referred to with function notation, $f(x) = x^0$, $f(x) = x^1$, $f(x) = x^2$, and $f(x) = \frac{1}{x}$

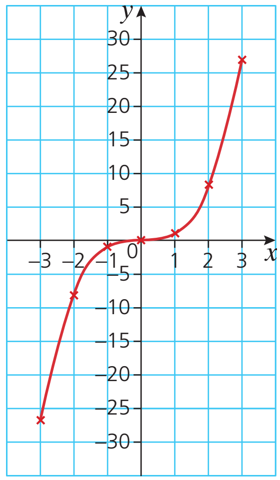


If $a = 1$ and $n = 2$, then $y = x^2$. This is a **quadratic** function giving a **parabola**.

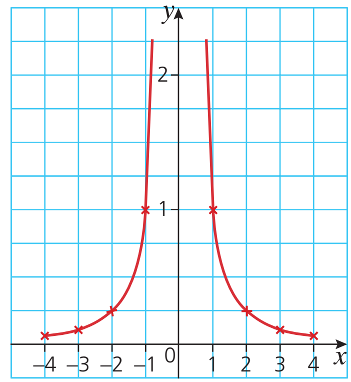


If $a = 1$ and $n = -1$, then $y = x^{-1}$ or $y = \frac{1}{x}$.

This is a **reciprocal** function giving a **hyperbola**.



If $a = 1$ and $n = 3$, then $f(x) = x^3$. This is a **cubic** function giving a **cubic curve**.

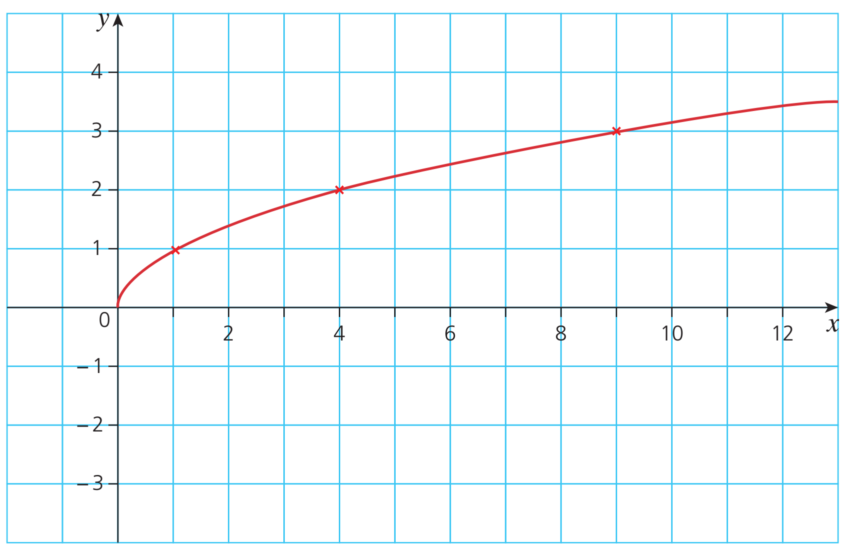


If $a = 1$ and $n = -2$, then $f(x) = x^{-2}$ or $f(x) = \frac{1}{x^2}$.

This is a **reciprocal** function, shown on the graph above.

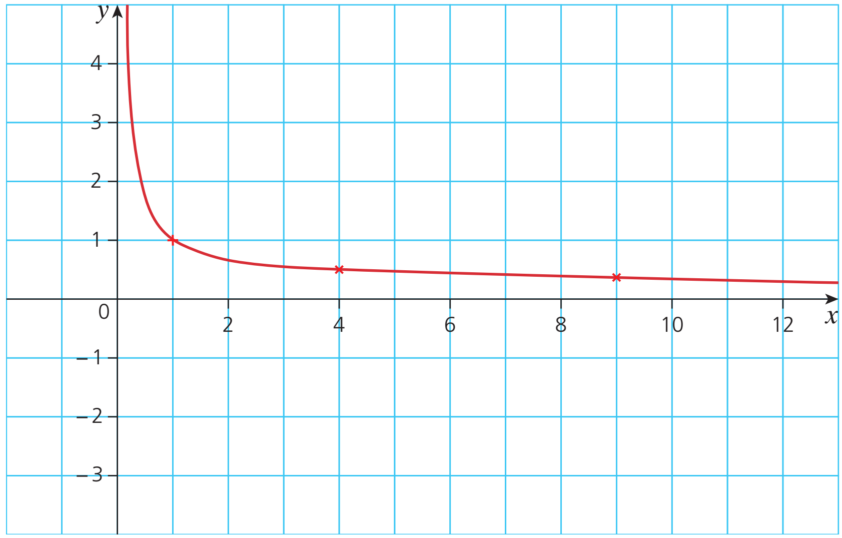
Note

In addition, there are three other types of graph at Extended level.



If $a = 1$ and $n = \frac{1}{2}$, then $f(x) = x^{\frac{1}{2}}$ or $f(x) = \sqrt{x}$.

This also produces a parabola.



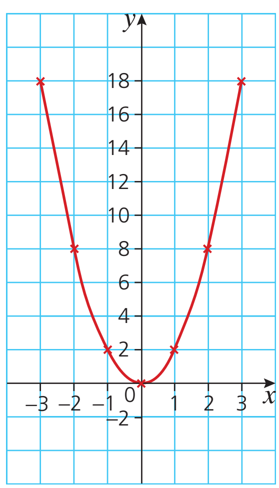
If $a = 1$ and $n = -\frac{1}{2}$, then $f(x) = x^{-\frac{1}{2}}$ or $f(x) = \frac{1}{\sqrt{x}}$.

This is also a reciprocal function.

Worked example

Draw a graph of the function $y = 2x^2$ for $-3 \leqslant x \leqslant 3$.

x	–3	–2	–1	0	1	2	3
y	18	8	2	0	2	8	18



Exercise 18.6

For each of the functions given below:

i draw up a table of values for x and $f(x)$,

ii plot the graph of the function.

1 $f(x) = \frac{1}{2}x + 4$, $-3 \leqslant x \leqslant 3$

2 $f(x) = -2x - 3$, $-4 \leqslant x \leqslant 2$

3 $f(x) = 2x^2 - 1$, $-3 \leqslant x \leqslant 3$

4 $f(x) = 0.5x^2 + x - 2$, $-5 \leqslant x \leqslant 3$

5 $f(x) = 3x^{-1}$, $-3 \leqslant x \leqslant 3$

6 $f(x) = \frac{1}{2}x^3 - 2x + 3$, $-3 \leqslant x \leqslant 3$

7 $f(x) = 2x^{-2}$, $-3 \leqslant x \leqslant 3$

8 $f(x) = \frac{1}{x^2} + 3x$, $-3 \leqslant x \leqslant 3$

9 $f(x) = \sqrt{x+2}$ $-2 \leqslant x \leqslant 14$ for $x = \{-2, 2, 7, 14\}$

10 $f(x) = \frac{1}{\sqrt{x}} - 4$ $-\frac{1}{16} \leqslant x \leqslant 9$ for $x = \left\{\frac{1}{16}, \frac{1}{4}, 1, 4, 9\right\}$

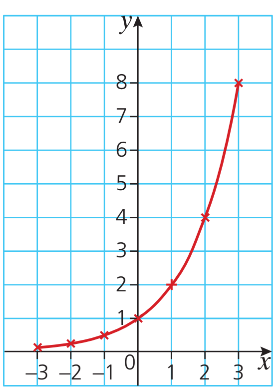
Exponential functions

Functions of the form $y = a^x$ are known as **exponential functions**. Plotting an exponential function is done in the same way as for other functions.

Worked example

Plot the graph of the function $y = 2^x$ for $-3 \leqslant x \leqslant 3$.

x	–3	–2	–1	0	1	2	3
y	0.125	0.25	0.5	1	2	4	8



Exercise 18.7

For each of the functions below:

i draw up a table of values of x and $f(x)$,

ii plot a graph of the function.

1 $f(x) = 3^x, \quad -3 \leqslant x \leqslant 3$

2 $f(x) = 1, \quad -3 \leqslant x \leqslant 3$

3 $f(x) = 2^x + 3, \quad -3 \leqslant x \leqslant 3$

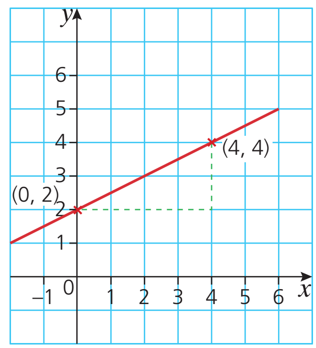
4 $f(x) = 3 \times 2^x + 2, \quad -3 \leqslant x \leqslant 3$

5 $f(x) = 2^x - x, \quad -3 \leqslant x \leqslant 3$

6 $f(x) = 3^x - x, \quad -3 \leqslant x \leqslant 3$

Gradients of curves

The gradient of a straight line is constant and is calculated by considering the coordinates of two of the points on the line and then carrying out the calculation $\frac{y_2 - y_1}{x_2 - x_1}$ as shown below:



$$\text{Gradient} = \frac{4-2}{4-0} = \frac{1}{2}$$

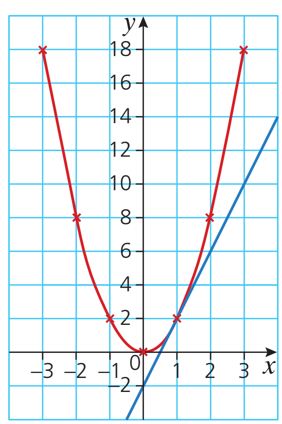
The gradient of a curve, however, is not constant: its slope changes. To calculate the gradient of a curve at a specific point, the following steps need to be taken:

- draw a tangent to the curve at that point,
- calculate the gradient of the tangent.

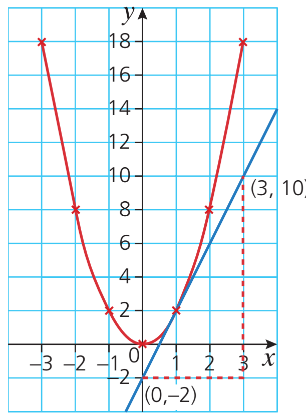
Worked example

For the function $y = 2x^2$, calculate the gradient of the curve at the point where $x = 1$.

On a graph of the function $y = 2x^2$, identify the point on the curve where $x = 1$ and then draw a tangent to that point. This gives:



Two points on the tangent are identified in order to calculate its gradient.



$$\text{Gradient} = \frac{10-(-2)}{3-0} = \frac{12}{3} = 4$$

Therefore the gradient of the function $y = 2x^2$ when $x = 1$ is 4.

Exercise 18.8

For each of the functions below:

i plot a graph,
ii calculate the gradient of the function at the specified point.

1 $y = x^2$, $-4 \leqslant x \leqslant 4$, gradient where $x = 1$

2 $y = \frac{1}{2}x^2$, $-4 \leqslant x \leqslant 4$, gradient where $x = -2$

3 $y = x^3$, $-3 \leqslant x \leqslant 3$, gradient where $x = 1$

4 $y = x^3 - 3x^2$, $-4 \leqslant x \leqslant 4$, gradient where $x = -2$

5 $y = 4x^{-1}$, $-4 \leqslant x \leqslant 4$, gradient where $x = -1$

6 $y = 2^x$, $-3 \leqslant x \leqslant 3$, gradient where $x = 0$

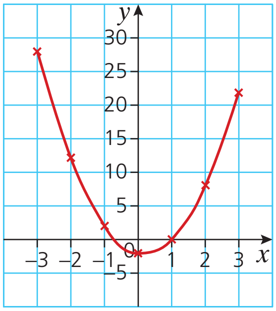
Solving equations by graphical methods

As shown earlier in this chapter, if a graph of a function is plotted, then it can be used to solve equations.

Worked examples

a i Plot a graph of $y = 3x^2 - x - 2$ for $-3 \leqslant x \leqslant 3$.

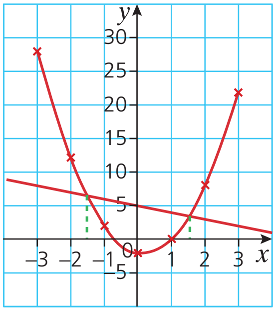
x	–3	–2	–1	0	1	2	3
y	28	12	2	–2	0	8	22



ii Use the graph to solve the equation $3x^2 - x - 2 = 0$.

To solve the equation, $y = 0$. Therefore where the curve intersects the x-axis gives the solution to the equation.

i.e. $3x^2 - x - 2 = 0$ when $x = -0.7$ and when $x = 1$.



iii Use the graph to solve the equation $3x^2 - 7 = 0$.
To be able to use the original graph, this equation needs to be manipulated in such a way that one side of the equation becomes:

$3x^2 - x - 2$.

Manipulating $3x^2 - 7 = 0$ gives:

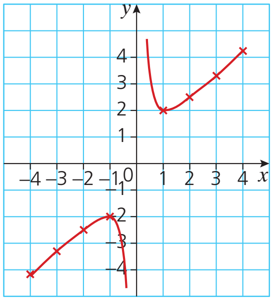
$3x^2 - x - 2 = -x + 5$ (subtracting x from both sides, and adding 5 to both sides).

Hence finding where the curve $y = 3x^2 - x - 2$ intersects the line $y = -x + 5$ gives the solution to the equation $3x^2 - 7 = 0$.

Therefore the solutions to $3x^2 - 7 = 0$ are $x \approx -1.5$ and $x \approx 1.5$.

b i Plot a graph of $y = \frac{1}{x} + x$ for $-4 \leqslant x \leqslant 4$.

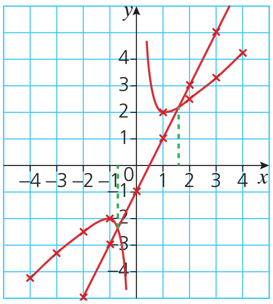
x	–4	–3	–2	–1	0	1	2	3	4
y	–4.25	–3.3	–2.5	–2	—	2	2.5	3.3	4.25



ii Use the graph to explain why $\frac{1}{x} + x = 0$ has no solution.

For $\frac{1}{x} + x = 0$, the graph will need to intersect the x-axis. From the plot opposite, it can be seen that the graph does not intersect the x-axis and hence the equation

$\frac{1}{x} + x = 0$ has no solution.



iii Use the graph to find the solution to $x^2 - x = 1$.
This equation needs to be manipulated in such a way that one side becomes $\frac{1}{x} + x$.

Manipulating $x^2 - x = 1$ gives:

$x - 1 = \frac{1}{x}$ (dividing both sides by x)

$2x - 1 = \frac{1}{x} + x$ (adding x to both sides)

Hence finding where the curve $y = \frac{1}{x} + x$ intersects the line $y = 2x - 1$ will give the solution to the equation $x^2 - x = 1$.

Therefore the solutions to the equation $x^2 - x = 1$ are $x \approx -0.6$ and $x \approx 1.6$
.

Exercise 18.9

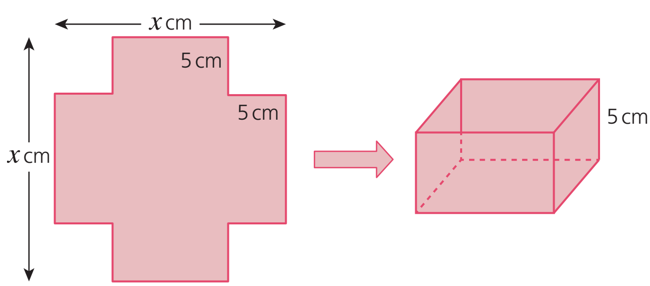
1 a Plot the function $y = \frac{1}{2}x^2 + 1$ for $-4 \leqslant x \leqslant 4$.
  b Showing your method clearly, use the graph to solve the equation $\frac{1}{2}x^2 = 4$.

2 a Plot the function $y = x^3 + x - 2$ for $-3 \leqslant x \leqslant 3$.
  b Showing your method clearly, use the graph to solve the equation $x^3 = 7 - x$.

3 a Plot the function $y = 2x^3 - x^2 + 3$ for $-2 \leqslant x \leqslant 2$.
  b Showing your method clearly, use the graph to solve the equation $2x^3 - 7 = 0$.

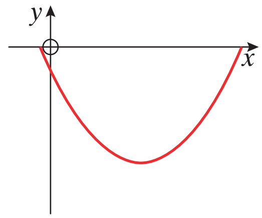
4 a Plot the function $y = \frac{2}{x^2} - x$ for $-4 \leqslant x \leqslant 4$.
  b Showing your method clearly, use the graph to solve the equation $4x^3 - 10x^2 + 2 = 0$.

5 An open box with a volume of $80\,\text{cm}^3$ is made by cutting 5 cm squares from each corner of a square piece of metal and then folding up the sides as shown.



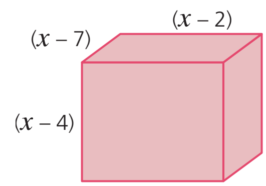
  a Write an expression for both the length and width of the box in terms of x.
  b Write an equation for the volume of the box in terms of x.
  c Calculate the possible dimensions of the original square metal sheet.

6 The cross-section of a bowl is shown on the axes below. y represents the bowl's depth and x its horizontal position.



The equation of the surface of the bowl is given as $y = \frac{1}{2}x^2 - 4x - 2$. Calculate the depth of the bowl.

7 The cuboid below has dimensions as shown.



a Write an equation for the volume V of the cuboid in terms of x.
b Sketch a graph of the volume V of the cuboid for values of x in the range $2 \leqslant x \leqslant 7$.
c Explain, using your graph, why the value of x must lie between 2 cm and 4 cm.
d Using trial and improvement, deduce the value of x (to 1 d.p.) which produces the cuboid with the largest volume.

8 **a** Plot the function $y = 2^x - x$ for $-2 \leqslant x \leqslant 5$.
b Showing your method clearly, use the graph to solve the equation $2^x = 2x + 2$.

9 A tap is dripping at a constant rate into a container. The level (l cm) of the water in the container is given by the equation $l = 2^t - 1$, where t is the time taken in hours.
a Calculate the level of the water after 3 hours.
b Calculate the level of the water in the container at the start.
c Calculate the time taken for the level of the water to reach 31 cm.
d Plot a graph showing the level of the water over the first 6 hours.
e From your graph, estimate the time taken for the water to reach a level of 45 cm.

10 Draw a graph of $y = 4^x$ for values of x between −1 and 3. Use your graph to find approximate solutions to the following equations:
a $4^x = 30$ **b** $4^x = \frac{1}{2}$

11 Draw a graph of $y = 2^x$ for values of x between −2 and 5. Use your graph to find approximate solutions to the following equations:
a $2^x = 20$ **b** $2(x + 2) = 40$

12 During an experiment, it is found that harmful bacteria grow at an exponential rate with respect to time. The approximate population of the bacteria, P, is modelled by the equation $P = 4^t + 100$, where t is the time in hours.
a Calculate the approximate number of harmful bacteria at the start of the experiment.
b Calculate the number of harmful bacteria after 5 hours. Give your answer to 3 significant figures.

Exercise 18.9 (cont)

c Draw a graph of $P = 4^t + 100$, for values of t from 0 to 6.
d Estimate from your graph the time taken for the bacteria population to reach 600.

13 The population of a type of insect is falling at an exponential rate. The population P is known to be modelled by the equation $P = 1000 \times \left(\frac{1}{2}\right)^t$, where t is the time in weeks.

a Copy and complete the following table of results, giving each value of P to the nearest whole number.

t	0	1	2	3	4	5	6	7	8	9	10
P			250								

b Plot a graph for the table of results above.
c Estimate from your graph the population of insects after $3\frac{1}{2}$ weeks.

Recognising and sketching functions

So far in this chapter, all graphs of functions have been plotted. In other words, values of x have been substituted into the equation, the corresponding y values have been calculated and the resulting (x, y) coordinates have been plotted.

However, plotting an accurate graph is time consuming and is not always necessary to answer a question. In many cases, a sketch of a graph is as useful as a plot and is considerably quicker.

When doing a sketch, certain key pieces of information need to be included. As a minimum, the points where the graph intersects both the x-axis and y-axis need to be given.

Sketching linear functions

Straight-line graphs can be sketched by working out where the line intersects both axes.

Worked example

Sketch the graph of $y = -3x + 5$

The graph intersects the y-axis when $x = 0$. This is therefore substituted into the equation

$$y = -3(0) + 5$$

so $y = 5$

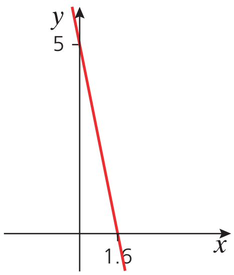
The graph intersects the x-axis when $y = 0$. This is then substituted into the equation and solved.

$$0 = -3x + 5$$

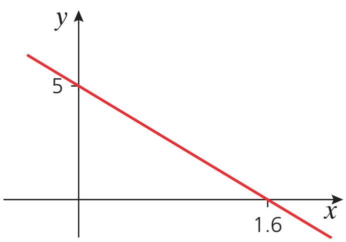
$$3x = 5$$

$$x = \frac{5}{3} \text{ (or 1.6)}$$

The sketch is therefore:



Note that the sketch below, although it looks very different to the one above, is also acceptable as it shows the same intersections with the axes.



Sketching quadratic functions

With a quadratic function, the sketch should be a smooth parabola shape. Once again, the important points to include are where it intersects the y-axis and, if applicable, the x-axis. If it does intersect the x-axis, giving the coordinates of the turning point is often not necessary unless asked for. However, if the graph does not intersect the x-axis, the coordinates of the turning point should be included.

Worked examples

a Sketch the graph of $y = x^2 - 8x + 15$.

The graph intersects the y-axis when $x = 0$.

Substituting $x = 0$ gives $y = 15$.

The graph intersects the x-axis when $y = 0$.

Substituting $y = 0$ gives $x^2 - 8x + 15 = 0$, which needs to be solved.

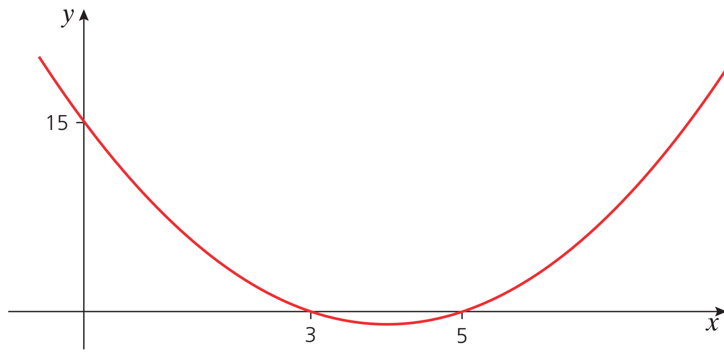
As the quadratic factorises, this is the quickest method to use.

$x^2 - 8x + 15 = 0$

$(x - 5)(x - 3) = 0$

Therefore $x = 5$ or $x = 3$.

As the x^2 term is positive, the graph will be U-shaped. A possible sketch is shown below.



Knowledge of turning points is not specifically required for the Core syllabus.

If the coordinate of the turning point is needed, this can be calculated at this stage. A parabola is symmetrical, so the minimum point must occur midway between the intersections with the x-axis, i.e. when $x = 4$.

Substituting $x = 4$ into the equation of the quadratic gives

$y = (4)^2 - 8(4) + 15 = -1$.

Therefore the coordinate of the minimum point is $(4, -1)$.

b Sketch the graph of the quadratic $y = -x^2 - 4x - 9$.

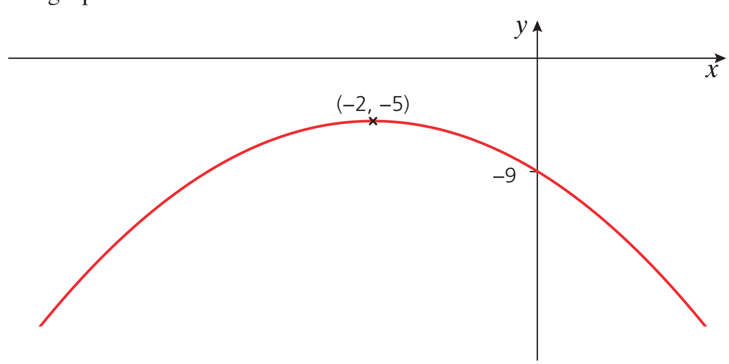
The graph intersects the y-axis when $x = 0$.

Substituting $x = 0$ gives $y = -9$.

The graph intersects the x-axis when $y = 0$. Substituting $y = 0$ produces the equation $-x^2 - 4x - 9 = 0$, which needs to be solved. The equation does not factorise. If the quadratic formula was used no solutions would be found either, implying that the graph does not intersect the x-axis. If this is the case, the coordinates of the turning point must be found and the completed square form of the equation is the most useful form to use.

In completed square form, the equation $y = -x^2 - 4x - 9$ is written as $y = -(x + 2)^2 - 5$. The coordinates of the turning point can be deduced from this as $(-2, -5)$

As the x^2 term is negative, the parabola is an inverted U-shape. A sketch of the graph is therefore:

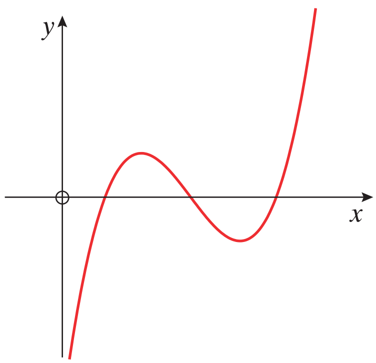


Sketching cubic functions

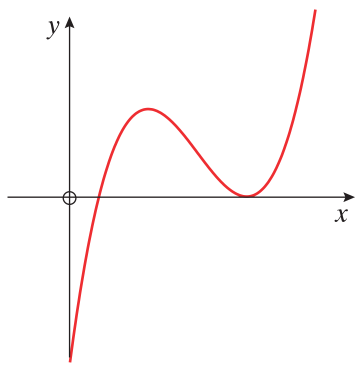
Generally, a cubic function takes the form $y = ax^3 + bx^2 + cx + d$ (where $a \neq 0$).

The usual shape of a cubic equation is  when the x^3 term is positive (i.e. $a > 0$), or  when the x^3 term is negative (i.e. $a < 0$).

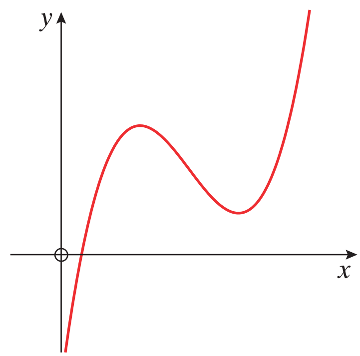
As a result of this, a cubic equation can intersect the x-axis up to three times.



Three points of intersection



Two points of intersection



One point of intersection

To sketch a cubic function, the intersections with both the y-axis and x-axis must be given.

Worked examples

a Sketch the function $y = (x - 2)(x - 3)(x - 5)$.

Where the graph intersects the y-axis, $x = 0$. Substituting this into the equation gives:

$$y = (0 - 2)(0 - 3)(0 - 5)$$
$$= (-2)(-3)(-5) = -30.$$

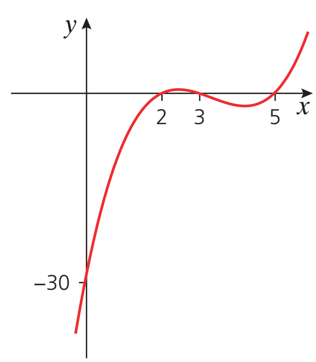
Where the graph intersects the x-axis, $y = 0$. Substituting this into the equation gives:

$$(x - 2)(x - 3)(x - 5) = 0$$

Therefore $x = 2$, 3 or 5.

As the x^3 would be positive if the brackets were expanded, the shape of the graph must be 

Using this information, the cubic function can be sketched as shown:



b Sketch the graph of $y = (-x+1)(x^2 - 6x + 9)$.

Where the graph intersects the y-axis, $x = 0$. Substituting this into the equation gives:

$y = (0+1)(0-0+9) = 9$.

To find where the graph intersects the x-axis substitute $y = 0$ into the equation.

$(-x+1)(x^2 - 6x + 9) = 0$

The quadratic expression in the second bracket is more useful if written in factorised form.

i.e. $(x^2 - 6x + 9) = (x-3)(x-3)$ or $(x-3)^2$.

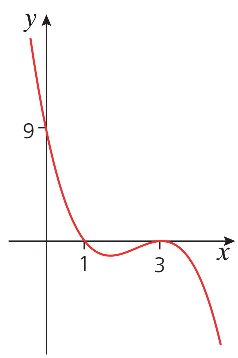
The equation to be solved can now be written as $(-x+1)(x-3)^2 = 0$.

Therefore, $x = 1, 3$ or 3. It is more usual to just give the distinct values i.e. $x = 1$ or $x = 3$.

As the x^3 term is negative, the shape of the cubic must be 

The repeated root of $x = 3$ implies that the graph just touches the x-axis at this point.

The sketch of the function is therefore:



Exercise 18.10

1 Sketch the following linear functions showing clearly where the lines intersect both axes.

a $y = 2x - 4$ **b** $y = \frac{1}{2}x + 6$ **c** $y = -2x - 3$

d $y = -\frac{1}{3}x + 9$ **e** $2y + x - 2 = 0$ **f** $x = \frac{2y+4}{3}$

2 Sketch the following quadratic functions, showing clearly where they intersect the y-axis and where/if they intersect the x-axis. Indicate also the coordinates of the turning point.

a $y = (x-4)(x-6)$ **b** $y = (x+2)(-x+3)$ **c** $y = (x-4)^2 + 1$

d $y = -x^2 + 3x$ **e** $y = -x^2 + 4x - 4$ **f** $y = -x^2 - 12x - 37$

g $y = x^2 + 6x - 5$ **h** $y = -3x^2 + 6x - 5$

3 Sketch the following cubic functions, showing clearly where they intersect the axes.

a $y = (x + 1)(x - 2)(x - 4)$ **b** $y = (x - 3)^2(x + 2)$

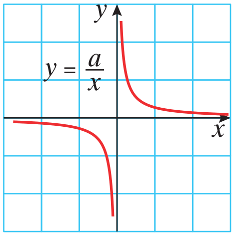
c $y = (x)(x^2 - 10x + 25)$ **d** $y = -x(-x + 4)(x - 6)$

e $y = (2x - 1)(-2x^2 - 5x - 3)$

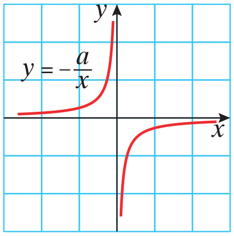
Sketching reciprocal functions

The reciprocal of x is $\frac{1}{x}$, similarly the reciprocal of x^2 is $\frac{1}{x^2}$. The reciprocal of an expression is 1 divided by that expression. In general therefore, reciprocal functions deal with functions of the form $y = \frac{1}{x^n}$. In this section we will look at how to sketch reciprocal functions.

Reciprocal functions of the form $y = \frac{1}{x}$ take one of two shapes.



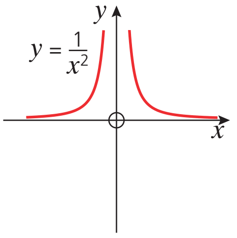
The shape of a positive function



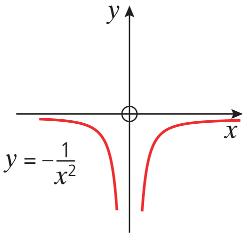
The shape of a negative function

The graph has special properties which need to be highlighted when the function is sketched. As can be seen above, the graphs do not intersect either axis. As x increases, the graph gets closer and closer to the x-axis (because $\frac{1}{x}$ gets smaller as x increases). The x-axis is known as an **asymptote**. As x gets closer to 0, then $\frac{1}{x}$ gets bigger and as a result the graph gets closer to the y-axis. The y-axis is therefore also an asymptote of the graph.

The graph of $y = \frac{1}{x^2}$ has similar properties and takes one of two shapes.



The shape of a positive function

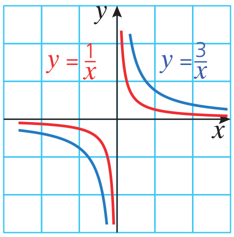


The shape of a negative function

Here too, both the x- and y-axes are asymptotes.

Worked examples

a Sketch on the same axes, the graphs of $y = \frac{1}{x}$ and $y = \frac{3}{x}$, labelling each graph clearly.

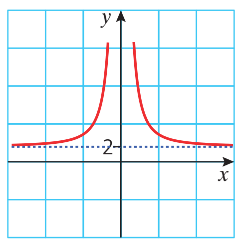


As the value of the numerator increases, the graph of $y = \frac{1}{x}$ is stretched in a direction parallel to the y-axis; this gives the appearance of the graph moving away from both axes.

b Sketch the graph of $y = \frac{1}{x^2} + 2$, stating clearly the equations of any asymptotes.

Compared with the graph of $y = \frac{1}{x^2}$, for a given x-value, the y-values for $y = \frac{1}{x^2} + 2$ have +2 added to them. This results in the whole graph of $y = \frac{1}{x^2}$ moving up two units in the y-direction (a translation of +2 in the y-direction).

The sketch is:



The asymptotes are therefore $y = 2$ and the y-axis.

Note that it is usual to indicate asymptotes other than the axes as a dotted line.

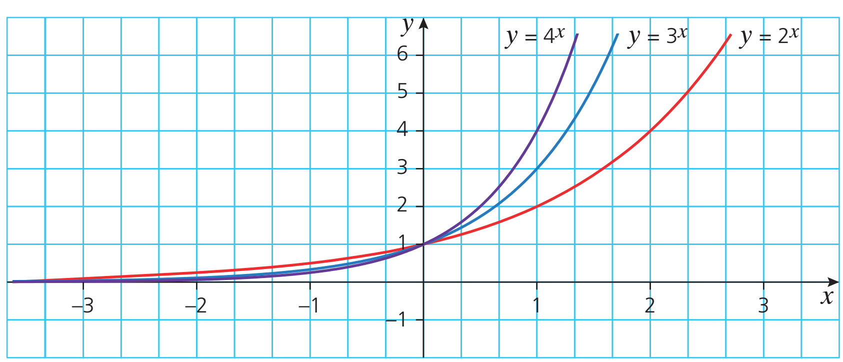
Exponential functions

Exponential functions take the general form $y = a^x$; examples therefore include $y = 2^x$ and $y = 3^x$.

These graphs also have a characteristic shape.

Worked example

a On the same axes, plot the graphs of $y = 2^x$, $y = 3^x$ and $y = 4^x$.



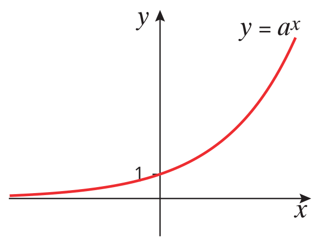
b Comment on any similarities between the three graphs.

The graphs all pass through the coordinate (0, 1) and the x-axis is an asymptote in all three cases.

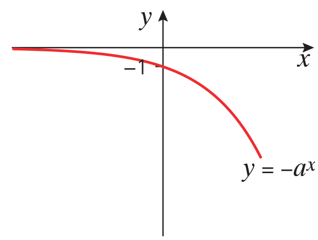
All the graphs of the form $y = a^x$ pass through the point (0, 1). This is because when $x = 0$ (the intercept with the y-axis), the equation becomes $y = a^0$. From your knowledge of indices you will know that any number raised to the power of zero is one.

Similarly, the reason the x-axis is an asymptote can also be explained using indices. When x is negative, a^{-x} can be written as $\frac{1}{a^x}$. Therefore, as a^x increases in value $\frac{1}{a^x}$ gets closer and closer to zero, hence closer and closer to the x-axis.

The graphs of $y = a^x$ and $y = -a^x$ therefore take the general shapes as shown below:

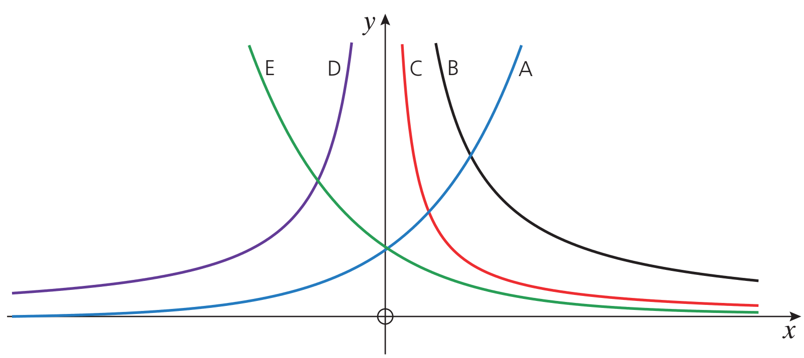


and



Exercise 18.11

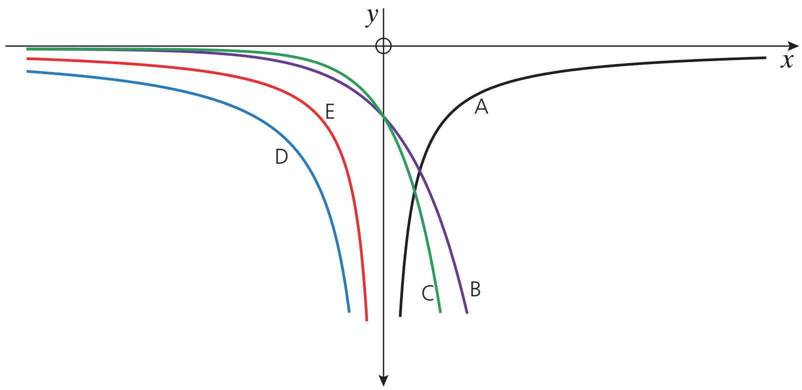
1 Match each of the graphs below with a possible equation.



$y = \frac{3}{x}$ $\quad$ $y = 2^{-x}$ $\quad$ $y = \frac{1}{x}$ $\quad$ $y = -\frac{2}{x}$ $\quad$ $y = 2^x$

Exercise 18.11 (cont)

2 Match each of the graphs below with a possible equation.



$y = \frac{1}{x}$ $y = -3^x$ $y = \frac{2}{x}$ $y = -\frac{1}{x}$ $y = -5^x$

Student assessment 1

1 Sketch the graph of the function $y = \frac{1}{x}$.

2 a Copy and complete the table below for the function $y = -x^2 - 7x - 12$.

x	–7	–6	–5	–4	–3	–2	–1	0	1	2
y		–6				–2				

b Plot a graph of the function.

3 Plot a graph of each of the functions below between the given limits of x.

a $y = x^2 - 3x - 10$, $-3 \leqslant x \leqslant 6$

b $y = -x^2 - 4x - 4$, $-5 \leqslant x \leqslant 1$

4 a Plot the graph of the quadratic equation $y = -x^2 - x + 15$ for $-6 \leqslant x \leqslant 4$.

b Showing your method clearly, use your graph to solve the following equations:

i $10 = x^2 + x$

ii $x^2 = x + 5$

5 a Plot the graph of $y = \frac{2}{x}$ for $-4 \leqslant x \leqslant 4$.

b Showing your method clearly, use your graph to solve the equation $x^2 + x = 2$.

6 In each of the following equations:

i State whether the turning point is maximum or a minimum, giving a reason for your answer.

ii State the coordinate of the turning point.

a $y = -(x + 3)^2 - 5$

b $y = \frac{1}{2}(x - 6)^2 + 2$

7 Sketch the function $y = \frac{1}{2}x^2 - 2$, showing clearly where it intersects both axes and indicating the coordinates of the turning point.

8 **a** Sketch the function $y = (3x + 2)(-x^2 - x + 2)$, indicating clearly the intersections with both axes.

b P is the point on the curve with coordinates (–1, –2).
A tangent to the curve is drawn at P and passes through (0, 3).

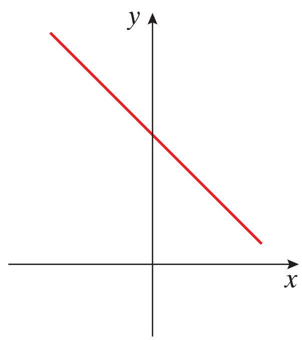
i Draw the tangent on your sketch above.

ii Calculate the coordinate of the point where the tangent meets the x-axis.

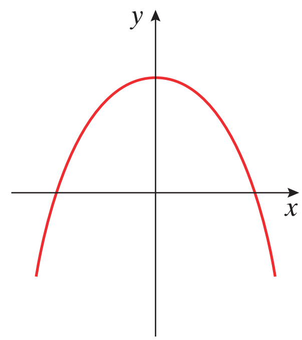
Student assessment 2

1 **a** Name the types of graph shown below:

i



ii



b Give a **possible** equation for each of the graphs drawn.

2 For each of the functions below:

i draw up a table of values,

ii plot a graph of the function.

a $f(x) = x^2 + 3x,\quad -5 \leqslant x \leqslant 2$

b $f(x) = \frac{1}{x} + 3x,\quad -3 \leqslant x \leqslant 3$

3 **a** Plot the function $y = \frac{1}{2}x^3 + 2x^2$ for $-5 \leqslant x \leqslant 2$.

b Calculate the gradient of the curve when:

i $x = 1$ **ii** $x = -1$

4 **a** Plot a graph of the function $y = 2x^2 - 5x - 5$ for $-2 \leqslant x \leqslant 5$.

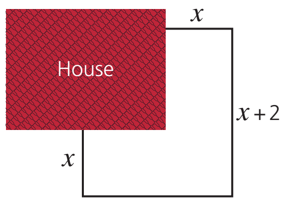
b Use the graph to solve the equation $2x^2 - 5x - 5 = 0$.

c Showing your method clearly, use the graph to solve the equation $2x^2 - 3x = 10$.

5 Sketch the function $\frac{y + 2}{3} = x$.

6 Sketch the function $y = -2x^2 + x + 15$, stating clearly where it intersects the axes and indicating the coordinates of the turning point.

7 A builder is constructing a fenced yard off the side of a house as shown below. The total length of the fence is 60 m.

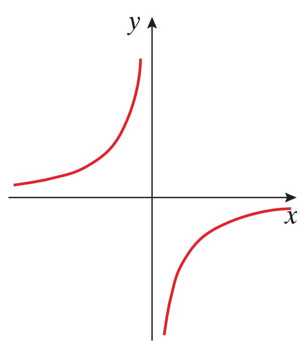


a Write an expression for the length of the unmarked side in terms of x.
b Write an equation for the area A of the yard.
c Sketch the graph of the function for the area A of the yard.
d From your graph deduce the value of x which gives the largest area for the yard.
e Calculate the largest area possible for the yard.

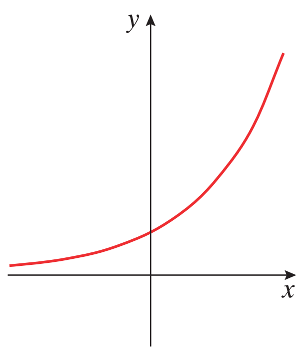
Student assessment 3

1 **a** Name the types of graph shown below:

i



ii



b Give a **possible** equation for each of the graphs drawn.

2 For each of the functions below:
   i draw up a table of values,
   ii plot a graph of the function.
a $f(x) = 2^x + x, \quad -3 \leqslant x \leqslant 3$
b $f(x) = 3^x - x^2, \quad -3 \leqslant x \leqslant 3$

3 **a** Plot the function $y = -x^3 - 4x^2 + 5$ for $-5 \leqslant x \leqslant 2$.
b Calculate the gradient of the curve when:
   i $x = 0$
   ii $x = -2$

4 **a** Copy and complete the table below for the function $y = \frac{1}{x^2} - 5$.

x	−3	−2	−1	−0.5	−0.25	0	0.25	0.5	1	2	3
y				−1		—					

b Plot a graph of the function.

c Use the graph to solve the equation $\frac{1}{x^2} = 5$.

d Showing your method clearly, use your graph to solve the equation $\frac{1}{x^2} + x^2 = 7$.

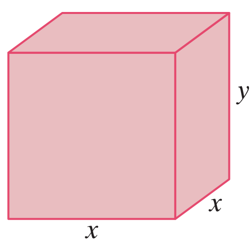
5 **a** Sketch the graph of $y = -2^x$ indicating clearly any intersections with either axis.

b Give the equation of any asymptote(s).

6 **a** Sketch the graph of $y = 2^x - 4$ indicating clearly any intersections with either axis.

b Give the equation of any asymptote(s).

7 A cuboid with side lengths x cm, x cm and y cm is shown below. Its total surface area is 392 cm^2



a Show that the length y can be written as $y = \frac{196 - x^2}{2x}$

b Write an equation for the volume V of the cuboid in terms of x.

c Sketch a graph for the volume V of the cuboid for values of x in the range
$0 \leqslant x \leqslant 14$

d Using your graph as a reference, calculate the integer value of x which will produce the greatest volume.

e Using your answer to part **d**, calculate the maximum volume of the cuboid.

19 Differentiation and the gradient function

Calculus is the cornerstone of much of the mathematics studied at a higher level. Differential calculus deals with finding the gradient of a function. In this chapter, you will look at functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$, where n is an integer.

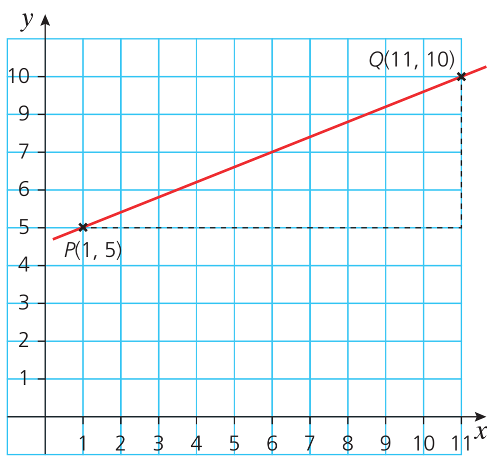
The gradient of a straight line

You will already be familiar with calculating the gradient of a straight line.

The gradient of the line passing through points (x_1, y_1) and (x_2, y_2) is calculated by $\frac{y_2 - y_1}{x_2 - x_1}$.

Therefore, the gradient of the line passing through points P and Q is:

$$\frac{10-5}{11-1} = \frac{5}{10}$$
$$= \frac{1}{2}$$

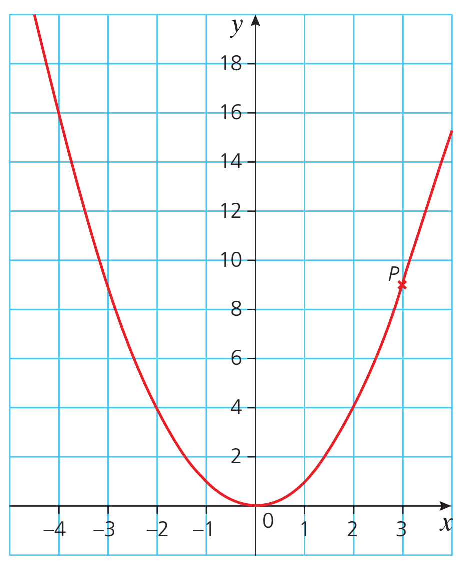


The gradient of a curve

The gradient of a straight line is constant, i.e. it is the same at any point on the line. However, not all functions are linear (straight lines). A function that produces a curved graph is more difficult to work with because the gradient of a curve is not constant.

The graph (right) shows the function $f(x) = x^2$.

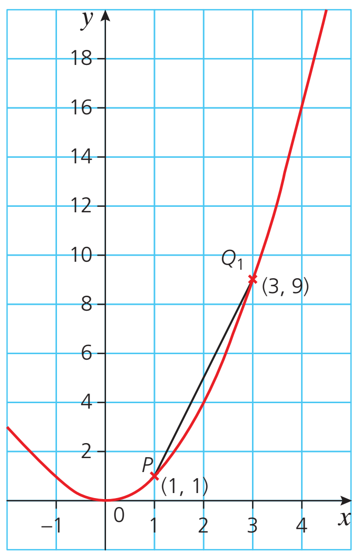
Point P is on the curve at (3, 9). If P moves along the curve to the right, the gradient of the curve becomes steeper.



If P moves along the curve towards the origin, the gradient of the curve becomes less steep.

The gradient of the function at the point $P(1, 1)$ can be calculated as follows:

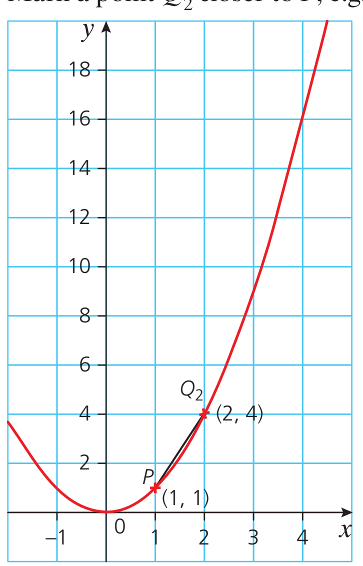
» Mark a point $Q_1(3, 9)$ on the graph and draw the line **segment** PQ_1.



The gradient of the line segment PQ_1 is an approximation of the gradient of the curve at P.

Gradient of PQ_1 is $\frac{9-1}{3-1} = 4$

» Mark a point Q_2 closer to P, e.g. (2, 4), and draw the line segment PQ_2.



The gradient of the line segment PQ_2 is still only an approximation of the gradient of the curve at P, but it is a better approximation than the gradient of PQ_1.

Gradient of PQ_2 is $\frac{4-1}{2-1} = 3$

» If a point $Q_3(1.5, 1.5^2)$ is chosen, the gradient PQ_3 will be an even better approximation.

Gradient of PQ_3 is $\frac{1.5^2-1}{1.5-1} = 2.5$

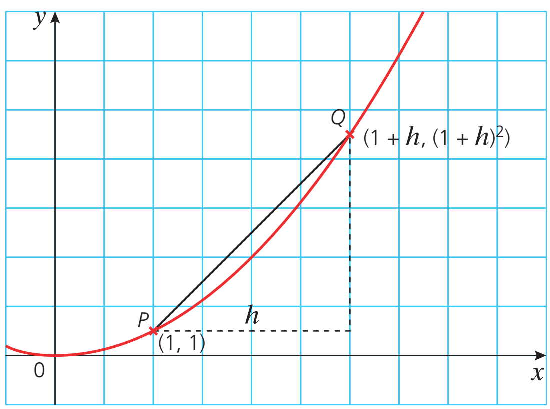
For the point $Q_4(1.25, 1.25^2)$, the gradient of PQ_4 is $\frac{1.25^2-1}{1.25-1} = 2.25$

For the point $Q_5(1.1, 1.1^2)$, the gradient of PQ_5 is $\frac{1.1^2-1}{1.1-1} = 2.1$

These results indicate that as point Q gets closer to P, the gradient of the line segment PQ gets closer to 2.

Worked example

Prove that the gradient of the function $f(x) = x^2$ is 2 when $x = 1$.
Consider points P and Q on the function $f(x) = x^2$.
P is at (1, 1) and Q, h units from P in the x-direction, has coordinates $(1+h, (1+h)^2)$.



Gradient of line segment PQ is $\frac{(1+h)^2-1}{1+h-1} = \frac{1+2h+h^2-1}{h}$

$$= \frac{h(2+h)}{h}$$

$$= 2+h$$

As Q gets closer to P, h gets smaller and smaller (tends to 0), and the value of $2+h$ becomes an even more accurate approximation of the gradient of the curve at point P.
As h tends to 0, the gradient $(2+h)$ of the line segment PQ tends to 2.
This can be written as:

The gradient at $P(1, 1) = \lim_{h\to 0}(2+h) = 2$

In other words, the limit of $2+h$ as h tends to 0 is 2.

Exercise 19.1

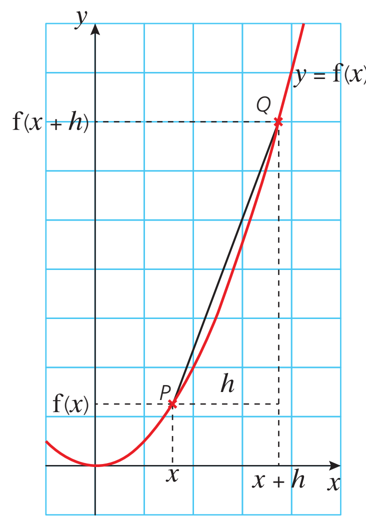
> **Note**
>
> You will already be familiar with the fact that the gradient of a curve at the point P is the same as the gradient of the tangent to the curve at P.

1. **a** Using the proof above as a guide, find the gradient of the function $f(x) = x^2$ when:
   i $x = 2$ **ii** $x = 3$ **iii** $x = -1$.
   b Make a table of the values of x and the corresponding values of the gradient of the function $f(x)$.
   c Looking at the pattern in your results, complete the sentence below.
   For the function $f(x) = x^2$, the gradient is …
2. **a** Find the gradient of the function $f(x) = 2x^2$ when:
   i $x = 1$ **ii** $x = 2$ **iii** $x = -2$.
   b Looking at the pattern in your results, complete the sentence below.
   For the function $f(x) = 2x^2$, the gradient is …
3. **a** Find the gradient of the function $f(x) = \frac{1}{2}x^2$ when:
   i $x = 1$ **ii** $x = 2$ **iii** $x = 3$.
   b Looking at the pattern in your results, complete the sentence below.
   For the function $f(x) = \frac{1}{2}x^2$, the gradient is …

The gradient function

The notation $f'(x)$ is shown here as it is used a lot in mathematics. However, knowledge of it is not part of the Extended syllabus.

You may have noticed a pattern in your answers to the previous exercise. In fact, there is a rule for calculating the gradient at any point on the particular curve. This rule is known as the **gradient function**, $f'(x)$ or $\frac{dy}{dx}$.



The function $f(x) = x^2$ has a gradient function $f'(x) = 2x$

or $\frac{dy}{dx} = 2x$.

The above proof can be generalised for other functions $f(x)$.

Gradient of line segment PQ

$$= \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$\text{Gradient at } P = \lim_{h \to 0}(\text{Gradient of line segment } PQ)$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is known as finding the gradient function from **first principles**.

> **Note**
>
> Limit notation and proof of first principles are beyond the requirements of the syllabus. However, they are included here for interest.

Worked example

Find, from first principles, the gradient function of $f(x) = x^2 + x$.

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0}\frac{((x+h)^2 + (x+h)) - (x^2+x)}{h}$$
$$= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}$$
$$= \lim_{h\to 0}\frac{2xh + h^2 + h}{h}$$
$$= \lim_{h\to 0}(2x + h + 1)$$
$$= 2x + 1$$

So the gradient at any point $P(x, y)$ on the curve $y = x^2 + x$ is given by $2x + 1$.

Exercise 19.2

1 Find, from first principles, the gradient function of each function. Use the worked example above as a guide.

a $f(x) = x^3$
b $f(x) = 3x^2$
c $f(x) = x^2 + 2x$
d $f(x) = x^2 - 2$
e $f(x) = 3x - 3$
f $f(x) = 2x^2 - x + 1$

2 Copy and complete the table below using your gradient functions from the previous question and Exercise 19.1 Q1–3.

Function $f(x)$	Gradient function $f'(x)$
x^2	
$2x^2$	
$\frac{1}{2}x^2$	
$x^2 + x$	$2x + 1$
x^3	
$3x^2$	
$x^2 + 2x$	
$x^2 - 2$	
$3x - 3$	
$2x^2 - x + 1$	

3 Look at your completed table for Q2. Describe any patterns you notice between a function and its gradient function.

The functions used so far have all been **polynomials**. There is a relationship between a polynomial function and its gradient function. This is best summarised as follows:

$$\textbf{If } f(x) = ax^n\textbf{, then } \frac{dy}{dx} = anx^{n-1}.$$

So, to work out the gradient function of a polynomial, multiply the coefficient of x by the power of x and subtract 1 from the power.

Note

Another way of writing $\frac{dy}{dx}$ is $f'(x)$

Worked examples

a Calculate the gradient function of $f(x) = 2x^3$.

$$\frac{dy}{dx} = 3 \times 2x^{(3-1)} = 6x^2$$

b Calculate the gradient function of $y = 5x^4$.

$$\frac{dy}{dx} = 4 \times 5x^{(4-1)} = 20x^3$$

Exercise 19.3

1 Calculate the gradient function of each of the following functions:

a $f(x) = x^4$ **b** $f(x) = x$ **c** $f(x) = 3x^2$

d $f(x) = 5x^3$ **e** $f(x) = 6x^3$ **f** $f(x) = 8x^7$

2 Calculate the gradient function of each of the following functions:

a $f(x) = \frac{1}{3}x^3$ **b** $f(x) = \frac{1}{4}x^4$ **c** $f(x) = \frac{1}{4}x^2$

d $f(x) = \frac{1}{2}x^4$ **e** $f(x) = \frac{2}{5}x^3$ **f** $f(x) = \frac{2}{9}x^3$

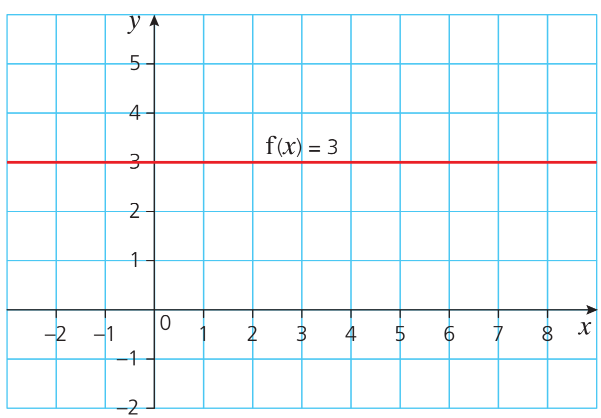
Differentiation

The process of finding the gradient function is known as **differentiation**. Differentiating a function produces the **derivative** or gradient function.

Worked examples

a Differentiate the function $f(x) = 3$ with respect to x.

The graph of $f(x) = 3$ is a horizontal line as shown:



A horizontal line has no gradient. Therefore

$$\Rightarrow \text{for } f(x) = 3, \frac{dy}{dx} = 0$$

This can also be calculated using the rule for differentiation.

$f(x) = 3$ can be written as $f(x) = 3x^0$.

So $\frac{dy}{dx} = 0 \times 3x^{(0-1)}$

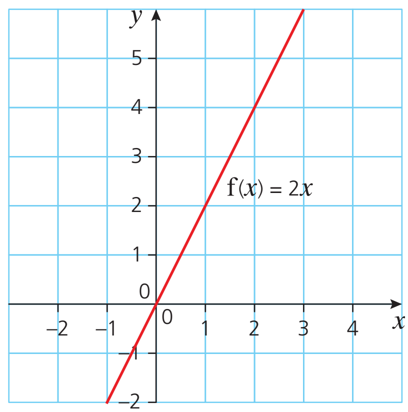
$= 0$

The derivative of a constant is zero.

If $\mathbf{f(x) = c \Rightarrow \frac{dy}{dx} = 0}$

b Differentiate the function $f(x) = 2x$ with respect to x.

The graph of $f(x) = 2x$ is a straight line as shown:



From earlier work on linear graphs, the gradient is known to be 2. Therefore

$\Rightarrow$ for $f(x) = 2x$, $\frac{dy}{dx} = 2$.

This too can be calculated using the rule for differentiation.

$f(x) = 2x$ can be written as $f(x) = 2x^1$.

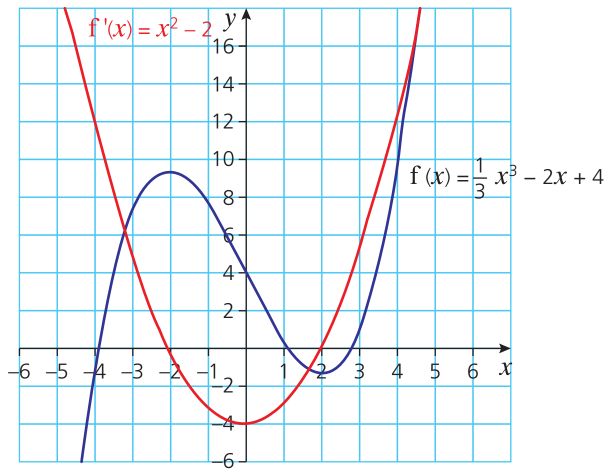
So $\frac{dy}{dx} = 1 \times 2x^{(1-1)}$

$= 2x^0$

But $x^0 = 1$, therefore $\frac{dy}{dx} = 2$.

If $\mathbf{f(x) = ax \Rightarrow \frac{dy}{dx} = a}$.

c Differentiate the function $f(x) = \frac{1}{3}x^3 - 2x + 4$ with respect to x.
The graphs of the function and its derivative are as follows:



It can be seen that the derivative of the function $f(x)$ is a quadratic. The equation of this quadratic is $y = x^2 - 2$. The derivative of $f(x)$ is therefore $f'(x) = x^2 - 2$.

In general, the derivative of a polynomial function with several terms can be found by differentiating each of the terms individually.

d Differentiate the function $f(x) = \frac{2x^3 + x^2}{x}$ with respect to x.

A common error here is to differentiate each of the terms individually.

The derivative of $\frac{2x^3 + x^2}{x}$ is NOT $\frac{6x^2 + 2x}{1}$.

$\frac{2x^3 + x^2}{x}$ can be written as $\frac{2x^3}{x} + \frac{x^2}{x}$ and simplified to $2x^2 + x$.

$$\text{Therefore } f(x) = \frac{2x^3 + x^2}{x}$$
$$= 2x^2 + x$$
$$\Rightarrow \frac{dy}{dx} = 4x + 1$$

In general, rewrite functions as sums of terms in powers of x before differentiating.

Exercise 19.4

1 Differentiate each expression with respect to x.

a $5x^3$ b $7x^2$ c $4x^6$

d $\frac{1}{4}x^2$ e $\frac{2}{3}x^6$ f $\frac{3}{4}x^5$

g 5 h $6x$ i $\frac{1}{8}$

Exercise 19.4 (cont)

2 Differentiate each expression with respect to x.

a $3x^2 + 4x$　b $5x^3 - 2x^2$　c $10x^3 - \frac{1}{2}x^2$

d $6x^3 - 3x^2 + x$　e $12x^4 - 2x^2 + 5$　f $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 4$

g $-3x^4 + 4x^2 - 1$　h $-6x^5 + 3x^4 - x + 1$　i $-\frac{3}{4}x^6 + \frac{2}{3}x^3 - 8$

3 Differentiate each expression with respect to x.

a $\frac{x^3 + x^2}{x}$　b $\frac{4x^3 - x^2}{x^2}$　c $\frac{6x^3 + 2x^2}{2x}$

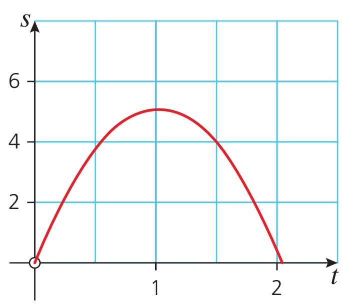
d $\frac{x^3 + 2x^2}{4x}$　e $3x(x + 1)$　f $2x^2(x - 2)$

g $(x + 5)^2$　h $(2x - 1)(x + 4)$　i $(x^2 + x)(x - 3)$

So far we have only used the variables x and y when finding the gradient function. This does not always need to be the case. Sometimes, as demonstrated below, it is more convenient or appropriate to use other variables.

If a stone is thrown vertically upwards from the ground with a speed of 10 m/s, its distance s from its point of release is given by the formula $s = 10t - 4.9t^2$, where t is the time in seconds after the stone's release.

A graph plotted to show distance against time is shown below.



The velocity (v) of the stone at any point can be found by calculating the rate of change of distance with respect to time, i.e. $\frac{ds}{dt}$.

Therefore if $s = 10t - 4.9t^2$

$$v = \frac{ds}{dt}$$
$$= 10 - 9.8t$$

Worked example

Calculate $\frac{ds}{dt}$ for the function $s = 6t^2 - 4t + 1$.

$$\frac{ds}{dt} = 12t - 4$$

Exercise 19.5

1 Differentiate each of the following with respect to t.

a $y = 3t^2 + t$ **b** $v = 2t^3 - t^2$ **c** $m = 5t^3 - t^2$

2 Calculate the derivative of each of the following functions.

a $y = x(x + 4)$ **b** $r = t(1 - t)$

c $v = t\left(\frac{1}{t} + t^2\right)$ **d** $p = r^2\left(\frac{2}{r} - 3\right)$

3 Differentiate each of the following with respect to t.

a $y = (t + 1)(t - 1)$ **b** $r = (t - 1)(2t + 2)$ **c** $v = \left(\frac{2t^2}{3} + 1\right)(t - 1)$

Calculating the second derivative

In the previous section we considered the position of a stone thrown vertically upwards. Its velocity (v) at any point was found by differentiating the equation for the distance (s) with respect to t, i.e. $v = \frac{ds}{dt}$.

In this section, we extend this to consider acceleration (a) which is the rate of change of velocity with time, i.e. $a = \frac{dv}{dt}$.

Therefore as $s = 10t - 4.9t^2$

$$v = \frac{ds}{dt}$$
$$= 10 - 9.8t$$

$$a = \frac{dv}{dt}$$
$$= -9.8$$

acceleration due to gravity

You will have noticed that the equation for the distance was differentiated twice to get the acceleration, i.e. the second derivative was obtained. Calculating the second derivative is a useful operation as will be seen later.

The notation used for the second derivative follows on from that used for the first derivative.

$$f(x) = ax^n \qquad \qquad y = ax^n$$

$$\Rightarrow f'(x) = anx^{n-1} \qquad \text{or} \qquad \frac{dy}{dx} = anx^{n-1}$$

$$\Rightarrow f''(x) = an(n-1)x^{n-2} \qquad \qquad \frac{d^2y}{dx^2} = an(n-1)x^{n-2}$$

Therefore either $f''(x)$ or $\frac{d^2y}{dx^2}$ are the most common forms of notation used for the second derivative, when differentiating with respect to x.

The notation f″(x) is shown here as it is used a lot in mathematics. However, knowledge of it is not part of the Extended syllabus.

Worked examples

a Find $\frac{d^2y}{dx^2}$ when $y = x^3 - 2x^2$.

$\frac{dy}{dx} = 3x^2 - 4x$

$\frac{d^2y}{dx^2} = 6x - 4$

b Work out $\frac{d^2s}{dt^2}$ for $s = 3t + \frac{1}{2}t^2$.

$\frac{ds}{dt} = 3 + t$

$\frac{d^2s}{dt^2} = 1$

Exercise 19.6

1 Find $\frac{d^2y}{dx^2}$ for each of the following.

a $y = 2x^3$ **b** $y = x^4 - \frac{1}{2}x^2$ **c** $y = \frac{1}{3}x^6$

d $y = 3x^2 - 2$ **e** $y = \frac{x^2}{4}$ **f** $y = 3x$

2 Find the second derivative of each function.

a $v = x^2(x - 3)$ **b** $P = \frac{1}{2}x^2(x^2 + x)$

c $t = x^{-1}(x + x^3)$ **d** $y = (x^2 + 1)(x^3 - x)$

Gradient of a curve at a point

You have seen that differentiating the equation of a curve gives the general equation for the gradient of any point on the curve. You can use this general equation to calculate the gradient at a specific point on the curve.

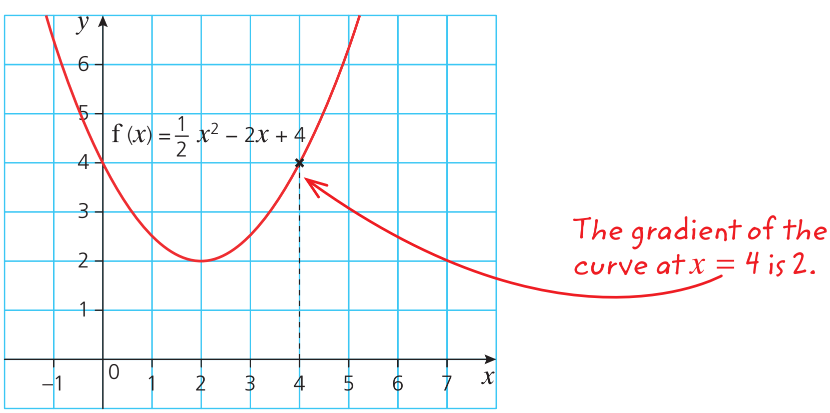
For the function $f(x) = \frac{1}{2}x^2 - 2x + 4$, the gradient function $f'(x) = x - 2$.

The gradient at any point on the curve can be calculated using this.

For example, when $x = 4$, $f'(4) = 4 - 2$

$= 2$

Therefore, the gradient of the curve $f(x) = \frac{1}{2}x^2 - 2x + 4$ is 2 when $x = 4$, as shown below.



Worked example

Calculate the gradient of the curve $f(x) = x^3 + x - 6$ when $x = -1$.

The gradient function $f'(x) = 3x^2 + 1$.

So when $x = -1$, $f'(-1) = 3(-1)^2 + 1$
$= 4$

Therefore the gradient is 4.

Exercise 19.7

1 Find the gradient of each function at the given value of x.

a $f(x) = x^2; x = 3$

b $f(x) = \frac{1}{2}x^2 - 2; x = -3$

c $f(x) = 3x^3 - 4x^2 - 2; x = 0$

d $f(x) = -x^2 + 2x - 1; x = 1$

e $f(x) = -\frac{1}{2}x^3 + x - 3; x = -1, x = 2$

f $f(x) = 6x; x = 5$

2 The number of newly infected people, N, on day t of a stomach bug outbreak is given by $N = 5t^2 - \frac{1}{2}t^3$.

a Calculate the number of new infections N when:
i $t = 1$ **ii** $t = 3$
iii $t = 6$ **iv** $t = 10$.

b Calculate the rate of new infections with respect to t, i.e. calculate $\frac{dN}{dt}$.

c Calculate the rate of new infections at the following times:
i $t = 1$ **ii** $t = 3$
iii $t = 6$ **iv** $t = 10$.

d Plot a graph of the equation $N = 5t^2 - \frac{1}{2}t^3$ for the values of t in the range $0 \leqslant t \leqslant 10$.

e Explain your answers to part **a**, using your graph to support your explanation.

f Explain your answers to part **c**, using your graph to support your explanation.

3 A weather balloon is released from the ground. Its height in metres, h, after time in hours t, is given by the formula:

$h = 30t^2 - t^3$ when $t \leqslant 20$.

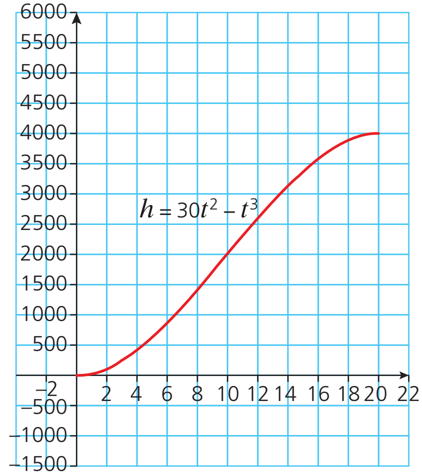
a Calculate the height of the balloon when:
i $t = 3$ **ii** $t = 10$.

b Calculate the rate at which the balloon is climbing with respect to time t.

c Calculate the rate of ascent when:
i $t = 2$ **ii** $t = 5$ **iii** $t = 20$.

Exercise 19.7 (cont)

d The graph of h against t is shown below.



Referring to your graph, explain your answers to part **c**:

e Estimate from your graph the time when the balloon was climbing at its fastest rate. Explain your answer.

Calculating the value of x when the gradient is given

So far you have calculated the gradient of a curve for a given value of x. It is also possible to work backwards and calculate the value of x when the gradient of a point is given.

Consider the function $f(x) = x^2 - 2x + 1$.

It is known that the gradient at a particular point on the curve is 4, but the x-coordinate of that point is not known.

The gradient function of the curve is $f'(x) = 2x - 2$.

Since the gradient at this particular point is 4, you can form an equation:

$$f'(x) = 4$$

So $2x - 2 = 4$

$$\Rightarrow 2x = 6$$

$$\Rightarrow \quad x = 3$$

Therefore $x = 3$ when the gradient of the curve is 4.

Worked example

The function $f(x) = x^3 - x^2 - 5$ has a gradient of 8 at a point P on the curve. Calculate the possible coordinates of point P.

The gradient function $f'(x) = 3x^2 - 2x$

At P, $3x^2 - 2x = 8$

This can be rearranged into the quadratic $3x^2 - 2x - 8 = 0$ and solved algebraically.

This requires the algebraic solution of the quadratic equation $3x^2 - 2x - 8 = 0$.

Factorising gives $(3x + 4)(x - 2) = 0$

Therefore $(3x + 4) = 0 \Rightarrow x = -\frac{4}{3}$ or $(x - 2) = 0 \Rightarrow x = 2$

The values of $f(x)$ can be calculated by substituting the x-values in to the equation as shown:

$$f\left(-\tfrac{4}{3}\right) = \left(-\tfrac{4}{3}\right)^3 - \left(-\tfrac{4}{3}\right)^2 - 5 = -9\tfrac{4}{27}$$

$f(2) = 2^3 - 2^2 - 5 = -1$

Therefore the possible coordinates of P are $\left(-1\tfrac{1}{3}, -9\tfrac{4}{27}\right)$ and $(2, -1)$

Exercise 19.8

1 Find the coordinate of the point P on each of the following curves, at the given gradient.
  - a $f(x) = x^2 - 3$, gradient $= 6$
  - b $f(x) = 3x^2 + 1$, gradient $= 15$
  - c $f(x) = 2x^2 - x + 4$, gradient $= 7$
  - d $f(x) = \frac{1}{2}x^2 - 3x - 1$, gradient $= -3$
  - e $f(x) = \frac{1}{3}x^2 + 4x$, gradient $= 6$
  - f $f(x) = -\frac{1}{5}x^2 + 2x + 1$, gradient $= 4$

2 Find the coordinate(s) of the point(s) on each of the following curves, at the given gradient.
  - a $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x$, gradient $= 6$
  - b $f(x) = \frac{1}{3}x^3 + 2x^2 + 6x$, gradient $= 3$
  - c $f(x) = \frac{1}{3}x^3 - 2x^2$, gradient $= -4$
  - d $f(x) = x^3 - x^2 + 4x$, gradient $= 5$

3 A stone is thrown vertically downwards off a tall cliff. The distance (s) it travels in metres is given by the formula $s = 4t + 5t^2$, where t is the time in seconds after the stone's release.
  - a What is the rate of change of distance with time, $\frac{ds}{dt}$? (This represents the velocity.)
  - b How many seconds after its release is the stone travelling at a velocity of 9 m/s?

Exercise 19.8 (cont)

c The speed of the stone as it hits the ground is 34 m/s. How many seconds after its release did the stone hit the ground?

d Using your answer to part **c**, calculate the distance the stone falls and hence the height of the cliff.

4 The temperature inside a pressure cooker (T) in degrees Celsius is given by the formula

$$T = 20 + 12t^2 - t^3$$

where t is the time in minutes after the cooking started and $t \leqslant 8$.

a Calculate the initial (starting) temperature of the pressure cooker.

b What is the rate of temperature increase with time?

c What is the rate of temperature increase when:

i $t = 1$ **ii** $t = 4$ **iii** $t = 8$?

d The pressure cooker was switched off when $\frac{dT}{dt} = 36$.

How long after the start could the pressure cooker have been switched off? Give both possible answers.

e What was the final temperature of the pressure cooker if it was switched off at the greater of the two times calculated in part **d**?

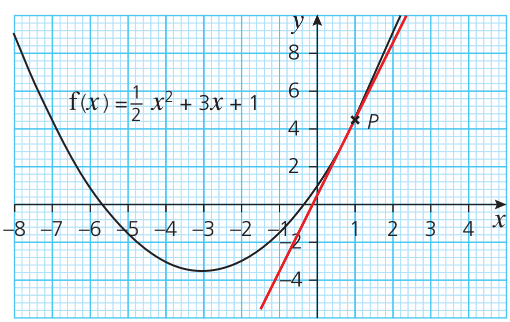
Equation of the tangent at a given point

You already know that the gradient of a tangent drawn at a point on a curve is equal to the gradient of the curve at that point.

Worked example

Find the equation of the tangent of $f(x) = \frac{1}{2}x^2 + 3x + 1$ at a point P, where $x = 1$.

The function $f(x) = \frac{1}{2}x^2 + 3x + 1$ has a gradient function of $f'(x) = x + 3$



At point P, where $x = 1$, the gradient of the curve is 4.

The tangent drawn to the curve at P also has a gradient of 4.

The equation of the tangent can also be calculated. As it is a straight line, it must take the form $y = mx + c$. The gradient m is 4 as shown above. Therefore $y = 4x + c$.

As the tangent passes through the point $P(1, 4\frac{1}{2})$, these values can be substituted for x and y so that c can be calculated.

$4\frac{1}{2} = 4 + c$

$\Rightarrow c = \frac{1}{2}$

The equation of the tangent is therefore $y = 4x + \frac{1}{2}$.

Exercise 19.9

1 For the function $f(x) = x^2 - 3x + 1$
  a Calculate the gradient function.
  b Calculate the gradient of the curve at the point $A(2, -1)$.
  A tangent is drawn to the curve at A.
  c What is the gradient of the tangent?
  d Calculate the equation of the tangent. Give your answer in the form $y = mx + c$.

2 For the function $f(x) = 2x^2 - 4x - 2$
  a Calculate the gradient of the curve at $x = 2$.
  A tangent is drawn to the curve at the point $(2, -2)$.
  b Calculate the equation of the tangent. Give your answer in the form $y = mx + c$.

3 A tangent is drawn to the curve $f(x) = \frac{1}{2}x^2 - 4x - 2$ at the point $P(0, -2)$.
  a Calculate the gradient of the tangent at P.
  b Calculate the equation of the tangent. Give your answer in the form $y = mx + c$.

4 A tangent, T_1, is drawn to the curve $f(x) = -x^2 + 4x + 1$ at point $A(4, 1)$.
  a Calculate the gradient of the tangent at A.
  b Calculate the equation of the tangent. Give your answer in the form $y = mx + c$.
  c Another tangent to the curve, T_2, is drawn at point $B(2, 5)$. Calculate the equation of T_2.

5 A tangent, T_1, is drawn to the curve $f(x) = -\frac{1}{4}x^2 - 3x + 1$ at point $P(-2, 6)$.
  a Calculate the equation of T_1.
  Another tangent to the curve, T_2, with equation $y = 10$, is drawn at point Q.
  b Calculate the coordinates of point Q.
  c T_1 and T_2 are extended so that they intersect. Calculate the coordinates of their point of intersection.

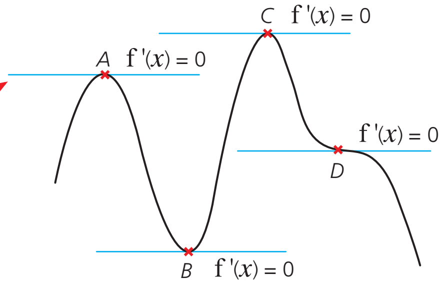
6 The equation of a tangent T, drawn to the curve $f(x) = -\frac{1}{2}x^2 - x - 4$ at P, has equation $y = -3x - 2$.
  a Calculate the gradient function of the curve.
  b What is the gradient of the tangent T?
  c What are the coordinates of point P?

Stationary points

There are times when the gradient of a point on a curve is zero, i.e. the tangent drawn at that point is horizontal. A point where the gradient of the curve is zero is known as a **stationary point**.

In the diagram, the stationary points A, B and C are also turning points (see page 200) because the gradient of the curve changes from positive to negative (or vice versa). Stationary point D is not a turning point because the gradient remains negative before and after the point.

There are different types of stationary point.



Points A and C are **local maxima**, point B is a **local minima** and point D is a **point of inflection**. This text covers local maximum and minimum points only.

As the worked example shows, it is not necessary to sketch a graph in order to find the position of any stationary points or to identify what type of stationary points they are.

Worked example

a A graph has equation $y = \frac{1}{3}x^3 - 4x + 5$. Find the coordinates of the stationary points on the graph.

If $y = \frac{1}{3}x^3 - 4x + 5$, $\frac{dy}{dx} = x^2 - 4$.

At a stationary point $\frac{dy}{dx} = 0$, so solve $x^2 - 4 = 0$ to find the x-coordinate of any stationary point.

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Substitute $x = 2$ and $x = -2$ into the equation of the curve to find the corresponding y-coordinates.

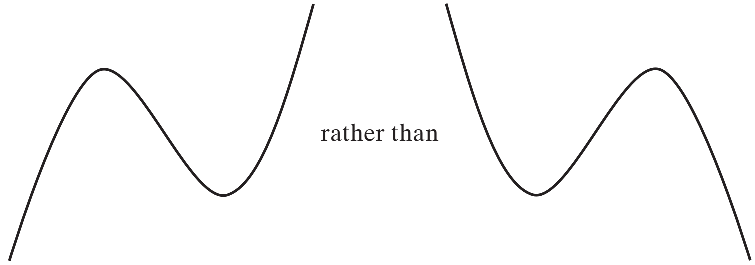
When $x = 2$, $y = \frac{1}{3}(2)^3 - 4(2) + 5 = -\frac{1}{3}$.

When $x = -2$, $y = \frac{1}{3}(-2)^3 - 4(-2) + 5 = 10\frac{1}{3}$.

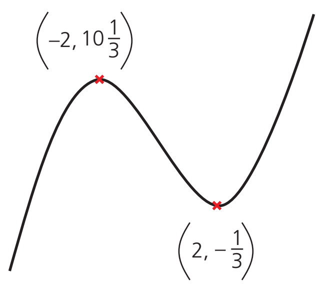
The coordinates of the stationary points are $\left(2, -\frac{1}{3}\right)$ and $\left(-2, 10\frac{1}{3}\right)$.

b Determine the nature of each of the stationary points.
There are several methods that can be used to establish the type of stationary point.

Graphical deduction
As the curve is cubic and the coefficient of the x^3 term is positive, the shape of the curve is of the form



Therefore it can be deduced that the positions of the stationary points are:



Hence $\left(-2, 10\frac{1}{3}\right)$ is a maximum point and $\left(2, -\frac{1}{3}\right)$ a minimum point.

Gradient inspection

The gradient of the curve either side of a stationary point can be calculated. At the stationary point where $x = 2$, consider the gradient at $x = 1$ and at $x = 3$.

$\frac{dy}{dx} = x^2 - 4,$ so when $x = 1$, $\frac{dy}{dx} = -3$

and when $x = 3$, $\frac{dy}{dx} = 5$

As x increases, the gradient changes from negative to positive, therefore the stationary point must be a minimum.

At the stationary point where $x = -2$, consider the gradient at $x = -3$ and at $x = -1$.

$\frac{dy}{dx} = x^2 - 4,$ so when $x = -3$, $\frac{dy}{dx} = 5$

and when $x = -1$, $\frac{dy}{dx} = -3$

As x increases, the gradient changes from positive to negative, therefore the stationary point must be a maximum.

The second derivative

The second derivative, $\frac{d^2y}{dx^2}$, is usually the most efficient way of determining whether a stationary point is a maximum or minimum.

The proof is beyond the scope of this book. However, the general rule is that:

$\frac{d^2y}{dx^2} < 0 \Rightarrow$ a maximum point

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ a minimum point.

In this example, $\frac{dy}{dx} = x^2 - 4$ and $\frac{d^2y}{dx^2} = 2x$.

Substituting the x-values (–2 and 2) of the stationary points into $\frac{d^2y}{dx^2}$ gives:

$\frac{d^2y}{dx^2} = 2(-2) = -4$ (a maximum point)

and $\frac{d^2y}{dx^2} = 2(2) = 4$ (a minimum point).

Note: When $\frac{d^2y}{dx^2} = 0$, the stationary point could either be a maximum or a minimum point, so another method should be used.

Exercise 19.10

1 For each function, calculate:
   i the gradient function
   ii the coordinates of any stationary points.

   a $f(x) = x^2 - 6x + 13$
   b $f(x) = x^2 + 12x + 35$
   c $f(x) = -x^2 + 8x - 13$
   d $f(x) = -6x + 7$

2 For each function, calculate:
   i the gradient function
   ii the coordinates of any stationary points.

   a $f(x) = x^3 - 12x^2 - 58$
   b $f(x) = x^3 - 12x$
   c $f(x) = x^3 - 3x^2 - 45x + 8$
   d $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x - 5$

For Questions 3–6:

a Calculate the gradient function.
b Calculate the coordinates of any stationary points.
c Determine the nature of each stationary point.
d Calculate the value of the y-intercept.
e Sketch the graph of the function.

3 $f(x) = 1 - 4x - x^2$

4 $f(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 3$

5 $f(x) = -\frac{2}{3}x^3 + 3x^2 - 4x$

6 $f(x) = x^3 - \frac{9}{2}x^2 - 30x + 4$

Student assessment 1

1 Find the gradient function of the following:

a $y = x^3$

b $y = 2x^2 - x$

c $y = -\frac{1}{2}x^2 + 2x$

d $y = \frac{2}{3}x^3 + 4x^2 - x$

2 Differentiate the following functions with respect to x.

a $f(x) = x(x+2)$

b $f(x) = (x+2)(x-3)$

c $f(x) = \frac{x^3 - x}{x}$

d $f(x) = \frac{x^3 + 2x^2}{2x}$

3 Find the values of p and q.

$y = 3x^p - qx^2$ if $\frac{dy}{dx} = 12x^{q+1} - 4x$

4 Find the second derivative of the following functions:

a $y = x^4 - 3x^2$

b $s = 2t^5 - t^3$

5 Find the gradient of the following curves at the given values of x.

a $f(x) = \frac{1}{2}x^2 + x;\ x = 1$

b $f(x) = -x^3 + 2x^2 + x;\ x = 0$

c $f(x) = (x-3)(x+8);\ x = \frac{1}{4}$

6 A stone is dropped from the top of a cliff. The distance (s) it falls is given by the equation $s = 5t^2$, where s is the distance in metres and t the time in seconds.

a Calculate the velocity v, by differentiating the distance s with respect to time t.

b Calculate the stone's velocity after 3 seconds.

c The stone hits the ground travelling at 42 m s^{-1}

i Calculate how long it took for the stone to hit the ground.

ii Calculate the height of the cliff.

Student assessment 2

1 The function $f(x) = x^3 + x^2 - 1$ has a gradient of zero at points P and Q, where the x coordinate of P is less than that of Q.

a Calculate the gradient function $f'(x)$.

b Calculate the coordinates of P.

c Calculate the coordinates of Q.

d Determine which of the points P or Q is a maximum. Explain your method clearly.

2 **a** Explain why the point $A(1, 1)$ lies on the curve $y = x^3 - x^2 + x$
b Calculate the gradient of the curve at A.
c Calculate the equation of the tangent to the curve at A.
d Calculate the equation of the normal to the curve at A.

3 For the function $f(x) = (x-2)^2 + 3$
a Calculate $f'(x)$.
b Determine the coordinates of the stationary point.

4 For the function $f(x) = x^4 - 2x^2$
a Calculate $f'(x)$.
b Determine the coordinates of any stationary points.
c Determine the nature of any stationary point.
d Find where the graph intersects or touches:
i the y-axis
ii the x-axis
e Sketch the graph of $f(x)$.

20 Functions

Functions as a mapping

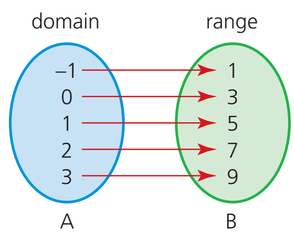
Consider the equation $y = 2x + 3$. It describes the relationship between two variables x and y. In this case, 3 is added to twice the value of x to produce y.

A function is a particular type of relationship between two variables. It has certain characteristics.

Consider the equation $y = 2x + 3$ for values of x within $-1 \leqslant x \leqslant 3$.

A table of results can be constructed and a mapping drawn.

x	y
–1	1
0	3
1	5
2	7
3	9



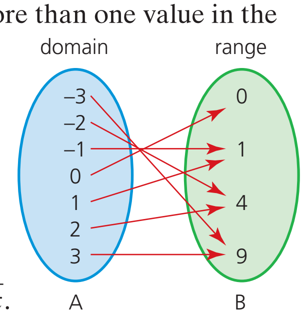
With a function, each value in set B (the **range**) is produced from one value in set A (the **domain**). The relationship can be written as a function:

$$f(x) = 2x + 3; -1 \leqslant x \leqslant 3$$

It is also usual to include the domain after the function, as a different domain will produce a different range.

The mapping from A to B can be a **one-to-one** mapping or a **many-to-one** mapping.

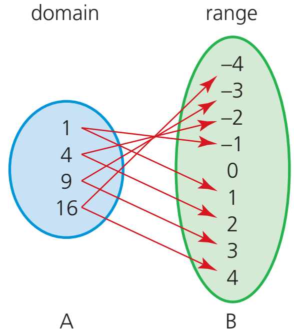
The function above, $f(x) = 2x + 3; -1 \leqslant x \leqslant 3$, is a one-to-one function as one value in the domain maps onto one value in the range. However, the function $f(x) = x^2$; where x is an integer, is a many-to-one function, as a value in the range can be generated by more than one value in the domain, as shown.



It is important to understand that one value in the domain (set A) maps to only one value in the range (set B). Therefore the mapping shown is the function $f(x) = x^2$; where x is an integer.

Some mappings will not represent functions; for example, consider the relationship $y = \pm\sqrt{x}$.

The following table and mapping diagram can be produced:



x	y
1	±1
4	±2
9	±3
16	±4

This relationship is not a function, as a value in the domain produces more than one value in the range.

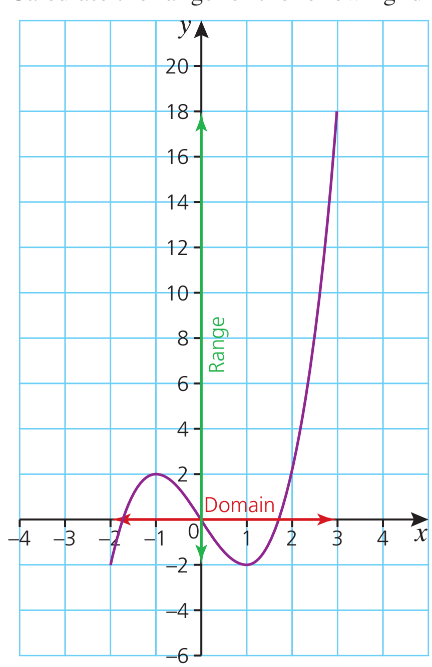
Note: If a domain is not specified then it is assumed to be all real values.

Calculating the range from the domain

The domain is the set of input values and the range is the set of output values for a function. (Note that the range is not the difference between the greatest and least values as in statistics.) The range is therefore not only dependent on the function itself, but also on the domain.

Worked example

Calculate the range for the following functions:



a $f(x) = x^3 - 3x; -2 \leqslant x \leqslant 3$

The graph of the function is shown above. As the domain is restricted to $-2 \leqslant x \leqslant 3$, the range is limited from -2 to 18.

This is written as: Range $-2 \leqslant f(x) \leqslant 18$.

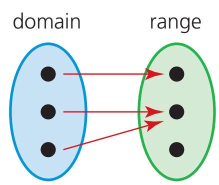
b $f(x) = x^3 - 3x$; x is a real number

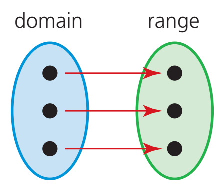
The graph will be similar to the one above except that the domain is not restricted. As the domain is for all real values of x, this implies that any real number can be an input value. As a result, the range will also be all real values.

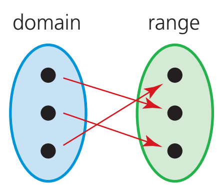
This is written as: Range $f(x)$ is all real numbers.

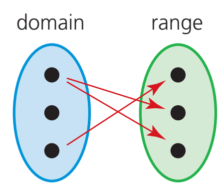
Exercise 20.1

1 Which of the following mappings show a function?

a


b


c


d


Give the domain and range of each of the functions in Questions 2–8.

2 $f(x) = 2x - 1; -1 \leqslant x \leqslant 3$

3 $f(x) = 3x + 2; -4 \leqslant x \leqslant 0$

4 $f(x) = -x + 4; -4 \leqslant x \leqslant 4$

5 $f(x) = x^2 + 2; -3 \leqslant x \leqslant 3$

6 $f(x) = x^2 + 2$; x is a real number

7 $f(x) = -x^2 + 2; 0 \leqslant x \leqslant 4$

8 $f(x) = x^3 - 2; -3 \leqslant x \leqslant 1$

Just as with formulas, values can be substituted into functions.

Worked examples

a For the function $f(x) = 3x - 5$, evaluate:

i $f(2)$

$f(2) = 3 \times 2 - 5$
$= 6 - 5$
$= 1$

ii $f(0)$

$f(0) = 3 \times 0 - 5$
$= 0 - 5$
$= -5$

iii $f(-2)$

$f(-2) = 3 \times (-2) - 5$
$= -6 - 5$
$= -11$

b For the function $f: x \mapsto \frac{2x+6}{3}$, evaluate:

i f(3)

$$f(3) = \frac{2 \times 3 + 6}{3} = \frac{6+6}{3} = 4$$

ii f(1.5)

$$f(1.5) = \frac{2 \times 1.5 + 6}{3} = \frac{3+6}{3} = 3$$

iii f(–1)

$$f(-1) = \frac{2 \times (-1) + 6}{3} = \frac{-2+6}{3} = \frac{4}{3}$$

c For the function $f(x) = x^2 + 4$, evaluate:

i f(2)

$$f(2) = 2^2 + 4 = 4 + 4 = 8$$

ii f(6)

$$f(6) = 6^2 + 4 = 36 + 4 = 40$$

iii f(–1)

$$f(-1) = (-1)^2 + 4 = 1 + 4 = 5$$

Exercise 20.2

1 If $f(x) = 2x + 2$, calculate:

a f(2) **b** f(4) **c** f(0.5) **d** f(1.5)
e f(0) **f** f(–2) **g** f(–6) **h** f(–0.5)

2 If $f(x) = 4x - 6$, calculate:

a f(4) **b** f(7) **c** f(3.5) **d** f(0.5)
e f(0.25) **f** f(–3) **g** f(–4.25) **h** f(0)

3 If $g(x) = -5x + 2$, calculate:

a g(0) **b** g(6) **c** g(4.5) **d** g(3.2)
e g(0.1) **f** g(–2) **g** g(–6.5) **h** g(–2.3)

4 If $h(x) = -3x - 7$, calculate:

a h(4) **b** h(6.5) **c** h(0) **d** h(0.4)
e h(–9) **f** h(–5) **g** h(–2) **h** h(–3.5)

Exercise 20.3

1 If $f(x) = \frac{3x+2}{4}$, calculate:

a f(2) **b** f(8) **c** f(2.5) **d** f(0)
e f(–0.5) **f** f(–6) **g** f(–4) **h** f(–1.6)

2 If $g(x) = \frac{5x-3}{3}$, calculate:

a g(3) **b** g(6) **c** g(0) **d** g(–3)
e g(–1.5) **f** g(–9) **g** g(–0.2) **h** g(–0.1)

3 If h: $x \mapsto \frac{-6x+8}{4}$, calculate:

a h(1) **b** h(0) **c** h(4) **d** h(1.5)
e h(–2) **f** h(–0.5) **g** h(–22) **h** h(–1.5)

4 If $f(x) = \frac{-5x-7}{-8}$, calculate:

a f(5) **b** f(1) **c** f(3) **d** f(–1)
e f(–7) **f** $f\left(-\frac{3}{5}\right)$ **g** f(–0.8) **h** f(0)

Exercise 20.4

1 If $f(x) = x^2 + 3$, calculate:
a f(4) **b** f(7) **c** f(1) **d** f(0)
e f(–1) **f** f(0.5) **g** f(–3) **h** $f(\sqrt{2})$

2 If $f(x) = 3x^2 - 5$, calculate:
a f(5) **b** f(8) **c** f(1) **d** f(0)
e f(–2) **f** $f(\sqrt{3})$ **g** $f\left(-\frac{1}{2}\right)$ **h** $f\left(-\frac{1}{3}\right)$

3 If $g(x) = -2x^2 + 4$, calculate:

a g(3) **b** $g\left(\frac{1}{2}\right)$ **c** g(0) **d** g(1.5)
e g(–4) **f** g(–1) **g** $g(\sqrt{5})$ **h** g(–6)

4 If $h(x) = \frac{-5x^2 + 15}{-2}$, calculate:
a h(1) **b** h(4) **c** $h(\sqrt{3})$ **d** h(0.5)
e h(0) **f** h(–3) **g** $h\left(\frac{1}{\sqrt{2}}\right)$ **h** h(–2.5)

5 If $f(x) = -6x(x - 4)$, calculate:
a f(0) **b** f(2) **c** f(4) **d** f(0.5)
e $f\left(-\frac{1}{2}\right)$ **f** $f\left(-\frac{1}{6}\right)$ **g** f(–2.5) **h** $f(\sqrt{2})$

6 If $g(x) = \frac{(x+2)(x-4)}{-x}$, calculate:
a g(1) **b** g(4) **c** g(8) **d** g(0)
e g(–2) **f** g(–10) **g** $g\left(-\frac{3}{2}\right)$ **h** g(–8)

Exercise 20.5

1 If $f(x) = 2x + 1$, write the following in their simplest form:
a $f(x + 1)$ **b** $f(2x - 3)$ **c** $f(x^2)$
d $f\left(\frac{x}{2}\right)$ **e** $f\left(\frac{x}{4}+1\right)$ **f** $f(x) - x$

2 If $g(x) = 3x^2 - 4$, write the following in their simplest form:

a $g(2x)$ **b** $g\left(\frac{x}{4}\right)$ **c** $g(\sqrt{2x})$
d $g(3x) + 4$ **e** $g(x - 1)$ **f** $g(2x + 2)$

3 If $f(x) = 4x^2 + 3x - 2$, write the following in their simplest form:
a $f(x) + 4$ **b** $f(2x) + 2$ **c** $f(x + 2) - 20$
d $f(x - 1) + 1$ **e** $f\left(\frac{x}{2}\right)$ **f** $f(3x + 2)$

Inverse functions

The **inverse** of a function is its reverse, i.e. it 'undoes' the function's effects. The inverse of the function $f(x)$ is written as $f^{-1}(x)$. To find the inverse of a function:

- rewrite the function, replacing $f(x)$ with y
- interchange x and y
- rearrange the equation to make y the subject.

Worked examples

a Find the inverse of each of the following functions:

i $f(x) = x + 2$

$y = x + 2$

$x = y + 2$

$y = x - 2$

So $f^{-1}(x) = x - 2$

ii $g(x) = 2x - 3$

$y = 2x - 3$

$x = 2y - 3$

$y = \frac{x+3}{2}$

So $g^{-1}(x) = \frac{x+3}{2}$

b If $f(x) = \frac{x-3}{3}$ calculate:

i $f^{-1}(2)$ **ii** $f^{-1}(-3)$

First calculate the **inverse function** $f^{-1}(x)$:

$y = \frac{x-3}{3}$

$x = \frac{y-3}{3}$

$y = 3x + 3$

So $f^{-1}(x) = 3x + 3$

i $f^{-1}(2) = 3(2) + 3 = 9$

ii $f^{-1}(-3) = 3(-3) + 3 = -6$

Exercise 20.6

Find the inverse of each of the following functions:

1 **a** $f(x) = x + 3$ **b** $f(x) = x + 6$
c $f(x) = x - 5$ **d** $g(x) = x$
e $h(x) = 2x$ **f** $p(x) = \frac{x}{3}$

2 **a** $f(x) = 4x$ **b** $f(x) = 2x + 5$
c $f(x) = 3x - 6$ **d** $f(x) = \frac{x+4}{2}$
e $g(x) = \frac{3x-2}{4}$ **f** $g(x) = \frac{8x+7}{5}$

3 **a** $f(x) = \frac{1}{2}x + 3$ **b** $g(x) = \frac{1}{4}x - 2$
c $h(x) = 4(3x - 6)$ **d** $p(x) = 6(x + 3)$
e $q(x) = -2(-3x + 2)$ **f** $f(x) = \frac{2}{3}(4x - 5)$

Exercise 20.7

1 If $f(x) = x - 4$, evaluate:
 a $f^{-1}(2)$ b $f^{-1}(0)$ c $f^{-1}(-5)$

2 If $f(x) = 2x + 1$, evaluate:
 a $f^{-1}(5)$ b $f^{-1}(0)$ c $f^{-1}(-11)$

3 If $g(x) = 6(x - 1)$, evaluate:
 a $g^{-1}(12)$ b $g^{-1}(3)$ c $g^{-1}(6)$

4 If $g(x) = \frac{2x+4}{3}$, evaluate:
 a $g^{-1}(4)$ b $g^{-1}(0)$ c $g^{-1}(-6)$

5 If $h(x) = \frac{1}{3}x - 2$, evaluate:
 a $h^{-1}\left(-\frac{1}{2}\right)$ b $h^{-1}(0)$ c $h^{-1}(-2)$

6 If $f(x) = \frac{4x-2}{5}$, evaluate:
 a $f^{-1}(6)$ b $f^{-1}(-2)$ c $f^{-1}(0)$

Composite functions

Worked examples

fg(x) implies substituting the function g(x) into the function f(x)

a If $f(x) = x + 2$ and $g(x) = x + 3$, find $fg(x)$.

$fg(x) = f(x + 3)$
$= (x + 3) + 2$
$= x + 5$

b If $f(x) = 2x - 1$ and $g(x) = x - 2$, find $fg(x)$.

$fg(x) = f(x - 2)$
$= 2(x - 2) - 1$
$= 2x - 4 - 1$
$= 2x - 5$

c If $f(x) = 2x + 3$ and $g(x) = 2x$, evaluate $fg(3)$.

$fg(x) = f(2x)$
$= 2(2x) + 3$
$= 4x + 3$
$fg(3) = 4 \times 3 + 3$
$= 15$

Exercise 20.8

1 Write a formula for $fg(x)$ in each of the following:
 a $f(x) = x - 3$ $g(x) = x + 5$
 b $f(x) = x + 4$ $g(x) = x - 1$
 c $f(x) = x$ $g(x) = 2x$
 d $f(x) = \frac{x}{2}$ $g(x) = 2x$

Exercise 20.8 (cont)

2 Write a formula for $pq(x)$ in each of the following:
 a $p(x) = 2x$ $q(x) = x + 4$
 b $p(x) = 3x + 1$ $q(x) = 2x$
 c $p(x) = 4x + 6$ $q(x) = (2x - 1)^2$
 d $p(x) = -x + 4$ $q(x) = (x + 2)^2$

3 Write a formula for $jk(x)$ in each of the following:
 a $j(x) = \frac{x-2}{4}$ $k(x) = 4x$
 b $j(x) = 3x + 2$ $k(x) = \frac{x-3}{2}$
 c $j(x) = \frac{2x+5}{3}$ $k(x) = \frac{1}{2}x + 1$
 d $j(x) = \frac{1}{4}(x - 3)$ $k(x) = \frac{8x+2}{5}$

4 Evaluate $fg(2)$ in each of the following:
 a $f(x) = x - 4$ $g(x) = x + 3$
 b $f(x) = 2x$ $g(x) = -x + 6$
 c $f(x) = 3x$ $g(x) = 6x + 1$
 d $f(x) = \frac{x}{2}$ $g(x) = -2x$

5 Evaluate $gh(-4)$ in each of the following:
 a $g(x) = 3x + 2$ $h(x) = -4x$
 b $g(x) = \frac{1}{2}(3x - 1)$ $h(x) = \frac{2x}{5}$
 c $g(x) = 4(-x + 2)$ $h(x) = \frac{2x+6}{4}$
 d $g(x) = \frac{4x+4}{5}$ $h(x) = -\frac{1}{3}(-x + 5)$

Student assessment 1

1 For the function $f(x) = 5x - 1$, evaluate:
 a $f(2)$ b $f(0)$ c $f(-3)$

2 For the function $g(x) = \frac{3x-2}{2}$, evaluate:
 a $g(4)$ b $g(0)$ c $g(-3)$

3 For the function $f(x) = \frac{(x+3)(x-4)}{2}$, evaluate:
 a $f(0)$ b $f(-3)$ c $f(-6)$

4 Find the inverse of each of the following functions:
 a $f(x) = -x + 4$ b $g(x) = \frac{3(x-6)}{2}$

5 If $h(x) = \frac{3}{2}(-x + 3)$, evaluate:
 a $h^{-1}(-3)$ b $h^{-1}\left(\frac{3}{2}\right)$

6 If $f(x) = 4x + 2$ and $g(x) = -x + 3$, find $fg(x)$.

7 A function is given as $f(x) = (x + 4)(x - 2)^2$; $-5 \leqslant x \leqslant 3$
 a Sketch $f(x)$ for the given domain.
 b Deduce the range of $f(x)$.

Student assessment 2

1 For the function $f(x) = 3x + 1$, evaluate:
  a $f(4)$ b $f(-1)$ c $f(0)$

2 For the function $g(x) = \frac{-x-2}{3}$, evaluate:
  a $g(4)$ b $g(-5)$ c $g(1)$

3 For the function $f(x) = x^2 - 3x$, evaluate:
  a $f(1)$ b $f(3)$ c $f(-3)$

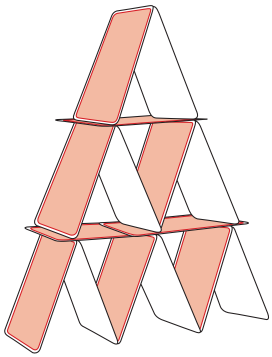
4 Find the inverse of the following functions:
  a $f(x) = -3x + 9$ b $g(x) = \frac{(x-2)}{4}$

5 If $h(x) = -5(-2x + 4)$, evaluate:
  a $h^{-1}(-10)$ b $h^{-1}(0)$

6 If $f(x) = 8x + 2$ and $g(x) = 4x - 1$, find $fg(x)$.

7 A function is given as $g(x) = -(x + 3)^2 + 5$; $-6 \leqslant x \leqslant 1$
  a Sketch $g(x)$ for the given domain.
  b Deduce the range of $g(x)$.

2 Mathematical investigations and ICT 2

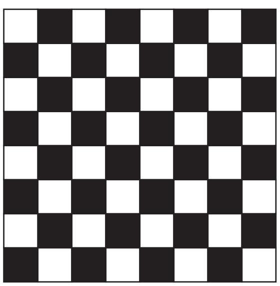


House of cards

The drawing shows a house of cards three layers high. Fifteen cards are needed to construct it.

1 How many cards are needed to construct a house ten layers high?
2 The world record is for a house 75 layers high. How many cards are needed to construct this house of cards?
3 Show that the general formula for a house n layers high requiring c cards is:

$c = \frac{1}{2}n(3n + 1)$

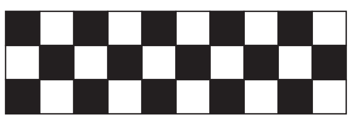


Chequered boards

A chessboard is an 8 × 8 square grid consisting of alternating black and white squares as shown:

There are 64 unit squares of which 32 are black and 32 are white.

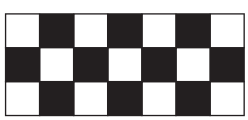
Consider boards of different sizes. The examples below show rectangular boards, each consisting of alternating black and white unit squares.



Total number of unit squares is 30.

Number of black squares is 15.

Number of white squares is 15.



Total number of unit squares is 21.

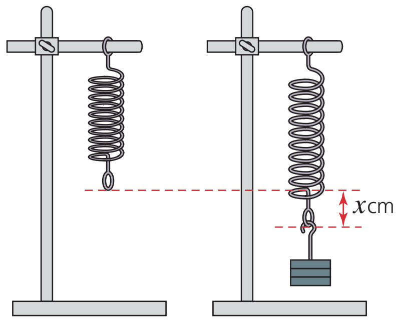
Number of black squares is 10.

Number of white squares is 11.

1 Investigate the number of black and white unit squares on different rectangular boards.
Note: For consistency you may find it helpful to always keep the bottom right-hand square the same colour.
2 What are the numbers of black and white squares on a board $m \times n$ units?

Modelling: Stretching a spring

A spring is attached to a clamp stand as shown below.



Different weights are attached to the end of the spring. The mass (m) in grams is noted as is the amount by which the spring stretches (x) in centimetres.

The data collected is shown in the table below:

Mass (g)	50	100	150	200	250	300	350	400	450	500
Extension (cm)	3.1	6.3	9.5	12.8	15.4	18.9	21.7	25.0	28.2	31.2

1 Plot a graph of mass against extension.
2 Describe the approximate relationship between the mass and the extension.
3 Draw a line of best fit through the data.
4 Calculate the equation of the line of best fit.
5 Use your equation to predict what the length of the spring would be for a mass of 275 g.
6 Explain why it is unlikely that the equation would be useful to find the extension if a mass of 5 kg was added to the spring.

ICT activity

You have seen that it is possible to solve some exponential equations by applying the laws of indices.

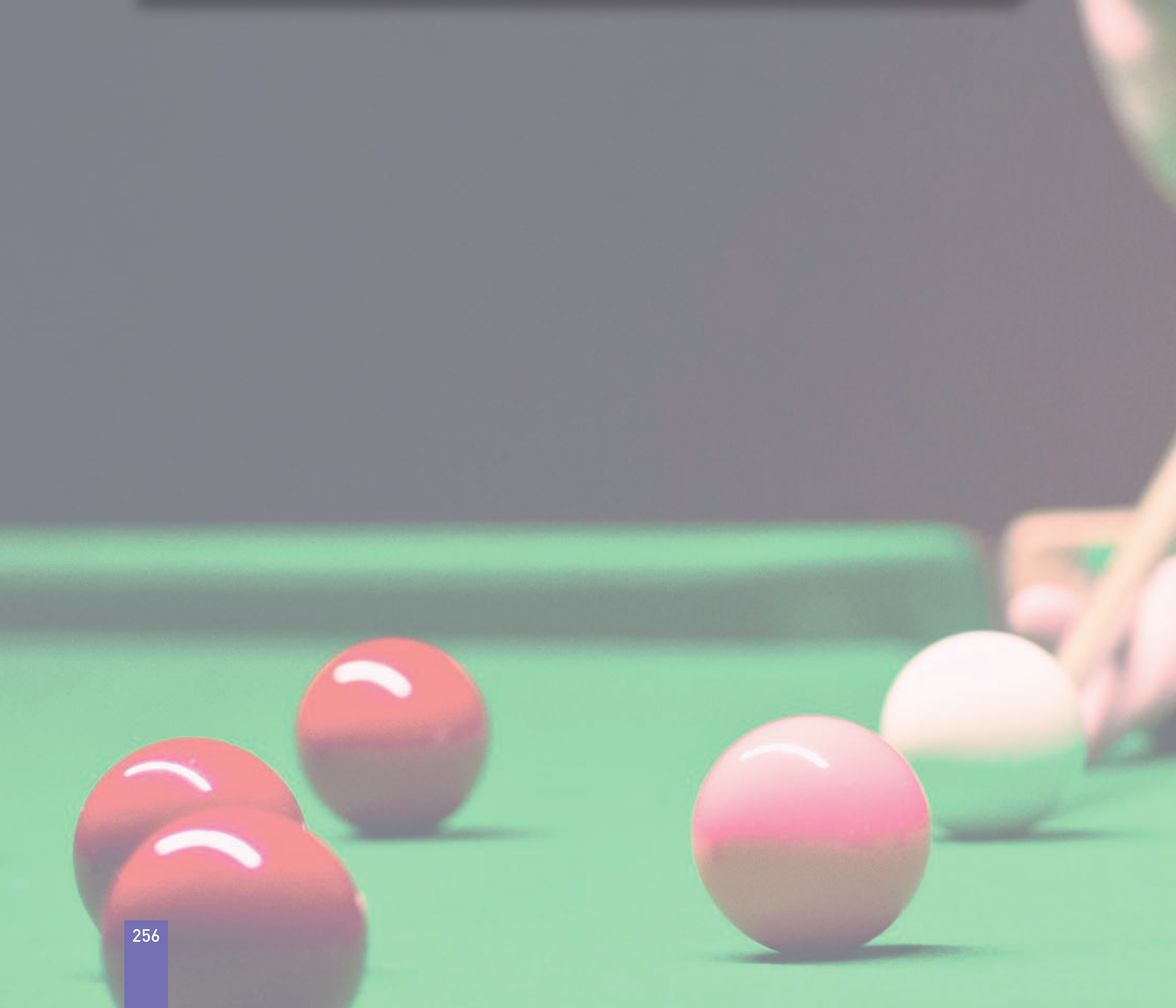
Use a graphics calculator and appropriate graphs to solve the following exponential equations:

1 $4^x = 40$ 2 $3^x = 17$ 3 $5^{x-1} = 6$ 4 $3^{-x} = 0.5$

TOPIC 3

Coordinate geometry

Contents



Learning objectives

E3.1 Coordinates
Use and interpret Cartesian coordinates in two dimensions.

E3.2 Drawing linear graphs
Draw straight-line graphs for linear equations.

E3.3 Gradient of linear graphs
1 Find the gradient of a straight line.
2 Calculate the gradient of a straight line from the coordinates of two points on it.

E3.4 Length and midpoint
1 Calculate the length of a line segment.
2 Find the coordinates of the midpoint of a line segment.

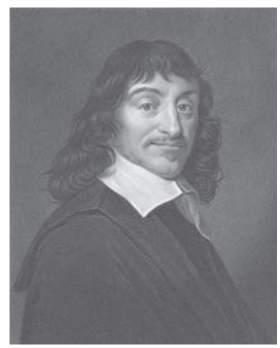
E3.5 Equations of linear graphs
Interpret and obtain the equation of a straight-line graph.

E3.6 Parallel lines
Find the gradient and equation of a straight line parallel to a given line.

E3.7 Perpendicular lines
Find the gradient and equation of a straight line perpendicular to a given line.

The French

In the middle of the seventeenth century there were three great French mathematicians, René Descartes, Blaise Pascal and Pierre de Fermat.



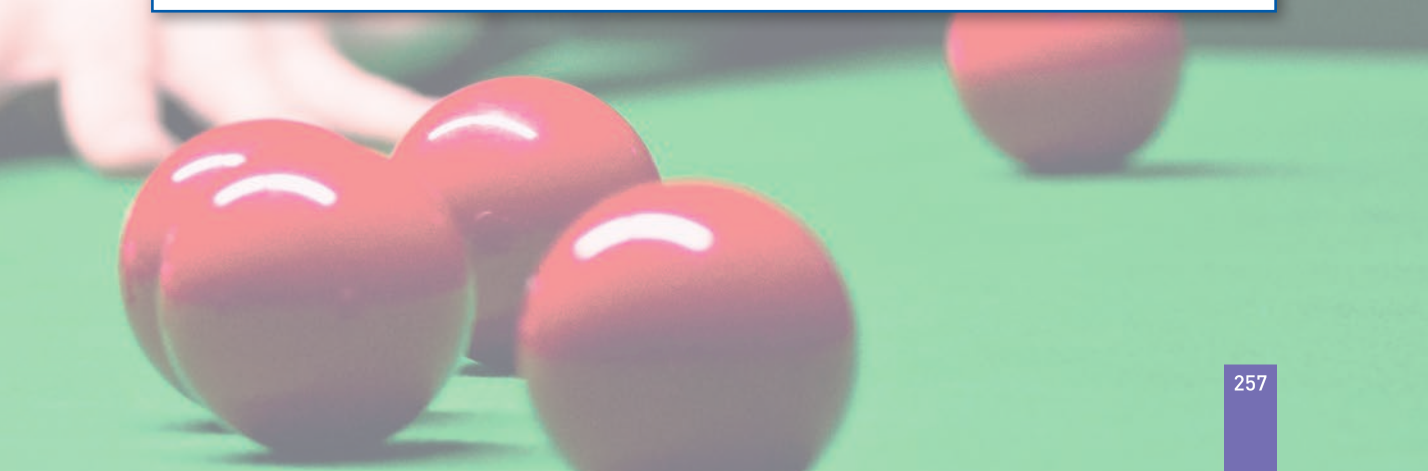
René Descartes (1596–1650)

René Descartes (1596–1650) was a philosopher and a mathematician. His book *The Meditations* asks 'How and what do I know?' His work in mathematics made a link between algebra and geometry. He thought that all nature could be explained in terms of mathematics. Although he was not considered as talented a mathematician as Pascal and Fermat, he has had greater influence on modern thought. The (x, y) coordinates we use are called Cartesian coordinates after Descartes.

Blaise Pascal (1623–1662) was a genius who studied geometry as a child. When he was 16 he stated and proved Pascal's Theorem, which relates any six points on any conic section. The Theorem is sometimes called the 'Cat's Cradle'. He founded probability theory and made contributions to the invention of calculus. He is best known for Pascal's Triangle.

Pierre de Fermat (1601–1665) was a brilliant mathematician and, along with Descartes, one of the most influential. Fermat invented number theory and worked on calculus. He discovered probability theory with his friend Pascal. It can be argued that Fermat was at least Newton's equal as a mathematician.

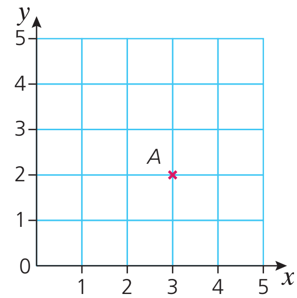
Fermat's most famous discovery in number theory includes 'Fermat's Last Theorem'. This theorem is derived from Pythagoras' theorem, which states that for a right-angled triangle, $x^2 = y^2 + z^2$ where x is the length of the hypotenuse. Fermat said that if the index (power) was greater than two and x, y, z were all whole numbers, then the equation was never true. (This theorem was only proved in 1995 by the English mathematician Andrew Wiles.)



21 Straight-line graphs

Coordinates

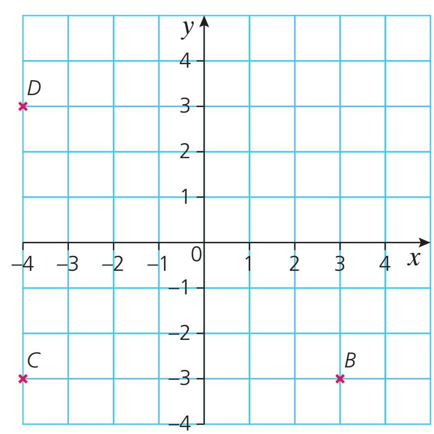
To fix a point in two dimensions (2D), its position is given in relation to a point called the **origin**. Through the origin, axes are drawn perpendicular to each other. The horizontal axis is known as the *x*-**axis**, and the vertical axis is known as the *y*-**axis**.



The *x*-axis is numbered from left to right. The *y*-axis is numbered from bottom to top.

The position of point *A* is given by two coordinates: the *x*-coordinate first, followed by the *y*-coordinate. So the coordinates of point *A* are (3, 2).

A number line can extend in both directions by extending the *x*- and *y*-axes below zero, as shown in the grid below:



Points *B*, *C* and *D* can be described by their coordinates:

Point *B* is at (3, −3),
Point *C* is at (−4, −3),
Point *D* is at (−4, 3).

Exercise 21.1

1 Draw a pair of axes with both x and y from –8 to +8.
Mark each of the following points on your grid:

a $A = (5, 2)$	**b** $B = (7, 3)$	**c** $C = (2, 4)$
d $D = (-8, 5)$	**e** $E = (-5, -8)$	**f** $F = (3, -7)$
g $G = (7, -3)$	**h** $H = (6, -6)$	

2 $A = (3, 2)$ $B = (3, -4)$ $C = (-2, -4)$ $D = (-2, 2)$

Draw a separate grid for each of Questions 2–4 with x- and y-axes from –6 to +6. Plot and join the points in order to name each shape drawn.

3 $E = (1, 3)$ $F = (4, -5)$ $G = (-2, -5)$

4 $H = (-6, 4)$ $I = (0, -4)$ $J = (4, -2)$ $K = (-2, 6)$

Exercise 21.2

Draw a pair of axes with both x and y from –10 to +10.

1 Plot the points $P = (-6, 4)$, $Q = (6, 4)$ and $R = (8, -2)$.
Plot point S such that $PQRS$ when drawn is a parallelogram.
- **a** Draw diagonals PR and QS. What are the coordinates of their point of intersection?
- **b** What is the area of $PQRS$?

2 On the same axes, plot point M at (–8, 4) and point N at (4, 4).
- **a** Join points $MNRS$. What shape is formed?
- **b** What is the area of $MNRS$?
- **c** Explain your answer to Question **2b**.

3 On the same axes, plot point J where point J has y-coordinate +10 and JRS, when joined, forms an isosceles triangle. What is the x-coordinate of all points on the line of symmetry of triangle JRS?

Exercise 21.3

1 **a** On a grid with axes numbered from –10 to +10 draw a hexagon $ABCDEF$ with centre (0, 0), points A(0, 8) and B(7, 4) and two lines of symmetry.
b Write down the coordinates of points C, D, E and F.

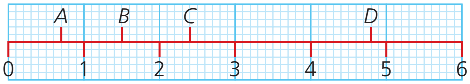
2 **a** On a similar grid to Question 1, draw an octagon $PQRSTUVW$ which has point P(2, –8), point Q(–6, –8) and point R(–7, –5). $PQ = RS = TU = VW$ and $QR = ST = UV = WP$.
b List the coordinates of points S, T, U, V and W.
c What are the coordinates of the centre of rotational symmetry of the octagon?

Reading scales

Exercise 21.4

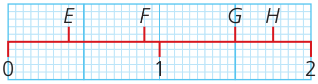
1 The points A, B, C and D are not at whole number points on the number line. Point A is at 0.7.

What are the positions of points B, C and D?



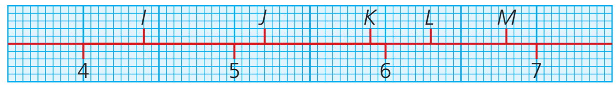
2 On this number line, point E is at 0.4.

What are the positions of points F, G and H?

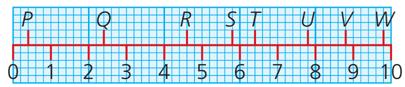


Exercise 21.4 (cont)

3 What are the positions of points *I*, *J*, *K*, *L* and *M*?

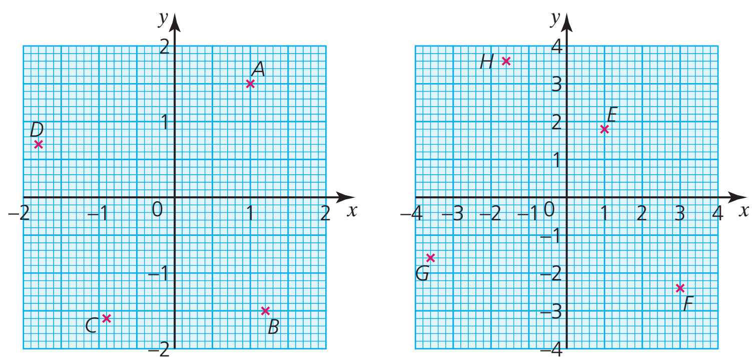


4 Point *P* is at position 0.4 and point *W* is at position 9.8.
What are the positions of points *Q*, *R*, *S*, *T*, *U* and *V*?



Exercise 21.5

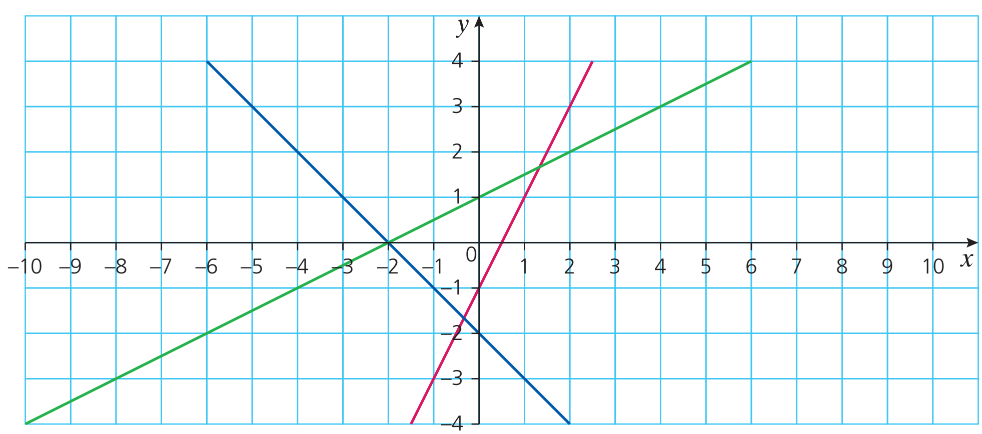
Give the coordinates of points *A*, *B*, *C*, *D*, *E*, *F*, *G* and *H*.



The gradient of a straight line

Lines are made of an infinite number of points. This chapter looks at those whose points form a straight line.

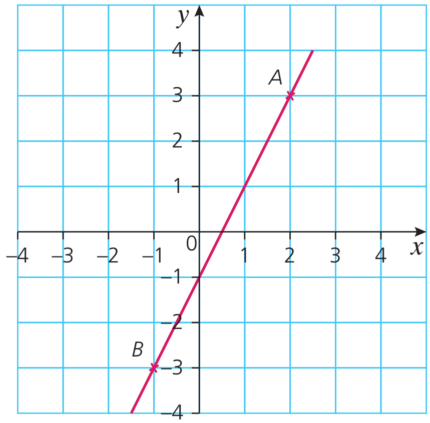
The graph below shows three straight lines.



The lines have some properties in common (e.g. they are straight), but also have differences. One of their differences is that they have different slopes. The slope of a line is called its **gradient**.

The gradient of a straight line is constant, i.e. it does not change. The gradient of a straight line can be calculated by considering the coordinates of any two points on the line.

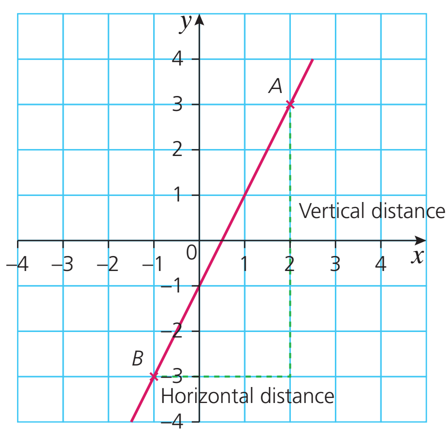
On the line below, two points A and B have been chosen.



The coordinates of the points are $A(2, 3)$ and $B(-1, -3)$. The gradient is calculated using the following formula:

$$\text{Gradient} = \frac{\text{vertical distance between two points}}{\text{horizontal distance between two points}}$$

Graphically this can be represented as follows:



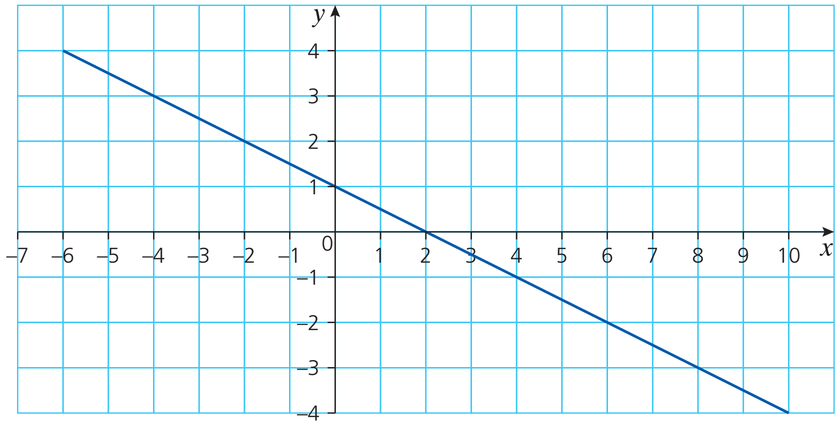
Therefore, gradient $= \frac{3-(-3)}{2-(-1)} = \frac{6}{3} = 2$

In general, therefore, if the two points chosen have coordinates (x_1, y_1) and (x_2, y_2), the gradient is calculated as:

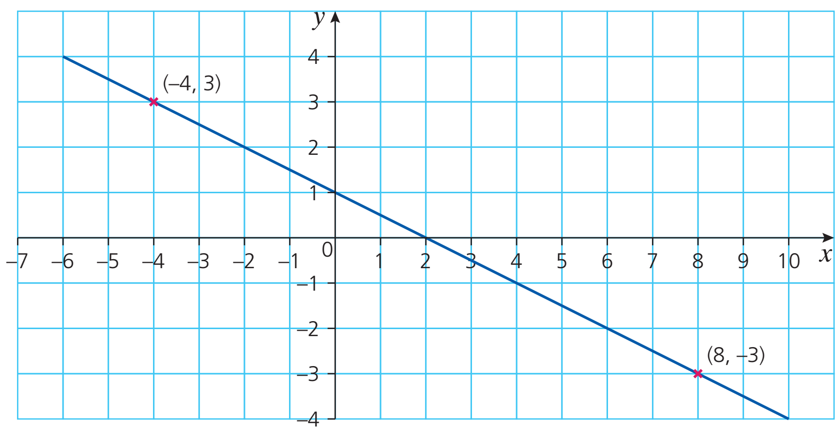
$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

Worked example

Calculate the gradient of the line shown below.



Choose two points on the line, e.g. (–4, 3) and (8, –3).



Let point 1 be (–4, 3) and point 2 be (8, –3).

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 3}{8 - (-4)}$$

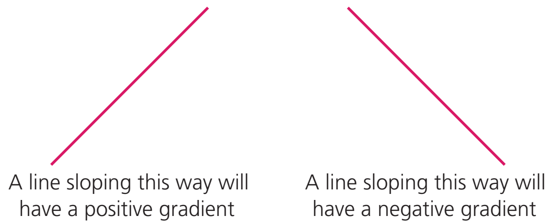
$$= \frac{-6}{12} = -\frac{1}{2}$$

Note: The gradient is not affected by which point is chosen as point 1 and which is chosen as point 2. In the example above, if point 1 was (8, –3) and point 2 (–4, 3), the gradient would be calculated as:

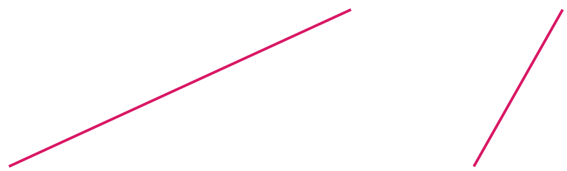
$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-3)}{-4 - 8}$$

$$= \frac{6}{-12} = -\frac{1}{2}$$

To check if the sign of the gradient is correct, the following guideline is useful.



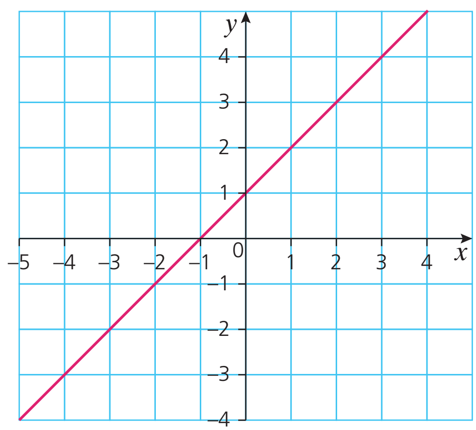
A large value for the gradient implies that the line is steep. The line on the right below will have a greater value for the gradient than the line on the left as it is steeper.



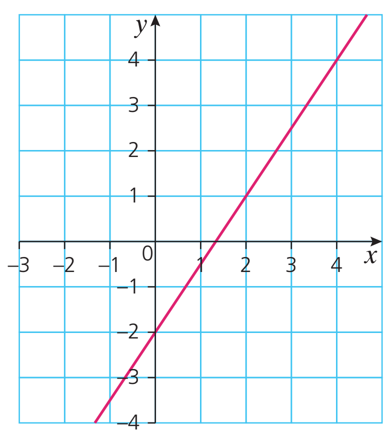
Exercise 21.6

1 For each of the following lines, select two points on the line and then calculate its gradient.

a

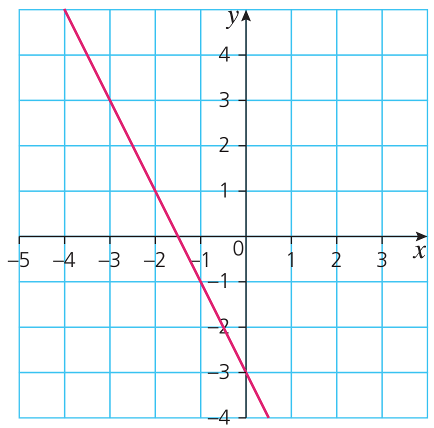


b

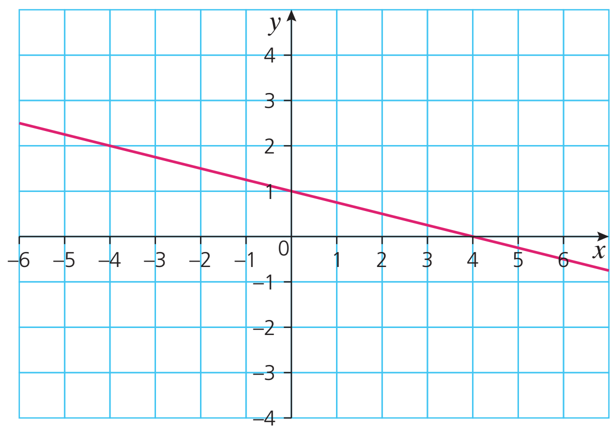


Exercise 21.6 (cont)

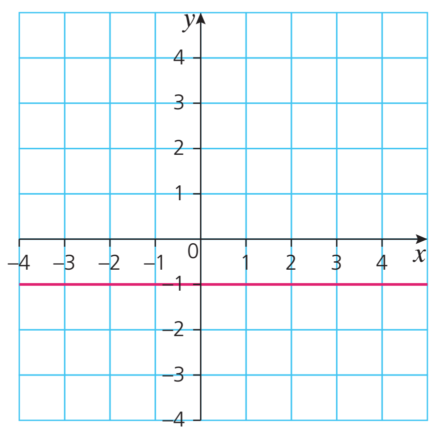
c



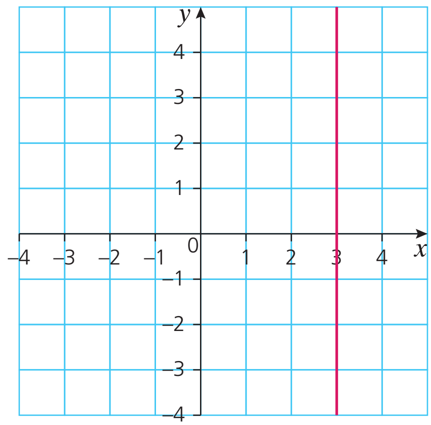
d



e



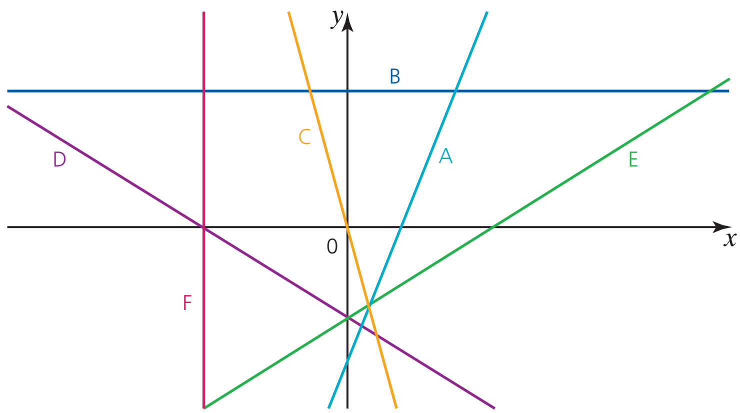
f



2 From your answers to Question 1e, what conclusion can you make about the gradient of any horizontal line?

3 From your answers to Question 1f, what conclusion can you make about the gradient of any vertical line?

4 The graph below shows six straight lines labelled A–F.



Six gradients are given below. Deduce which line has which gradient.

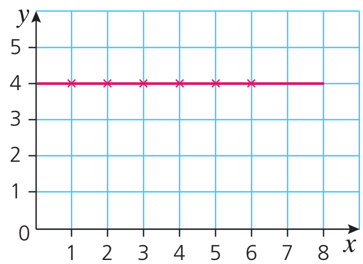
Gradient = $\frac{1}{2}$	Gradient is undefined	Gradient = 2
Gradient = –3	Gradient = 0	Gradient = $-\frac{1}{2}$

The equation of a straight line

The coordinates of every point on a straight line all have a common relationship. This relationship when expressed algebraically as an equation in terms of x and/or y is known as the **equation of the straight line**.

Worked examples

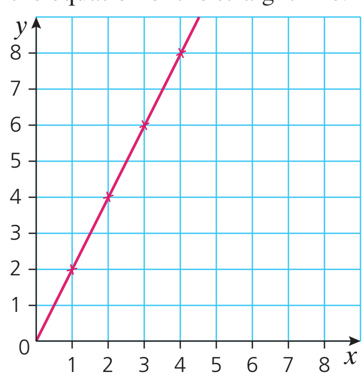
a By looking at the coordinates of some of the points on the line below, establish the equation of the straight line.



x	y
1	4
2	4
3	4
4	4
5	4
6	4

Some of the points on the line have been identified and their coordinates entered in a table above. By looking at the table it can be seen that the only rule all the points have in common is that $y = 4$.
Hence the equation of the straight line is $y = 4$.

b By looking at the coordinates of some of the points on the line, establish the equation of the straight line.



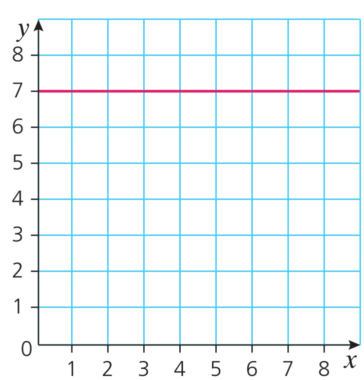
x	y
1	2
2	4
3	6
4	8

Once again, by looking at the table it can be seen that the relationship between the x- and y-coordinates is that each y-coordinate is twice the corresponding x-coordinate.
Hence the equation of the straight line is $y = 2x$.

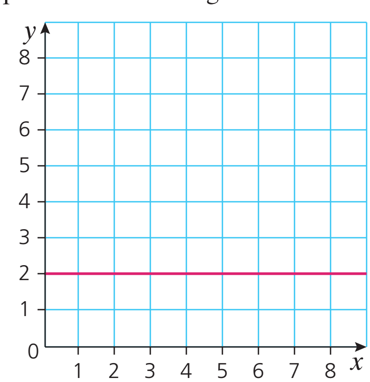
Exercise 21.7

1 In each of the following, identify the coordinates of some of the points on the line and use these to find the equation of the straight line.

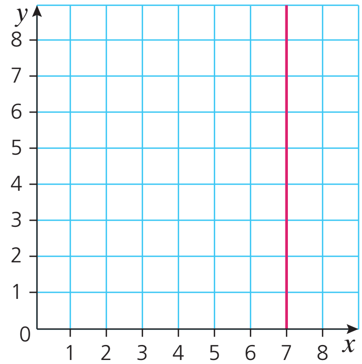
a



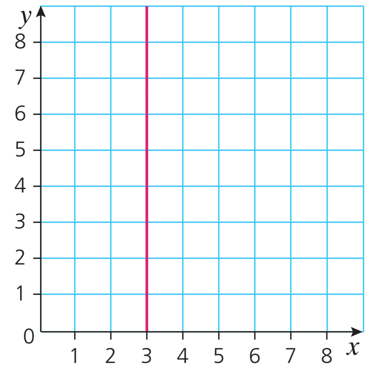
b



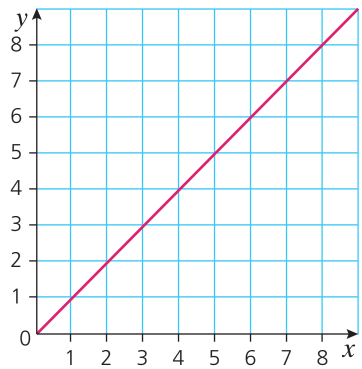
c


y
8
7
6
5
4
3
2
1
0
1 2 3 4 5 6 7 8 x

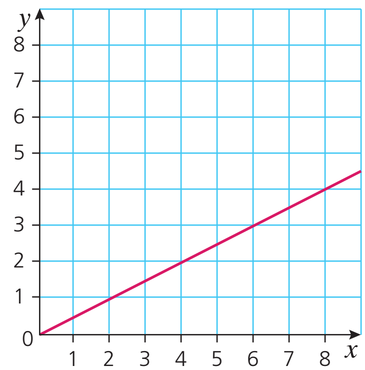
d


y
8
7
6
5
4
3
2
1
0
1 2 3 4 5 6 7 8 x

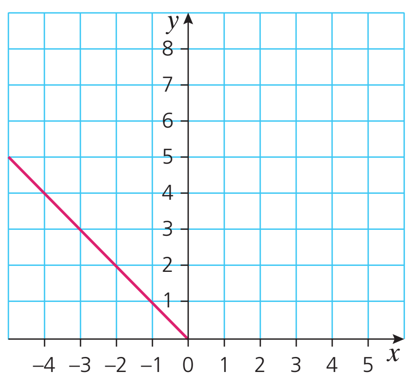
e


y
8
7
6
5
4
3
2
1
0
1 2 3 4 5 6 7 8 x

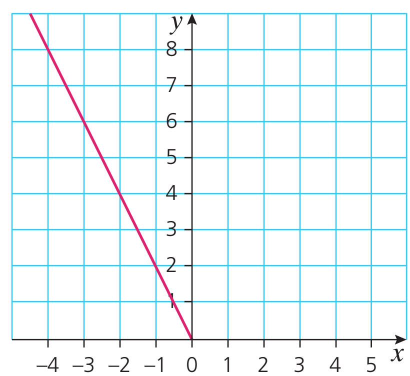
f


y
8
7
6
5
4
3
2
1
0
1 2 3 4 5 6 7 8 x

g


y
8
7
6
5
4
3
2
1
–4 –3 –2 –1 0 1 2 3 4 5 x

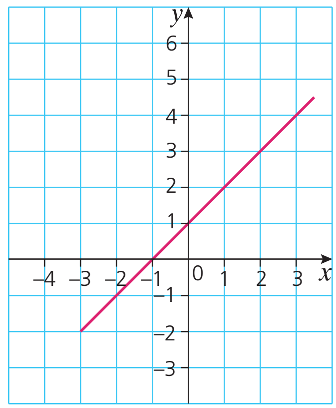
h


y
8
7
6
5
4
3
2
1
–4 –3 –2 –1 0 1 2 3 4 5 x

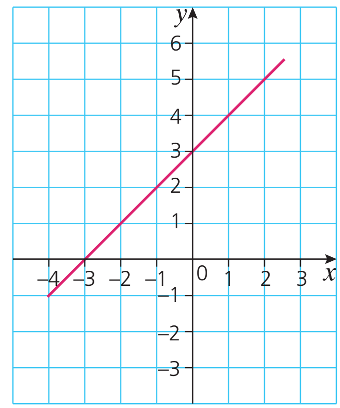
Exercise 21.8

1 In each of the following, identify the coordinates of some of the points on the line and use these to find the equation of the straight line.

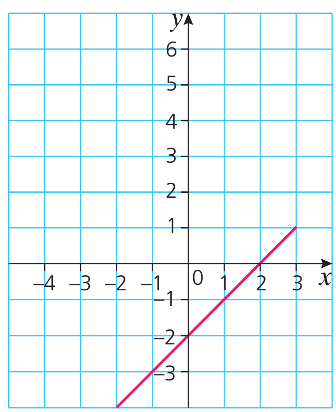
a



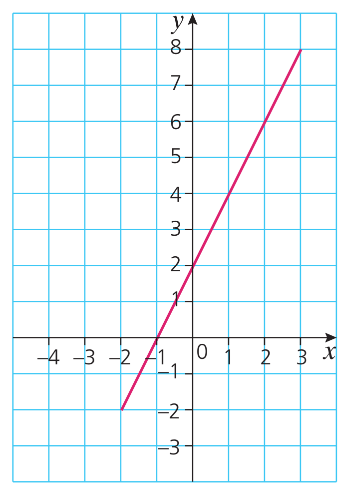
b



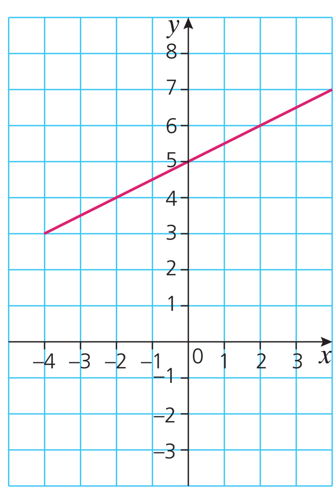
c



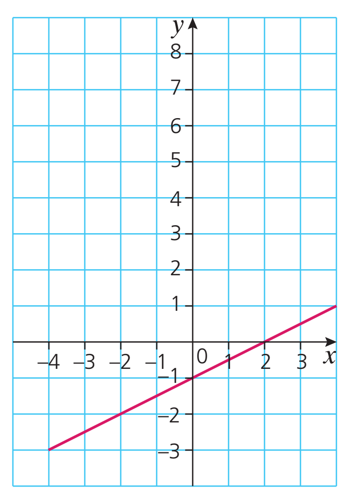
d



e

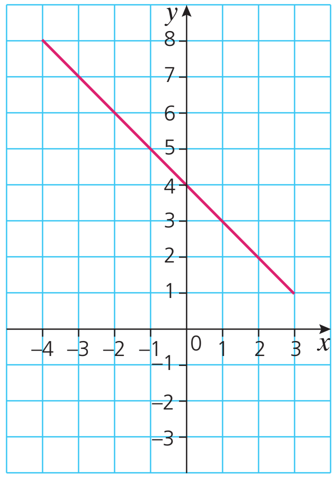


f

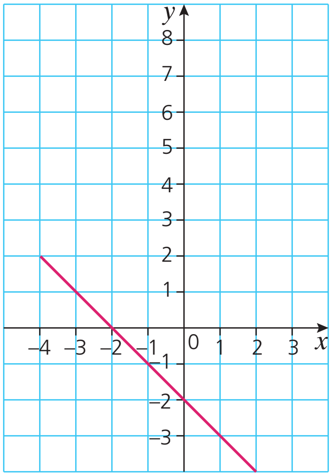


2 In each of the following, identify the coordinates of some of the points on the line and use these to find the equation of the straight line.

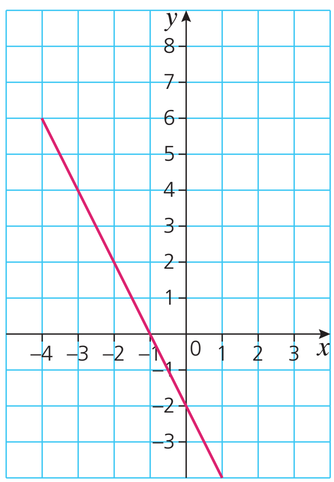
a



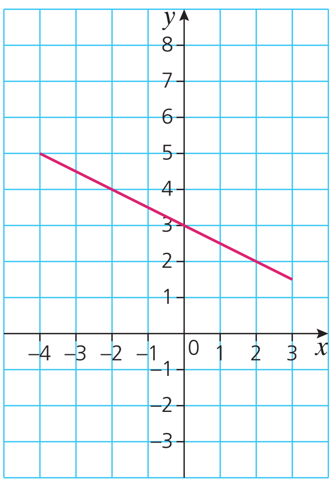
b



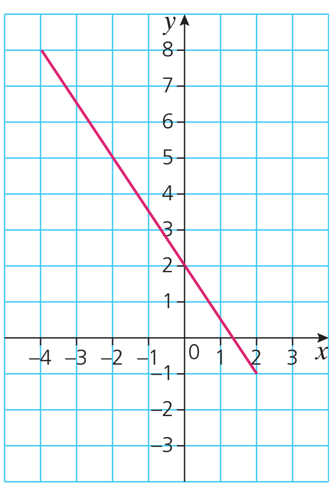
c



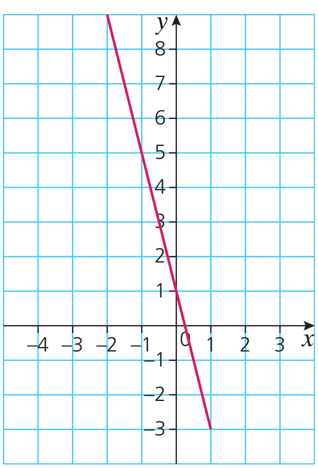
d



e



f



Exercise 21.8 (cont)

3 a For each of the graphs in Questions 1 and 2, calculate the gradient of the straight line.
  b What do you notice about the gradient of each line and its equation?
  c What do you notice about the equation of the straight line and where the line intersects the y-axis?

4 Copy the diagrams in Question 1. Draw two lines on each diagram parallel to the given line.
  a Write the equation of these new lines in the form $y = mx + c$.
  b What do you notice about the equations of these new parallel lines?

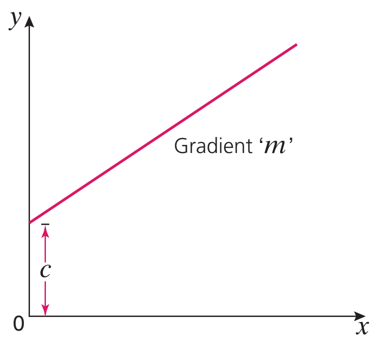
5 In Question 2, you have an equation for these lines in the form $y = mx + c$. Change the value of the intercept c and then draw the new line. What do you notice about this new line and the first line?

The general equation of a straight line

In general the equation of any straight line can be written in the form:

$$y = mx + c$$

where m represents the gradient of the straight line and c the intercept with the y-axis. This is shown in the diagram below.



By looking at the equation of a straight line written in the form $y = mx + c$, it is therefore possible to deduce the line's gradient and intercept with the y-axis without having to draw it.

Worked examples

a Calculate the gradient and y-intercept of the following straight lines:

  i $y = 3x - 2$ — gradient $= 3$, y-intercept $= -2$

  ii $y = -2x + 6$ — gradient $= -2$, y-intercept $= 6$

b Calculate the gradient and y-intercept of the following straight lines:

  i $2y = 4x + 2$

  This needs to be rearranged into **gradient-intercept** form (i.e. $y = mx + c$).

  $y = 2x + 1$ — gradient $= 2$, y-intercept $= 1$

ii $y - 2x = -4$

Rearranging into gradient-intercept form, we have:

$y = 2x - 4$ gradient $= 2$

y-intercept $= -4$

iii $-4y + 2x = 4$

Rearranging into gradient-intercept form, we have:

$y = \frac{1}{2}x - 1$ gradient $= \frac{1}{2}$

y-intercept $= -1$

iv $\frac{y+3}{4} = -x + 2$

Rearranging into gradient-intercept form, we have:

$y + 3 = -4x + 8$

$y = -4x + 5$ gradient $= -4$

y-intercept $= 5$

Equation of a straight line in the form $ax + by + c = 0$

The general equation of a straight line takes the form $y = mx + c$; however, this is not the only form the equation of a straight line can take.

A different form is to rearrange the equation so that all the terms are on one side of the equation and equal to zero and to write it so that there are no fractions.

Worked examples

a The equation of a straight line is given as $y = \frac{1}{2}x + 3$. Write this in the form $ax + by + c = 0$, where a, b and c are integers.

Rearranging the equation so that all the terms are on one side produces $\frac{1}{2}x - y + 3 = 0$.

However, there is still a fraction in the equation and the question stated that a, b and c are integers.

To eliminate the fraction, both sides of the equation are multiplied by two. This gives: $x - 2y + 6 = 0$

b The equation of a straight line is given as $y = \frac{2}{5}x - \frac{1}{3}$. Write this in the form $ax + by + c = 0$, where a, b and c are integers.

Rearranging the equation so that all the terms are on the same side gives:

$\frac{2}{5}x - y - \frac{1}{3} = 0$

To eliminate the fractions, multiply both sides by 15 to give $6x - 15y - 5 = 0$.

Exercise 21.9

For the following linear equations, calculate both the gradient and y-intercept in each case.

1 **a** $y = 2x + 1$ **b** $y = 3x + 5$ **c** $y = x - 2$
d $y = \frac{1}{2}x + 4$ **e** $y = -3x + 6$ **f** $y = -\frac{2}{3}x + 1$
g $y = -x$ **h** $y = -x - 2$ **i** $y = -(2x - 2)$

2 **a** $y - 3x = 1$ **b** $y + \frac{1}{2}x - 2 = 0$ **c** $y + 3 = -2x$
d $y + 2x + 4 = 0$ **e** $y - \frac{1}{4}x - 6 = 0$ **f** $-3x + y = 2$
g $2 + y = x$ **h** $8x - 6 + y = 0$ **i** $-(3x + 1) + y = 0$

3 **a** $2y = 4x - 6$ **b** $2y = x + 8$ **c** $\frac{1}{2}y = x - 2$
d $\frac{1}{4}y = -2x + 3$ **e** $3y - 6x = 0$ **f** $\frac{1}{3}y + x = 1$
g $6y - 6 = 12x$ **h** $4y - 8 + 2x = 0$ **i** $2y - (4x - 1) = 0$

4 **a** $2x - y = 4$ **b** $x - y + 6 = 0$ **c** $-2y = 6x + 2$
d $12 - 3y = 3x$ **e** $5x - \frac{1}{2}y = 1$ **f** $-\frac{2}{3}y + 1 = 2x$
g $9x - 2 = -y$ **h** $-3x + 7 = -\frac{1}{2}y$ **i** $-(4x - 3) = -2y$

5 **a** $\frac{y + 2}{4} = \frac{1}{2}x$ **b** $\frac{y - 3}{x} = 2$ **c** $\frac{y - x}{8} = 0$
d $\frac{2y - 3x}{2} = 6$ **e** $\frac{3y - 2}{x} = -3$ **f** $\frac{\frac{1}{2}y - 1}{x} = -2$
g $\frac{3x - y}{2} = 6$ **h** $\frac{6 - 2y}{3} = 2$ **i** $\frac{-(x + 2y)}{5x} = 1$

6 **a** $\frac{3x - y}{y} = 2$ **b** $\frac{-x + 2y}{4} = y + 1$ **c** $\frac{y - x}{x + y} = 2$
d $\frac{1}{y} = \frac{1}{x}$ **e** $\frac{-(6x + y)}{2} = y + 1$ **f** $\frac{2x - 3y + 4}{4} = 4$

7 **a** $\frac{y + 1}{x} + \frac{3y - 2}{2x} = -1$ **b** $\frac{x}{y + 1} + \frac{1}{2y + 2} = 3$
c $\frac{-(-y + 3x)}{-(6x - 2y)} = 1$ **d** $\frac{-(x - 2y) - (-x - 2y)}{4 + x - y} = -2$

8 Write each of the following equations in the form $ax + by + c = 0$ where a, b and c are integers.

a $y = \frac{1}{3}x + 1$ **b** $y = \frac{2}{5}x - 2$ **c** $3y = \frac{3}{2}x - \frac{1}{2}$
d $-\frac{1}{2}y = x + \frac{1}{3}$ **e** $\frac{y + 1}{2} = \frac{3}{4}x - 1$ **f** $-\frac{3}{5}y - 4 = x$

9 Write each of the following equations in the form $ax + by + c = 0$ where a, b and c are integers.

a $\frac{y - 1}{x} - \frac{y - 2}{3x} = 1$ **b** $\frac{2y + x}{x - y} = \frac{1}{2}$ **c** $\frac{3}{y} = -\frac{2}{5x}$
d $\frac{-(2x + 3y)}{3x} = \frac{2}{3}$ **e** $\frac{3(x - 2y)}{x} - 4 = 0$ **f** $\frac{2x}{5y} + \frac{x}{-y} = -1$

Parallel lines and their equations

Lines that are parallel, by their very definition, must have the same gradient. Similarly, lines with the same gradient must be parallel. So a straight line with equation $y = -3x + 4$ must be parallel to a line with equation $y = -3x - 2$ as both have a gradient of –3.

Worked examples

a A straight line has equation $y = 2x + 4$. Another straight line has equation $y = -2x + 4$. Explain, giving reasons, whether the two lines are parallel to each other or not.

They are not parallel as one has a gradient of 2, the other has a gradient of –2.

b A straight line has equation $4x - 2y + 1 = 0$.

Another straight line has equation $\frac{2x - 4}{y} = 1$.

Explain, giving reasons, whether the two lines are parallel to each other or not.

Rearranging the equations into gradient-intercept form gives:

$$4x - 2y + 1 = 0 \qquad\qquad \frac{2x-4}{y} = 1$$

$$2y = 4x + 1 \qquad\qquad y = 2x - 4$$

$$y = 2x + \frac{1}{2}$$

With both equations written in gradient-intercept form, it is possible to see that both lines have a gradient of 2 and are therefore parallel.

c A straight line A has equation $y = -3x + 6$. A second line B is parallel to line A and passes through the point with coordinates (–4, 10). Calculate the equation of line B.

As line B is a straight line it must take the form $y = mx + c$.

As it is parallel to line A, its gradient must be –3.
Because line B passes through the point (–4, 10), these values can be substituted into the general equation of the straight line to give:

$$10 = -3 \times (-4) + c$$

Rearranging to find c gives: $c = -2$
The equation of line B is therefore $y = -3x - 2$.

Exercise 21.10

1 A straight line has equation $y = x + \frac{4}{3}$. Write down the equation of another straight line parallel to it.

2 A straight line has equation $y = -x + 6$. Which of the following lines is/are parallel to it?

a $y = -x - 2$ **b** $y = 8 - x$
c $y = x - 6$ **d** $y = -x$

3 A straight line has equation $3y - 3x = 4$. Write down the equation of another straight line parallel to it.

4 A straight line has equation $y = -x + 6$. Which of the following lines is/are parallel to it?

a $2(y + x) = -5$ **b** $-3x - 3y + 7 = 0$
c $2y = -x + 12$ **d** $y + x = \frac{1}{10}$

Exercise 21.10 (cont)

5 Find the equation of the line parallel to $y = 4x - 1$ that passes through (0, 0).

6 Find the equations of lines parallel to $y = -3x + 1$ that pass through each of the following points:

a (0, 4) b (–2, 4) c $\left(-\frac{5}{2}, 4\right)$

7 Find the equations of lines parallel to $x - 2y = 6$ that pass through each of the following points:

a (–4, 1) b $\left(\frac{1}{2}, 0\right)$

Drawing straight-line graphs

To draw a straight-line graph only two points need to be known. Once these have been plotted, the line can be drawn between them and extended if necessary at both ends.

Worked examples

a Plot the line $y = x + 3$.

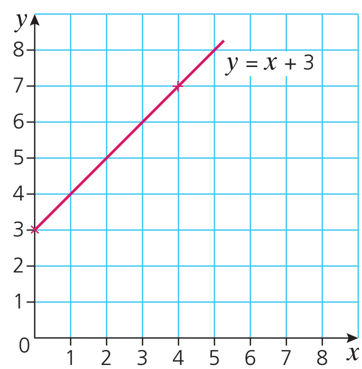
To identify two points, simply choose two values of x. Substitute these into the equation and calculate their corresponding y values.

When $x = 0$, $y = 3$

When $x = 4$, $y = 7$

Therefore two of the points on the line are (0, 3) and (4, 7).

The straight line $y = x + 3$ is plotted below.



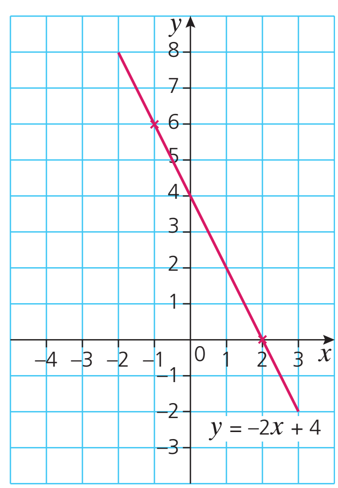
b Plot the line $y = -2x + 4$.

When $x = 2$, $y = 0$

When $x = -1$, $y = 6$

The coordinates of two points on the line are (2, 0) and (–1, 6).

Note that, in questions of this sort, it is often easier to rearrange the equation into gradient-intercept form first.



Exercise 21.11

1 Plot the following straight lines:

a $y = 2x + 3$ b $y = x - 4$ c $y = 3x - 2$
d $y = -2x$ e $y = -x - 1$ f $-y = x + 1$
g $-y = 3x - 3$ h $2y = 4x - 2$ i $y - 4 = 3x$

2 Plot the following straight lines:

a $-2x + y = 4$ b $-4x + 2y = 12$ c $3y = 6x - 3$
d $2x = x + 1$ e $3y - 6x = 9$ f $2y + x = 8$
g $x + y + 2 = 0$ h $3x + 2y - 4 = 0$ i $4 = 4y - 2x$

3 Plot the following straight lines:

a $\frac{x + y}{2} = 1$ b $x + \frac{y}{2} = 1$ c $\frac{x}{3} + \frac{y}{2} = 1$
d $y + \frac{x}{2} = 3$ e $\frac{y}{5} + \frac{x}{3} = 0$ f $\frac{-(2x + y)}{4} = 1$
g $\frac{y - (x - y)}{3x} = -1$ h $\frac{y}{2x + 3} - \frac{1}{2} = 0$ i $-2(x + y) + 4 = -y$

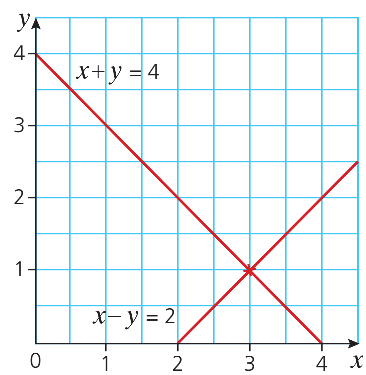
Graphical solution of simultaneous equations

When solving two equations simultaneously, the aim is to find a solution which works for both equations. In Chapter 13 it was shown how to arrive at the solution algebraically. It is, however, possible to arrive at the same solution graphically.

Worked example

a By plotting both of the following equations on the same axes, find a common solution.

$x + y = 4$ (1)
$x - y = 2$ (2)



When both lines are plotted, the point at which they cross gives the common solution as it is the only point which lies on both lines.

Therefore the common solution is the point (3, 1).

b Check the result obtained above by solving the equations algebraically.

$x + y = 4$ (1)

$x - y = 2$ (2)

Adding equations (1) + (2) $\rightarrow$ $2x = 6$

$x = 3$

Substituting $x = 3$ into equation (1) we have:

$3 + y = 4$

$y = 1$

Therefore the common solution occurs at (3, 1) so $x = 3$, $y = 1$.

Exercise 21.12

Solve the simultaneous equations below:

i by graphical means,

ii by algebraic means.

1 **a** $x + y = 5$, $x - y = 1$ **b** $x + y = 7$, $x - y = 3$ **c** $2x + y = 5$, $x - y = 1$

d $2x + 2y = 6$, $2x - y = 3$ **e** $x + 3y = -1$, $x - 2y = -6$ **f** $x - y = 6$, $x + y = 2$

2 **a** $3x - 2y = 13$, $2x + y = 4$ **b** $4x - 5y = 1$, $2x + y = -3$ **c** $x + 5 = y$, $2x + 3y - 5 = 0$

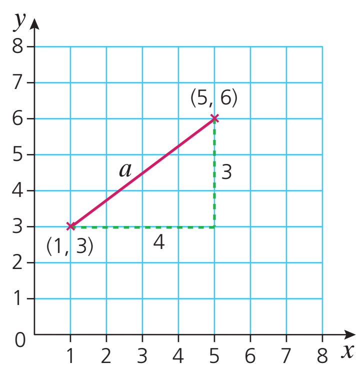
d $x = y$, $x + y + 6 = 0$ **e** $2x + y = 4$, $4x + 2y = 8$ **f** $y - 3x = 1$, $y = 3x - 3$

Calculating the length of a line segment

A line segment is formed when two points are joined by a straight line. To calculate the **distance between two points**, and therefore the length of the line segment, their coordinates need to be given. Once these are known, **Pythagoras' theorem** can be used to calculate the distance.

Worked example

The coordinates of two points are (1, 3) and (5, 6). Draw a pair of axes, plot the given points and calculate the distance between them.



By dropping a vertical line from the point (5, 6) and drawing a horizontal line from (1, 3), a right-angled triangle is formed. The length of the hypotenuse of the triangle is the length we wish to find.

Using Pythagoras' theorem, we have:

$$a^2 = 3^2 + 4^2 = 25$$

$$a = \sqrt{25} = 5$$

The length of the line segment is 5 units.

To find the distance between two points directly from their coordinates, use the following formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Worked example

Without plotting the points, calculate the distance between the points (1, 3) and (5, 6).

$$d = \sqrt{(1-5)^2 + (3-6)^2}$$
$$= \sqrt{(-4)^2 + (-3)^2}$$
$$= \sqrt{25} = 5$$

The distance between the two points is 5 units.

The midpoint of a line segment

To find the **midpoint** of a line segment, use the coordinates of its end points. To find the *x*-coordinate of the midpoint, find the mean of the *x*-coordinates of the end points. Similarly, to find the *y*-coordinate of the midpoint, find the mean of the *y*-coordinates of the end points.

Worked examples

a Find the coordinates of the midpoint of the line segment AB where A is (1, 3) and B is (5, 6).

The *x*-coordinate of the midpoint will be $\frac{1+5}{2} = 3$

The *y*-coordinate of the midpoint will be $\frac{3+6}{2} = 4.5$

So the coordinates of the midpoint are (3, 4.5).

b Find the coordinates of the midpoint of a line segment PQ where P is (–2, –5) and Q is (4, 7).

The *x*-coordinate of the midpoint will be $\frac{-2+4}{2} = 1$

The *y*-coordinate of the midpoint will be $\frac{-5+7}{2} = 1$

So the coordinates of the midpoint are (1, 1).

Exercise 21.13

1 **i** Plot each of the following pairs of points.
ii Calculate the distance between each pair of points.
iii Find the coordinates of the midpoint of the line segment joining the two points.

a (5, 6) (1, 2)
b (6, 4) (3, 1)
c (1, 4) (5, 8)
d (0, 0) (4, 8)
e (2, 1) (4, 7)
f (0, 7) (–3, 1)
g (–3, –3) (–1, 5)
h (4, 2) (–4, –2)
i (–3, 5) (4, 5)
j (2, 0) (2, 6)
k (–4, 3) (4, 5)
l (3, 6) (–3, –3)

Exercise 21.13 (cont)

2 Without plotting the points:
  i calculate the distance between each of the following pairs of points
  ii find the coordinates of the midpoint of the line segment joining the two points.

a	(1, 4) (4, 1)	**b**	(3, 6) (7, 2)
c	(2, 6) (6, –2)	**d**	(1, 2) (9, –2)
e	(0, 3) (–3, 6)	**f**	(–3, –5) (–5, –1)
g	(–2, 6) (2, 0)	**h**	(2, –3) (8, 1)
i	(6, 1) (–6, 4)	**j**	(–2, 2) (4, –4)
k	(–5, –3) (6, –3)	**l**	(3, 6) (5, –2)

The equation of a line through two points

The equation of a straight line can be deduced once the coordinates of two points on the line are known.

Worked example

Calculate the equation of the straight line passing through the points (–3, 3) and (5, 5).
The equation of any straight line can be written in the general form $y = mx + c$.
Here we have:

$$\text{gradient} = \frac{5-3}{5-(-3)} = \frac{2}{8}$$

$$\text{gradient} = \frac{1}{4}$$

The equation of the line now takes the form $y = \frac{1}{4}x + c$.

Since the line passes through the two given points, their coordinates must satisfy the equation. So to calculate the value of 'c' the x and y coordinates of one of the points are substituted into the equation. Substituting (5, 5) into the equation gives:

$$5 = \frac{1}{4} \times 5 + c$$

$$5 = \frac{5}{4} + c$$

Therefore $c = 5 - 1\frac{1}{4} = 3\frac{3}{4}$

The equation of the straight line passing through (–3, 3) and (5, 5) is:

$$y = \frac{1}{4}x + 3\frac{3}{4}$$

Exercise 21.14

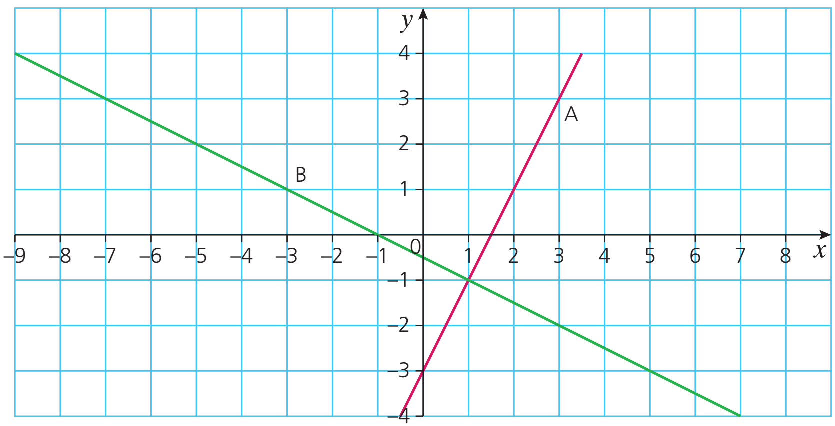
Find the equation of the straight line which passes through each of the following pairs of points:

1	**a** (1, 1) (4, 7)		**b** (1, 4) (3, 10)		**c** (1, 5) (2, 7)	
	d (0, –4) (3, –1)		**e** (1, 6) (2, 10)		**f** (0, 4) (1, 3)	
	g (3, –4) (10, –18)		**h** (0, –1) (1, –4)		**i** (0, 0) (10, 5)	

2 **a** (–5, 3) (2, 4) **b** (–3, –2) (4, 4) **c** (–7, –3) (–1, 6)
d (2, 5) (1, –4) **e** (–3, 4) (5, 0) **f** (6, 4) (–7, 7)
g (–5, 2) (6, 2) **h** (1, –3) (–2, 6) **i** (6, –4) (6, 6)

Perpendicular lines

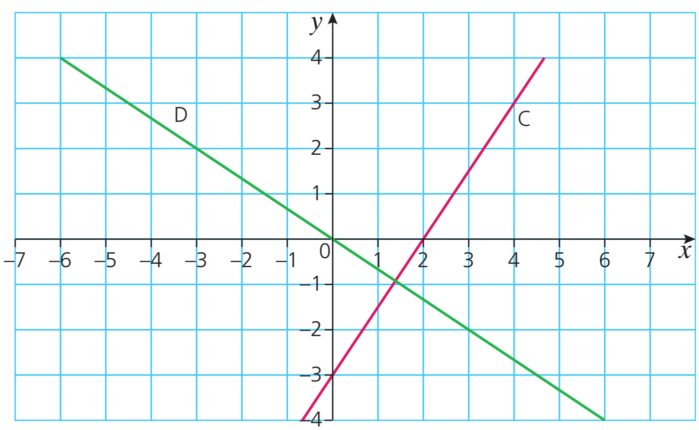
The two lines shown below are perpendicular to each other.



Line A has a gradient of 2.

Line B has a gradient of $-\frac{1}{2}$.

The diagram below also shows two lines perpendicular to each other.



Line C has a gradient of $\frac{3}{2}$.

Line D has a gradient of $-\frac{2}{3}$.

Notice that in both cases, the product of the two gradients is equal to –1.

In the first example $2 \times (-\frac{1}{2}) = -1$.

In the second example $\frac{3}{2} \times (-\frac{2}{3}) = -1$.

This is in fact the case for the gradients of any two perpendicular lines.

If two lines L_1 and L_2 are perpendicular to each other, the product of their gradients m_1 and m_2 is –1.

i.e. $m_1m_2 = -1$

Therefore the gradient of one line is the negative reciprocal of the other line.

i.e. $m_1 = -\frac{1}{m_2}$

Worked examples

a **i** Calculate the gradient of the line joining the two points (3, 6) and (1, –6).

Gradient $= \frac{6-(-6)}{3-1} = \frac{12}{2} = 6$

ii Calculate the gradient of a line perpendicular to the one in part **i** above.

$m_1 = -\frac{1}{m_2}$, therefore the gradient of the perpendicular line is $-\frac{1}{6}$.

iii The perpendicular line also passes through the point (–1, 6). Calculate the equation of the perpendicular line.

The equation of the perpendicular line will take the form $y = mx + c$.

As its gradient is $-\frac{1}{6}$ and it passes through the point (–1, 6), this can be substituted into the equation to give:

$6 = -\frac{1}{6} \times (-1) + c$

Therefore $c = \frac{35}{6}$.

The equation of the perpendicular line is $y = -\frac{1}{6}x + \frac{35}{6}$.

b **i** Show that the point (–4, –1) lies on the line $y = -\frac{1}{4}x - 2$.

If the point (–4, –1) lies on the line, its values of x and y will satisfy the equation. Substituting the values of x and y into the equation gives:

$$-1 = -\frac{1}{4} \times (-4) - 2$$

$$-1 = -1$$

Therefore the point lies on the line.

ii Deduce the gradient of a line perpendicular to the one given in part **i** above.

$$m_1 = -\frac{1}{m_2} \quad \text{therefore} \quad m_1 = -\frac{1}{-\frac{1}{4}} = 4$$

Therefore the gradient of the perpendicular line is 4.

iii The perpendicular line also passes through the point (–4, –1). Calculate its equation.

The equation of the perpendicular line takes the general form $y = mx + c$.

Substituting in the values of x, y and m gives:

$$-1 = 4 \times (-4) + c$$

Therefore $c = 15$.

The equation of the perpendicular line is $y = 4x + 15$.

Exercise 21.15

1 Calculate:

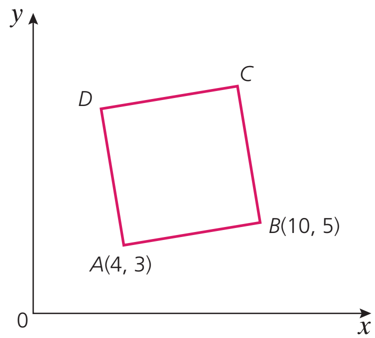
  i the gradient of the line joining the following pairs of points
  ii the gradient of a line perpendicular to this line
  iii the equation of the perpendicular line if it passes through the second point each time.

a (1, 4) (4, 1) **b** (3, 6) (7, 2) **c** (2, 6) (6, –2)
d (1, 2) (9, –2) **e** (0, 3) (–3, 6) **f** (–3, –5) (–5, –1)
g (–2, 6) (2, 0) **h** (2, –3) (8, 1) **i** (6, 1) (–6, 4)
j (–2, 2) (4, –4) **k** (–5, –3) (6, –3) **l** (3, 6) (5, –2)

2 Calculate the gradient of lines perpendicular to each of the following straight lines.

a $2y = 3x + 2$ **b** $4x - 5y = 10$ **c** $\frac{1}{2}y + 4x - 3 = 0$ **d** $\frac{2}{3}y - \frac{1}{2}x = 1$

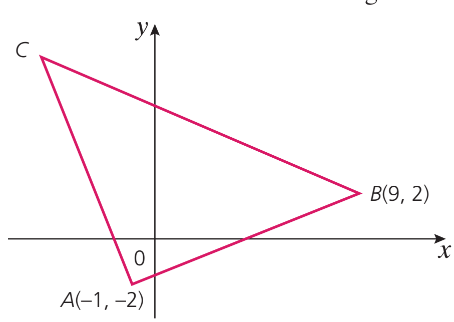
3 The diagram below shows a square $ABCD$. The coordinates of A and B are given.



Calculate:

a the gradient of the line AB
b the equation of the line passing through A and B
c the gradient of the line AD
d the equation of the line passing through A and D
e the equation of the line passing through B and C
f the coordinates of C
g the coordinates of D
h the equation of the line passing through C and D
i the length of the sides of the square to 1 d.p.
j the coordinates of the midpoint of the line segment AC.

4 The diagram below shows a right-angled isosceles triangle ABC, where $AB = AC$.
The coordinates of A and B are given.



Exercise 21.15 (cont)

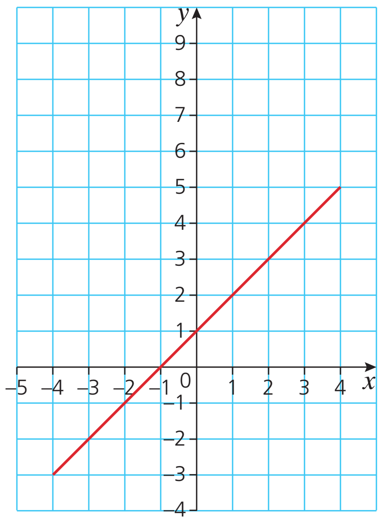
a Calculate:
- **i** the equation of the line passing through the points A and B
- **ii** the equation of the line passing through A and C
- **iii** the length of the line segment BC to 1 d.p.
- **iv** the coordinates of the midpoints of all three sides of the triangle.

b A perpendicular bisector of the line AB is a line which is at right angles to AB and passes through its midpoint.
Calculate the equation of the perpendicular bisector of AB.

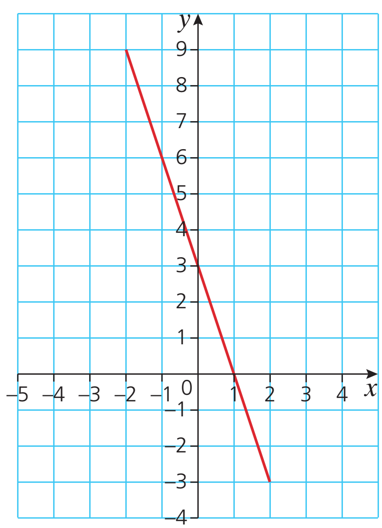
Student assessment 1

1 For each of the following lines, select two points on the line and then calculate its gradient.

a

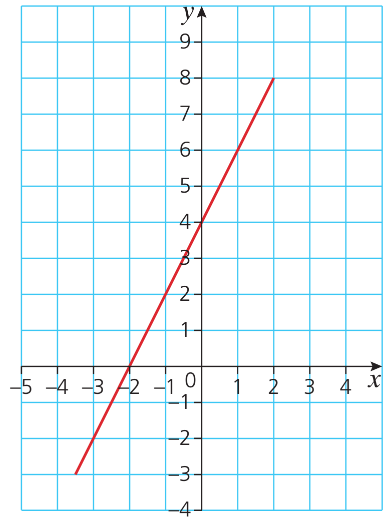


b

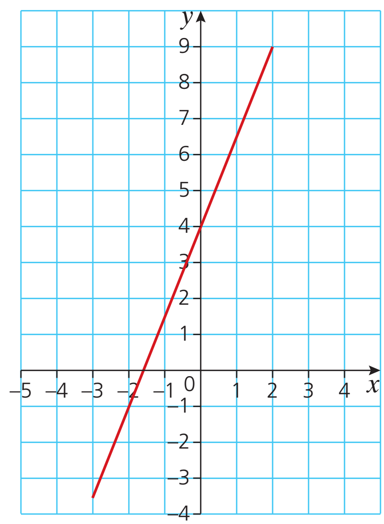


2 Find the equation of the straight line for each of the following:

a



b



3 Write down the equation of the line parallel to the line $y = -\frac{2}{3}x + 4$ which passes through the point (6, 2).

4 Plot the following graphs on the same pair of axes, labelling each clearly.

a $x = -2$ **b** $y = 3$

c $y = 2x$ **d** $y = -\frac{x}{2}$

5 Calculate the gradient and y-intercept for each of the following linear equations:

a $y = -3x + 4$

b $\frac{1}{3}y - x = 2$

c $2x + 4y - 6 = 0$

6 Solve the following pairs of simultaneous equations graphically:

a $x + y = 4$
$x - y = 0$

b $3x + y = 2$
$x - y = 2$

c $y + 4x + 4 = 0$
$x + y = 2$

d $x - y = -2$
$3x + 2y + 6 = 0$

7 The coordinates of the end points of two line segments are given below.
For each line segment calculate:

i the length
ii the midpoint.

a (–6, –1) (6, 4)

b (1, 2) (7, 10)

8 Find the equation of the straight line which passes through each of the following pairs of points:

i in the form $y = mx + c$
ii in the form $ax + by + c = 0$

a (1, –1) (4, 8)

b (0, 7) (3, 1)

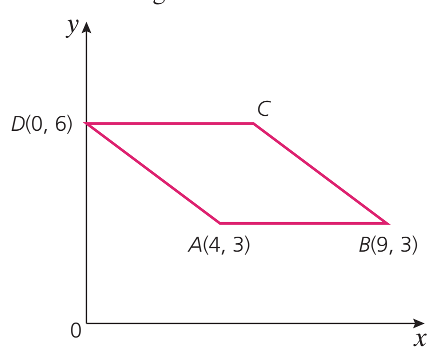
9 A line L_1 passes through the points (–2, 5) and (5, 3).

a Write down the equation of the line L_1.

Another line L_2 is perpendicular to L_1 and also passes through the point (–2, 5).

b Write down the equation of the line L_2.

10 The diagram below shows a **rhombus** $ABCD$. The coordinates of A, B and D are given.



a Calculate:

i the coordinate of the point C
ii the equation of the line passing through A and C
iii the equation of the line passing through B and D.

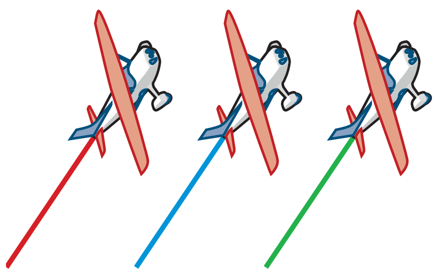
b Are the diagonals of the rhombus perpendicular to each other? Justify your answer.

3 Mathematical investigations and ICT 3

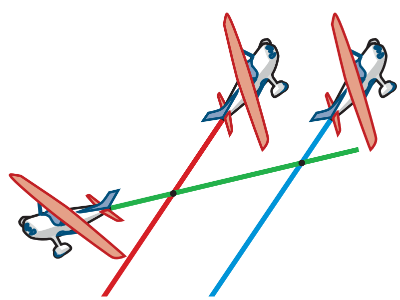
Plane trails

In an aircraft show, planes are made to fly with a coloured smoke trail. Depending on the formation of the planes, the trails can intersect in different ways.

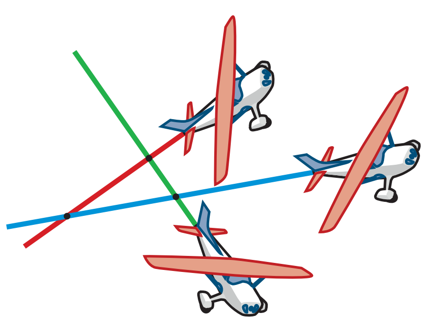
In the diagram below, the three smoke trails do not cross, as they are parallel.



In the following diagram, there are two crossing points.



By flying differently, the three planes can produce trails that cross at three points.

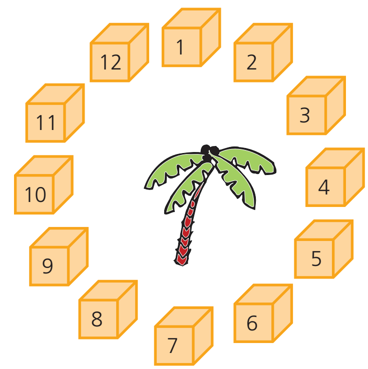


1 Investigate the connection between the maximum number of crossing points and the number of planes.

2 Record the results of your investigation in an ordered table.

3 Write an algebraic rule linking the number of planes (p) and the maximum number of crossing points (n).

Hidden treasure



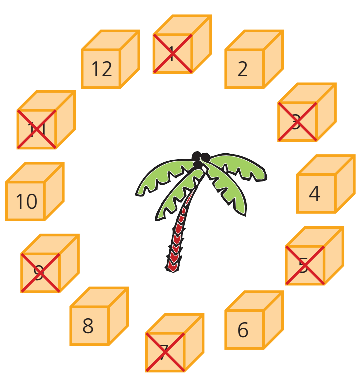
A television show sets up a puzzle for its contestants to try and solve. Some buried treasure is hidden on a 'treasure island'. The treasure is hidden in one of the 12 treasure chests shown (right). Each contestant stands by one of the treasure chests.

The treasure is hidden according to the following rule:

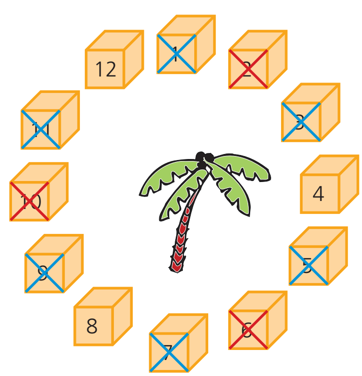
It is not hidden in chest 1.
Chest 2 is left empty for the time being.
It is not hidden in chest 3.
Chest 4 is left empty for the time being.
It is not hidden in chest 5.

The pattern of crossing out the first chest and then alternate chests is continued until only one chest is left. This will involve going around the circle several times, continuing the pattern. The treasure is hidden in the last chest left.

The diagrams below show how the last chest is chosen:



After the first round, chests 1, 3, 5, 7, 9 and 11 have been discounted.



After the second round, chests 2, 6 and 10 have also been discounted.

After the third round, chests 4 and 12 have also been discounted. This leaves only chest 8.

The treasure is therefore hidden in chest 8.

Unfortunately for participants, the number of contestants changes each time.

1 Investigate which treasure chest you would choose if there are:
- **a** 4 contestants
- **b** 5 contestants
- **c** 8 contestants
- **d** 9 contestants
- **e** 15 contestants.

2 Investigate the winning treasure chest for other numbers of contestants and enter your results in an ordered table.

3 State any patterns you notice in your table of results.

4 Use your patterns to predict the winning chest for 31, 32 and 33 contestants.

5 Write a rule linking the winning chest x and the number of contestants n.

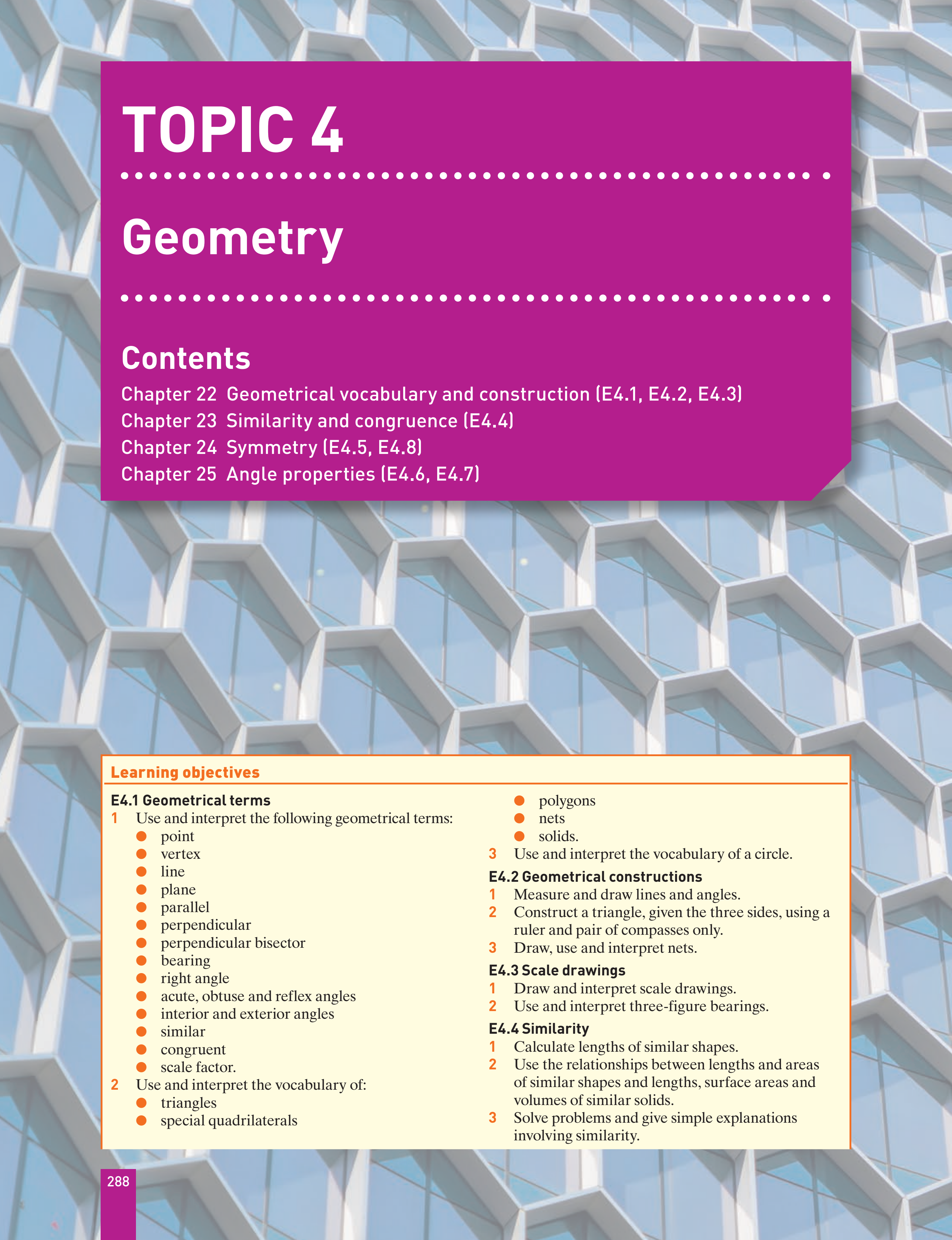
ICT activity

For each question, use a graphing package to plot the inequalities on the same pair of axes. Leave unshaded the region which satisfies all of them simultaneously.

1 $y \leqslant x$ $\quad y > 0$ $\quad x \leqslant 3$

2 $x + y > 3$ $\quad y \leqslant 4$ $\quad y - x > 2$

3 $2y + x \leqslant 5$ $\quad y - 3x - 6 < 0$ $\quad 2y - x > 3$



TOPIC 4

Geometry

Contents

Learning objectives

E4.1 Geometrical terms

1 Use and interpret the following geometrical terms:
- point
- vertex
- line
- plane
- parallel
- perpendicular
- perpendicular bisector
- bearing
- right angle
- acute, obtuse and reflex angles
- interior and exterior angles
- similar
- congruent
- scale factor.

2 Use and interpret the vocabulary of:
- triangles
- special quadrilaterals
- polygons
- nets
- solids.

3 Use and interpret the vocabulary of a circle.

E4.2 Geometrical constructions

1 Measure and draw lines and angles.

2 Construct a triangle, given the three sides, using a ruler and pair of compasses only.

3 Draw, use and interpret nets.

E4.3 Scale drawings

1 Draw and interpret scale drawings.

2 Use and interpret three-figure bearings.

E4.4 Similarity

1 Calculate lengths of similar shapes.

2 Use the relationships between lengths and areas of similar shapes and lengths, surface areas and volumes of similar solids.

3 Solve problems and give simple explanations involving similarity.

E4.5 Symmetry

1 Recognise line symmetry and order of rotational symmetry in two dimensions.
2 Recognise symmetry properties of prisms, cylinders, pyramids and cones.

E4.6 Angles

1 Calculate unknown angles and give simple explanations using the following geometrical properties:
   - sum of angles at a point = 360°
   - sum of angles at a point on a straight line = 180°
   - vertically opposite angles are equal
   - angle sum of a triangle = 180° and angle sum of a quadrilateral = 360°.
2 Calculate unknown angles and give geometric explanations for angles formed within parallel lines:
   - corresponding angles are equal
   - alternate angles are equal
   - co-interior angles sum to 180° (supplementary).
3 Know and use angle properties of regular and irregular polygons.

E4.7 Circle theorems I

Calculate unknown angles and give explanations using the following geometrical properties of circles:

- angle in a semicircle = 90°
- angle between tangent and radius = 90°
- angle at the centre is twice the angle at the circumference
- angles in the same segment are equal
- opposite angles of a cyclic quadrilateral sum to 180° (supplementary)
- alternate segment theorem.

E4.8 Circle theorems II

Use the following symmetry properties of circles:

- equal chords are equidistant from the centre
- the perpendicular bisector of a chord passes through the centre
- tangents from an external point are equal in length.

The Greeks

Many of the great Greek mathematicians came from the Greek Islands, from cities such as Ephesus or Miletus (which are in present day Turkey) or from Alexandria in Egypt. This section briefly mentions some of the Greek mathematicians of 'The Golden Age'. You may wish to find out more about them.

Archimedes (287–212BCE)

Thales of Alexandria invented the 365-day calendar and predicted the dates of eclipses of the Sun and the Moon.

Pythagoras of Samos founded a school of mathematicians and worked with geometry. His successor as leader was Theano, the first woman to hold a major role in mathematics.

Eudoxus of Asia Minor (Turkey) worked with irrational numbers like pi and discovered the formula for the volume of a cone.

Euclid of Alexandria formed what would now be called a university department. His book became the set text in schools and universities for 2000 years.

Apollonius of Perga (Turkey) worked on, and gave names to, the parabola, the hyperbola and the ellipse.

Archimedes is accepted today as the greatest mathematician of all time. However, he was so far ahead of his time that his influence on his contemporaries was limited by their lack of understanding.