22 Geometrical vocabulary and construction

Angles and lines

Different types of angle have different names:

acute angles lie between 0° and 90°

right angles are exactly 90°

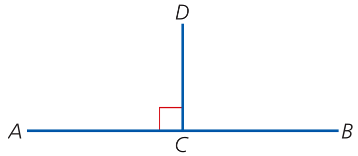
obtuse angles lie between 90° and 180°

reflex angles lie between 180° and 360°

To find the shortest distance between two points, you measure the length of the **straight line** which joins them.

Two lines which meet at right angles are **perpendicular** to each other.

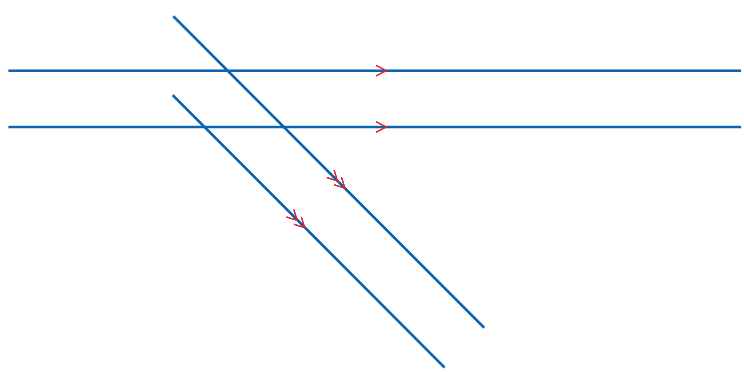
So in the diagram below, CD is **perpendicular** to AB and AB is perpendicular to CD.



If the lines AD and BD are drawn to form a triangle, the line CD can be called the **height** or **altitude** of the triangle ABD.

Parallel lines are straight lines which can be continued to infinity in either direction without meeting.

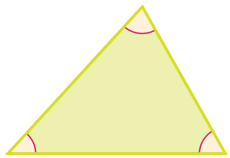
Railway lines are an example of parallel lines. Parallel lines are marked with arrows as shown:



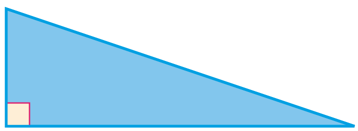
Triangles

Triangles can be described in terms of their sides or their angles, or both.

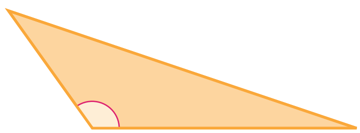
An **acute-angled** triangle has all its angles less than 90°.



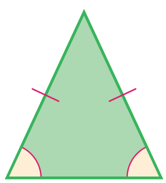
A **right-angled** triangle has an angle of 90°.



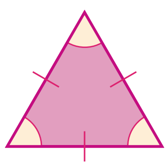
An **obtuse-angled** triangle has one angle greater than 90°.



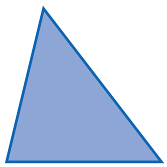
An **isosceles** triangle has two sides of equal length, and the angles opposite the equal sides are equal.



An **equilateral** triangle has three sides of equal length and three equal angles.



A **scalene** triangle has three sides of different lengths and all three angles are different.



Congruent triangles

Congruent triangles are **identical**. They have corresponding sides of the same length and **corresponding angles** which are equal.

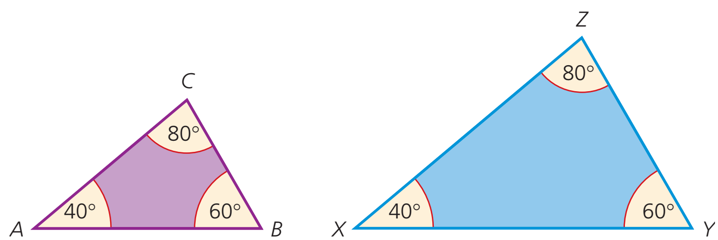
Note

All diagrams are not drawn to scale.

Similar triangles

If the angles of two triangles are the same, then their corresponding sides will also be in proportion to each other. When this is the case, the triangles are said to be **similar**.

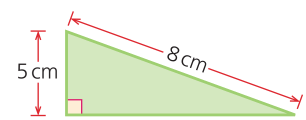
In the diagram below, triangle ABC is similar to triangle XYZ. Similar shapes are covered in more detail in Chapter 23.



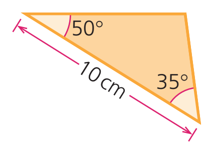
Exercise 22.1

1 In the diagrams below, identify pairs of congruent triangles.

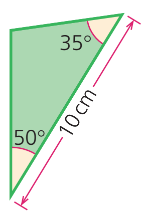
a



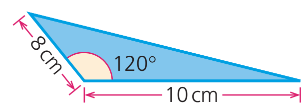
b



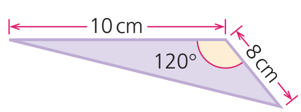
c



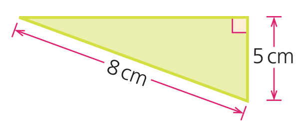
d



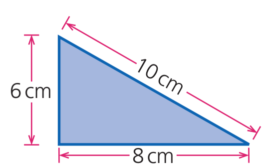
e



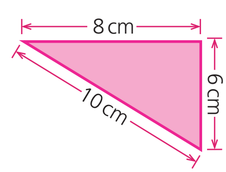
f



g



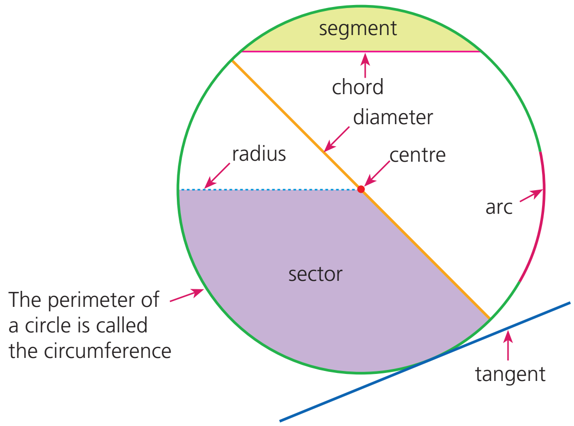
h



Circles

Note

Half a circle is known as a semicircle.



The diagram shows two arcs. The major arc is green and the minor arc is pink.

Quadrilaterals

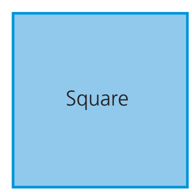
A **quadrilateral** is a plane (two-dimensional) shape consisting of four angles and four sides. There are several types of quadrilateral. The main ones and their properties are described below.

Two pairs of parallel sides.

All sides are equal.

All angles are equal.

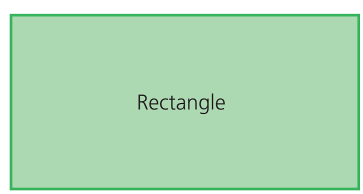
Diagonals intersect at right angles.



Two pairs of parallel sides.

Opposite sides are equal.

All angles are equal.

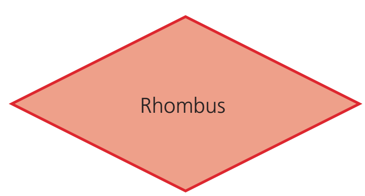


Two pairs of parallel sides.

All sides are equal.

Opposite angles are equal.

Diagonals intersect at right angles.



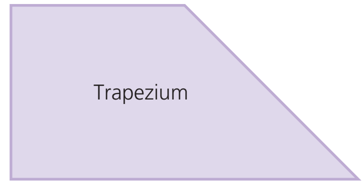
Two pairs of parallel sides.

Opposite sides are equal.

Opposite angles are equal.

One pair of parallel sides.

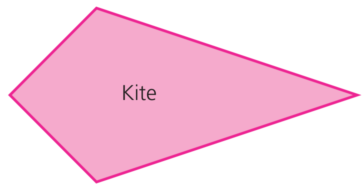
An **isosceles trapezium** has one pair of parallel sides and the other pair of sides are equal in length.



Two pairs of equal sides.

One pair of equal angles.

Diagonals intersect at right angles.



Exercise 22.2

1 Copy and complete the following table. The first line has been started for you.

	Rectangle	Square	Parallelogram	Kite	Rhombus	Equilateral triangle
Opposite sides equal in length	Yes	Yes				
All sides equal in length						
All angles right angles						
Both pairs of opposite sides parallel						
Diagonals equal in length						
Diagonals intersect at right angles						
All angles equal						

Polygons

Any two-dimensional closed figure made up of straight lines is called a **polygon**.

If the sides are the same length and the interior angles are equal, the figure is called a **regular polygon**. If the sides and angles are not all of equal length and size, it is known as an **irregular polygon**.

> **Note**
>
> Heptagon, nonagon and dodecagon are not part of the syllabus, but are included here for interest.

The names of the common polygons are:

3 sides	**tri**angle
4 sides	**quad**rilateral
5 sides	**penta**gon
6 sides	**hexa**gon
7 sides	**hepta**gon
8 sides	**octa**gon
9 sides	**nona**gon
10 sides	**deca**gon
12 sides	**dodeca**gon

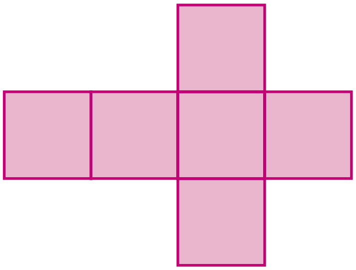
Two polygons are said to be **similar** if

a their angles are the same

b corresponding sides are in proportion.

Nets

The diagram below is the **net** of a cube. It shows the faces of the cube opened out into a two-dimensional plan. The net of a three-dimensional shape can be folded up to make that shape.



Exercise 22.3

Draw the following on squared paper:

1 Two other possible nets of a cube.

2 The net of a cuboid (rectangular prism).

3 The net of a triangular prism.

4 The net of a cylinder.

5 The net of a square-based **pyramid**.

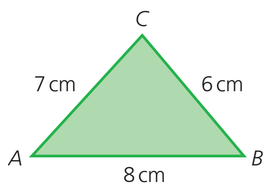
6 The net of a tetrahedron.

Constructing triangles

Triangles can be drawn accurately by using a ruler and a pair of compasses. This is called **constructing** a triangle.

Worked example

The sketch shows the triangle ABC.

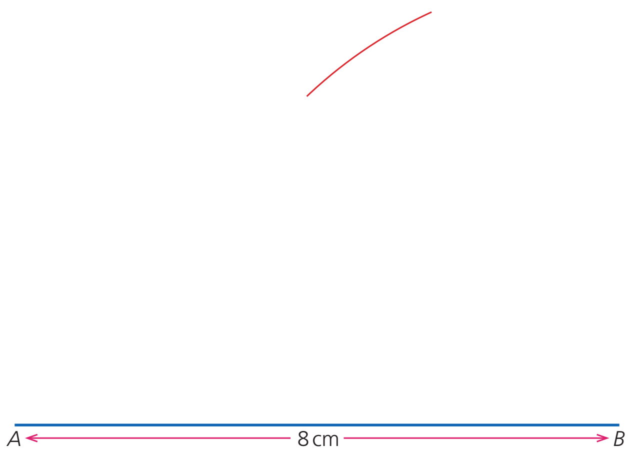


Construct the triangle ABC given that:

$AB = 8$ cm, $BC = 6$ cm and $AC = 7$ cm

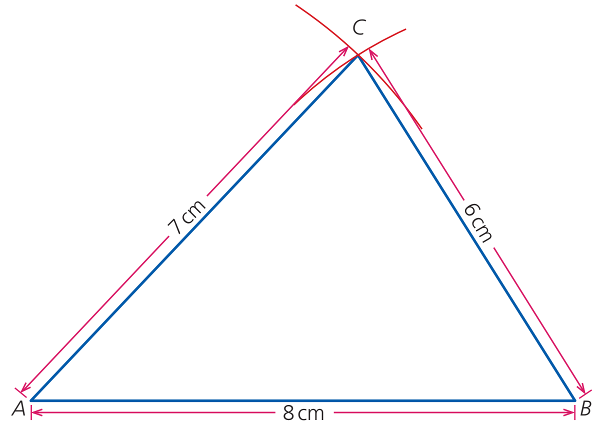
- Draw the line AB using a ruler:

- Open up a pair of compasses to 6 cm. Place the compass point on B and draw an arc:



- Note that every point on the arc is 6 cm away from B.
- Open up the pair of compasses to 7 cm. Place the compass point on A and draw another arc, with centre A and radius 7 cm, ensuring that it intersects with the first arc. Every point on the second arc is 7 cm from A. Where the two arcs intersect is point C, as it is both 6 cm from B and 7 cm from A.

- Join C to A and C to B:



Exercise 22.4

Using only a ruler and a pair of compasses, construct the following triangles:

1 Triangle ABC where $AB = 10$ cm, $AC = 7$ cm and $BC = 9$ cm

2 Triangle LMN where $LM = 4$ cm, $LN = 8$ cm and $MN = 5$ cm

3 Triangle PQR, an equilateral triangle of side length 7 cm

4 **a** Triangle ABC where $AB = 8$ cm, $AC = 4$ cm and $BC = 3$ cm
b Is this triangle possible? Explain your answer.

Scale drawings

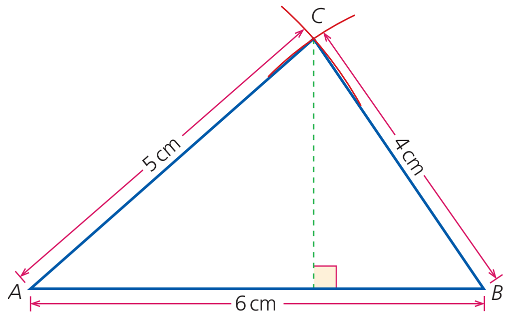
Scale drawings are used when an accurate diagram, drawn in proportion, is needed. Common uses of scale drawings include maps and plans. The use of scale drawings involves understanding how to scale measurements.

Worked examples

a A map is drawn to a scale of 1 : 10 000. If two objects are 1 cm apart on the map, how far apart are they in real life? Give your answer in metres.
A scale of 1 : 10 000 means that 1 cm on the map represents 10 000 cm in real life.
Therefore the distance = 10 000 cm
= 100 m

b A model boat is built to a scale of 1 : 50. If the length of the real boat is 12 m, calculate the length of the model boat in cm.
A scale of 1 : 50 means that 50 cm on the real boat is 1 cm on the model boat.
12 m = 1200 cm
Therefore the length of the model boat = 1200 ÷ 50 cm
= 24 cm

c **i** Construct, to a scale of 1 : 1, a triangle ABC such that $AB = 6$ cm, $AC = 5$ cm and $BC = 4$ cm.



ii Measure the perpendicular length of C from AB.
Perpendicular length is 3.3 cm.

iii Calculate the area of the triangle.

$$\text{Area} = \frac{\text{base length} \times \text{perpendicular height}}{2}$$

$$\text{Area} = \frac{6 \times 3.3}{2}\ \text{cm} = 9.9\ \text{cm}^2$$

Exercise 22.5

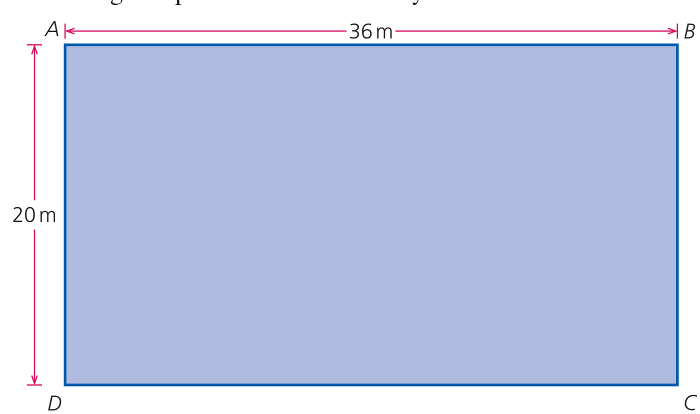
1 In the following questions, both the scale to which a map is drawn and the distance between two objects on the map are given.
Find the real distance between the two objects, giving your answer in metres.

a 1 : 10 000 3 cm **b** 1 : 10 000 2.5 cm
c 1 : 20 000 1.5 cm **d** 1 : 8000 5.2 cm

2 In the following questions, both the scale to which a map is drawn and the true distance between two objects are given.
Find the distance between the two objects on the map, giving your answer in cm.

a 1 : 15 000 1.5 km **b** 1 : 50 000 4 km
c 1 : 10 000 600 m **d** 1 : 25 000 1.7 km

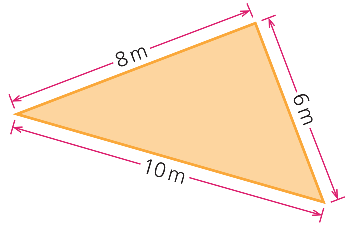
3 A rectangular pool measures 20 m by 36 m as shown below:



a Construct a scale drawing of the pool, using 1 cm for every 4 m.

b A boy swims across the pool from D in a straight line so that he arrives at a point which is 40 m from D and 30 m from C. Work out the distance the boy swam.

4 A triangular enclosure is shown in the diagram below:



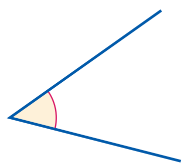
a Using a scale of 1 cm for each metre, construct a scale drawing of the enclosure.

b Calculate the true area of the enclosure.

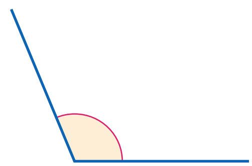
Student assessment 1

1 Are the angles below acute, obtuse, reflex or right angles?

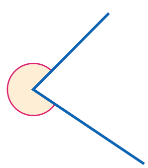
a



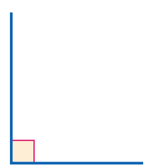
b



c



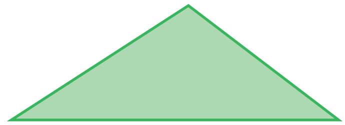
d



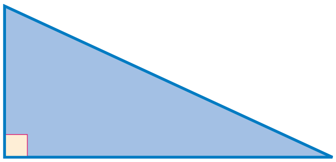
2 Draw and label two pairs of intersecting parallel lines.

3 Identify the types of triangles below in two ways (for example, obtuse-angled scalene triangle):

a



b



4 Draw a circle of radius 3 cm. Mark on it:

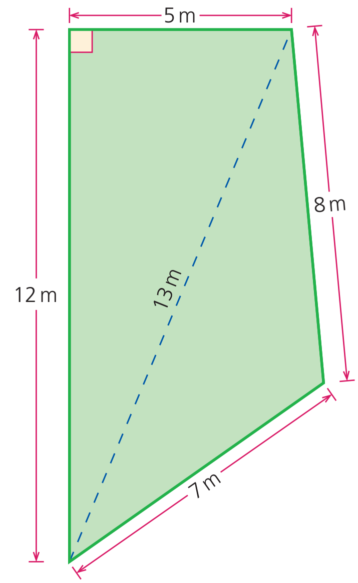
a a diameter b a chord c a sector.

5 Draw a rhombus and write down three of its properties.

6 On squared paper, draw the net of a triangular prism.

7 Construct triangle ABC such that $AB = 8\,\text{cm}$, $AC = 6\,\text{cm}$ and $BC = 12\,\text{cm}$.

8 A plan of a living room is shown below:



a Using a pair of compasses, construct a scale drawing of the room using 1 cm for every metre.

b Using a set square if necessary, calculate the total area of the actual living room.

9 In the following questions, both the scale to which a map is drawn and the true distance between two objects are given. Find the distance between the two objects on the map, giving your answer in cm.

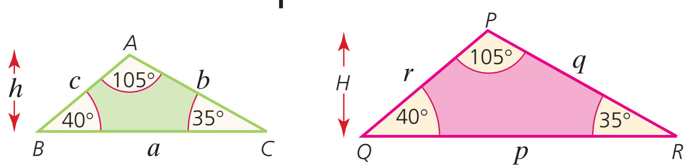
a 1:20000 4.4 km **b** 1:50000 12.2 km

23 Similarity and congruence

Similar shapes

Note

All diagrams are not drawn to scale.



Two polygons are said to be **similar** if a) they are equi-angular and b) corresponding sides are in proportion.

For triangles, being equi-angular implies that corresponding sides are in proportion. The converse is also true.

In the diagrams triangle ABC and triangle PQR are similar.

For similar figures the ratios of the lengths of the sides are the same and represent the **scale factor**, i.e.

$\frac{p}{a} = \frac{q}{b} = \frac{r}{c} = k$ (where k is the scale factor of enlargement)

The heights of similar triangles are proportional also:

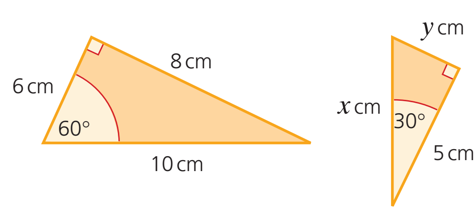
$$\frac{H}{h} = \frac{p}{a} = \frac{q}{b} = \frac{r}{c} = k$$

The ratio of the areas of similar triangles (the **area factor**) is equal to the square of the scale factor.

$$\frac{\text{Area of } PQR}{\text{Area of } ABC} = \frac{\frac{1}{2}H \times p}{\frac{1}{2}h \times a} = \frac{H}{h} \times \frac{p}{a} = k \times k = k^2$$

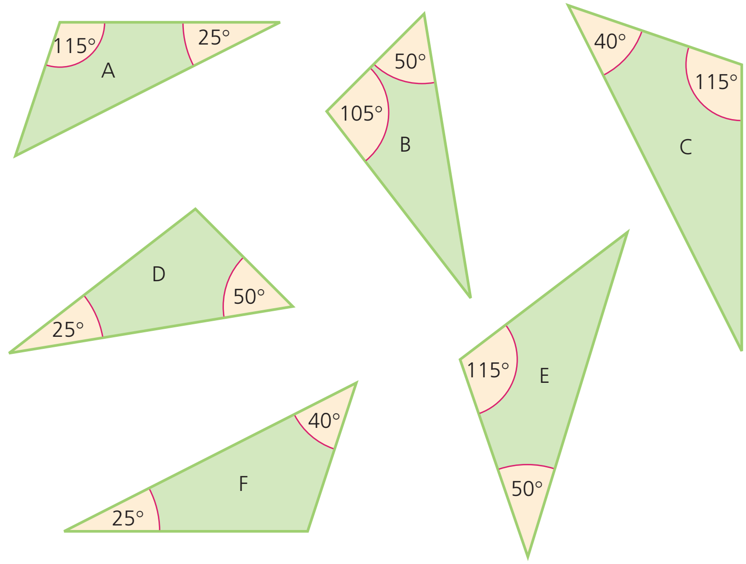
Exercise 23.1

1 a Explain why the two triangles (below) are similar.
  b Calculate the scale factor which reduces the larger triangle to the smaller one.
  c Calculate the value of x and the value of y.

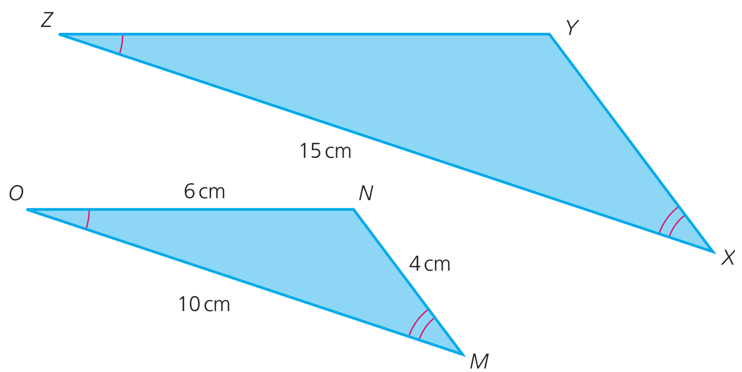


Exercise 23.1 (cont)

2 Which of the triangles below are similar?

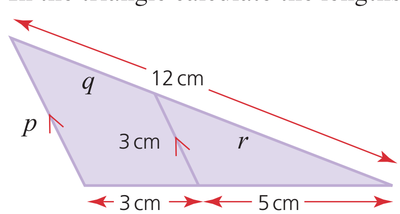


3 The triangles below are similar.

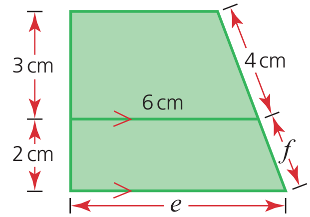


a Calculate the length XY.
b Calculate the length YZ.

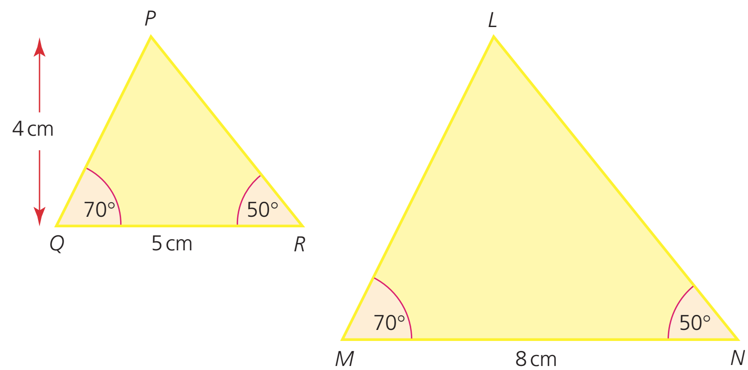
4 In the triangle calculate the lengths of sides p, q and r.



5 In the trapezium calculate the lengths of the sides *e* and *f*.



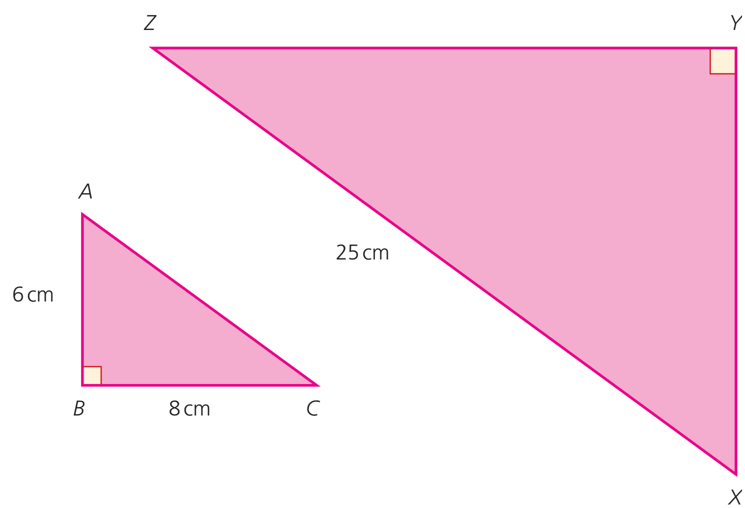
6 The triangles PQR and LMN are similar.



Calculate:

- **a** the area of triangle PQR
- **b** the scale factor of enlargement
- **c** the area of triangle LMN.

7 The triangles ABC and XYZ below are similar.



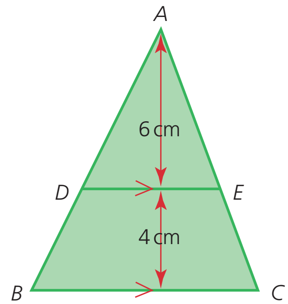
- **a** Using Pythagoras' theorem calculate the length of AC.
- **b** Calculate the scale factor of enlargement.
- **c** Calculate the area of triangle XYZ.

Exercise 23.1 (cont)

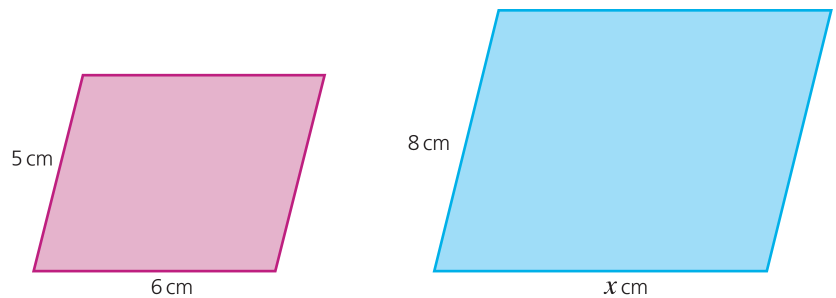
8 The triangle ADE shown has an area of $12\,\text{cm}^2$.

a Calculate the area of triangle ABC.

b Calculate the length BC.

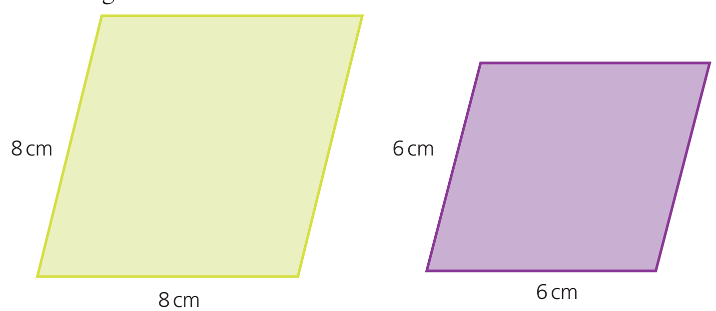


9 The parallelograms below are similar.



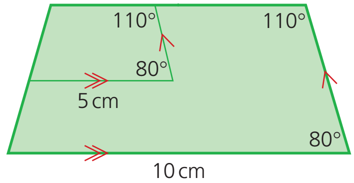
Calculate the length of the side marked x.

10 The diagram below shows two rhombuses.



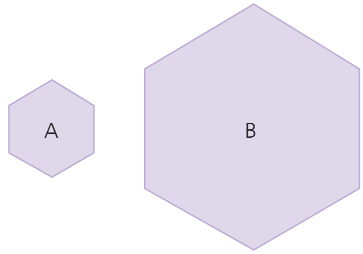
Explain, giving reasons, whether the two rhombuses are similar.

11 The diagram shows a trapezium within a trapezium. Explain, giving reasons, whether the two trapezia are similar.



Exercise 23.2

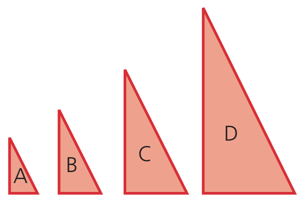
1 In the hexagons below, hexagon B is an enlargement of hexagon A by a scale factor of 2.5.



If the area of A is $8\,\text{cm}^2$, calculate the area of B.

2 P and Q are two regular pentagons. Q is an enlargement of P by a scale factor of 3. If the area of pentagon Q is $90\,\text{cm}^2$, calculate the area of P.

3 The diagram below shows four triangles A, B, C and D. Each is an enlargement of the previous one by a scale factor of 1.5.



a If the area of C is $202.5\,\text{cm}^2$, calculate the area of:
   i triangle D   ii triangle B   iii triangle A.

b If the triangles were to continue in this sequence, which letter triangle would be the first to have an area greater than $15\,000\,\text{cm}^2$?

4 A square is enlarged by increasing the length of its sides by 10%. If the length of its sides was originally 6 cm, calculate the area of the enlarged square.

5 A square of side length 4 cm is enlarged by increasing the length of its sides by 25% and then increasing the lengths again by a further 50%. Calculate the area of the final square.

6 An equilateral triangle has an area of $25\,\text{cm}^2$. If the length of its sides is reduced by 15%, calculate the area of the reduced triangle.

Area and volume of similar shapes

Earlier in the chapter we found the following relationship between the scale factor and the area factor of enlargement:

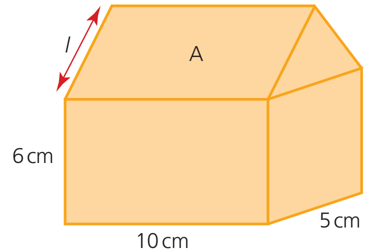
Area factor = (scale factor)^2

A similar relationship can be stated for volumes of similar shapes:

i.e. **Volume factor** = (scale factor)^3

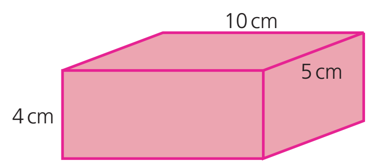
Exercise 23.3

1 The diagram shows a scale model of a garage. Its width is 5 cm, its length 10 cm and the height of its walls 6 cm.



a If the width of the real garage is 4 m, calculate:
  i the length of the real garage
  ii the real height of the garage wall.
b If the apex of the roof of the real garage is 2 m above the top of the walls, use Pythagoras' theorem to find the real slant length l.
c What is the area of the roof section A on the model?

2 A cuboid has dimensions as shown in the diagram.

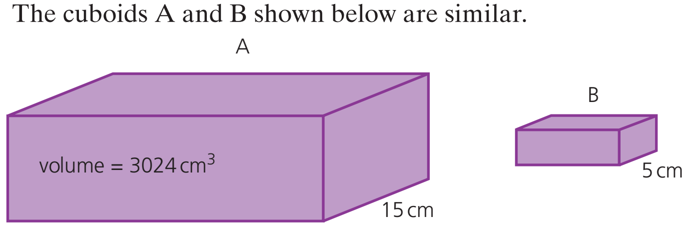


If the cuboid is enlarged by a scale factor of 2.5, calculate:
a the total surface area of the original cuboid
b the total surface area of the enlarged cuboid
c the volume of the original cuboid
d the volume of the enlarged cuboid.

3 A cube has side length 3 cm.
a Calculate its total surface area.
b If the cube is enlarged and has a total surface area of 486 cm^2, calculate the scale factor of enlargement.
c Calculate the volume of the enlarged cube.

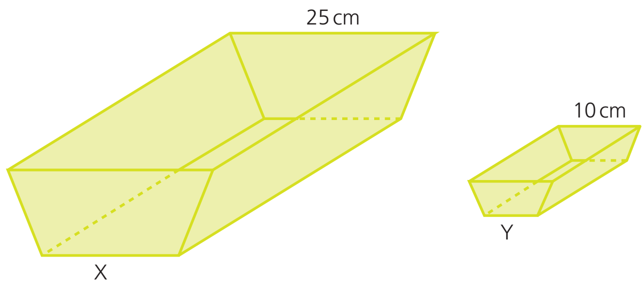
4 Two cubes P and Q are of different sizes. If n is the ratio of their corresponding sides, express in terms of n:
a the ratio of their surface areas
b the ratio of their volumes.

5 The cuboids A and B shown below are similar.



Calculate the volume of cuboid B.

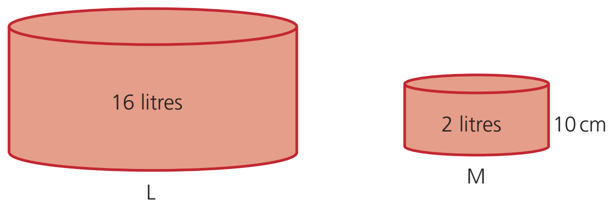
6 Two similar troughs X and Y are shown below.



If the capacity of X is 10 litres, calculate the capacity of Y.

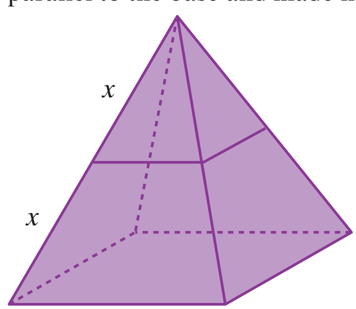
Exercise 23.4

1 The two cylinders L and M shown below are similar.



If the height of cylinder M is 10 cm, calculate the height of cylinder L.

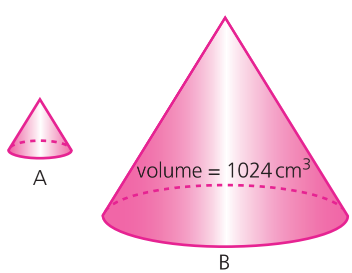
2 A square-based pyramid (below) is cut into two shapes by a cut running parallel to the base and made halfway up.



a Calculate the ratio of the volume of the smaller pyramid to that of the original one.

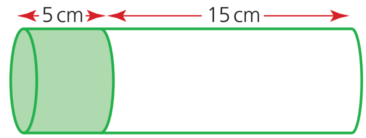
b Calculate the ratio of the volume of the small pyramid to that of the truncated base.

3 The two cones A and B are similar. Cone B is an enlargement of A by a scale factor of 4.
If the volume of cone B is 1024 cm^3, calculate the volume of cone A.



Exercise 23.4 (cont)

4 **a** Stating your reasons clearly, decide whether the two cylinders shown are similar or not.



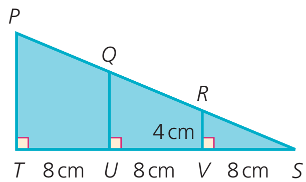
b What is the ratio of the curved surface area of the shaded cylinder to that of the unshaded cylinder?

5 The diagram shows a triangle.

a Calculate the area of triangle *RSV*.

b Calculate the area of triangle *QSU*.

c Calculate the area of triangle *PST*.



6 The area of an island on a map is 30 cm². The scale used on the map is 1 : 100 000.

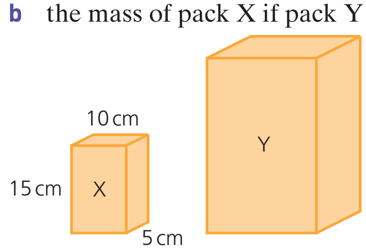
a Calculate the area in square kilometres of the real island.

b An airport on the island is on a rectangular piece of land measuring 3 km by 2 km. Calculate the area of the airport on the map in cm².

7 The two packs of cheese X and Y are similar.
The total surface area of pack Y is four times that of pack X.
Calculate:

a the dimensions of pack Y

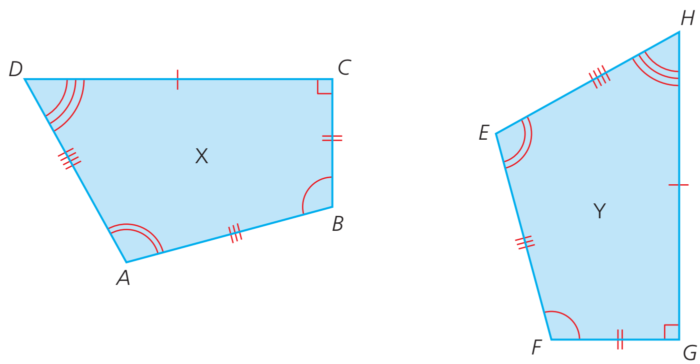
b the mass of pack X if pack Y has a mass of 800 g.



Congruent shapes

Two shapes are **congruent** if their corresponding sides are the same length and their corresponding angles are the same size, i.e. the shapes are exactly the same size and shape.

Shapes X and Y are congruent:



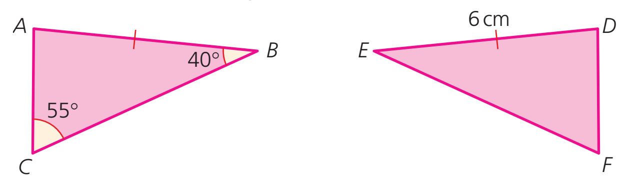
They are congruent as $AB = EF$, $BC = FG$, $CD = GH$ and $DA = HE$. Also angle DAB = angle HEF, angle ABC = angle EFG, angle BCD = angle FGH and angle CDA = angle GHE.

Congruent shapes can therefore be reflections and rotations of each other.

Note: Congruent shapes are, by definition, also similar, but similar shapes are not necessarily congruent.

Worked example

Triangles ABC and DEF are congruent:



In the triangle DEF, the angle at D is defined as angle FDE.

a Calculate the size of angle FDE.

As the two triangles are congruent angle FDE = angle CAB

angle $CAB = 180° - 40° - 55° = 85°$

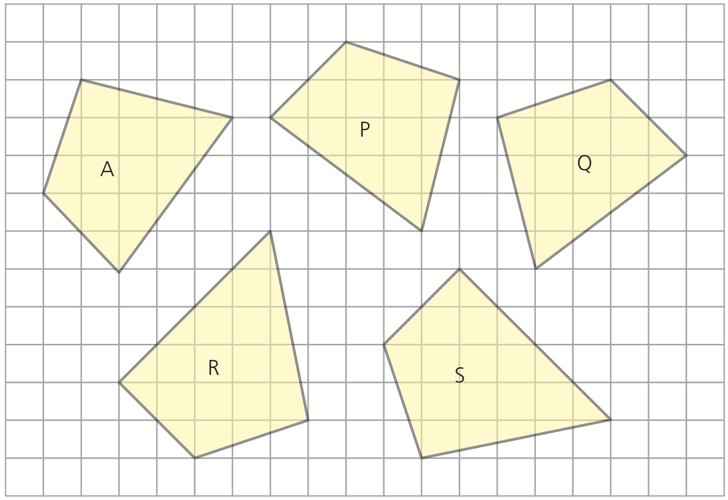
Therefore angle $FDE = 85°$

b Deduce the length of AB.

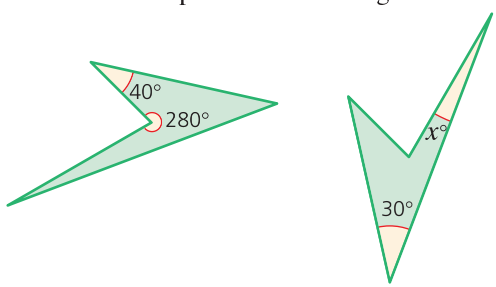
As $AB = DE$, $AB = 6$ cm

Exercise 23.5

1 Look at the shapes on the grid below. Which shapes are congruent to shape A?



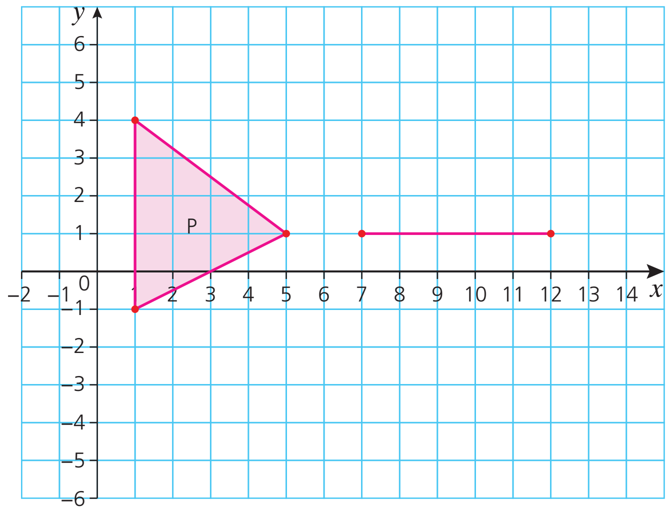
2 The two shapes below are congruent:



Calculate the size of x.

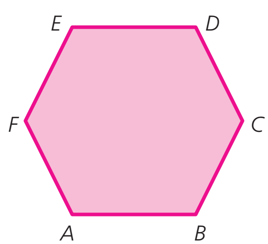
3 A quadrilateral is plotted on a pair of axes. The coordinates of its four vertices are (0, 1), (0, 5), (3, 4) and (3, 3).
Another quadrilateral, congruent to the first, is also plotted on the same axes. Three of its vertices have coordinates (6, 5), (5, 2) and (4, 2).
Calculate the coordinates of the fourth vertex.

4 Triangle P is drawn on a graph. One side of another triangle, Q, is also shown.



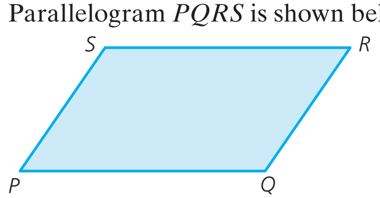
If triangles P and Q are congruent, give all the possible coordinates for the position of the missing vertex.

5 Regular hexagon *ABCDEF* is shown below:



a Identify two triangles congruent to triangle *ACD*.
b Identify two triangles congruent to triangle *DEF*.

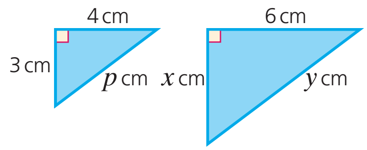
6 Parallelogram *PQRS* is shown below:



Explain, justifying your answer, whether triangle *PQS* is congruent to triangle *QRS*.

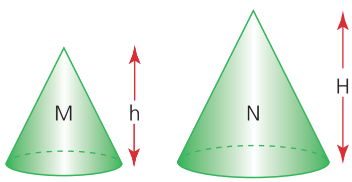
Student assessment 1

1 The two triangles (below) are similar.

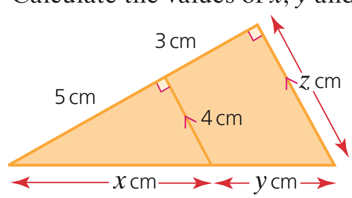


a Using Pythagoras' theorem, calculate the value of p.
b Calculate the values of x and y.

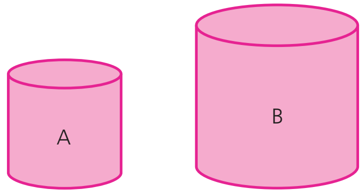
2 Cones M and N are similar.
a Express the ratio of their surface areas in the form, area of M : area of N.
b Express the ratio of their volumes in the form, volume of M : volume of N.



3 Calculate the values of x, y and z in the triangle below.

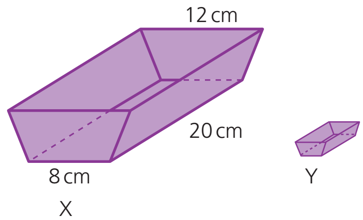


4 The tins A and B are similar. The capacity of tin B is three times that of tin A. If the label on tin A has an area of $75\,\text{cm}^2$, calculate the area of the label on tin B.



5 A cube of side 4 cm is enlarged by a scale factor of 2.5.
a Calculate the volume of the enlarged cube.
b Calculate the surface area of the enlarged cube.

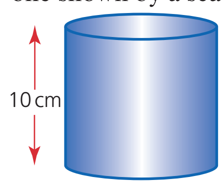
6 The two troughs X and Y are similar. The scale factor of enlargement from Y to X is 4. If the capacity of trough X is $1200\,\text{cm}^3$, calculate the capacity of trough Y.



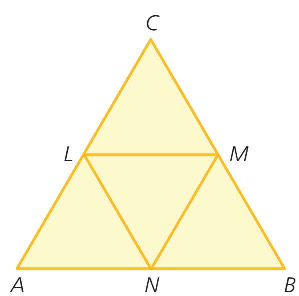
7 The rectangular floor plan of a house measures 8 cm by 6 cm. If the scale of the plan is 1 : 50, calculate:

a the dimensions of the actual floor,
b the area of the actual floor in m^2.

8 The volume of the cylinder below is $400\,\text{cm}^3$.
Calculate the volume of a similar cylinder formed by enlarging the one shown by a scale factor of 2.



9 The diagram below shows an equilateral triangle ABC. The midpoints L, M and N of each side are also joined.



a Identify a trapezium congruent to trapezium $BCLN$.
b Identify a triangle similar to triangle LMN.

10 Decide whether each of the following statements is true or false.

a All circles are similar.
b All squares are similar.
c All rectangles are similar.
d All equilateral triangles are congruent.

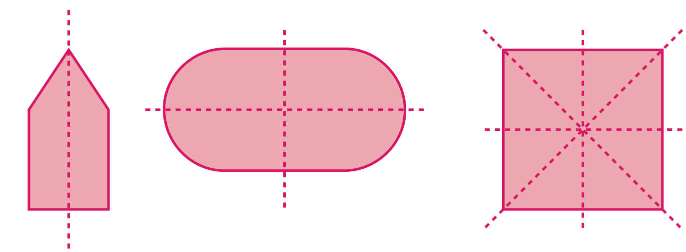
24 Symmetry

Symmetry in two- and three-dimensional shapes

> **Note**
>
> All diagrams are not drawn to scale.

A **line of symmetry** divides a two-dimensional (flat) shape into two congruent (identical) shapes.

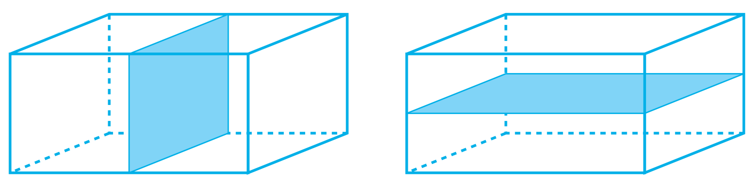
e.g.



1 line of symmetry 2 lines of symmetry 4 lines of symmetry

A **plane of symmetry** divides a three-dimensional (solid) shape into two congruent solid shapes.

e.g.

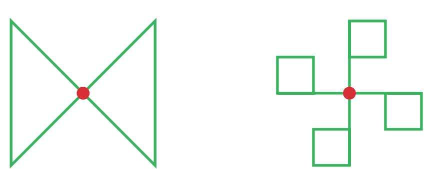


A cuboid has at least three planes of symmetry, two of which are shown above.

A shape has **reflective symmetry** if it has one or more lines or planes of symmetry.

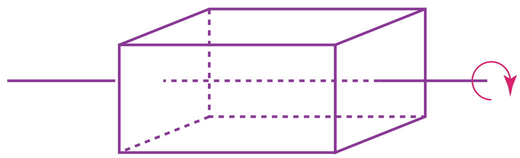
A two-dimensional shape has **rotational symmetry** if, when rotated about a central point, it is identical to the original shape and orientation. The number of times it does this during a complete revolution is called the **order of rotational symmetry**.

e.g.



rotational symmetry of order 2 rotational symmetry of order 4

A three-dimensional shape has **rotational symmetry** if, when rotated about a central axis, it looks the same at certain intervals.
e.g.



This cuboid has rotational symmetry of order 2 about the axis shown.

Exercise 24.1

Note

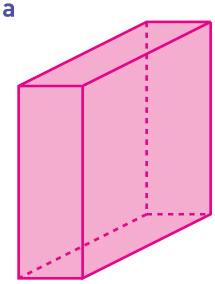
Heptagon and nonagon are not part of the Extended syllabus, but are included here for interest.

1 Copy and complete the following table of the symmetry properties for the given regular polygons. It has been started for you.

Regular polygon	Number of lines of symmetry	Order of rotational symmetry
Equilateral triangle	3	
Square		
Pentagon		
Hexagon		6
Heptagon		
Octagon		
Nonagon		
Decagon		

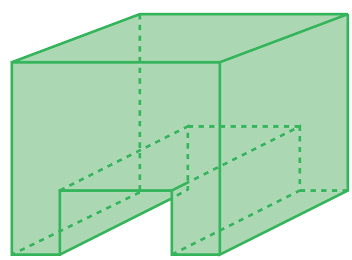
2 Draw each of the solid shapes below twice, then:
- i on each drawing of the shape, draw a different plane of symmetry,
- ii state how many planes of symmetry the shape has in total.

a



cuboid

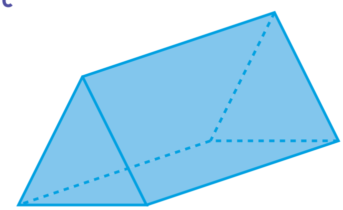
b



prism

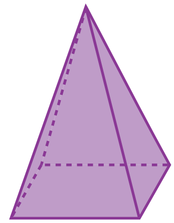
Exercise 24.1 (cont)

c



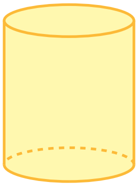
equilateral triangular prism

d



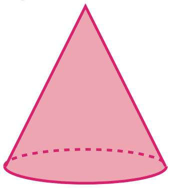
square-based pyramid

e



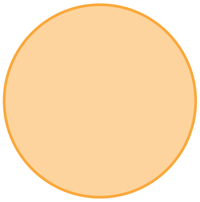
cylinder

f



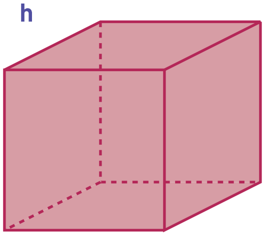
cone

g



sphere

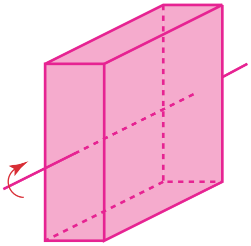
h



cube

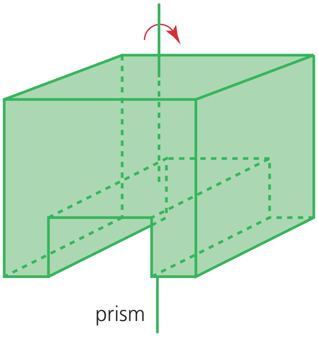
3 For each of the solid shapes shown below, determine the order of rotational symmetry about the axis shown.

a



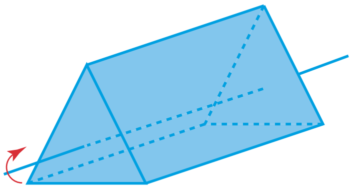
cuboid

b



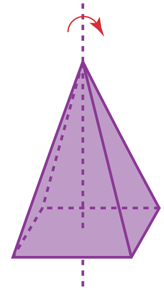
prism

c



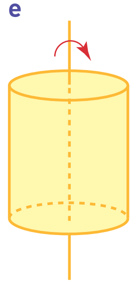
equilateral triangular prism

d



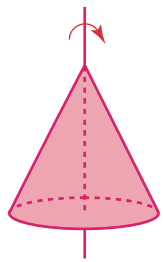
square-based pyramid

e



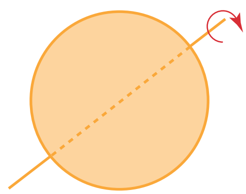
cylinder

f



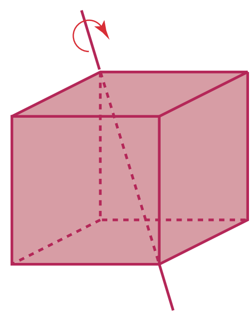
cone

g



sphere

h

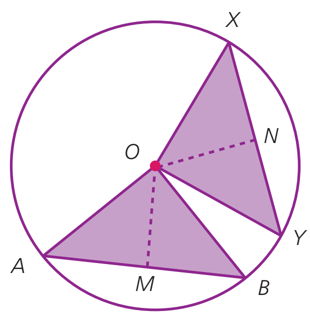


cube

Circle properties

Equal chords and perpendicular bisectors

If chords AB and XY are of equal length, then, since OA, OB, OX and OY are radii, the triangles OAB and OXY are congruent isosceles triangles.



It follows that:

- the section of a line of symmetry OM through triangle OAB is the same length as the section of a line of symmetry ON through triangle OXY,
- OM and ON are **perpendicular bisectors** of AB and XY respectively.

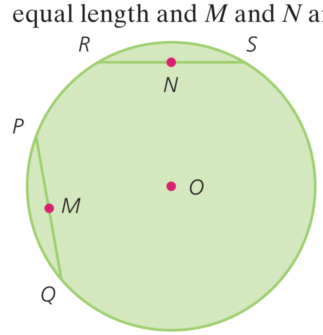
This is one of the circle theorems, namely, that the perpendicular bisector of a chord passes through the centre of the circle.

This also demonstrates the fact that the perpendicular distance from a point to a line is the shortest distance to the line.

Therefore it can be deduced that equal chords are equidistant from the centre.

Exercise 24.2

1 In the diagram, O is the centre of the circle, PQ and RS are chords of equal length and M and N are their respective midpoints.

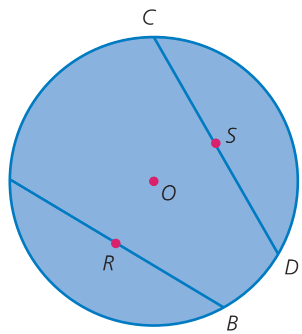


a What kind of triangle is triangle POQ?
b Describe the line ON in relation to RS.
c If angle POQ is 80°, calculate angle OQP.
d Calculate angle ORS.

Exercise 24.2 (cont)

e If PQ is 6 cm, calculate the length OM.
f Calculate the diameter of the circle.

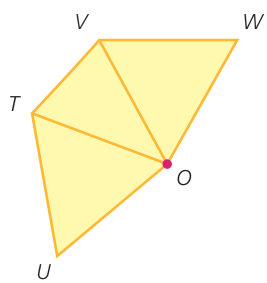
2 In the diagram, O is the centre of the circle. AB and CD are equal chords and the points R and S are their midpoints respectively.



State whether the statements below are true or false, giving reasons for your answers.

a angle $COD = 2 \times$ angle AOR.
b $OR = OS$.
c If angle ROB is 60° then triangle AOB is equilateral.
d OR and OS are perpendicular bisectors of AB and CD respectively.

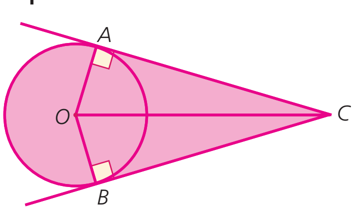
3 Using the diagram, state whether the following statements are true or false, giving reasons for your answer.



a If triangle VOW and triangle TOU are isosceles triangles, then T, U, V and W would all lie on the circumference of a circle with its centre at O.
b If triangle VOW and triangle TOU are congruent isosceles triangles, then T, U, V and W would all lie on the circumference of a circle with its centre at O.

Tangents from an external point

Triangles OAC and OBC are congruent since angle OAC and angle OBC are right angles, $OA = OB$ because they are both radii, and OC is common to both triangles. Hence $AC = BC$.

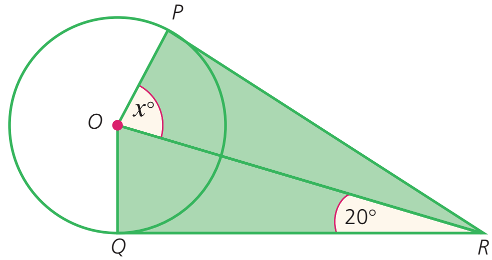


In general, therefore, tangents being drawn to the same circle from an external point are equal in length.

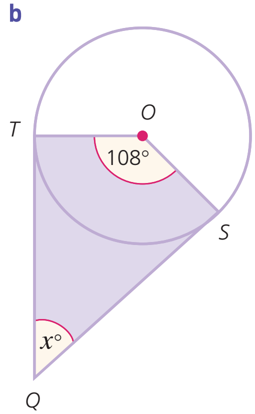
Exercise 24.3

1 Copy each of the diagrams below and calculate the size of the angle marked $x°$ in each case. Assume that the lines drawn from points on the circumference are tangents.

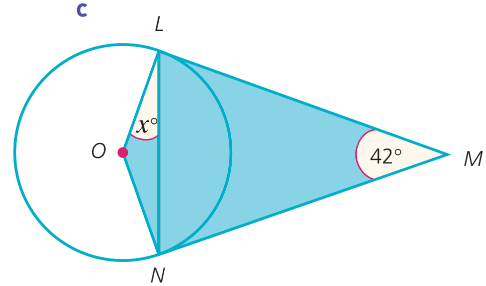
a



b

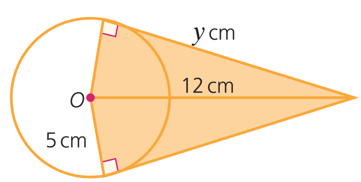


c

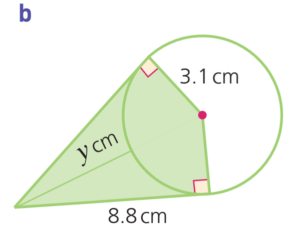


2 Copy each of the diagrams below and calculate the length of the side marked y cm in each case. Assume that the lines drawn from points on the circumference are tangents.

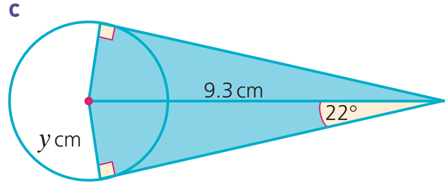
a



b



c

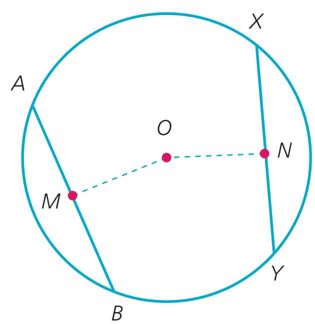


Student assessment 1

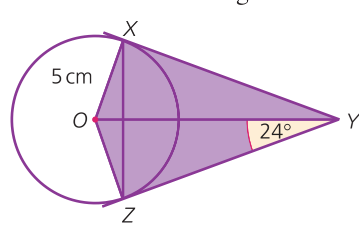
1 Draw a two-dimensional shape with exactly:
 a rotational symmetry of order 2,
 b rotational symmetry of order 4,
 c rotational symmetry of order 6.

2 Draw and name a three-dimensional shape with the following orders of rotational symmetry around one axis. Mark the position of this axis of symmetry clearly.
 a Order 2
 b Order 3
 c Order 8

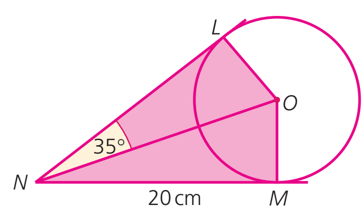
3 In the diagram, OM and ON are perpendicular bisectors of AB and XY respectively. $OM = ON$.
Prove that AB and XY are chords of equal length.



4 In the diagram, XY and YZ are both tangents to the circle with centre O.
 a Calculate angle OZX.
 b Calculate the length XZ.



5 In the diagram, LN and MN are both tangents to the circle centre O. If angle LNO is 35°, calculate the circumference of the circle.



25 Angle properties

Angles at a point and on a line

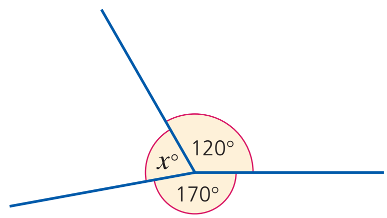
One complete revolution is equivalent to a rotation of 360° about a point. Similarly, half a complete revolution is equivalent to a rotation of 180° about a point. These facts can be seen clearly by looking at either a circular angle measurer or a semicircular protractor.

Note

All diagrams are not drawn to scale.

Worked examples

a Calculate the size of the angle x in the diagram below:



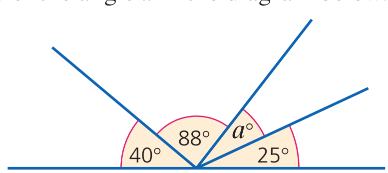
The sum of all the angles around a point is 360°. Therefore:

$$120 + 170 + x = 360$$
$$x = 360 - 120 - 170$$
$$x = 70$$

Therefore angle x is 70°.

Note that the size of the angle x is **calculated** and **not measured**.

b Calculate the size of the angle a in the diagram below:



The sum of all the **angles at a point** on a straight line is 180°. Therefore:

$$40 + 88 + a + 25 = 180$$
$$a = 180 - 40 - 88 - 25$$
$$a = 27$$

Therefore angle a is 27°.

Note

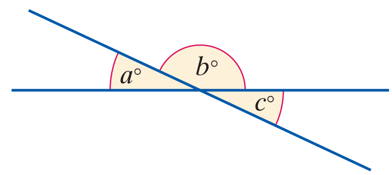
Pairs of angles which add up to 180° are also called **supplementary** angles.

Angles formed within parallel lines

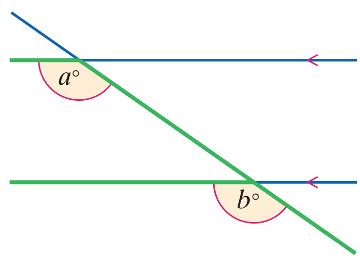
When two straight lines cross, it is found that the angles opposite each other are the same size. They are known as **vertically opposite angles**. By using the fact that angles at a point on a straight line add up to 180°, it can be shown why vertically opposite angles must always be equal in size.

$a + b = 180°$
$c + b = 180°$

Therefore, a is equal to c.

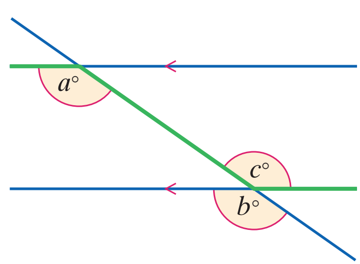


When a line intersects two parallel lines, as in the diagram below, it is found that certain angles are the same size.



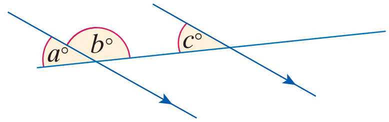
The angles a and b are equal and are known as **corresponding angles**. Corresponding angles can be found by looking for an 'F' formation in a diagram.

A line intersecting two parallel lines also produces another pair of equal angles, known as **alternate angles**. These can be shown to be equal by using the fact that both vertically opposite and corresponding angles are equal.



In the diagram above, $a = b$ (corresponding angles). But $b = c$ (vertically opposite). It can therefore be deduced that $a = c$.

Angles a and c are alternate angles. These can be found by looking for a 'Z' formation in a diagram.



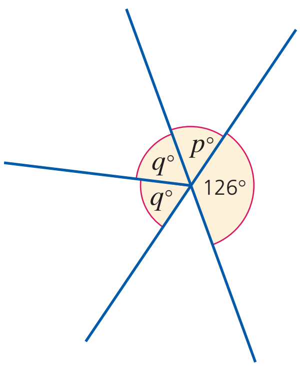
As $a° = c°$ (corresponding angles) and $a° + b° = 180°$ (angles on a straight line add up to 180°) then $b° + c° = 180°$.

b and *c* are **co-interior** angles as they face each other between parallel lines. Co-interior angles therefore add up to 180°.

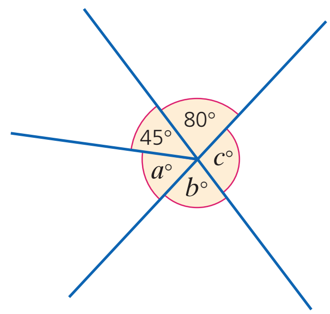
Exercise 25.1

In each of the following questions, some of the angles are given. Deduce, giving your reasons, the size of the other labelled angles.

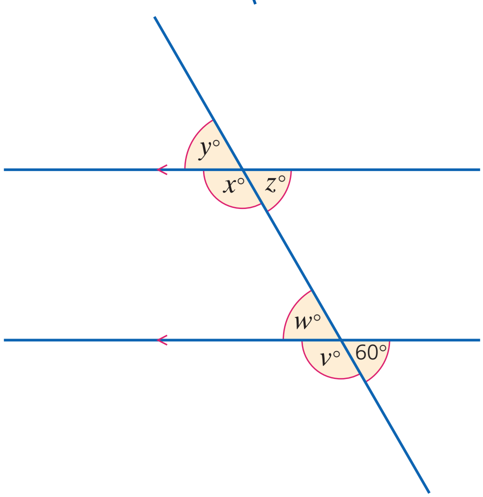
1



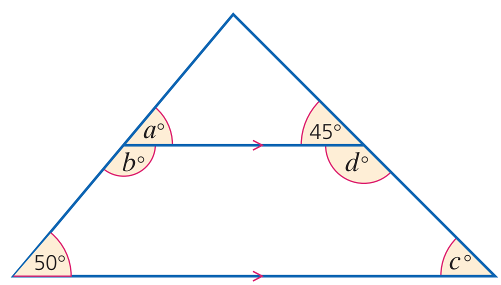
2



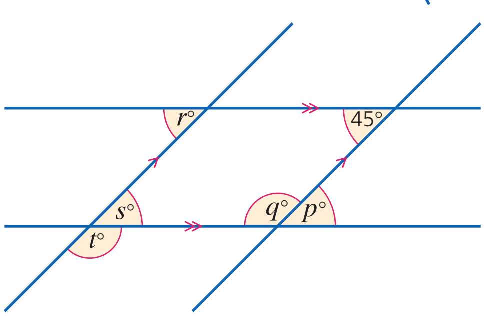
3



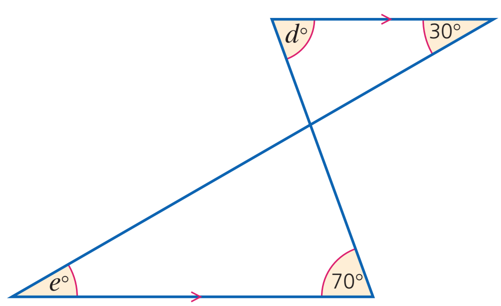
4



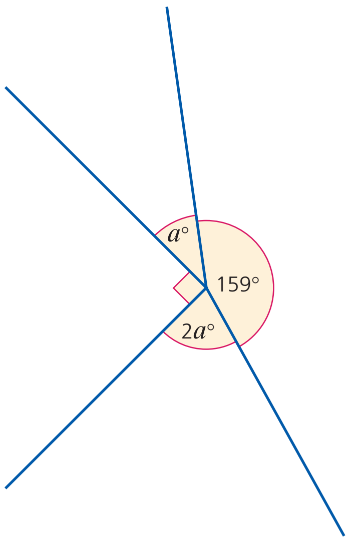
5



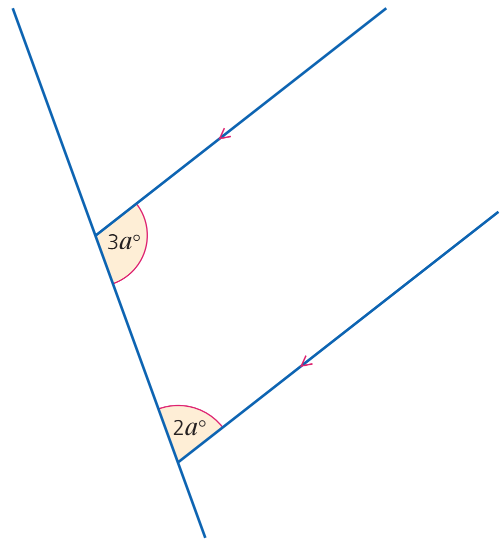
6



7



8

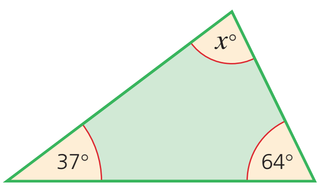


Angles in a triangle

The sum of the angles in a triangle is 180°.

Worked example

Calculate the size of the angle x in the triangle below:



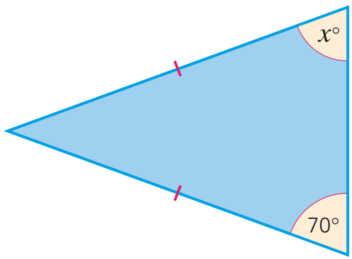
$$37 + 64 + x = 180$$
$$x = 180 - 37 - 64$$

Therefore angle x is 79°.

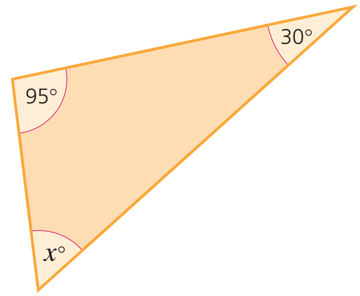
Exercise 25.2

1 For each of the triangles below, use the information given to calculate the size of angle x.

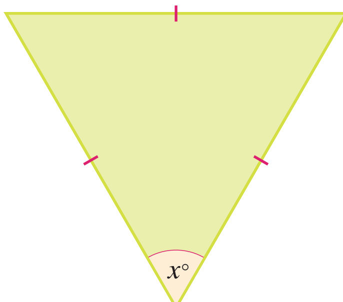
a



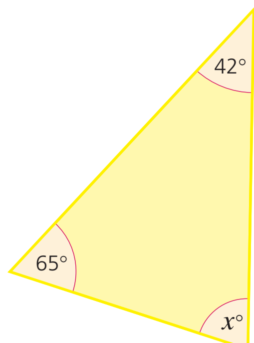
b



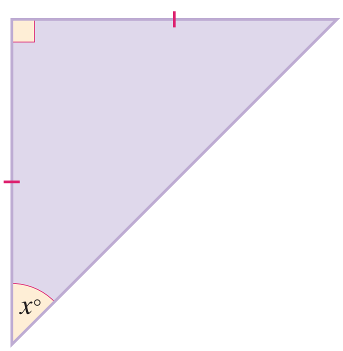
c



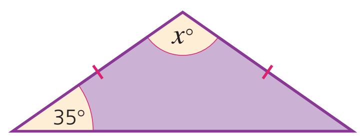
d



e

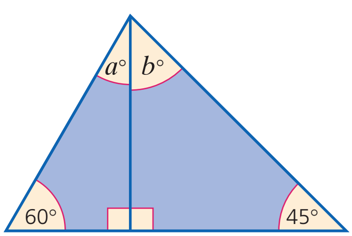


f

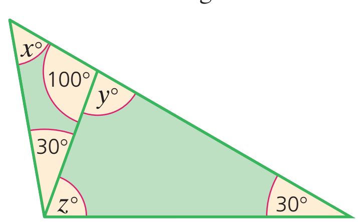


2 In each of the diagrams below, calculate the size of the labelled angles.

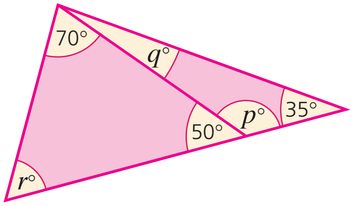
a



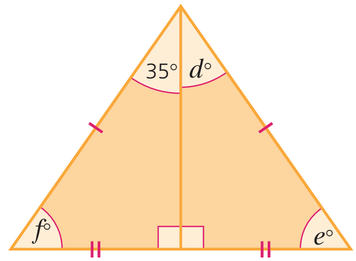
b



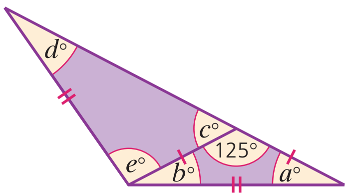
c



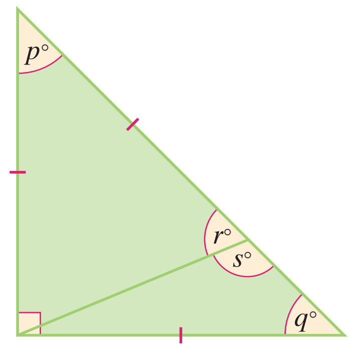
d



e

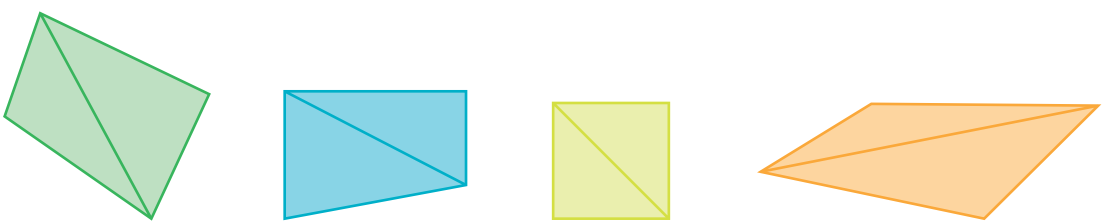


f



Angles in a quadrilateral

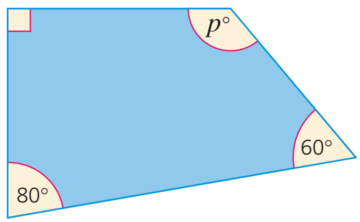
In the quadrilaterals below, a straight line is drawn from one of the corners (vertices) to the opposite corner. The result is to split the quadrilaterals into two triangles.



The sum of the angles of a triangle is 180°. Therefore, as a quadrilateral can be drawn as two triangles, the sum of the four angles of any quadrilateral must be 360°.

Worked example

Calculate the size of angle p in the quadrilateral below:



$90 + 80 + 60 + p = 360$

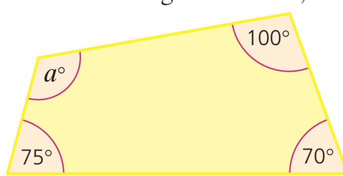
$p = 360 - 90 - 80 - 60$

Therefore angle p is 130°.

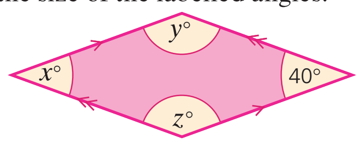
Exercise 25.3

In each of the diagrams below, calculate the size of the labelled angles.

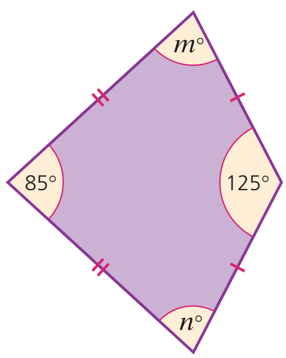
1



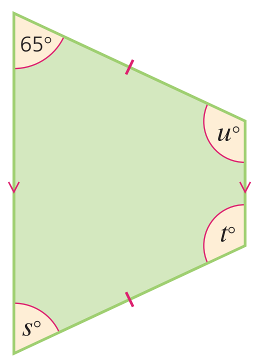
2



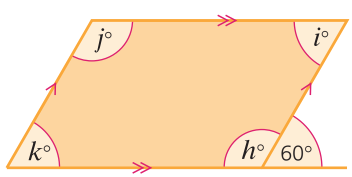
3



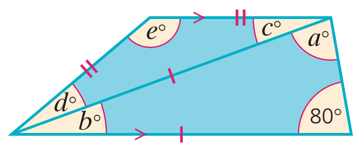
4



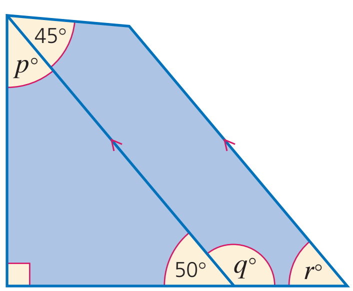
5



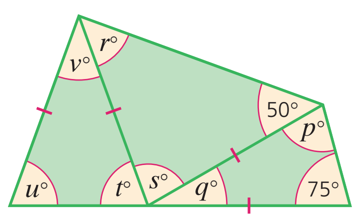
6



7

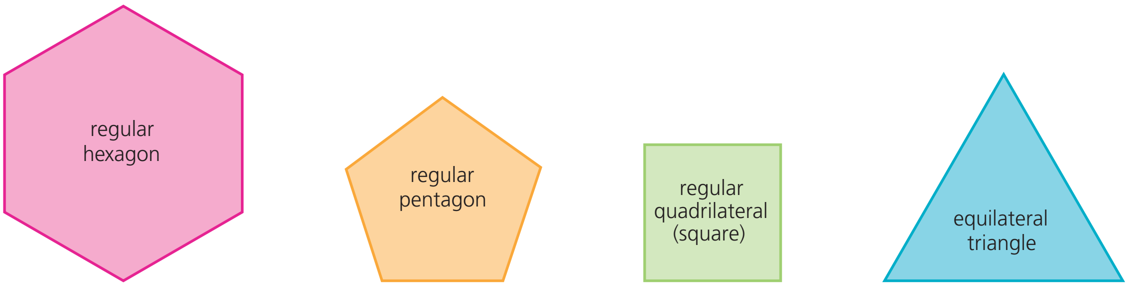


8



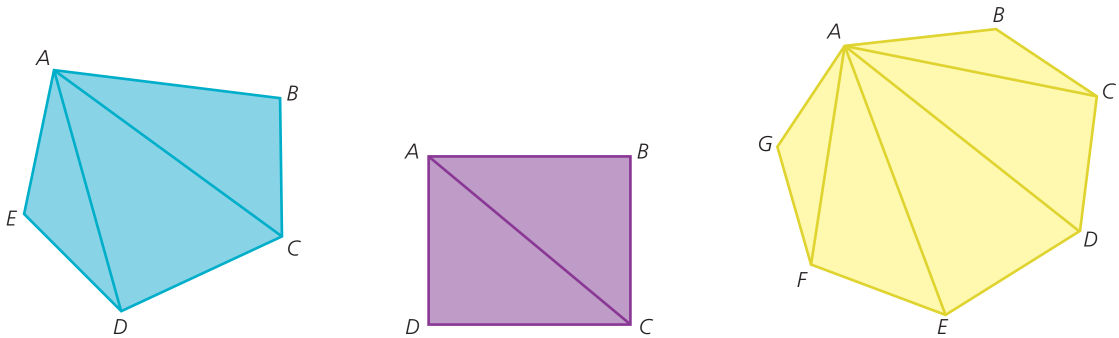
Polygons

A **regular polygon** is distinctive in that all its sides are of equal length and all its angles are of equal size. Below are some examples of regular polygons.



The sum of the interior angles of a polygon

In the polygons below, a straight line is drawn from each vertex to vertex A.



As can be seen, the number of triangles is always two less than the number of sides the polygon has, i.e. if there are n sides, there will be $(n - 2)$ triangles.

Since the angles of a triangle add up to 180°, the sum of the interior angles of a polygon is therefore $180(n - 2)$ degrees.

Worked example

Find the sum of the interior angles of a regular pentagon and hence the size of each interior angle.

For a pentagon, $n = 5$.

Therefore the sum of the interior angles $= 180(5 - 2)°$

$= 180 \times 3°$

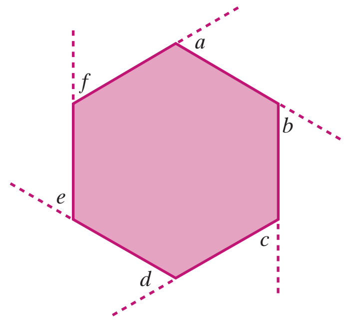
$= 540°$

For a regular pentagon the interior angles are of equal size.

Therefore each angle $= \frac{540°}{5} = 108°$.

The sum of the exterior angles of a polygon

The angles marked a, b, c, d, e and f in the diagram below represent the exterior angles of a regular hexagon.



For any convex polygon the sum of the exterior angles is 360°.
If the polygon is regular and has n sides, then each exterior angle $= \frac{360°}{n}$.

Worked examples

a Find the size of an exterior angle of a regular nine-sided polygon.

$$\frac{360}{9} = 40°$$

b Calculate the number of sides a regular polygon has if each exterior angle is 15°.

$$n = \frac{360}{15}$$
$$= 24$$

The polygon has 24 sides.

Exercise 25.4

Note

Heptagon, nonagon and dodecagon are not part of the syllabus. They have 7, 9 and 12 sides respectively.

1 Find the sum of the interior angles of the following polygons:
a a hexagon **b** a nonagon **c** a heptagon

2 Find the value of each interior angle of the following regular polygons:
a an octagon **b** a square
c a decagon **d** a dodecagon

3 Find the size of each exterior angle of the following regular polygons:
a a pentagon **b** a dodecagon **c** a heptagon

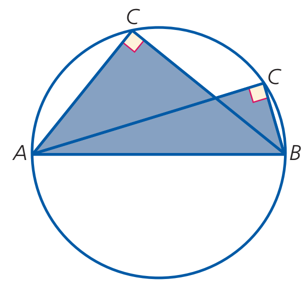
4 The exterior angles of regular polygons are given below. In each case calculate the number of sides the polygon has.
a 20° **b** 36° **c** 10°
d 45° **e** 18° **f** 3°

5 The interior angles of regular polygons are given below. In each case calculate the number of sides the polygon has.
a 108° **b** 150° **c** 162°
d 156° **e** 171° **f** 179°

6 Calculate the number of sides a regular polygon has if an interior angle is five times the size of an exterior angle.

The angle in a semicircle

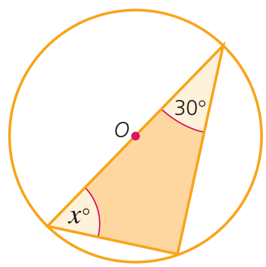
In the diagram below, if AB represents the diameter of the circle, then the angle at C is 90°.



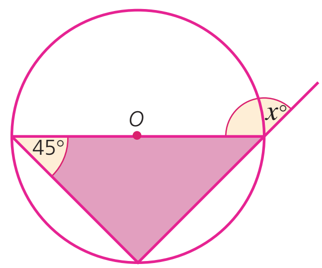
Exercise 25.5

In each of the following diagrams, O marks the centre of the circle. Calculate the value of x in each case.

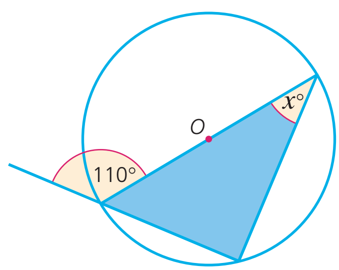
1



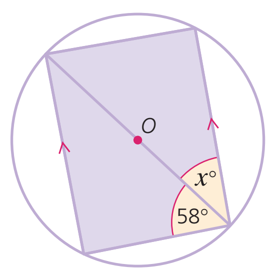
2



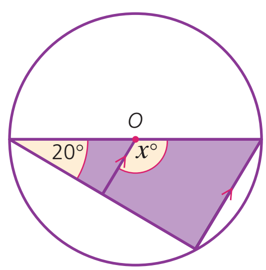
3



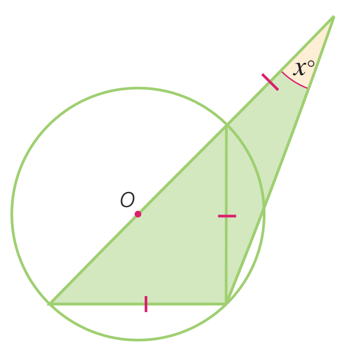
4



5



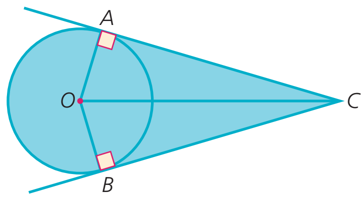
6



The angle between a tangent and a radius of a circle

The angle between a tangent at a point and the radius to the same point on the circle is a right angle.

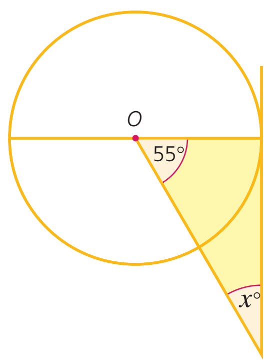
In the diagram below, triangles OAC and OBC are congruent as angle OAC and angle OBC are right angles, $OA = OB$ because they are both radii, and OC is common to both triangles.



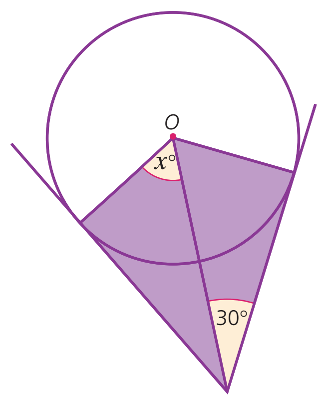
Exercise 25.6

In each of the following diagrams, O marks the centre of the circle. Calculate the value of x in each case.

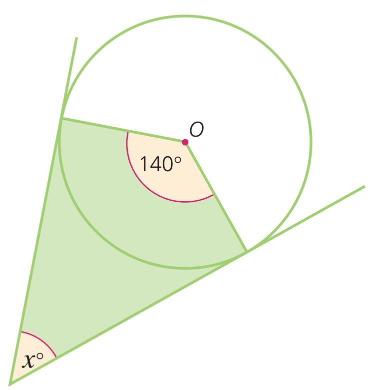
1



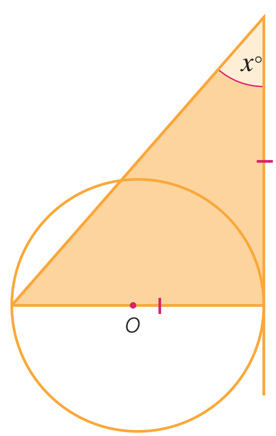
2



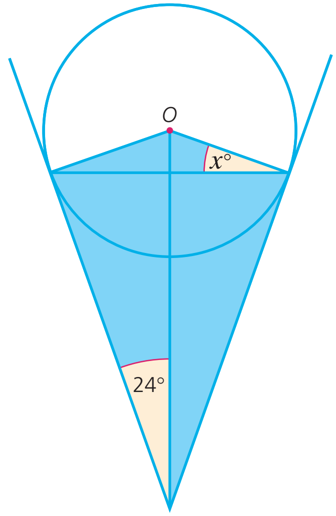
3



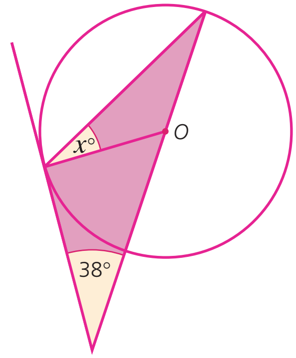
4



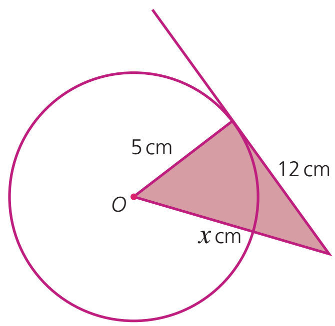
5



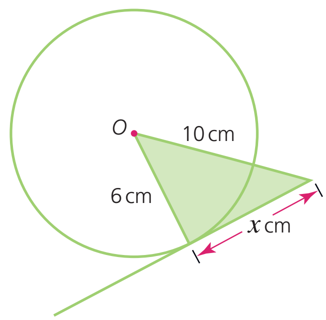
6



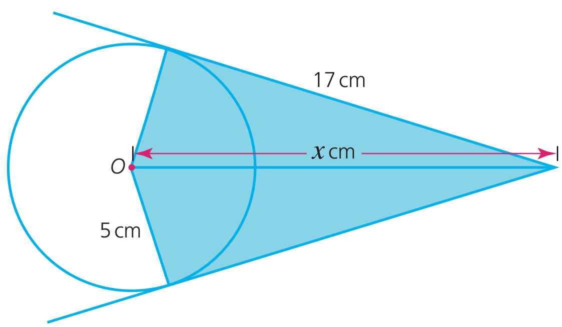
7



8



9



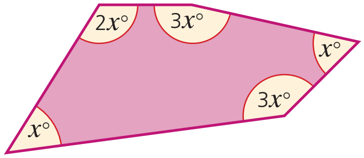
Angle properties of irregular polygons

As explained earlier in this chapter, the sum of the interior angles of a polygon is given by $180(n - 2)°$, where n represents the number of sides of the polygon. The sum of the exterior angles of any polygon is 360°.

Both of these rules also apply to irregular polygons, i.e. those where the lengths of the sides and the sizes of the interior angles are not all equal.

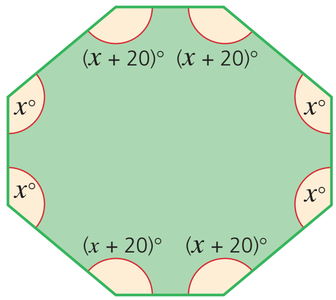
Exercise 25.7

1 For the pentagon:

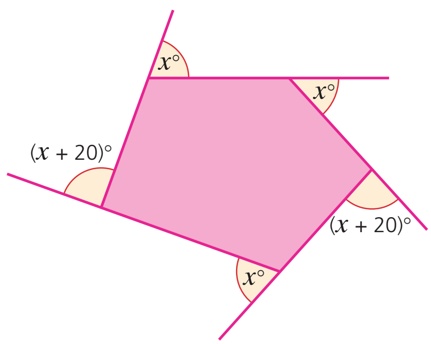


a calculate the value of x,
b calculate the size of each of the angles.

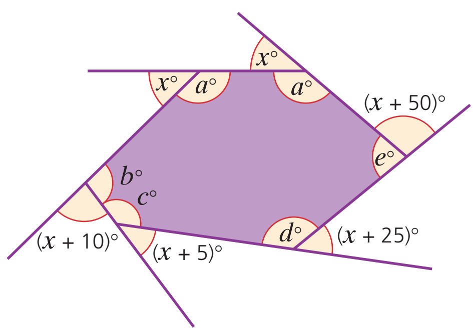
2 Find the size of each angle in the octagon (below).



3 Calculate the value of x for the pentagon shown.



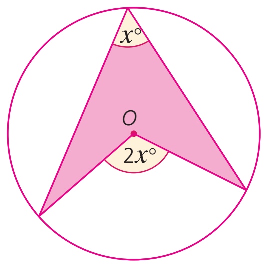
4 Calculate the size of each of the angles a, b, c, d and e in the hexagon.

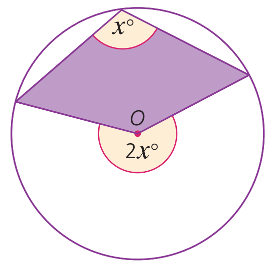


Angle at the centre of a circle

The angle subtended at the centre of a circle by an arc is twice the size of the angle on the circumference subtended by the same arc.

These diagrams illustrate this theorem:

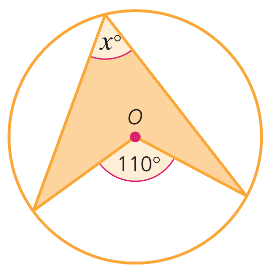




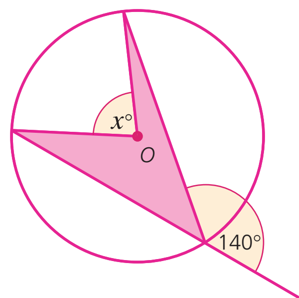
Exercise 25.8

In each of the following diagrams, O marks the centre of the circle. Calculate the size of the marked angles:

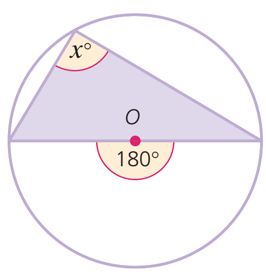
1



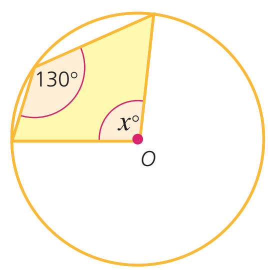
2



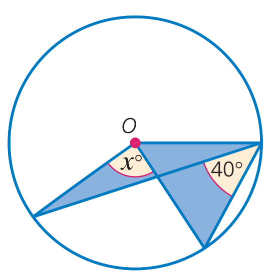
3



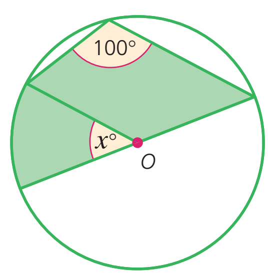
4



5

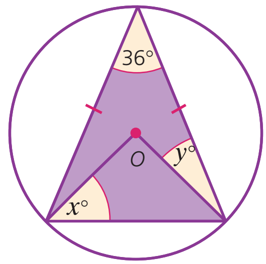


6

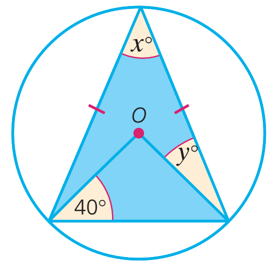


Exercise 25.8 (cont)

7



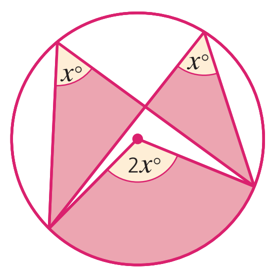
8



Angles in the same segment

Angles in the same segment of a circle are equal.

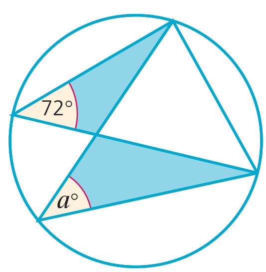
This can be explained simply by using the theorem that the angle subtended at the centre is twice the angle on the circumference. Looking at the diagram, if the angle at the centre is $2x°$, then each of the angles at the circumference must be equal to $x°$.



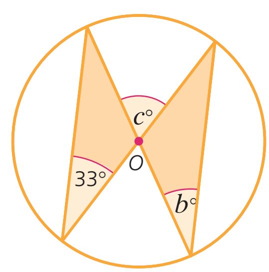
Exercise 25.9

Calculate the marked angles in the following diagrams:

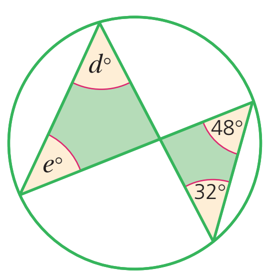
1



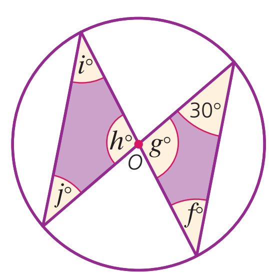
2



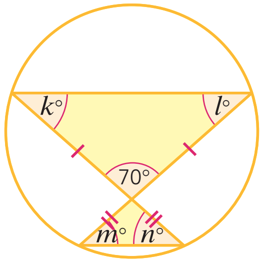
3



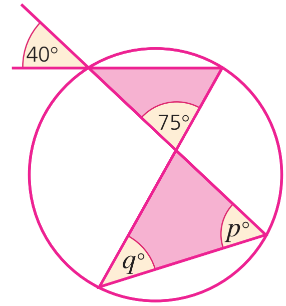
4



5



6

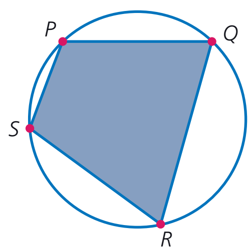


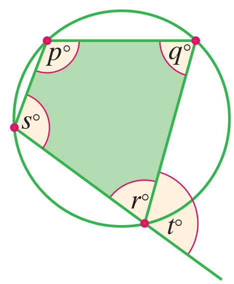
Angles in opposite segments

Points P, Q, R and S all lie on the circumference of the circle (below). They are called concyclic points. Joining the points P, Q, R and S produces a **cyclic quadrilateral**.

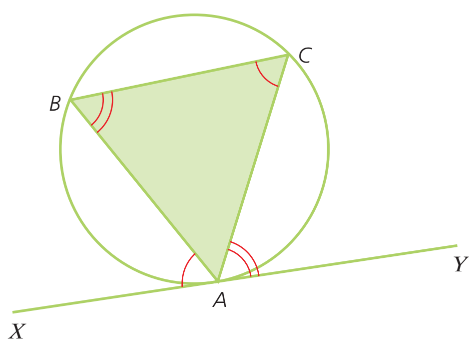
The opposite angles are **supplementary**, i.e. they add up to 180°. Since $p° + r° = 180°$ (supplementary angles) and $r° + t° = 180°$ (**angles on a straight line**) it follows that $p° = t°$.

Therefore the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.





Alternate segment theorem

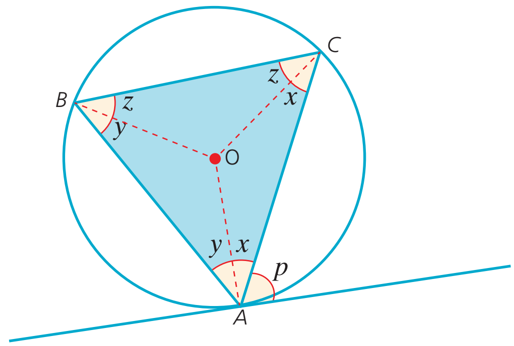


XY is a tangent to the circle at point A. AB is a chord of the circle.

The alternate segment theorem states that the angle between a tangent and a chord at their point of contact is equal to the angle in the alternate segment.

i.e. angle BAX = angle ACB, similarly, angle CAY = angle ABC

This can be proved as follows:



Point O is the centre of the circle.

Drawing radii to points A, B and C creates isosceles triangles AOC, BOC and AOB.

Angle p is the angle between the chord AC and the tangent to the circle at A.

$p = 90 - x$ because the angle between a radius and a tangent to a circle at a point is a right angle.

In triangle ABC, $(x + y) + (y + z) + (x + z) = 180°$

Therefore $2x + 2y + 2z = 180°$

$$2(x + y + z) = 180°$$

$$x + y + z = 90°$$

If $x + y + z = 90°$ then $y + z = 90 - x$

As angle $ABC = y + z$, then angle $ABC = 90 - x$

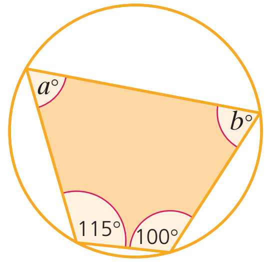
As $p = 90 - x$

Then angle $ABC = p$

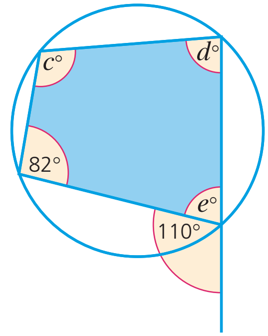
Exercise 25.10

For each Question 1–8, calculate the size of the marked angles.

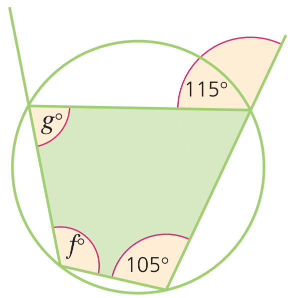
1



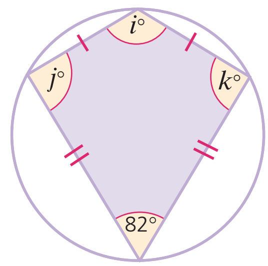
2



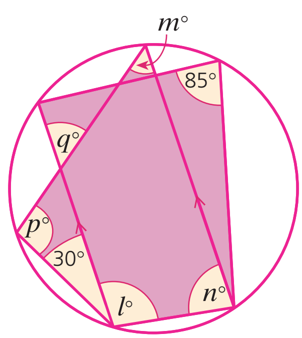
3



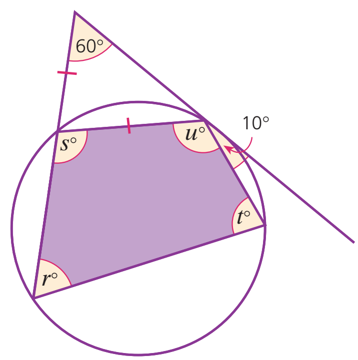
4



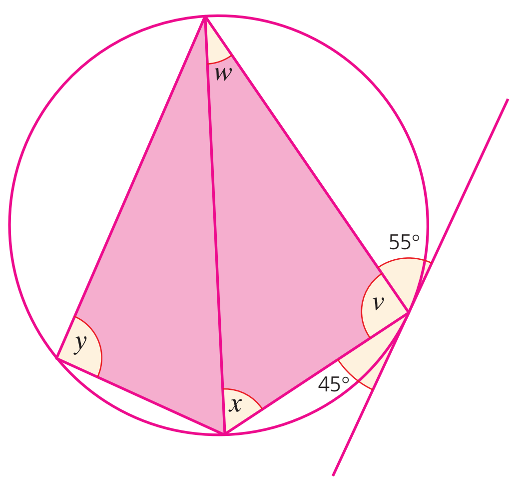
5



6

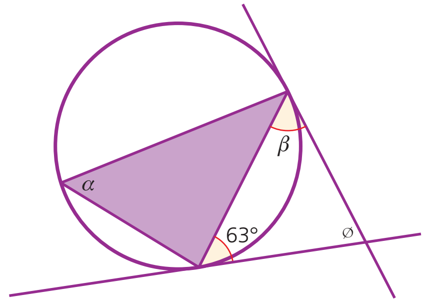


7



Exercise 25.10 (cont)

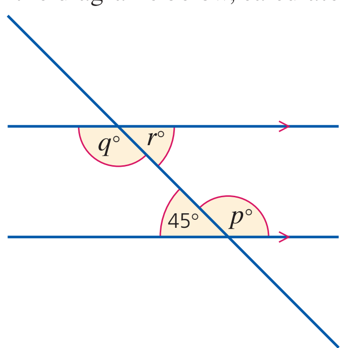
8



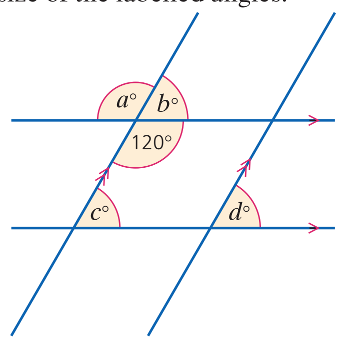
Student assessment 1

1 For the diagrams below, calculate the size of the labelled angles.

a

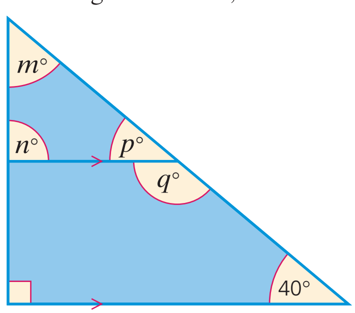


b

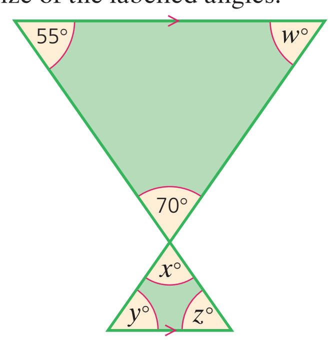


2 For the diagrams below, calculate the size of the labelled angles.

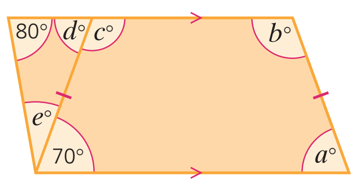
a



b



c



3 Find the size of each interior angle of a 20-sided regular polygon.

4 What is the sum of the interior angles of a nine-sided polygon?

5 What is the sum of the exterior angles of a regular polygon?

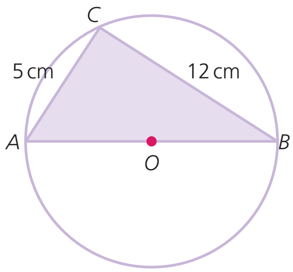
6 What is the size of each exterior angle of a regular pentagon?

7 A regular polygon has interior angles of size 156°. Calculate the number of sides it has.

8 If AB is the diameter of the circle and $AC = 5\,\text{cm}$ and $BC = 12\,\text{cm}$, calculate:

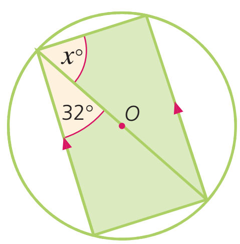
a the size of angle ACB,

b the length of the radius of the circle.

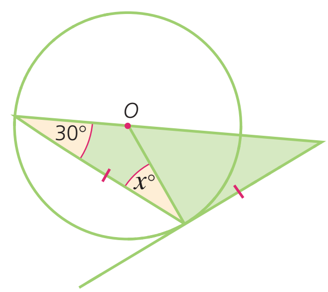


In Questions 9–12, O marks the centre of the circle. Calculate the size of the angle marked x in each case.

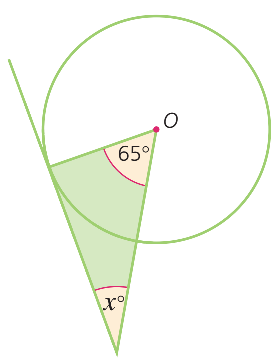
9



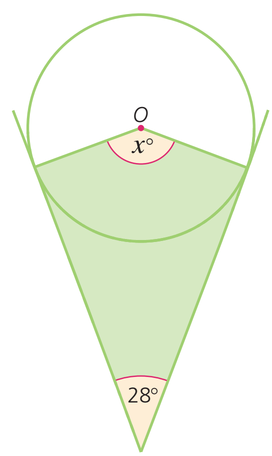
10



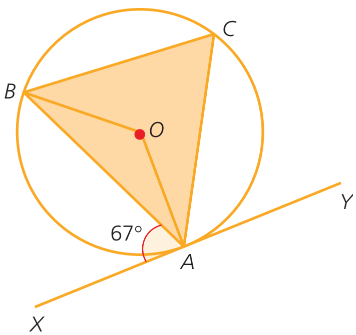
11



12



13



In the diagram above, XY is a tangent to the circle at A. O is the centre of the circle. Calculate each of the following angles, justifying each of your answers.

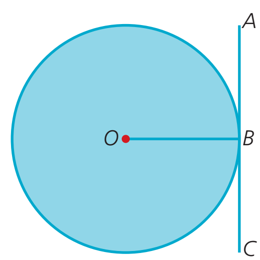
a angle ACB.
b reflex angle AOB.
c angle ABO.

Student assessment 2

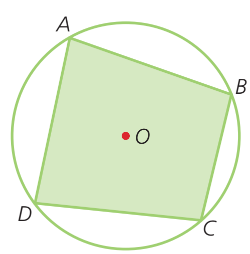
1 In the following diagrams, O marks the centre of the circle. Identify which angles are:

i supplementary angles,
ii right angles,
iii equal.

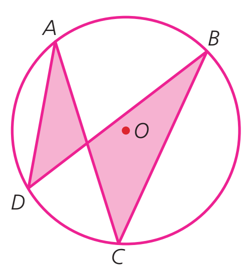
a



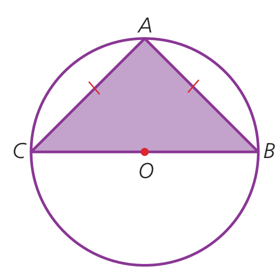
b



c

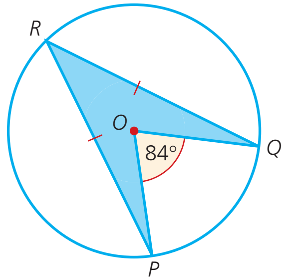


d

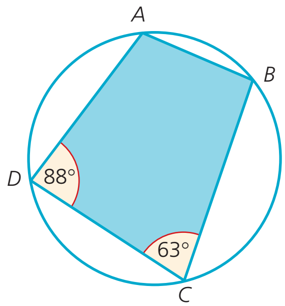


2 If angle $POQ = 84°$ and O marks the centre of the circle in the diagram, calculate the following:

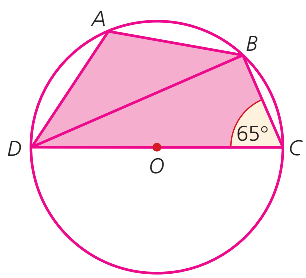
a angle PRQ **b** angle OQR



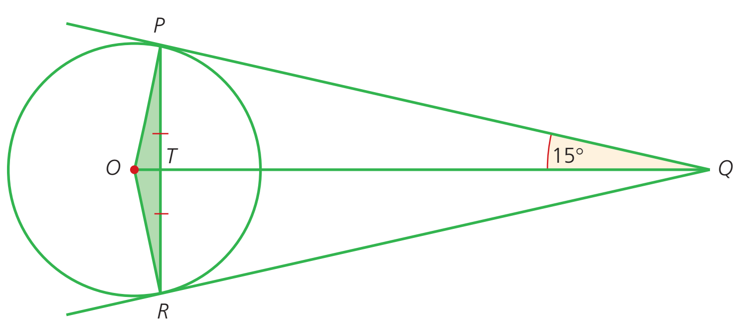
3 Calculate angle DAB and angle ABC in the diagram below.



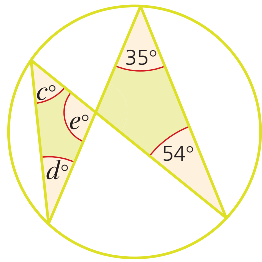
4 If DC is a diameter and O marks the centre of the circle, calculate angles BDC and DAB.



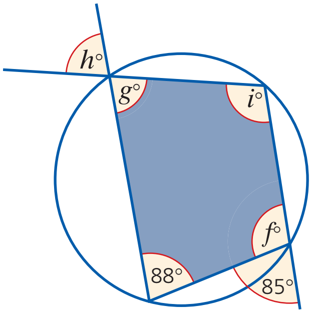
5 Calculate as many angles as possible in the diagram below. O marks the centre of the circle.



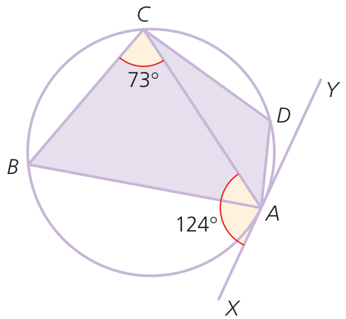
6 Calculate the values of c, d and e.



7 Calculate the values of f, g, h and i.



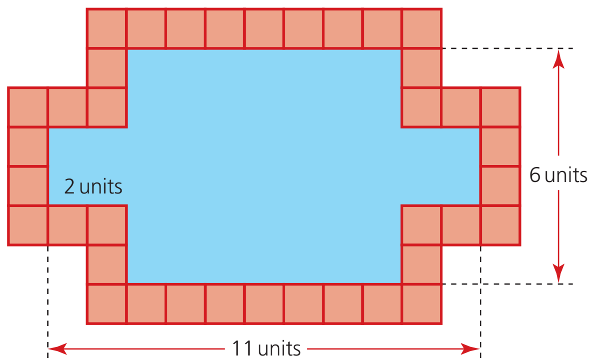
8 In the diagram below, XY is a tangent to the circle at A. All four vertices of the quadrilateral $ABCD$ lie on the circumference of the circle.
If angle $XAC = 124°$ and angle $BCA = 73°$, calculate, giving reasons, the size of the angle BAC.



4 Mathematical investigations and ICT 4

Fountain borders

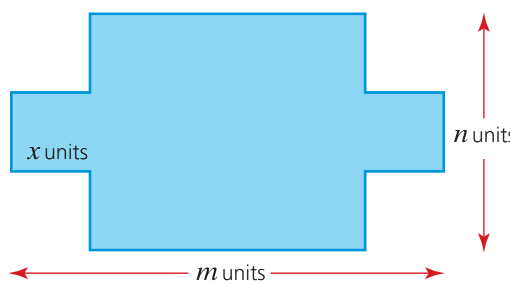
The Alhambra Palace in Granada, Spain, has many fountains which pour water into pools. Many of the pools are surrounded by beautiful ceramic tiles. This investigation looks at the number of square tiles needed to surround a particular shape of pool.



The diagram above shows a rectangular pool 11×6 units, in which a square of dimension 2×2 units is taken from each corner.

The total number of unit square tiles needed to surround the pool is 38.

The shape of the pools can be generalised as shown below:



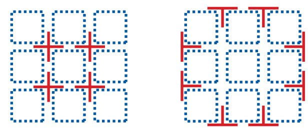
1 Investigate the number of unit square tiles needed for different-sized pools. Record your results in an ordered table.

2 From your results write an algebraic rule in terms of m, n and x for the number of tiles T needed to surround a pool.

3 Justify, in words and using diagrams, why your rule works.

Tiled walls

Many cultures have used tiles to decorate buildings. Putting tiles on a wall takes skill. These days, to make sure that each tile is in the correct position, 'spacers' are used between the tiles.



You can see from the diagrams that there are + shaped spacers (used where four tiles meet) and ⊤ shaped spacers (used at the edges of a pattern).

1 Draw other-sized squares and rectangles and investigate the relationship between the dimensions of the shape (length and width) and the number of + shaped and ⊤ shaped spacers.

2 Record your results in an ordered table.

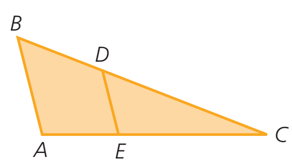
3 Write an algebraic rule for the number of + shaped spacers c in a rectangle l tiles long by w tiles wide.

4 Write an algebraic rule for the number of ⊤ shaped spacers t in a rectangle l tiles long by w tiles wide.

ICT activity 1

In this activity, you will be using a dynamic geometry package such as Cabri or GeoGebra to demonstrate that for the triangle below:

$$\frac{AB}{ED} = \frac{AC}{EC} = \frac{BC}{DC}$$



1 a Using the geometry package, construct the triangle ABC.
  b Construct the line segment ED such that it is parallel to AB. (You will need to construct a line parallel to AB first and then attach the line segment ED to it.)
  c Using a 'measurement' tool, measure each of the lengths AB, AC, BC, ED, EC and DC.
  d Using a 'calculator' tool, calculate the ratios $\frac{AB}{ED}$, $\frac{AC}{EC}$, $\frac{BC}{DC}$.

2 Comment on your answers to Question 1d.

3 **a** Grab vertex *B* and move it to a new position. What happens to the ratios you calculated in Question 1d?

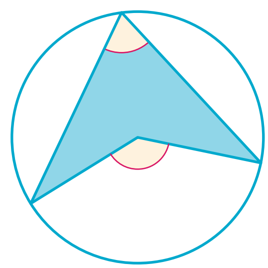
b Grab the vertices *A* and *C* in turn and move them to new positions. What happens to the ratios? Explain why this happens.

4 Grab point *D* and move it to a new position along the side *BC*. Explain, giving reasons, what happens to the ratios.

ICT activity 2

Using a geometry package, such as Cabri or GeoGebra, demonstrate the following angle properties of a circle:

1 The angle subtended at the centre of a circle by an arc is twice the size of the angle on the circumference subtended by the same arc. The diagram below demonstrates the construction that needs to be formed:



2 The angles in the same segment of a circle are equal.

3 The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

TOPIC 5

Mensuration

Contents

Learning objectives

E5.1 Units of measure
Use metric units of mass, length, area, volume and capacity in practical situations and convert quantities into larger or smaller units.

E5.2 Area and perimeter
Carry out calculations involving the perimeter and area of a rectangle, triangle, parallelogram and trapezium.

E5.3 Circles, arcs and sectors

1. Carry out calculations involving the circumference and area of a circle.
2. Carry out calculations involving arc length and sector area as fractions of the circumference and area of a circle.

E5.4 Surface area and volume
Carry out calculations and solve problems involving the surface area and volume of a:

- cuboid
- prism
- cylinder
- sphere
- pyramid
- cone.

E5.5 Compound shapes and parts of shapes

1. Carry out calculations and solve problems involving perimeters and areas of:
   - compound shapes
   - parts of shapes.
2. Carry out calculations and solve problems involving surface areas and volumes of:
   - compound solids
   - parts of solids.

The Egyptians

The Egyptians must have been very talented architects and surveyors to have planned the many large buildings and monuments built in that country thousands of years ago.

Evidence of the use of mathematics in Egypt in the Old Kingdom (about 2500BCE) is found on a wall near Meidum (south of Cairo); it gives guidelines for the slope of the stepped pyramid built there. The lines in the diagram are spaced at a distance of one cubit. A cubit is the distance from the tip of the finger to the elbow (about 50 cm). These lines show the use of that unit of measurement.

The earliest true mathematical documents date from about 1800BCE. The Moscow Mathematical Papyrus, the Egyptian Mathematical Leather Roll, the Kahun Papyri and the Berlin Papyrus all date to this period.

The Rhind Mathematical Papyrus

The Rhind Mathematical Papyrus, which was written in about 1650BCE, is said to be based on an older mathematical text. The Moscow Mathematical Papyrus and Rhind Mathematical Papyrus are so-called mathematical problem texts. They consist of a collection of mainly mensuration problems with solutions. These could have been written by a teacher for students to solve similar problems to the ones you will work on in this topic.

During the New Kingdom (about 1500–100BCE) papyri record land measurements. In the worker's village of Deir el-Medina, several records have been found that record volumes of dirt removed while digging the underground tombs.

26 Measures

Metric units

The metric system uses a variety of units for length, mass and capacity.

- The common **units of length** are: kilometre (km), metre (m), centimetre (cm) and millimetre (mm).
- The common **units of mass** are: tonne (t), kilogram (kg), gram (g) and milligram (mg).
- The common **units of capacity** are: litre (L or l) and millilitre (ml).

Note: 'centi' comes from the Latin *centum* meaning hundred (a centimetre is one hundredth of a metre);

'milli' comes from the Latin *mille* meaning thousand (a millimetre is one thousandth of a metre);

'kilo' comes from the Greek *khilloi* meaning thousand (a kilometre is one thousand metres).

It may be useful to have some practical experience of estimating lengths, volumes and capacities before starting the following exercises.

Exercise 26.1

Copy and complete the sentences below:

1. a There are ... centimetres in one metre.
   b There are ... millimetres in one metre.
   c One metre is one ... of a kilometre.
   d One milligram is one ... of a gram.
   e One thousandth of a litre is one ... .

2. Which of the units below would be used to measure the following?
   mm, cm, m, km, mg, g, kg, t, ml, litres
   a your height
   b the length of your finger
   c the mass of a shoe
   d the amount of liquid in a cup
   e the height of a van
   f the mass of a ship
   g the capacity of a swimming pool
   h the length of a highway
   i the mass of an elephant
   j the capacity of the petrol tank of a car

Converting from one unit to another

Length

1 km = 1000 m

Therefore 1 m = $\frac{1}{1000}$ km

1 m = 1000 mm

Therefore 1 mm = $\frac{1}{1000}$ m

1 m = 100 cm

Therefore 1 cm = $\frac{1}{100}$ m

1 cm = 10 mm

Therefore 1 mm = $\frac{1}{10}$ cm

Worked examples

a Change 5.8 km into m.
Since 1 km = 1000 m,
5.8 km is 5.8 × 1000 m
5.8 km = 5800 m

b Change 4700 mm to m.
Since 1 m is 1000 mm,
4700 mm is 4700 ÷ 1000 m
4700 mm = 4.7 m

c Convert 2.3 km into cm.
2.3 km is 2.3 × 1000 m = 2300 m
2300 m is 2300 × 100 cm
2.3 km = 230 000 cm

Exercise 26.2

1 Put in the missing unit to make the following statements correct:
a 300 ... = 30 cm **b** 6000 mm = 6 ... **c** 3.2 m = 3200 ...
d 4.2 ... = 4200 mm **e** 2.5 km = 2500 ...

2 Convert the following to millimetres:
a 8.5 cm **b** 23 cm **c** 0.83 m
d 0.05 m **e** 0.004 m

3 Convert the following to metres:
a 560 cm **b** 6.4 km **c** 96 cm
d 0.004 km **e** 12 mm

4 Convert the following to kilometres:
a 1150 m **b** 250 000 m **c** 500 m
d 70 m **e** 8 m

Mass

1 tonne is 1000 kg

Therefore 1 kg = $\frac{1}{1000}$ tonne

1 kilogram is 1000 g

Therefore 1 g = $\frac{1}{1000}$ kg

1 g is 1000 mg

Therefore 1 mg = $\frac{1}{1000}$ g

Worked examples

a Convert 8300 kg to tonnes.
Since 1000 kg = 1 tonne, 8300 kg is 8300 ÷ 1000 tonnes
8300 kg = 8.3 tonnes

b Convert 2.5 g to mg.
Since 1 g is 1000 mg, 2.5 g is 2.5 × 1000 mg
2.5 g = 2500 mg

Exercise 26.3

1 Convert the following:
- **a** 3.8 g to mg
- **b** 28 500 kg to tonnes
- **c** 4.28 tonnes to kg
- **d** 320 mg to g
- **e** 0.5 tonnes to kg

Capacity

1 litre is 1000 millilitres

Therefore 1 ml = $\frac{1}{1000}$ litre

$1\ \text{m}^3 = 1000$ litres

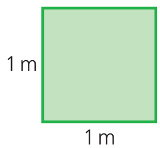
$1\ \text{cm}^3 = 1$ ml

Exercise 26.4

1 Calculate the following and give the totals in ml:
- **a** 3 litres + 1500 ml
- **b** 0.88 litre + 650 ml
- **c** 0.75 litre + 6300 ml
- **d** 450 ml + 0.55 litre

2 Calculate the following and give the total in litres:
- **a** 0.75 litre + 450 ml
- **b** 850 ml + 490 ml
- **c** 0.6 litre + 0.8 litre
- **d** 80 ml + 620 ml + 0.7 litre

Area and volume conversions

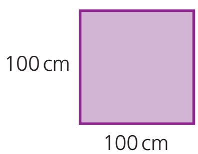


Converting between units for area and volume is not as straightforward as converting between units for length.

The diagram (left) shows a square of side length 1 m.

Area of the square = $1\,\text{m}^2$

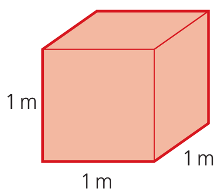
However, if the lengths of the sides are written in cm, each of the sides are 100 cm.



Area of the square = $100 \times 100 = 10\,000\,\text{cm}^2$

Therefore an area of $1\,\text{m}^2 = 10\,000\,\text{cm}^2$.

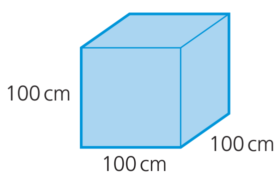
Similarly, a square of side length 1 cm is the same as a square of side length 10 mm. Therefore an area of $1\,\text{cm}^2$ is equivalent to an area of $100\,\text{mm}^2$.



The diagram (left) shows a cube of side length 1 m.

Volume of the cube = $1\,\text{m}^3$

Once again, if the lengths of the sides are written in cm, each of the sides are 100 cm.



Volume of the cube = $100 \times 100 \times 100 = 1\,000\,000\,\text{cm}^3$

Therefore a volume of $1\,\text{m}^3 = 1\,000\,000\,\text{cm}^3$.

Similarly, a cube of side length 1 cm is the same as a cube of side length 10 mm.

Therefore a volume of $1\,\text{cm}^3$ is equivalent to a volume of $1000\,\text{mm}^3$.

Exercise 26.5

1 Convert the following areas:
- a $10\,\text{m}^2$ to cm^2
- b $2\,\text{m}^2$ to mm^2
- c $5\,\text{km}^2$ to m^2
- d $3.2\,\text{km}^2$ to m^2
- e $8.3\,\text{cm}^2$ to mm^2

2 Convert the following areas:
- a $500\,\text{cm}^2$ to m^2
- b $15000\,\text{mm}^2$ to cm^2
- c $1000\,\text{m}^2$ to km^2
- d $40\,000\,\text{mm}^2$ to m^2
- e $2\,500\,000\,\text{cm}^2$ to km^2

3 Convert the following volumes:
- a $2.5\,\text{m}^3$ to cm^3
- b $3.4\,\text{cm}^3$ to mm^3
- c $2\,\text{km}^3$ to m^3
- d $0.2\,\text{m}^3$ to cm^3
- e $0.03\,\text{m}^3$ to mm^3

4 Convert the following volumes:
- a $150\,000\,\text{cm}^3$ to m^3
- b $24000\,\text{mm}^3$ to cm^3
- c $850\,000\,\text{m}^3$ to km^3
- d $300\,\text{mm}^3$ to cm^3
- e $15\,\text{cm}^3$ to m^3

5 Convert the following volumes and capacities:
- a 1.2 litres to cm^3
- b $0.5\,\text{m}^3$ to litres
- c 4250 ml to cm^3
- d 220 litres to m^3

6 A water tank in the shape of a cuboid has dimensions $120 \times 80 \times 50$ cm. It is 75% full. How many litres of water are in the tank?

Student assessment 1

1 Convert the following lengths into the units indicated:
   a 2.6 cm to mm b 0.88 m to cm
   c 6800 m to km d 0.875 km to m

2 Convert the following masses into the units indicated:
   a 4.2 g to mg b 3940 g to kg
   c 4.1 kg to g d 0.72 tonnes to kg

3 Convert the following liquid measures into the units indicated:
   a 1800 ml to litres b 0.083 litre to ml
   c 3.2 litres to ml d 250 000 ml to litres

4 Convert the following areas:
   a 56 cm^2 to mm^2 b 2.05 m^2 to cm^2

5 Convert the following volumes:
   a 8670 cm^3 to m^3 b 444 000 cm^3 to m^3

Student assessment 2

1 Convert the following lengths into the units indicated:
   a 3100 mm to cm b 6.4 km to m
   c 0.4 cm to mm d 460 mm to cm

2 Convert the following masses into the units indicated:
   a 3.6 mg to g b 550 mg to g
   c 6500 g to kg d 1510 kg to tonnes

3 Convert the following measures of capacity to the units indicated:
   a 3400 ml to litres b 6.7 litres to ml
   c 0.73 litre to ml d 300 000 ml to litres

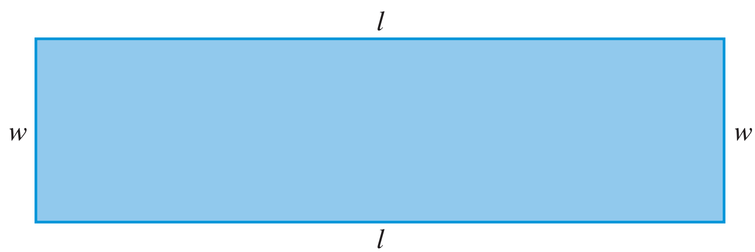
4 Convert the following areas:
   a 0.03 m^2 to mm^2 b 0.005 km^2 to m^2

5 Convert the following volumes:
   a 100 400 cm^3 to m^3 b 5005 m^3 to km^3

27 Perimeter, area and volume

The perimeter and area of a rectangle

The **perimeter** of a shape is the distance around the outside of the shape. Perimeter can be measured in mm, cm, m, km, etc.



The perimeter of the rectangle above of length l and width w is therefore:

$$\text{Perimeter} = l + w + l + w$$

This can be rearranged to give:

$$\text{Perimeter} = 2l + 2w$$

This in turn can be factorised to give:

$$\text{Perimeter} = 2(l + w)$$

The **area** of a shape is the amount of surface that it covers.

Area is measured in mm^2, cm^2, m^2, km^2, etc.

The area A of the rectangle above is given by the formula:

$$A = lw$$

Worked example

Calculate the width of a rectangle of area $200\,cm^2$ and length 25 cm.

$$A = lw$$

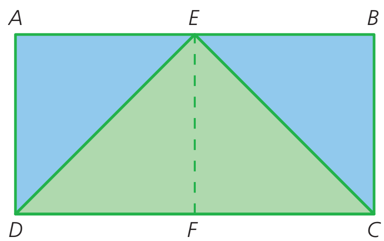
$$200 = 25w$$

$$w = 8$$

So the width is 8 cm.

The area of a triangle

Rectangle $ABCD$ has a triangle CDE drawn inside it.



Point E is said to be a **vertex** of the triangle.

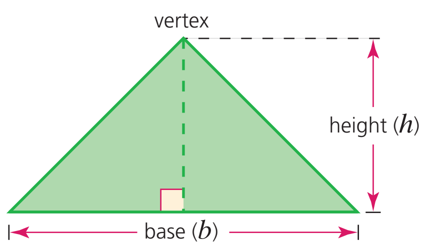
EF is the **height** or **altitude** of the triangle.

CD is the **length** of the rectangle, but is called the **base** of the triangle.

It can be seen from the diagram that triangle DEF is half the area of the rectangle $AEFD$.

Also triangle CFE is half the area of rectangle $EBCF$.

It follows that triangle CDE is half the area of rectangle $ABCD$.



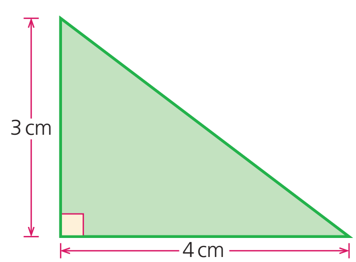
The **area of a triangle** $A = \frac{1}{2}bh$, where b is the base and h is the height.

Note: It does not matter which side is called the base, but the height must be measured at right angles from the base to the opposite vertex.

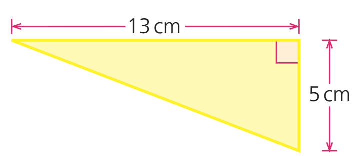
Exercise 27.1

1 Calculate the areas of the triangles below:

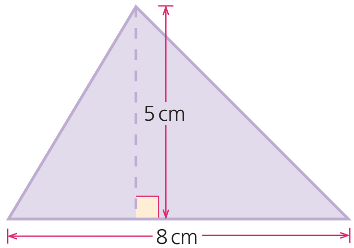
a



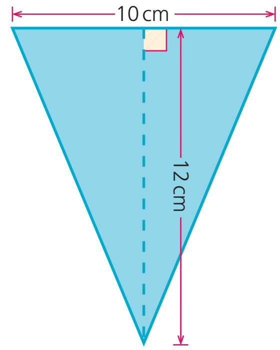
b



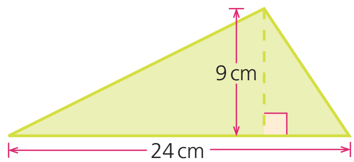
c



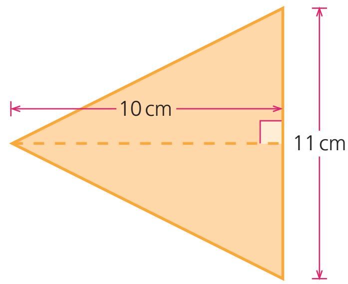
d



e

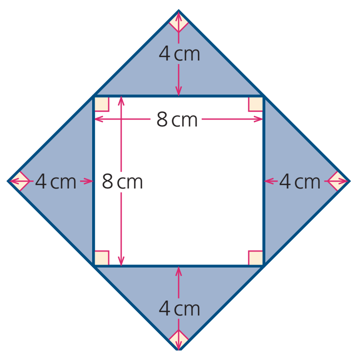


f

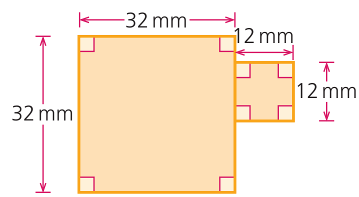


2 Calculate the areas of the shapes below:

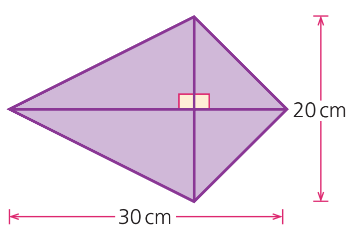
a



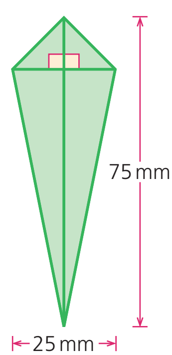
b



c

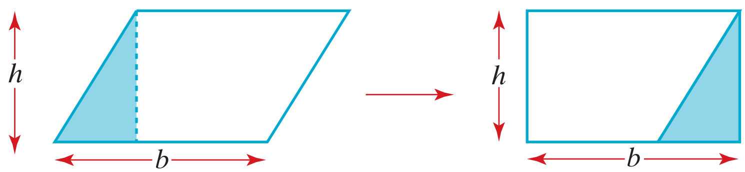


d



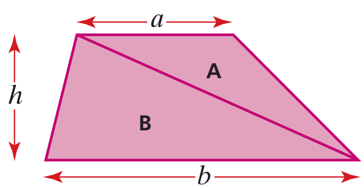
The area of a parallelogram and a trapezium

A **parallelogram** can be rearranged to form a rectangle as shown below:



Therefore: **area of parallelogram** = base length × perpendicular height.

A **trapezium** can be visualised as being split into two triangles as shown:



Area of triangle A $= \frac{1}{2} \times a \times h$

Area of triangle B $= \frac{1}{2} \times b \times h$

Area of the trapezium

= area of triangle A + area of triangle B

$= \frac{1}{2}ah + \frac{1}{2}bh$

$= \frac{1}{2}h(a + b)$

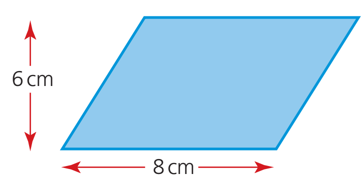
Worked examples

a Calculate the area of the parallelogram.

Area = base length × perpendicular height

= 8 × 6

= 48 cm^2



b Calculate the shaded area in the shape (below).

Area of rectangle $= 12 \times 8$

$= 96\,\text{cm}^2$

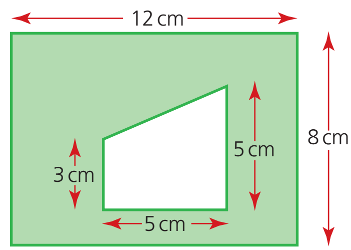
Area of trapezium $= \frac{1}{2} \times 5(3 + 5)$

$= 2.5 \times 8$

$= 20\,\text{cm}^2$

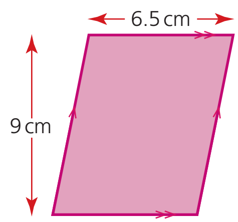
Shaded area $= 96 - 20$

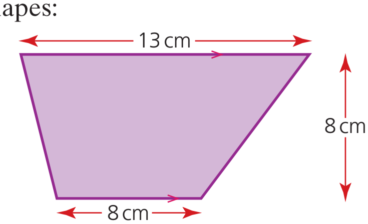
$= 76\,\text{cm}^2$

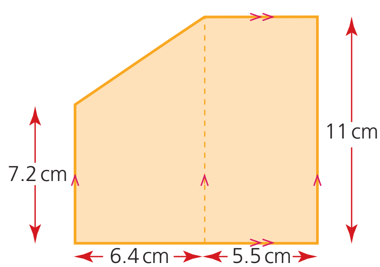


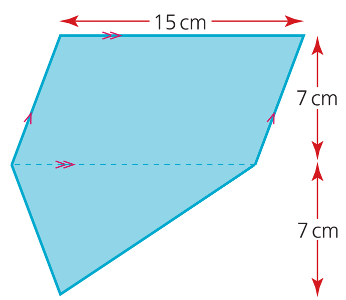
Exercise 27.2

Find the area of each of the following shapes:

1


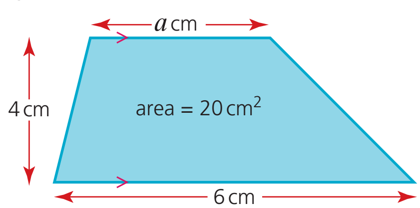
2


3


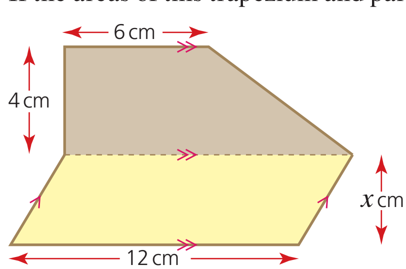
4


Exercise 27.3

1 Calculate a.

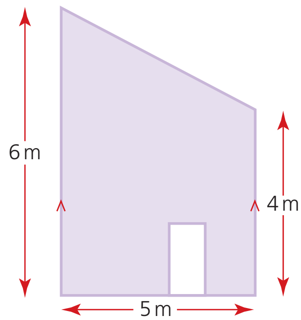


2 If the areas of this trapezium and parallelogram are equal, calculate x.



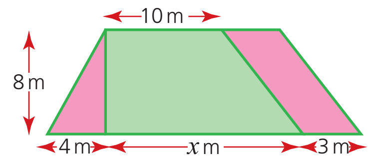
Exercise 27.3 (cont)

3 The end view of a house is as shown in the diagram (below). If the door has a width and height of 0.75 m and 2 m respectively, calculate the area of brickwork.

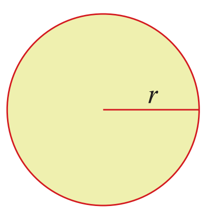


4 A garden in the shape of a trapezium is split into three parts: a flower bed in the shape of a triangle, a flower bed in the shape of a parallelogram and a section of grass in the shape of a trapezium, as shown (right). The area of the grass is two and a half times the total area of flower beds. Calculate:

a the area of each flower bed,
b the area of grass,
c the value of x.

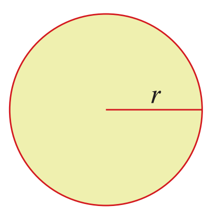


The circumference and area of a circle



The circumference is $2\pi r$.

$C = 2\pi r$



The area is πr^2.

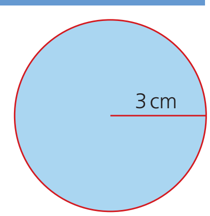
$A = \pi r^2$

Note

You should always use the π button on your calculator unless a question says otherwise.

Worked examples

a Calculate the circumference of this circle, giving your answer to 3 s.f.



$C = 2\pi r$

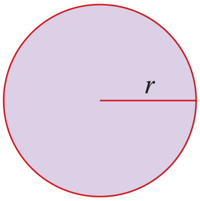
$= 2\pi \times 3 = 18.8$

The circumference is 18.8 cm.

Note: The answer 18.8 cm is only correct to 3 s.f. and therefore only an approximation. An **exact** answer involves leaving the answer in terms of π.

i.e. $C = 2\pi r$
$= 2\pi \times 3$
$= 6\pi$ cm

b If the circumference of this circle is 12 cm, calculate the radius, giving your answer



i to 3 s.f.
ii in terms of π

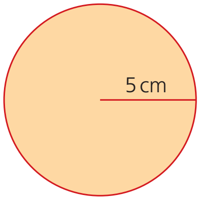
i $C = 2\pi r$

$r = \frac{12}{2\pi}$
$= 1.91$

The radius is 1.91 cm.

ii $r = \frac{C}{2\pi} = \frac{12}{2\pi}$
$= \frac{6}{\pi}$ cm

c Calculate the area of this circle, giving your answer



i to 3 s.f.
ii in exact form

i $A = \pi r^2$
$= \pi \times 5^2 = 78.5$

The area is 78.5 cm^2.

ii $A = \pi r^2$
$= \pi \times 5^2$
$= 25\pi$ cm^2

d The area of a circle is 34 cm^2, calculate the radius, giving your answer

i to 3 s.f.
ii in terms of π

i $A = \pi r^2$

$r = \sqrt{\frac{A}{\pi}}$

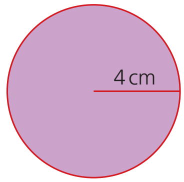
$r = \sqrt{\frac{34}{\pi}} = 3.29$

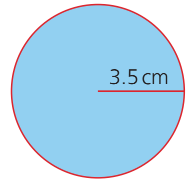
The radius is 3.29 cm.

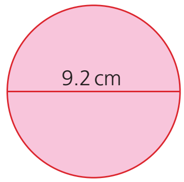
ii $r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{34}{\pi}}$ cm

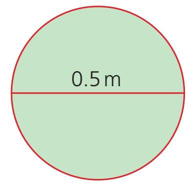
Exercise 27.4

1 Calculate the circumference of each circle, giving your answer to 3 s.f.

a


b


c


d


2 Calculate the area of each of the circles in Question 1. Give your answers to 3 s.f.

3 Calculate the radius of a circle when the circumference is:
a 15 cm
b π cm
c 4 m
d 8 mm

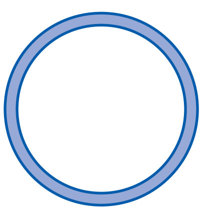
4 Calculate the diameter of a circle when the area is:
a 16 cm^2
b 9π cm^2
c 8.2 m^2
d 14.6 mm^2

Exercise 27.5

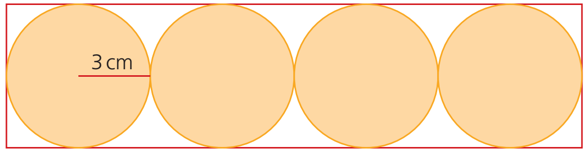
1 The wheel of a car has an outer radius of 25 cm. Calculate:
a how far the car has travelled after one complete turn of the wheel,
b how many times the wheel turns for a journey of 1 km.

2 The back wheel of a wheelchair has a diameter of 60 cm. Calculate how far a the wheelchair has travelled after it has rotated 100 times.

3 A circular ring has a cross-section as shown. If the outer radius is 22 mm and the inner radius 20 mm, calculate the cross-sectional area of the ring. Give your answer in terms of π.



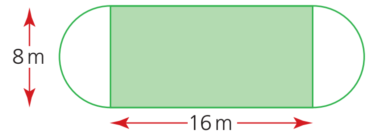
4 Four circles are drawn in a line and enclosed by a rectangle as shown (below). If the radius of each circle is 3 cm, calculate the unshaded area within the rectangle. Give your answer in exact form.



5 A garden is made up of a rectangular patch of grass and two semicircular vegetable patches. If the dimensions of the rectangular patch are 16 m (length) and 8 m (width) respectively, calculate in exact form:

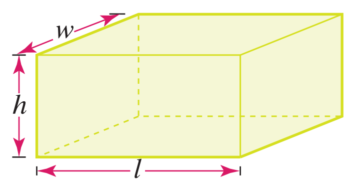
a the perimeter of the garden,

b the total area of the garden.



The surface area of a cuboid and a cylinder

To calculate the surface area of a **cuboid**, start by looking at its individual faces. These are either squares or rectangles. The surface area of a cuboid is the sum of the areas of its faces.



Area of top = wl Area of bottom = wl

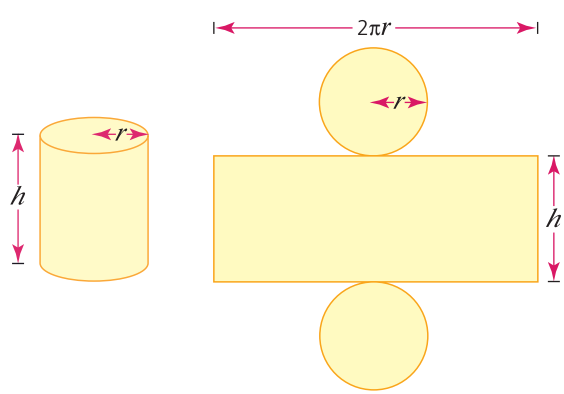
Area of front = lh Area of back = lh

Area of one side = wh Area of other side = wh

Total surface area

$$= 2wl + 2lh + 2wh$$
$$= 2(wl + lh + wh)$$

For the surface area of a **cylinder**, it is best to visualise the net of the solid: it is made up of one rectangular piece and two circular pieces.



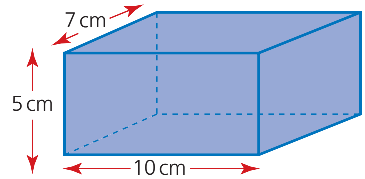
Area of circular pieces = $2 \times \pi r^2$

Area of rectangular piece = $2\pi r \times h$

Total surface area = $2\pi r^2 + 2\pi rh$

$= 2\pi r(r + h)$

Worked examples



a Calculate the surface area of the cuboid shown.

Total area of top and bottom $= 2 \times 7 \times 10 = 140\,\text{cm}^2$

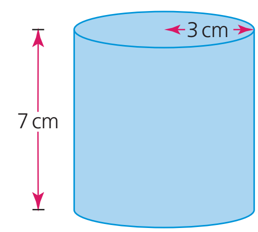
Total area of front and back $= 2 \times 5 \times 10 = 100\,\text{cm}^2$

Total area of both sides $= 2 \times 5 \times 7 = 70\,\text{cm}^2$

Total surface area $= 310\,\text{cm}^2$

If the answer is left in terms of π, it is an exact answer. The answer of $188\,\text{cm}^2$ is only an approximation correct to three significant figures.

b If the height of a cylinder is 7 cm and the radius of its circular top is 3 cm, calculate its surface area.



Total surface area $= 2\pi r(r + h)$

$= 2\pi \times 3 \times (3 + 7)$

$= 6\pi \times 10$

$= 60\pi$

$= 188\,\text{cm}^2$ (3 s.f.)

The total surface area is $188\,\text{cm}^2$.

Exercise 27.6

1 Calculate the surface area of each of the following cuboids:

a $l = 12\,\text{cm}$, $w = 10\,\text{cm}$, $h = 5\,\text{cm}$
b $l = 4\,\text{cm}$, $w = 6\,\text{cm}$, $h = 8\,\text{cm}$
c $l = 4.2\,\text{cm}$, $w = 7.1\,\text{cm}$, $h = 3.9\,\text{cm}$
d $l = 5.2\,\text{cm}$, $w = 2.1\,\text{cm}$, $h = 0.8\,\text{cm}$

2 Calculate the height of each of the following cuboids:

a $l = 5\,\text{cm}$, $w = 6\,\text{cm}$, surface area $= 104\,\text{cm}^2$
b $l = 2\,\text{cm}$, $w = 8\,\text{cm}$, surface area $= 112\,\text{cm}^2$
c $l = 3.5\,\text{cm}$, $w = 4\,\text{cm}$, surface area $= 118\,\text{cm}^2$
d $l = 4.2\,\text{cm}$, $w = 10\,\text{cm}$, surface area $= 226\,\text{cm}^2$

3 Calculate the surface area of each of the following cylinders.

i Give your answer in terms of π.
ii Give your answer correct to 3 s.f.

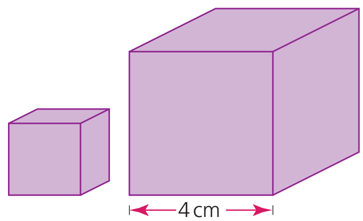
a $r = 2\,\text{cm}$, $h = 6\,\text{cm}$
b $r = 4\,\text{cm}$, $h = 7\,\text{cm}$
c $r = 3.5\,\text{cm}$, $h = 9.2\,\text{cm}$
d $r = 0.8\,\text{cm}$, $h = 4.3\,\text{cm}$

4 Calculate the height of each of the following cylinders. Give your answers to 1 d.p.

a $r = 2.0\,\text{cm}$, surface area $= 40\,\text{cm}^2$
b $r = 3.5\,\text{cm}$, surface area $= 88\,\text{cm}^2$
c $r = 5.5\,\text{cm}$, surface area $= 250\,\text{cm}^2$
d $r = 3.0\,\text{cm}$, surface area $= 189\,\text{cm}^2$

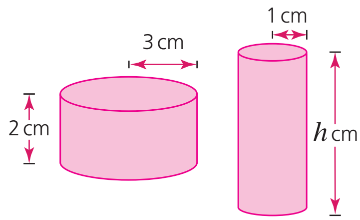
Exercise 27.7

1. Two cubes are placed next to each other. The length of each of the edges of the larger cube is 4 cm. If the ratio of their surface areas is 1 : 4, calculate:
   a the surface area of the small cube,
   b the length of an edge of the small cube.



2. A cube and a cylinder have the same surface area. If the cube has an edge length of 6 cm and the cylinder a radius of 2 cm, calculate:
   a the surface area of the cube,
   b the height of the cylinder.

3. Two cylinders have the same surface area.
   The shorter of the two has a radius of 3 cm and a height of 2 cm, and the taller cylinder has a radius of 1 cm. Calculate:

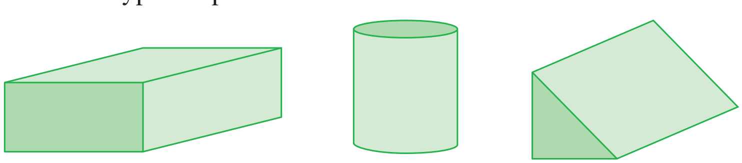


   a the surface area of one of the cylinders in terms of π,
   b the height of the taller cylinder.

4. Two cuboids have the same surface area. The dimensions of one of them are: length = 3 cm, width = 4 cm and height = 2 cm.
   Calculate the height of the other cuboid if its length is 1 cm and its width is 4 cm.

The volume and surface area of a prism

A prism is any three-dimensional object which has a constant cross-sectional area. Below are a few examples of some of the more common types of prism.



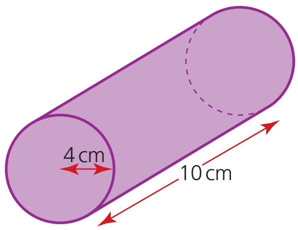
Rectangular prism (cuboid) | Circular prism (cylinder) | Triangular prism

When each of the shapes is cut parallel to the shaded face, the cross-section is constant and the shape is therefore classified as a prism.

Volume of a prism = area of cross-section × length

Surface area of a prism = sum of the areas of each of its faces

Worked examples



a Calculate the volume of the cylinder in the diagram.

Volume = cross-sectional area × length

$= \pi \times 4^2 \times 10$

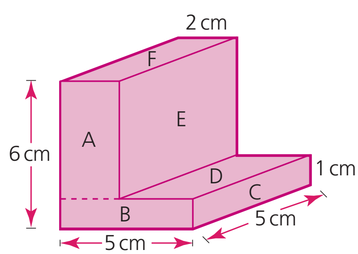
Volume = $503\,\text{cm}^3$ (3 s.f.)

As an exact value, the volume would be left as $160\pi\,\text{cm}^3$

b Calculate the

i volume and

ii surface area of the 'L' shaped prism in the diagram.



The cross-sectional area can be split into two rectangles:

i Area of rectangle A = 5×2
$= 10\,\text{cm}^2$

Area of rectangle B = 5×1
$= 5\,\text{cm}^2$

Total cross-sectional area = $(10\,\text{cm}^2 + 5\,\text{cm}^2) = 15\,\text{cm}^2$

Volume of prism = 15×5
$= 75\,\text{cm}^3$

ii Area of rectangle A = $5 \times 2 = 10$ cm^2
Area of rectangle B = $5 \times 1 = 5$ cm^2
Area of rectangle C = $5 \times 1 = 5$ cm^2
Area of rectangle D = $3 \times 5 = 15$ cm^2
Area of rectangle E = $5 \times 5 = 25$ cm^2
Area of rectangle F = $2 \times 5 = 10$ cm^2
Area of back is the same as A + B = 15 cm^2
Area of left face is the same as C + E = 30 cm^2
Area of base = $5 \times 5 = 25$ cm^2
Total surface area = $10 + 5 + 5 + 15 + 25 + 10 + 15 + 30 + 25 = 140$ cm^2

Exercise 27.8

1 Calculate the volume of each of the following cuboids, where w, l and h represent the width, length and height respectively.

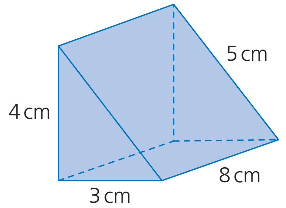
a $w = 2$ cm, $l = 3$ cm, $h = 4$ cm
b $w = 6$ cm, $l = 1$ cm, $h = 3$ cm
c $w = 6$ cm, $l = 23$ mm, $h = 2$ cm
d $w = 42$ mm, $l = 3$ cm, $h = 0.007$ m

2 Calculate the volume of each of the following cylinders, where r represents the radius of the circular face and h the height of the cylinder.

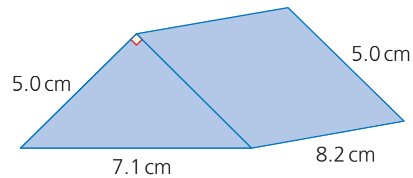
a $r = 4$ cm, $h = 9$ cm
b $r = 3.5$ cm, $h = 7.2$ cm
c $r = 25$ mm, $h = 10$ cm
d $r = 0.3$ cm, $h = 17$ mm

3 Calculate the volume and total surface area of each of the following right-angled triangular prisms.

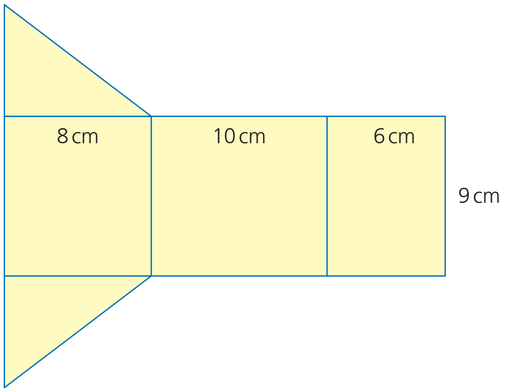
a



b



4 The diagram below shows the net of a right-angled triangular prism.



Exercise 27.8 (cont)

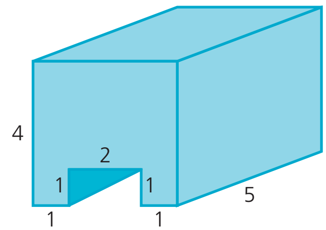
Calculate:

a The surface area of the prism.

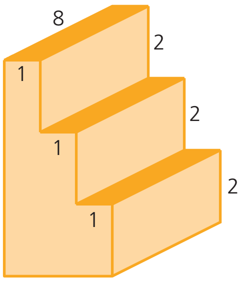
b The volume of the prism.

5 Calculate the volume of each of the following prisms. All dimensions are given in centimetres.

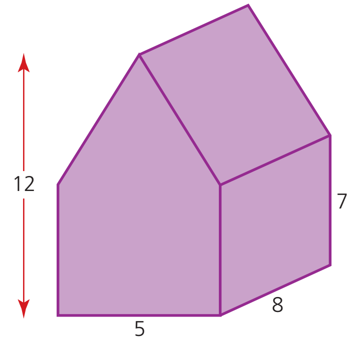
a



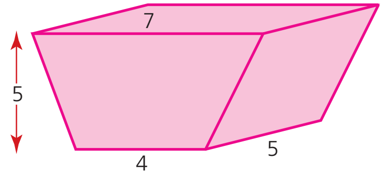
b



c

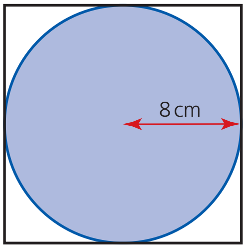


d



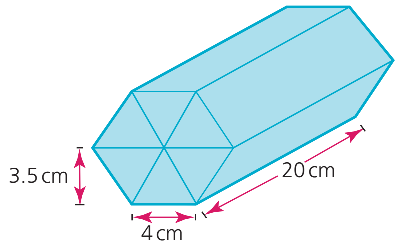
Exercise 27.9

1 The diagram shows a plan view of a cylinder inside a box the shape of a cube. If the radius of the cylinder is 8 cm, calculate the percentage volume of the cube not occupied by the cylinder.



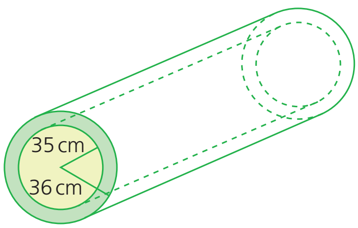
2 A chocolate bar is made in the shape of a triangular prism. The triangular face of the prism is equilateral and has an edge length of 4 cm and a perpendicular height of 3.5 cm. The manufacturer also sells these in special packs of six bars arranged as a hexagonal prism. If the prisms are 20 cm long, calculate:

a the cross-sectional area of the pack,
b the volume of the pack.



3 A cuboid and a cylinder have the same volume. The radius and height of the cylinder are 2.5 cm and 8 cm respectively. If the length and width of the cuboid are each 5 cm, calculate its height to 1 d.p.

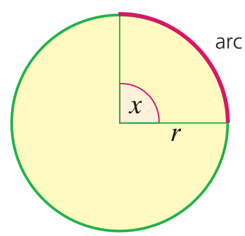
4 A section of steel pipe is shown in the diagram. The inner radius is 35 cm and the outer radius 36 cm.



Calculate the volume of steel used in making the pipe if it has a length of 130 m. Give your answer in terms of π.

Arc length

An **arc** is part of the circumference of a circle between two radii.



Its length is proportional to the size of the angle x between the two radii. The length of the arc as a fraction of the circumference of the whole circle is therefore equal to the fraction that x is of 360°.

$$\text{Arc length} = \frac{x}{360} \times 2\pi r$$

Worked examples

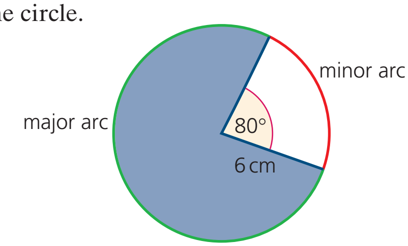
a Find the length of the minor arc in the circle.

i Give your answer to 3 s.f.

$$\text{Arc length} = \frac{80}{360} \times 2 \times \pi \times 6$$
$$= 8.38\text{ cm}$$

ii Give your answer in terms of π

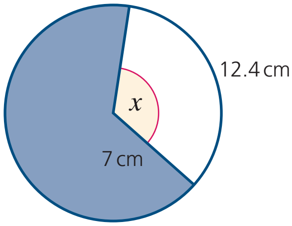
$$\text{Arc length} = \frac{80}{360} \times 2 \times \pi \times 6$$
$$= \frac{8}{3}\pi\text{ cm}$$



Note

The Core syllabus only requires calculations involving factors of 360.

b In the circle the length of the minor arc is 12.4 cm and the radius is 7 cm.



i Calculate the angle x.

Arc length $= \frac{x}{360} \times 2\pi r$

$$12.4 = \frac{x}{360} \times 2 \times \pi \times 7$$

$$\frac{12.4 \times 360}{2 \times \pi \times 7} = x$$

$$= 101.5° \text{ (1 d.p.)}$$

ii Calculate the length of the major arc.

$C = 2\pi r$

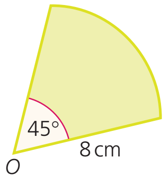
$= 2 \times \pi \times 7 = 44.0$ cm (3 s.f.)

Major arc = circumference − minor arc = (44.0 − 12.4) = 31.6 cm

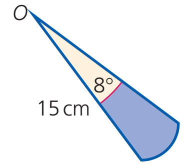
Exercise 27.10

1 For each of the following, give the length of the arc to 3 s.f. O is the centre of the circle.

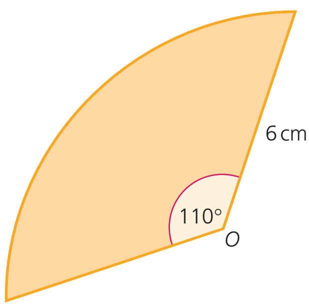
a



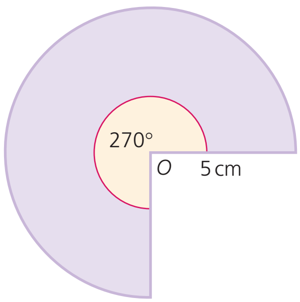
b



c



d



2 A sector is the region of a circle enclosed by two radii and an arc. Calculate the angle x for each of the following sectors. The radius r and arc length a are given in each case.

a $r = 14$ cm, $a = 8$ cm
b $r = 4$ cm, $a = 16$ cm
c $r = 7.5$ cm, $a = 7.5$ cm
d $r = 6.8$ cm, $a = 13.6$ cm

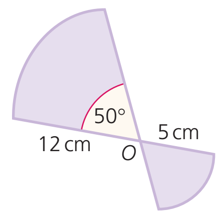
3 Calculate the radius r for each of the following sectors. The angle x and arc length a are given in each case.

a $x = 75°$, $a = 16$ cm
b $x = 300°$, $a = 24$ cm
c $x = 20°$, $a = 6.5$ cm
d $x = 243°$, $a = 17$ cm

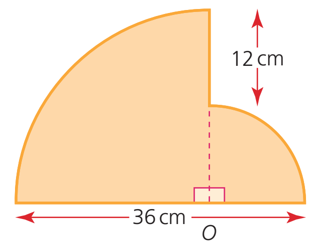
Exercise 27.11

1 Calculate the perimeter of each of these shapes. Give your answers in exact form.

a



b

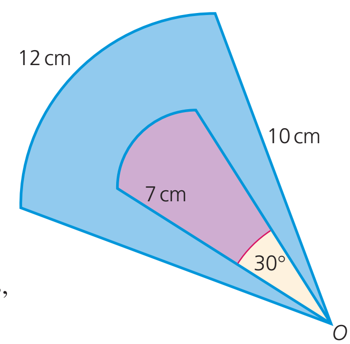


2 A shape is made from two sectors arranged in such a way that they share the same centre.
The radius of the smaller sector is 7 cm and the radius of the larger sector is 10 cm. If the angle at the centre of the smaller sector is 30° and the arc length of the larger sector is 12 cm, calculate:

a the arc length of the smaller sector,

b the total perimeter of the two sectors,

c the angle at the centre of the larger sector.

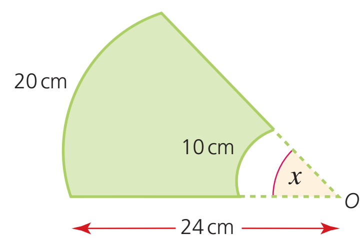


3 For the diagram (right), calculate:

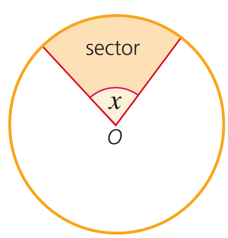
a the radius of the smaller sector,

b the perimeter of the shape,

c the angle x.



The area of a sector

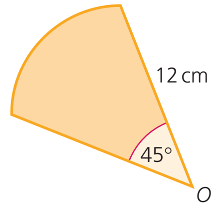


A **sector** is the region of a circle enclosed by two radii and an arc. Its area is proportional to the size of the angle x between the two radii. The area of the sector as a fraction of the area of the whole circle is therefore equal to the fraction that x is of 360°.

Area of sector $= \frac{x}{360} \times \pi r^2$

Worked examples

a Calculate the area of the sector, giving your answer
i to 3 s.f. **ii** in terms of π

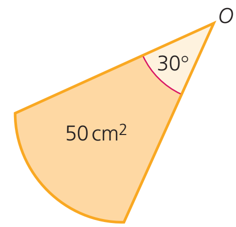


i Area $= \frac{45}{360} \times \pi \times 12^2$

$= 56.5\,\text{cm}^2$

ii Area $= 18\pi\,\text{cm}^2$

b Calculate the radius of the sector, giving your answer to 3 s.f.



Area $= \frac{x}{360} \times \pi r^2$

$50 = \frac{30}{360} \times \pi \times r^2$

$\frac{50 \times 360}{30\pi} = r^2$

$r = 13.8$

The radius is 13.8 cm.

Exercise 27.12

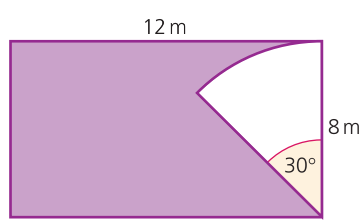
1 Calculate the area of each of the following sectors, using the values of the angle x and radius r in each case.
a $x = 60°$, $r = 8\,\text{cm}$
b $x = 120°$, $r = 14\,\text{cm}$
c $x = 2°$, $r = 18\,\text{cm}$
d $x = 320°$, $r = 4\,\text{cm}$

2 Calculate the radius for each of the following sectors, using the values of the angle x and the area A in each case.
a $x = 40°$, $A = 120\,\text{cm}^2$
b $x = 12°$, $A = 42\,\text{cm}^2$
c $x = 150°$, $A = 4\,\text{cm}^2$
d $x = 300°$, $A = 400\,\text{cm}^2$

3 Calculate the value of the angle x, to the nearest degree, for each of the following sectors, using the values of r and A in each case.
a $r = 12\,\text{cm}$, $A = 60\,\text{cm}^2$
b $r = 26\,\text{cm}$, $A = 0.02\,\text{m}^2$
c $r = 0.32\,\text{m}$, $A = 180\,\text{cm}^2$
d $r = 38\,\text{mm}$, $A = 16\,\text{cm}^2$

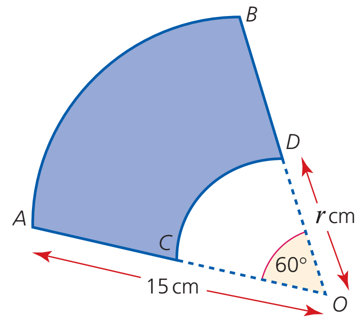
Exercise 27.13

1 A rotating sprinkler is placed in one corner of a garden (below). If it has a reach of 8 m and rotates through an angle of 30°, calculate the area of garden not being watered. Give your answer in terms of π.



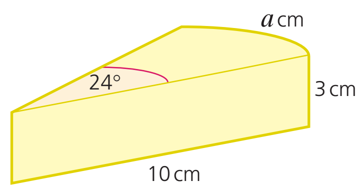
2 Two sectors AOB and COD share the same centre O. The area of AOB is three times the area of COD. Calculate:

a the area of sector AOB,
b the area of sector COD,
c the radius r cm of sector COD.



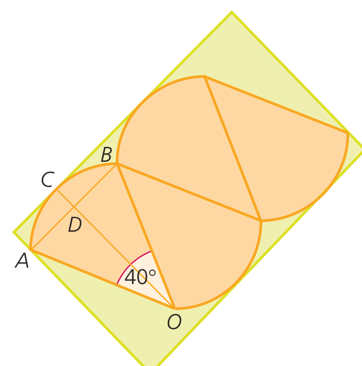
3 A circular cake is cut. One of the slices is shown. Calculate:

a the length a cm of the arc,
b the total surface area of all the sides of the slice,
c the volume of the slice.



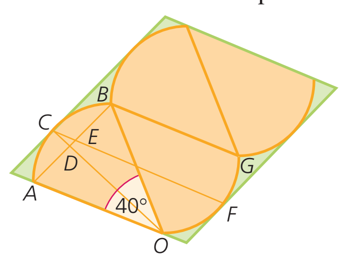
4 The diagram shows a plan view of four tiles in the shape of sectors placed in the bottom of a box. C is the midpoint of the arc AB and intersects the chord AB at point D. If the length OB is 10 cm, calculate:

a the length OD,
b the length CD,
c the area of the sector AOB,
d the length and width of the box,
e the area of the base of the box not covered by the tiles.



5 The tiles in Question 4 are repackaged and are now placed in a box, the base of which is a parallelogram. Given that C and F are the midpoints of arcs AB and OG respectively, calculate:

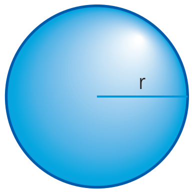
a the angle OCF,
b the length CE,
c the length of the sides of the box,
d the area of the base of the box not covered by the tiles.



The volume of a sphere

You should know how to use this formula for the volume of a sphere, but you do not need to memorise it.

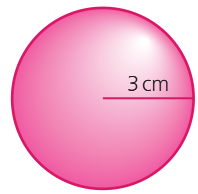
Volume of sphere $= \frac{4}{3}\pi r^3$



Worked examples

a Calculate the volume of the sphere, giving your answer:

i to 3 s.f. **ii** in terms of π



i Volume of sphere $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times 3^3$

$= 113.1$

The volume is $113\,\text{cm}^3$.

ii Volume of sphere $= \frac{4}{3}\pi \times 3^3$

$= 36\pi\,\text{cm}^3$

b Given that the volume of a sphere is $150\,\text{cm}^3$, calculate its radius to 3 s.f.

$V = \frac{4}{3}\pi r^3$

$r^3 = \frac{3V}{4\pi}$

$r^3 = \frac{3 \times 150}{4 \times \pi}$

$r = \sqrt[3]{35.8} = 3.30$

The radius is 3.30 cm.

Exercise 27.14

1 Calculate the volume of each of the following spheres. The radius r is given in each case.

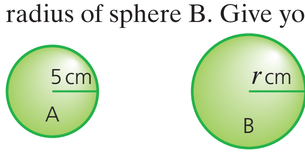
a $r = 6\,\text{cm}$ **b** $r = 9.5\,\text{cm}$
c $r = 8.2\,\text{cm}$ **d** $r = 0.7\,\text{cm}$

2 Calculate the radius of each of the following spheres. Give your answers in centimetres and to 1 d.p. The volume V is given in each case.

a $V = 130\,\text{cm}^3$ **b** $V = 720\,\text{cm}^3$
c $V = 0.2\,\text{m}^3$ **d** $V = 1000\,\text{mm}^3$

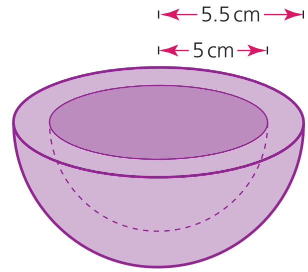
Exercise 27.15

1 Given that sphere B has twice the volume of sphere A, calculate the radius of sphere B. Give your answer to 1 d.p.

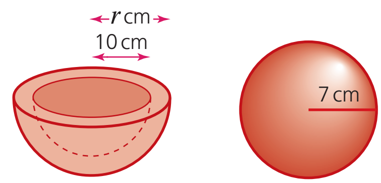


> **Note**
>
> A hemisphere is half a sphere.

2 Calculate the volume of material used to make the hemispherical bowl if the inner radius of the bowl is 5 cm and its outer radius 5.5 cm. Give your answer in terms of π.



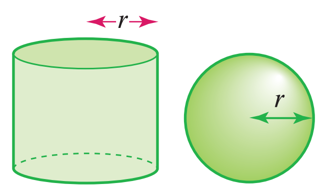
3 The volume of the material used to make the sphere and hemispherical bowl are the same. Given that the radius of the sphere is 7 cm and the inner radius of the bowl is 10 cm, calculate, to 1 d.p., the outer radius r cm of the bowl.



4 A steel ball is melted down to make eight smaller identical balls. If the radius of the original steel ball was 20 cm, calculate the radius of each of the smaller balls.

5 A steel ball of volume 600 cm^3 is melted down and made into three smaller balls A, B and C. If the volumes of A, B and C are in the ratio 7 : 5 : 3, calculate to 1 d.p. the radius of each of A, B and C.

6 The cylinder and sphere shown have the same radius and the same height. Calculate the ratio of their volumes, giving your answer in the form volume of cylinder : volume of sphere.



The surface area of a sphere

You should know how to use this formula for the surface area of a sphere, but you do not need to memorise it. →

Surface area of sphere $= 4\pi r^2$

Exercise 27.16

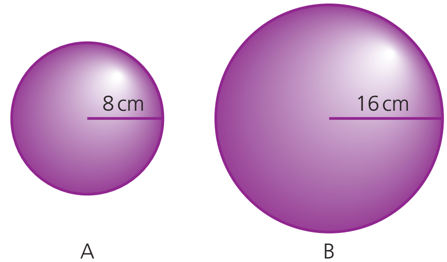
1 Calculate the surface area of each of the following spheres when:

a $r = 6$ cm **b** $r = 4.5$ cm
c $r = 12.25$ cm **d** $r = \frac{1}{\sqrt{\pi}}$ cm

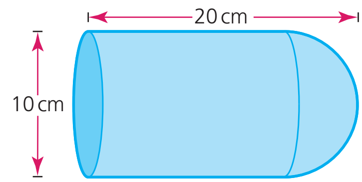
2 Calculate the radius of each of the following spheres, given the surface area in each case.

a $A = 50$ cm^2 **b** $A = 16.5$ cm^2
c $A = 120$ mm^2 **d** $A = \pi$ cm^2

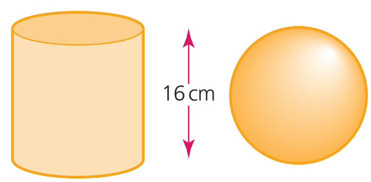
3 Sphere A has a radius of 8 cm and sphere B has a radius of 16 cm. Calculate the ratio of their surface areas in the form $1:n$.



4 A hemisphere of diameter 10 cm is attached to a cylinder of equal diameter as shown. If the total length of the shape is 20 cm, calculate the surface area of the whole shape.

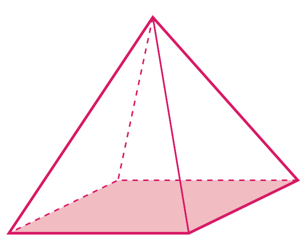


5 A sphere and a cylinder both have the same surface area and the same height of 16 cm. Calculate the radius of the cylinder.

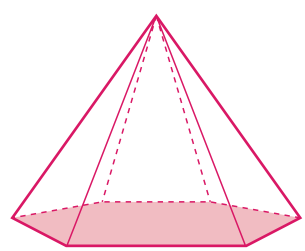


The volume of a pyramid

A pyramid is a three-dimensional shape in which each of its faces must be plane. A pyramid has a polygon for its base and the other faces are triangles with a common vertex, known as the **apex**. Its individual name is taken from the shape of the base.



Square-based pyramid

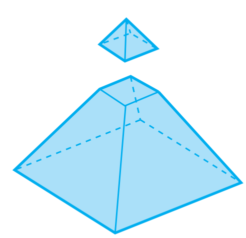


Hexagonal-based pyramid

You should know how to use this formula for the volume of a pyramid, but you do not need to memorise it.

Volume of any pyramid

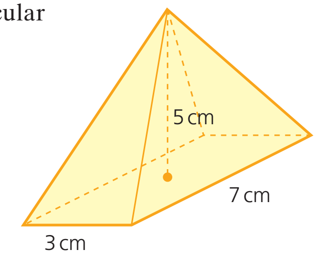
$= \frac{1}{3} \times$ area of base $\times$ perpendicular height



With a pyramid, if a cut is made parallel to the base and the top part of the pyramid removed, the shape that is left is known as a '**frustum**' or a '**truncated pyramid**'.

Worked examples

a A rectangular-based pyramid has a perpendicular height of 5 cm and base dimensions as shown. Calculate the volume of the pyramid.

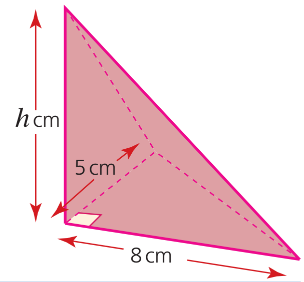


$$\text{Volume} = \frac{1}{3} \times \text{base area} \times \text{height}$$

$$= \frac{1}{3} \times 3 \times 7 \times 5 = 35$$

The volume is 35 cm^3.

b The pyramid shown has a volume of 60 cm^3. Calculate its perpendicular height h cm.



$$\text{Volume} = \frac{1}{3} \times \text{base area} \times \text{height}$$

$$\text{Height} = \frac{3 \times \text{volume}}{\text{base area}}$$

$$h = \frac{3 \times 60}{\frac{1}{2} \times 8 \times 5}$$

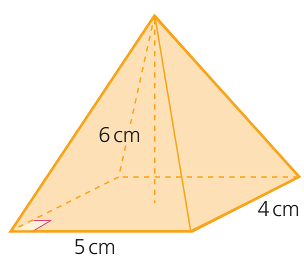
$$h = 9$$

The height is 9 cm.

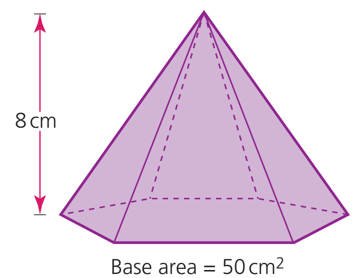
Exercise 27.17

Find the volume of each of the following pyramids:

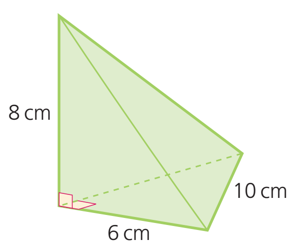
1



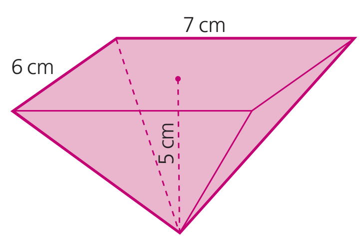
2



3

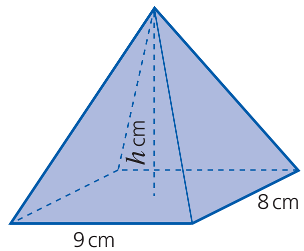


4

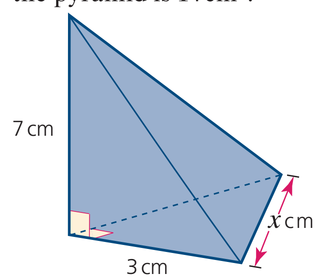


Exercise 27.18

1 Calculate the perpendicular height h cm for the pyramid, given that it has a volume of $168\,cm^3$.

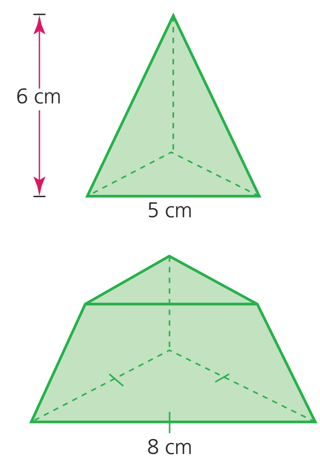


2 Calculate the length of the edge marked x cm, given that the volume of the pyramid is $14\,cm^3$.



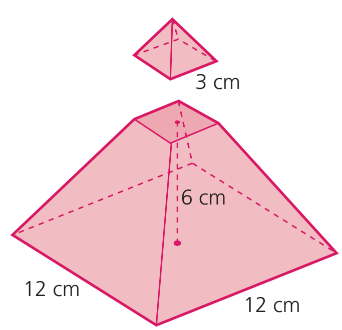
3 The top of a triangular-based pyramid is cut off. The cut is made parallel to the base. If the vertical height of the top is 6 cm, calculate:

a the height of the a truncated pyramid,
b the volume of the small pyramid,
c the volume of the original pyramid.



4 The top of a square-based pyramid (right) is cut off. The cut is made parallel to the base. If the base of the smaller pyramid has a side length of 3 cm and the vertical height of the truncated pyramid is 6 cm, calculate:

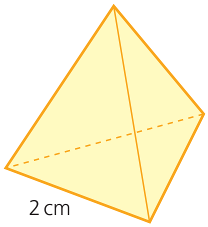
a the height of the original pyramid,
b the volume of the original pyramid,
c the volume of the truncated pyramid.



The surface area of a pyramid

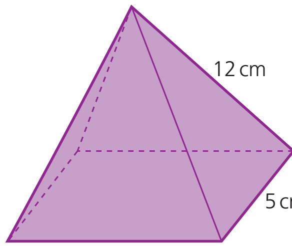
The surface area of a pyramid is found simply by adding together the areas of all of its faces.

Exercise 27.19

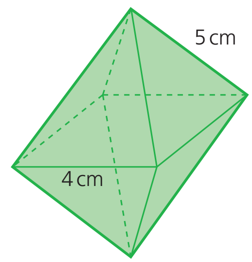


1 Calculate the surface area of a regular tetrahedron with edge length 2 cm.

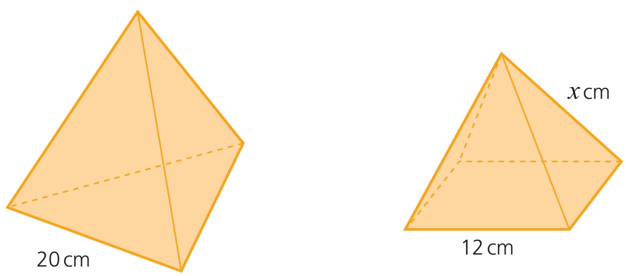
2 The rectangular-based pyramid shown has a sloping edge length of 12 cm. Calculate its surface area.



3 Two square-based pyramids are glued together as shown (right). Given that all the triangular faces are congruent, calculate the surface area of the whole shape.



4 The two pyramids shown below have the same surface area.



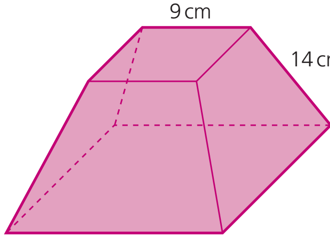
Calculate:

a the surface area of the regular tetrahedron,

b the area of one of the triangular faces on the square-based pyramid,

c the value of x.

5 Calculate the surface area of the frustum shown. Assume that all the sloping faces are congruent.

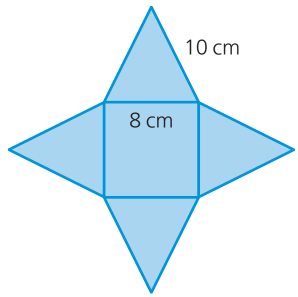


Exercise 27.19 (cont)

Note

To help you answer this question, you can refer to the section on Pythagoras' theorem applied to three dimensions, in Chapter 30.

6 The diagram below shows the net of a square-based pyramid, with dimensions as shown.



Calculate, giving your answers to 3 s.f.:

a the total surface area of the pyramid,

b the total volume of the pyramid.

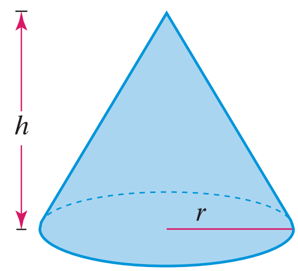
The volume of a cone

A cone is a pyramid with a circular base. The formula for its volume is therefore the same as for any other pyramid.

You should know how to use the formula for the volume of a cone, but you do not need to memorise it.

$$\text{Volume} = \frac{1}{3} \times \text{base area} \times \text{height}$$

$$= \frac{1}{3}\pi r^2 h$$



Worked examples

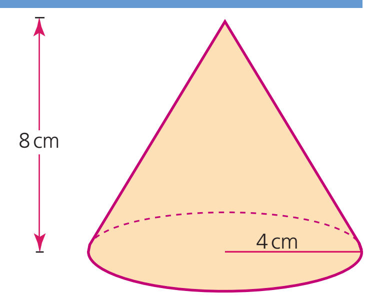
a Calculate the **volume of the cone**.

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3} \times \pi \times 4^2 \times 8$$

$$= 134.0 \text{ (1 d.p.)}$$

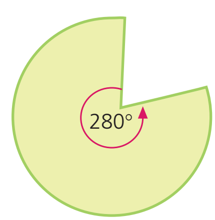
The volume is 134 cm^3 (3 s.f.).

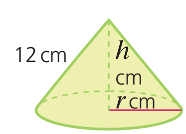


b The sector below is assembled to form a cone as shown.

i Calculate, in terms of π, the base circumference of the cone.

The base circumference of the cone is equal to the arc length of the sector.
The radius of the sector is equal to the slant height of the cone (i.e. 12 cm).





Sector arc length $= \frac{\phi}{360} \times 2\pi r$

$= \frac{280}{360} \times 2\pi \times 12 = \frac{56}{3}\pi$

So the base circumference is $\frac{56}{3}\pi$ cm.

ii Calculate, in exact form, the base radius of the cone.

The base of a cone is circular, therefore:

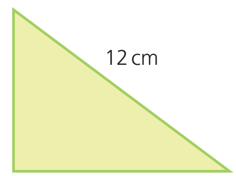
$$C = 2\pi r$$

$$r = \frac{C}{2\pi}$$

$$= \frac{\frac{56}{3}\pi}{2\pi} = \frac{56}{6} = \frac{28}{3}$$

So the radius is $\frac{28}{3}$ cm.

iii Calculate the exact height of the cone.



The vertical height of the cone can be calculated using Pythagoras' theorem on the right-angled triangle enclosed by the base radius, vertical height and the sloping face, as shown (right).

Note that the length of the sloping face is equal to the radius of the sector.

$$12^2 = h^2 + \left(\frac{28}{3}\right)^2$$

$$h^2 = 12^2 - \left(\frac{28}{3}\right)^2$$

$$h^2 = \frac{512}{9}$$

$$h = \frac{\sqrt{512}}{3} = \frac{16\sqrt{2}}{3}$$

Therefore the vertical height is $\frac{16\sqrt{2}}{3}$ cm.

iv Calculate the volume of the cone, leaving your answer both in terms of π and to 3 s.f.

Volume $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times \left(\frac{28}{3}\right)^2 \times \frac{16\sqrt{2}}{3}$

$= \frac{12544\sqrt{2}}{81}\pi \text{ cm}^3$

$= 688 \text{ cm}^3$

Note

In the worked examples, the previous answer was used to calculate the next stage of the question. By using exact values each time, you will avoid introducing rounding errors into the calculation.

Exercise 27.20

1 Calculate the volume of each of the following cones. Use the values for the base radius r and the vertical height h given in each case.

a $r = 3$ cm, $h = 6$ cm
b $r = 6$ cm, $h = 7$ cm
c $r = 8$ mm, $h = 2$ cm
d $r = 6$ cm, $h = 44$ mm

2 Calculate the base radius of each of the following cones. Use the values for the volume V and the vertical height h given in each case.

a $V = 600\,\text{cm}^3$, $h = 12$ cm
b $V = 225\,\text{cm}^3$, $h = 18$ mm
c $V = 1400\,\text{mm}^3$, $h = 2$ cm
d $V = 0.04\,\text{m}^3$, $h = 145$ mm

3 The base circumference C and the length of the sloping face l are given for each of the following cones. Calculate

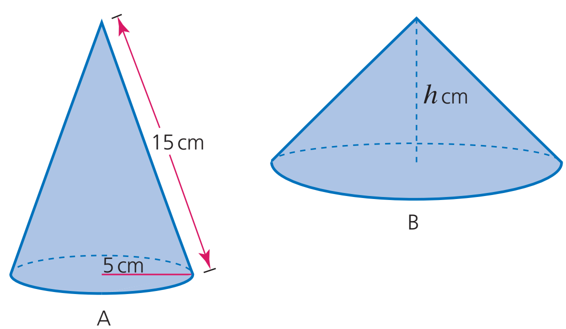
i the base radius,
ii the vertical height,
iii the volume in each case.

Give all answers to 3 s.f.

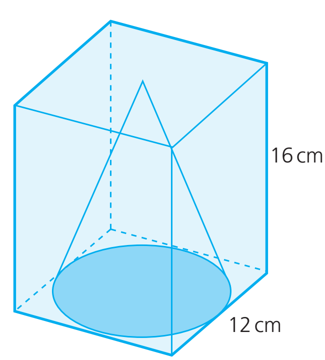
a $C = 50$ cm, $l = 15$ cm
b $C = 100$ cm, $l = 18$ cm
c $C = 0.4$ m, $l = 75$ mm
d $C = 240$ mm, $l = 6$ cm

Exercise 27.21

1 The two cones A and B shown below have the same volume. Using the dimensions shown and given that the base circumference of cone B is 60 cm, calculate the height h cm.

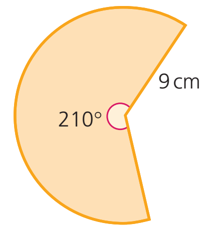


2 A cone is placed inside a cuboid as shown. If the base diameter of the cone is 12 cm and the height of the cuboid is 16 cm, calculate the volume of the cuboid not occupied by the cone.

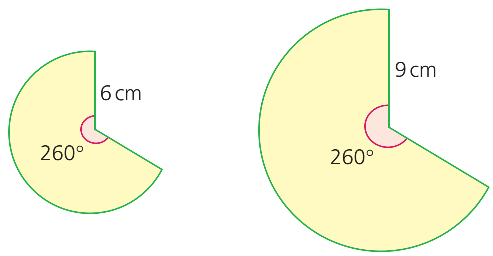


3 The sector shown is assembled to form a cone. Calculate the following, giving your answers to parts **a**, **b** and **c** in exact form:

a the base circumference of the cone,
b the base radius of the cone,
c the vertical height of the cone,
d the volume of the cone. Give your answer correct to 3 s.f.

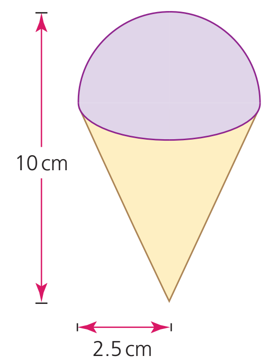


4 Two similar sectors are assembled into cones (below). Calculate the ratio of their volumes.

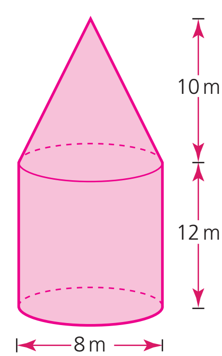


Exercise 27.22

1 An ice cream consists of a hemisphere and a cone (right). Calculate, in exact form, its total volume.

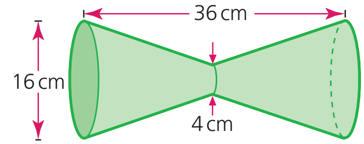


2 A cone is placed on top of a cylinder. Using the dimensions given (right), calculate the total volume of the shape.

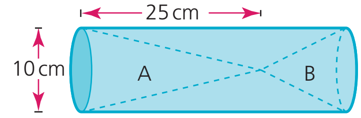


Exercise 27.22 (cont)

3 Two identical truncated cones are placed end to end as shown. Calculate the total volume of the shape.



4 Two cones A and B are placed either end of a cylindrical tube as shown. Given that the volumes of A and B are in the ratio 2 : 1, calculate:
a the volume of cone A,
b the height of cone B,
c the volume of the cylinder.

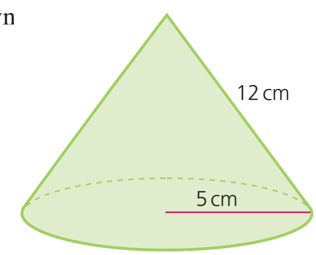


The surface area of a cone

The surface area of a cone comprises the area of the circular base and the area of the curved face. The area of the curved face is equal to the area of the sector from which it is formed.

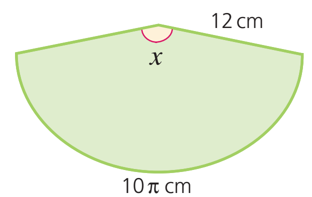
Worked examples

Calculate the total surface area of the cone shown



Surface area of base $= \pi r^2 = 25\pi\,\text{cm}^2$

The curved surface area can best be visualised if drawn as a sector as shown in the diagram:



The radius of the sector is equivalent to the slant height of the cone. The curved perimeter of the sector is equivalent to the base circumference of the cone.

$$\frac{x}{360} = \frac{10\pi}{24\pi}$$

Therefore $x = 150°$

$$\text{Area of sector} = \frac{150}{360} \times \pi \times 12^2 = 60\pi\,\text{cm}^2$$

$$\begin{aligned}\text{Total surface area} &= 60\pi + 25\pi \\ &= 85\pi \\ &= 267 \text{ (3 s.f.)}\end{aligned}$$

The total surface area is $267\,\text{cm}^2$.

The area of the sector was calculated to be $60\pi\,\text{cm}^2$. This is therefore also the area of the curved surface of the cone.

The curved surface of a cone can also be calculated using the formula Area $= \pi rl$, where r represents the radius of the circular base and l represents the slant length of the cone.

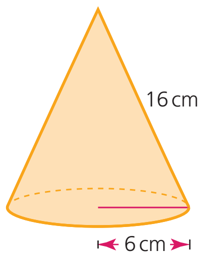
You should know how to use the formula for the curved surface of a cone, i.e. the area of the sector, but you do not need to memorise it.

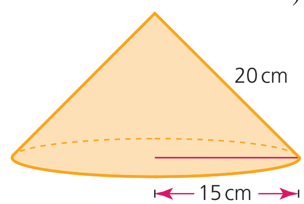
In the example above, the curved surface area $= \pi \times 5 \times 12$

$$= 60\pi\,\text{cm}^2$$

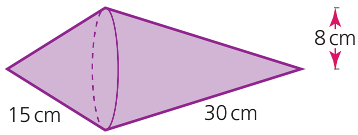
Exercise 27.23

1 Calculate the surface area of each of the following cones (you may use the formula to work out the area of the curved surface of each cone):

a 

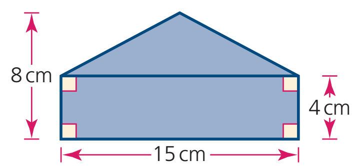
b 

2 Two cones with the same base radius are stuck together as shown. Calculate the surface area of the shape.



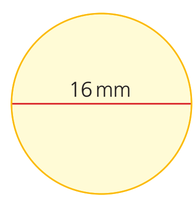
Student assessment 1

1 Calculate the area of the shape below.

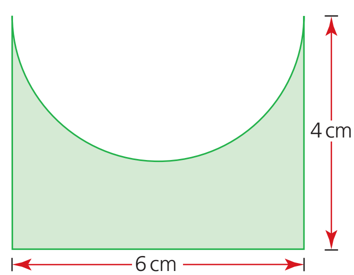


2 Calculate the circumference and area of each of the following circles. Give your answers to 3 s.f.

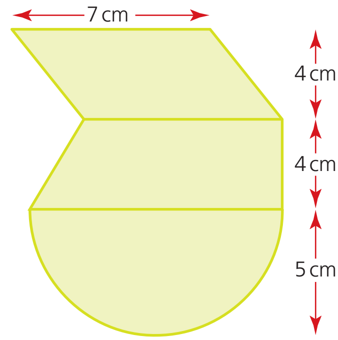
a 

b 

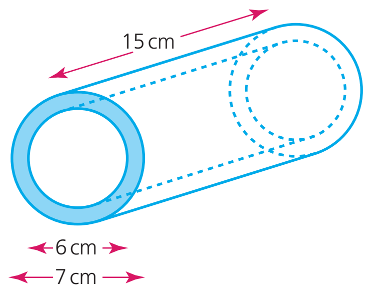
3 A semicircular shape is cut out of the side of a rectangle as shown. Calculate the shaded area, giving your answer in exact form.



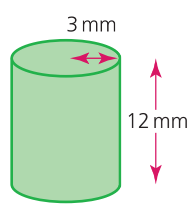
4 For the diagram (right), calculate the area of:
  a the semicircle,
  b the trapezium,
  c the whole shape.

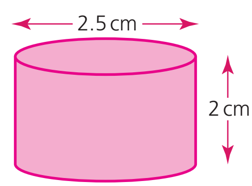


5 A cylindrical tube has an inner diameter of 6 cm, an outer diameter of 7 cm and a length of 15 cm. Calculate the following to 3 s.f.:
  a the surface area of the shaded end,
  b the inside surface area of the tube,
  c the total surface area of the tube.



6 Calculate the volume of each of the following cylinders. Give your answers in terms of π.

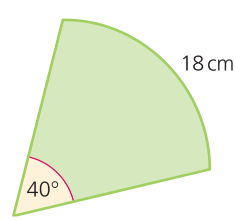
a


b


Student assessment 2

1 Calculate the arc length of each of the following sectors. The angle x and radius r are given in each case.
  a $x = 45°$, $r = 15$ cm
  b $x = 150°$, $r = 13.5$ cm

2 Calculate the area of the sector shown, giving your answer in exact form.

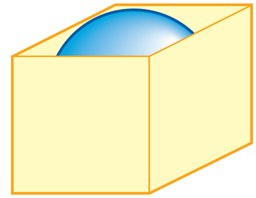


3 A sphere has a radius of 6.5 cm. Calculate to 3 s.f.:
  a its total surface area,
  b its volume.

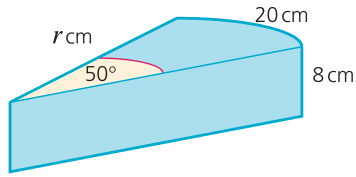
4 Calculate the angle x in each of the following sectors. The radius r and arc length a are given in each case.
  a $r = 20$ mm, $a = 95$ mm
  b $r = 9$ cm, $a = 9$ mm

Student assessment 3

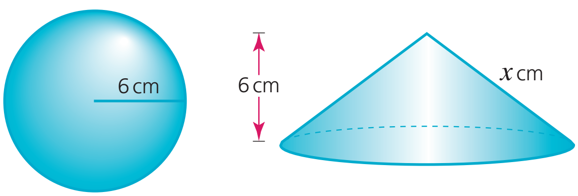
1 A ball is placed inside a box into which it will fit tightly. If the radius of the ball is 10 cm, calculate the percentage volume of the box not occupied by the ball.



2 The prism has a cross-sectional area in the shape of a sector.
Calculate:
a the radius r cm,
b the cross-sectional area of the prism,
c the total surface area of the prism,
d the volume of the prism.

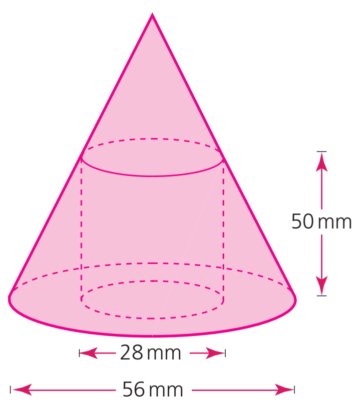


3 The cone and sphere shown (below) have the same volume.



If the radius of the sphere and the height of the cone are both 6 cm, calculate each of the following. Give your answers in exact form:
a the volume of the sphere,
b the base radius of the cone,
c the slant height x cm,
d the surface area of the cone.

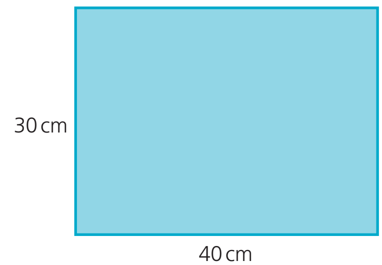
4 The top of a cone is cut off and a cylindrical hole is drilled out of the remaining frustum as shown (right).
Calculate:
a the height of the original cone,
b the volume of the original cone,
c the volume of the solid frustum,
d the volume of the cylindrical hole,
e the volume of the remaining part of the frustum.



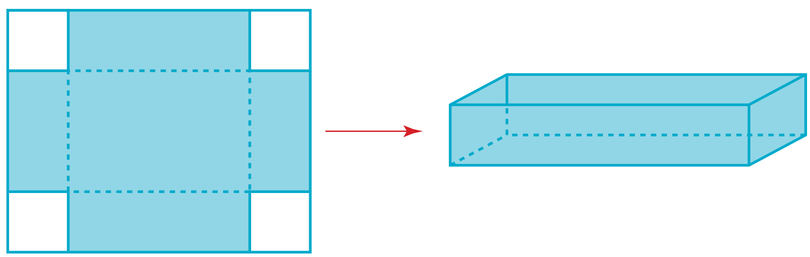
5 Mathematical investigations and ICT 5

Metal trays

A rectangular sheet of metal measures 30 cm by 40 cm.



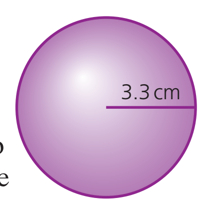
The sheet has squares of equal size cut from each corner. It is then folded to form a metal tray as shown.



1 a Calculate the length, width and height of the tray if a square of side length 1 cm is cut from each corner of the sheet of metal.
  b Calculate the volume of this tray.
2 a Calculate the length, width and height of the tray if a square of side length 2 cm is cut from each corner of the sheet of metal.
  b Calculate the volume of this tray.
3 Using a spreadsheet if necessary, investigate the relationship between the volume of the tray and the size of the square cut from each corner. Enter your results in an ordered table.
4 Calculate, to 1 d.p., the side length of the square that produces the tray with the greatest volume.
5 State the greatest volume to the nearest whole number.

Tennis balls

Tennis balls are spherical and have a radius of 3.3 cm.



A manufacturer wishes to make a cuboidal container with a lid that holds 12 tennis balls. The container is to be made of cardboard. The manufacturer wishes to use as little cardboard as possible.

1 Sketch some of the different containers that might be considered.
2 For each container, calculate the total area of cardboard used and therefore decide on the most economical design.

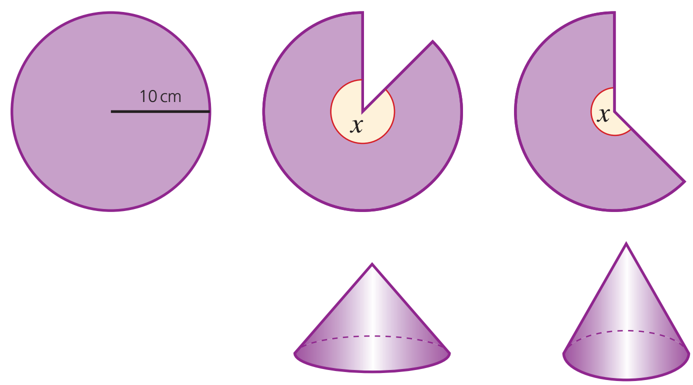
The manufacturer now considers the possibility of using other flat-faced containers.

3 Sketch some of the different containers that might be considered.
4 Investigate the different amounts of cardboard used for each design.
5 Which type of container would you recommend to the manufacturer?

ICT activity

In this activity you will be using a spreadsheet to investigate the maximum possible volume of a cone constructed from a sector of fixed radius.

Circles of radius 10 cm are cut from paper and used to construct cones. Different sized sectors are cut from the circles and then arranged to form a cone, e.g.



1 Using a spreadsheet like the one below, calculate the maximum possible volume, for a cone constructed from one of these circles:

	A	B	C	D	E	F
1	Angle of sector (θ)	Sector arc length (cm)	Base circumference of cone (cm)	Base radius of cone (cm)	Vertical height of cone (cm)	Volume of cone (cm^3)
2	5	0.873	0.873	0.139	9.999	0.202
3	10	1.745	1.745	0.278	9.996	0.808
4	15	2.618	2.618	0.417	9.991	1.816
5	20					
6	25					
7	30					
8	Continue to 355^0	Enter formulae here to calculate the results for each column				

2 Plot a graph to show how the volume changes as x increases. Comment on your graph.

TOPIC 6

Trigonometry

Contents

Learning objectives

E6.1 Pythagoras' theorem

Know and use Pythagoras' theorem.

E6.2 Right-angled triangles

1 Know and use the sine, cosine and tangent ratios for acute angles in calculations involving sides and angles of a right-angled triangle.
2 Solve problems in two dimensions using Pythagoras' theorem and trigonometry.
3 Know that the perpendicular distance from a point to a line is the shortest distance to the line.
4 Carry out calculations involving angles of elevation and depression.

E6.3 Exact trigonometric values

Know the exact values of:

1 $\sin x$ and $\cos x$ for $x = 0°, 30°, 45°, 60°$ and $90°$
2 $\tan x$ for $x = 0°, 30°, 45°, 60°$.

E6.4 Trigonometric functions

1 Recognise, sketch and interpret the following graphs for $0° \leqslant x \leqslant 360°$:
- $y = \sin x$
- $y = \cos x$
- $y = \tan x$.

2 Solve trigonometric equations involving $\sin x$, $\cos x$ or $\tan x$, for $0° \leqslant x \leqslant 360°$.

E6.5 Non-right-angled triangles

1 Use the sine and cosine rules in calculations involving lengths and angles for any triangle.
2 Use the formula

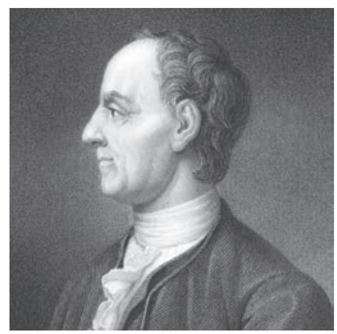
area of triangle $= \frac{1}{2}ab \sin C$.

E6.6 Pythagoras' theorem and trigonometry in 3D

Carry out calculations and solve problems in three dimensions using Pythagoras' theorem and trigonometry, including calculating the angle between a line and a plane.

The Swiss

Leonhard Euler



Leonhard Euler (1707–1783)

Euler, like Newton before him, was the greatest mathematician of his generation. He studied all areas of mathematics and continued to work hard after he had gone blind.

As a young man, Euler discovered and proved:

the sum of the infinite series $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) = \frac{\pi^2}{6}$

(i.e.) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} = \frac{\pi^2}{6}$

This brought him to the attention of other mathematicians.

Euler made discoveries in many areas of mathematics, especially calculus and trigonometry. He also developed the ideas of Newton and Leibniz.

Euler worked on graph theory and functions and was the first to prove several theorems in geometry. He studied relationships between a triangle's height, midpoint and circumscribing and inscribing circles, and he also discovered an expression for the volume of a tetrahedron (a triangular pyramid) in terms of its sides.

He also worked on number theory and found the largest prime number known at the time.

Some of the most important constant symbols in mathematics, π, e and i (the square root of -1), were introduced by Euler.

The Bernoulli family

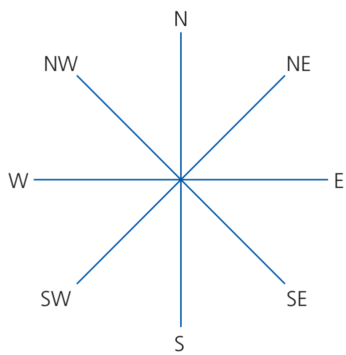
The Bernoullis were a family of Swiss merchants who were friends of Euler. The two brothers, Johann and Jacob, were very gifted mathematicians and scientists, as were their children and grandchildren. They made discoveries in calculus, trigonometry and probability theory in mathematics. In science, they worked on astronomy, magnetism, mechanics, thermodynamics and more.

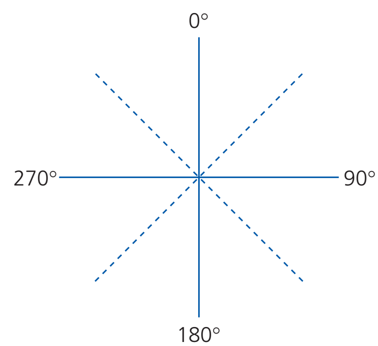
Unfortunately, many members of the Bernoulli family were not pleasant people. The older members of the family were jealous of each other's successes and often stole the work of their sons and grandsons and pretended that it was their own.

28 Bearings

Note
All diagrams are not drawn to scale.

Bearings

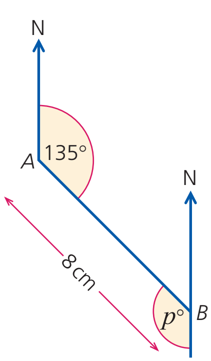




In the days when sailing ships travelled the oceans of the world, compass **bearings** like the ones in the diagram above were used.

As the need for more accurate direction arose, extra points were added to N, S, E, W, NE, SE, SW and NW. Midway between North and North East was North North East, and midway between North East and East was East North East, and so on. This gave 16 points of the compass. This was later extended even further, eventually to 64 points.

As the speed of travel increased, a new system was required. The new system was the **three-figure bearing** system. North was given the bearing zero. 360° in a clockwise direction was one full rotation.



Back bearings

The bearing of B from A is 135° and the distance from A to B is 8 cm, as shown (left). The bearing of A from B is called the **back bearing**.

Since the two North lines are parallel:

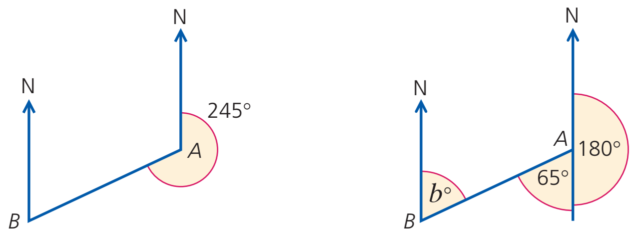
$p = 135°$ (alternate angles), so the back bearing is $(180 + 135)°$.

That is, 315°.

(There are a number of methods of solving this type of problem.)

Worked example

The bearing of B from A is 245°.
What is the bearing of A from B?



Since the two North lines are parallel:
b = 65° (alternate angles), so the bearing is (245 – 180)°. That is, 065°.

Exercise 28.1

1 Draw a diagram to show the following compass bearings and journey. Use a scale of 1 cm : 1 km. North can be taken to be a line vertically up the page.
Start at point A. Travel a distance of 7 km on a bearing of 135° to point B. From B, travel 12 km on a bearing of 250° to point C. Measure the distance and bearing of A from C.

2 Given the following bearings of point B from point A, draw diagrams and use them to calculate the bearings of A from B.
a bearing 163° b bearing 214°

3 Given the following bearings of point D from point C, draw diagrams and use them to calculate the bearings of C from D.
a bearing 300° b bearing 282°

Student assessment 1

1 A climber gets to the top of Mont Blanc. He can see in the distance a number of ski resorts. He uses his map to find the bearing and distance of the resorts, and records them as shown below:

Val d'Isère 30 km	bearing 082°
Les Arcs 40 km	bearing 135°
La Plagne 45 km	bearing 205°
Méribel 35 km	bearing 320°

Choose an appropriate scale and draw a diagram to show the position of each resort. What are the distance and bearing of the following?
a Val d'Isère from La Plagne. b Méribel from Les Arcs.

2 A coastal radar station picks up a distress call from a ship. It is 50 km away on a bearing of 345°. The radar station contacts a lifeboat at sea which is 20 km away on a bearing of 220°.
Make a scale drawing and use it to find the distance and bearing of the ship from the lifeboat.

3 An aircraft is seen on radar at airport A. The aircraft is 210 km away from the airport on a bearing of 065°. The aircraft is diverted to airport B, which is 130 km away from A on a bearing of 215°.
Choose an appropriate scale and make a scale drawing to find the distance and bearing of airport B from the aircraft.

29 Trigonometry

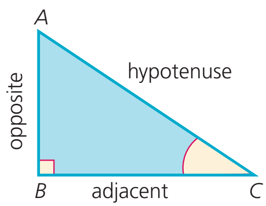
Note

All diagrams are not drawn to scale.

In this chapter, unless instructed otherwise, give your answers exactly or correct to three significant figures as appropriate. Answers in degrees should be given to one decimal place.

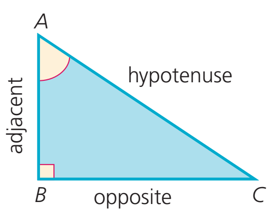
There are three basic trigonometric ratios: sine, cosine and tangent.

Each of these relates an angle of a right-angled triangle to a ratio of the lengths of two of its sides.



The sides of the triangle have names, two of which are dependent on their position in relation to a specific angle.

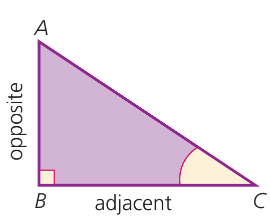
The longest side (always opposite the right angle) is called the **hypotenuse**. The side opposite the angle is called the **opposite** side and the side next to the angle is called the **adjacent** side.



Note that, when the chosen angle is at A, the sides labelled opposite and adjacent change.

Tangent

$$\tan C = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$



Worked examples

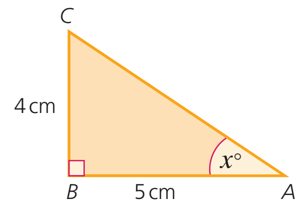
a Calculate the size of angle BAC in each of the triangles.

i $\tan x° = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{5}$

$x = \tan^{-1}\left(\frac{4}{5}\right)$

$x = 38.7$ (1 d.p.)

angle $BAC = 38.7°$

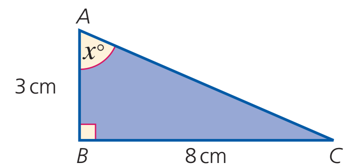


ii $\tan x° = \frac{8}{3}$

$x = \tan^{-1}\left(\frac{8}{3}\right)$

$x = 69.4$ (1 d.p.)

angle $BAC = 69.4°$



Answers involving angles should be given to 1 d.p. unless stated otherwise.

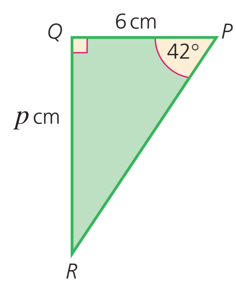
b Calculate the length of the opposite side QR.

$\tan 42° = \frac{p}{6}$

$6 \times \tan 42° = p$

$p = 5.40$ (3 s.f.)

$QR = 5.40$ cm (3 s.f.)



c Calculate the length of the adjacent side XY.

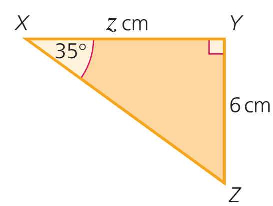
$\tan 35° = \frac{6}{z}$

$z \times \tan 35° = 6$

$z = \frac{6}{\tan 35°}$

$z = 8.57$ (3 s.f.)

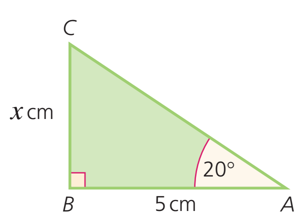
$XY = 8.57$ cm (3 s.f.)



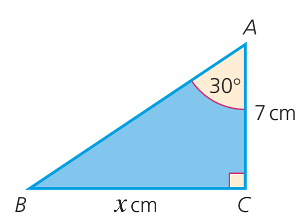
Exercise 29.1

Calculate the length of the side marked x cm in each of the diagrams in Questions 1 and 2.

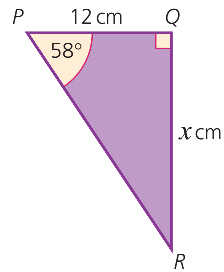
1 a

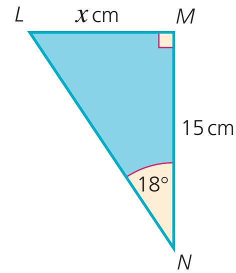


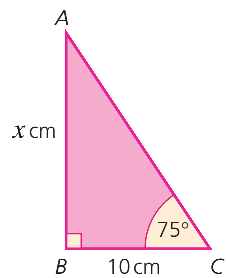
b

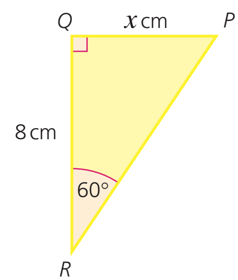


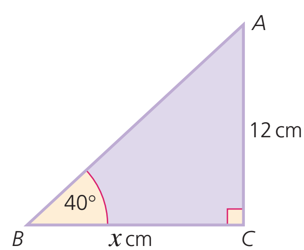
Exercise 29.1 (cont)

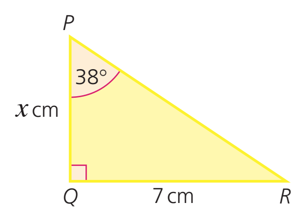
c 

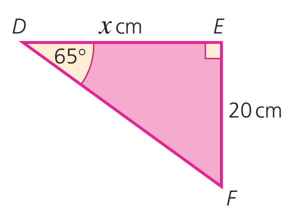
d 

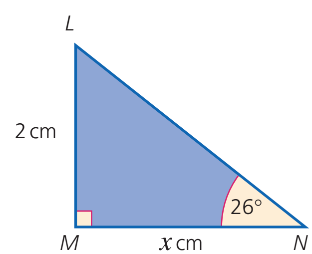
e 

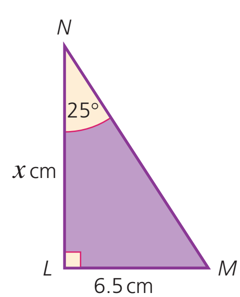
f 

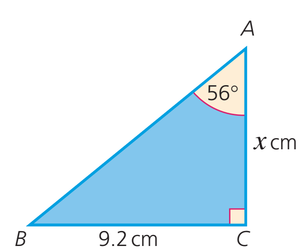
2 a 

b 

c 

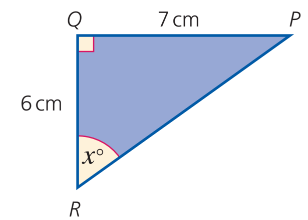
d 

e 

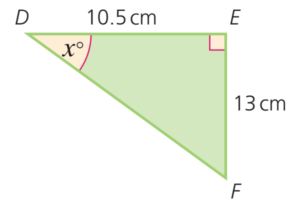
f 

3 Calculate the size of the angle marked $x°$ in each of the following diagrams.

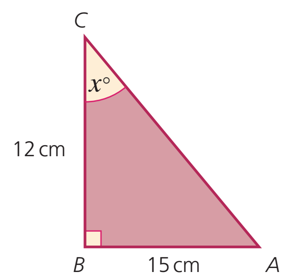
a



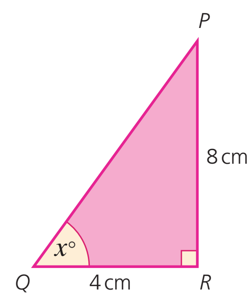
b



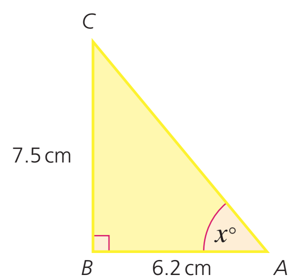
c



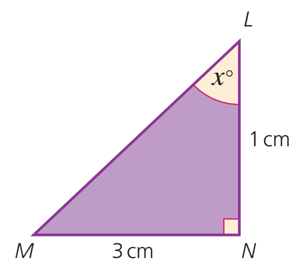
d



e

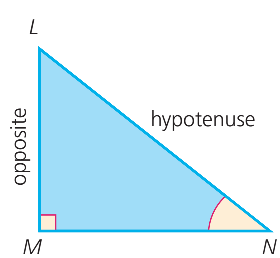


f



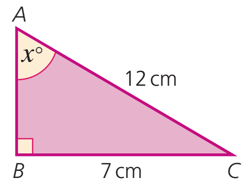
Sine

$$\sin N = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$



Worked examples

a Calculate the size of angle BAC.



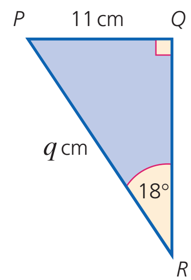
$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{7}{12}$$

$$x = \sin^{-1}\left(\frac{7}{12}\right)$$

$$x = 35.7 \text{ (1 d.p.)}$$

$$\text{angle } BAC = 35.7° \text{ (1 d.p.)}$$

b Calculate the length of the hypotenuse PR.



$$\sin 18° = \frac{11}{q}$$

$$q \times \sin 18° = 11$$

$$q = \frac{11}{\sin 18°}$$

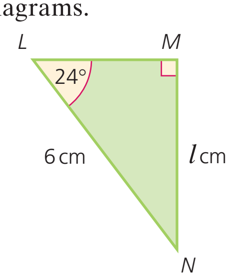
$$q = 35.6 \text{ (3 s.f.)}$$

$$PR = 35.6 \text{ cm (3 s.f.)}$$

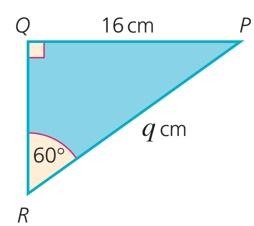
Exercise 29.2

1 Calculate the length of the marked side in each of the following diagrams.

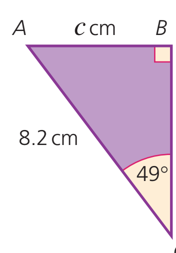
a



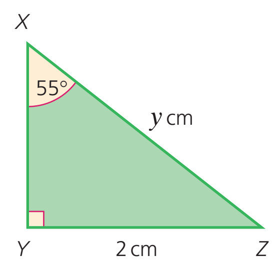
b



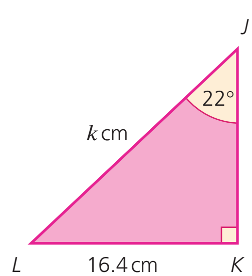
c



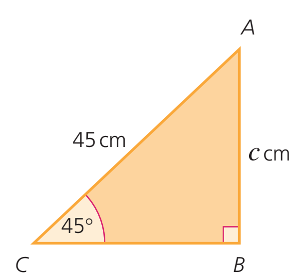
d



e

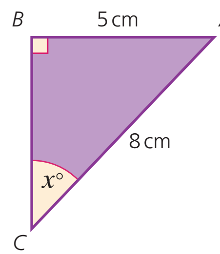


f

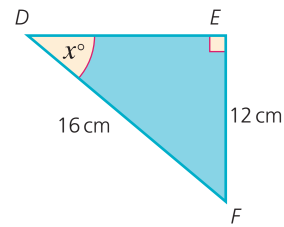


2 Calculate the size of the angle marked x in each of the following diagrams.

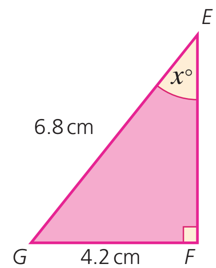
a



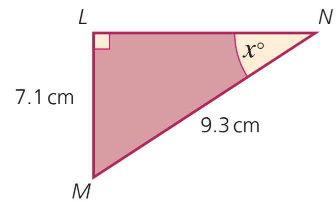
b



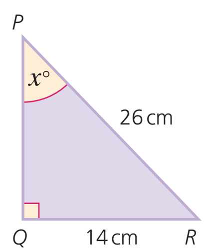
c



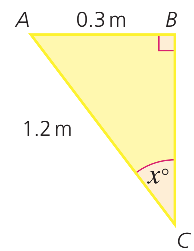
d



e

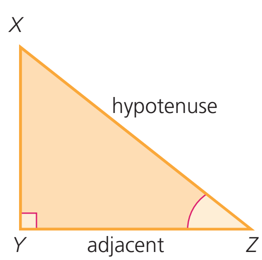


f



Cosine

$$\cos Z = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$



Worked examples

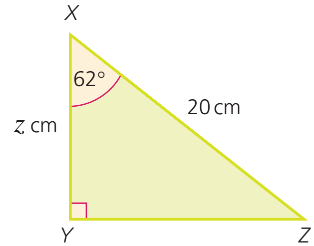
a Calculate the length XY.

$$\cos 62^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{z}{20}$$

$$z = 20 \times \cos 62^\circ$$

$$z = 9.39 \text{ (3 s.f.)}$$

$$XY = 9.39\,\text{cm (3 s.f.)}$$



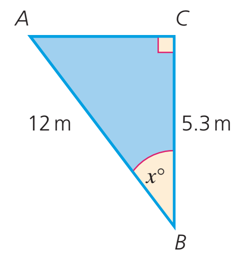
b Calculate the size of angle ABC.

$$\cos x = \frac{5.3}{12}$$

$$x = \cos^{-1}\left(\frac{5.3}{12}\right)$$

$$x = 63.8 \text{ (1 d.p.)}$$

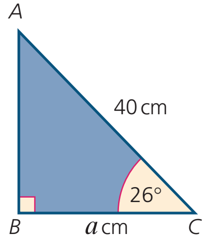
$$\text{angle } ABC = 63.8^\circ \text{ (1 d.p.)}$$



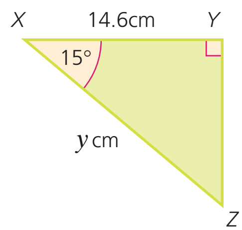
Exercise 29.3

1 Calculate the marked side or angle in each of the following diagrams.

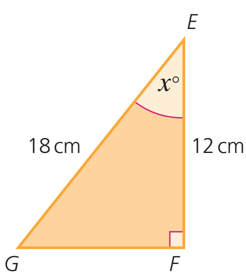
a



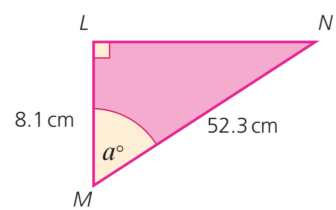
b



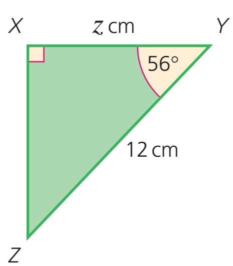
c



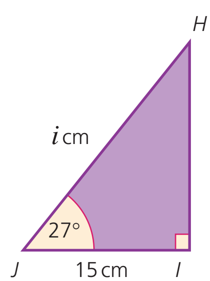
d



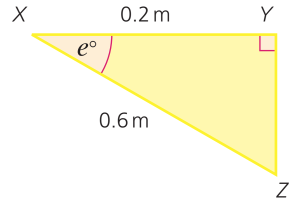
e



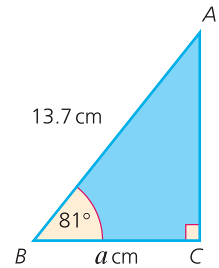
f



g



h

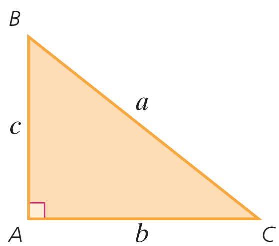


Pythagoras' theorem

Pythagoras' theorem states the relationship between the lengths of the three sides of a right-angled triangle.

Pythagoras' theorem states that:

$a^2 = b^2 + c^2$

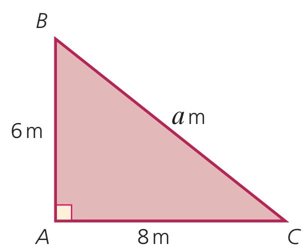


Worked examples

a Calculate the length of the side BC.

Using Pythagoras:

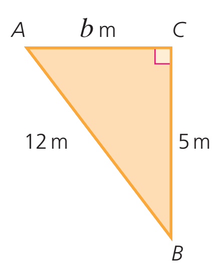
$$a^2 = b^2 + c^2$$
$$a^2 = 8^2 + 6^2$$
$$a^2 = 64 + 36 = 100$$
$$a = \sqrt{100}$$
$$a = 10$$
$$BC = 10\,\text{m}$$



b Calculate the length of the side AC.

Using Pythagoras:

$$a^2 = b^2 + c^2$$
$$12^2 = b^2 + 5^2$$
$$b^2 = 144 - 25 = 119$$
$$b = \sqrt{119}$$
$$b = 10.9 \quad (3\text{ s.f.})$$
$$AC = 10.9\,\text{m}\ (3\text{ s.f.})$$

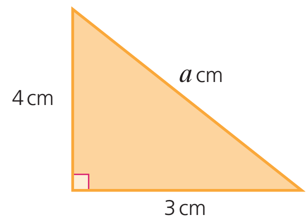


Note that if the answer is left in surd form as $\sqrt{119}$, this is known as leaving your answer in exact form. However, leaving an answer as a surd is in the Extended syllabus only.

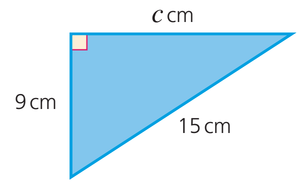
Exercise 29.4

In each of the diagrams in Questions 1 and 2, use Pythagoras' theorem to calculate the length of the marked side. Where the answer is not an integer, leave it in exact form.

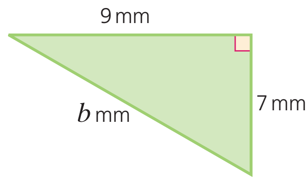
1 **a**



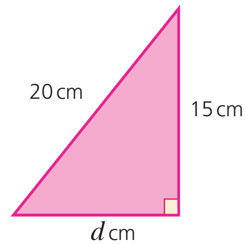
b



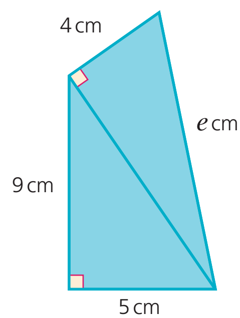
c



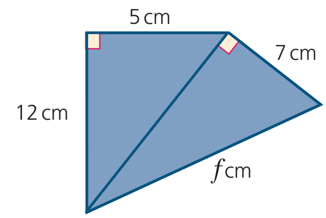
d



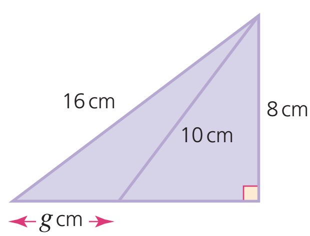
2 **a**



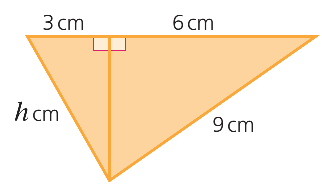
b



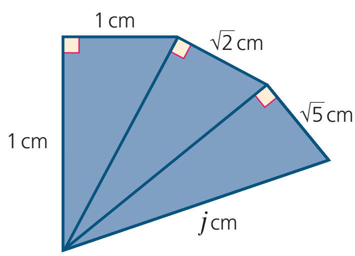
c



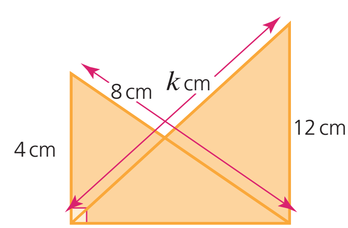
d



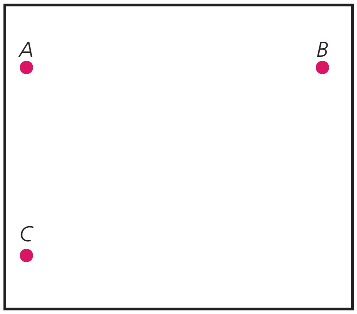
e



f



3 Villages A, B and C lie on the edge of the Namib Desert. Village A is 30 km due north of village C. Village B is 65 km due east of A. Calculate the shortest distance between villages C and B, giving your answer to the nearest 0.1 km.

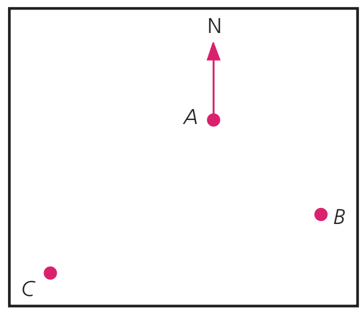


4 Town X is 54 km due west of town Y. The shortest distance between town Y and town Z is 86 km. If town Z is due south of X, calculate the distance between X and Z, giving your answer to the nearest kilometre.

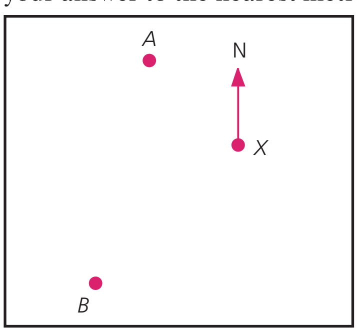
5 Village B is on a bearing of 135° and at a distance of 40 km from village A, as shown. Village C is on a bearing of 225° and a distance of 62 km from village A.

a Show that triangle ABC is right-angled.

b Calculate the distance from B to C, giving your answer to the nearest 0.1 km.



6 Two boats set off from X at the same time (below). Boat A sets off on a bearing of 325° and with a velocity of 14 km/h. Boat B sets off on a bearing of 235° with a velocity of 18 km/h. Calculate the distance between the boats after they have been travelling for 2.5 hours. Give your answer to the nearest metre.



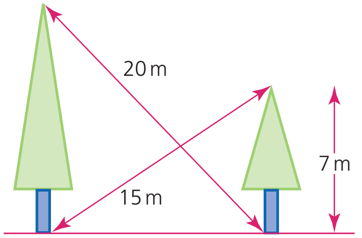
7 A boat sets off on a trip from S. It heads towards B, a point 6 km away and due north. At B it changes direction and heads towards point C, 6 km away from and due east of B. At C it changes direction once again and heads on a bearing of 135° towards D, which is 13 km from C.

a Calculate the distance between S and C to the nearest 0.1 km.

b Calculate the distance the boat will have to travel if it is to return to S from D.

Exercise 29.4 (cont)

8 Two trees are standing on flat ground.



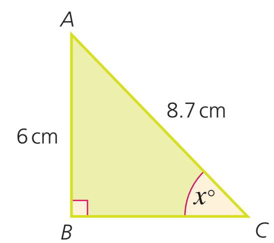
The height of the smaller tree is 7 m. The distance between the top of the smaller tree and the base of the taller tree is 15 m.
The distance between the top of the taller tree and the base of the smaller tree is 20 m.

a Calculate the horizontal distance between the two trees.
b Calculate the height of the taller tree.

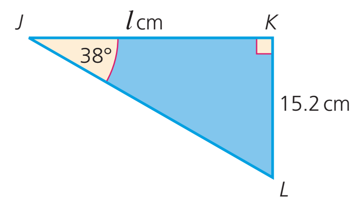
Exercise 29.5

1 By using Pythagoras' theorem, trigonometry or both, calculate the marked value in each of the following diagrams. In each case give your answer to 1 d.p.

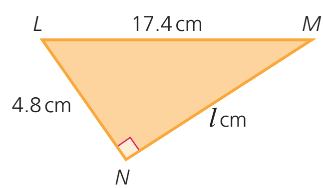
a



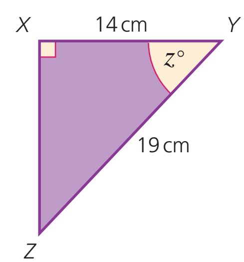
b



c



d

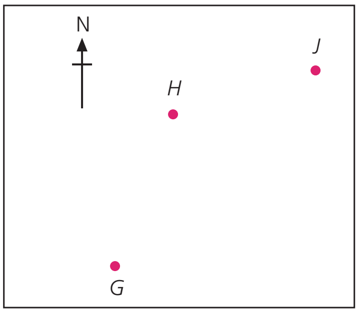


2 A sailing boat sets off from a point X and heads towards Y, a point 17 km north. At point Y, it changes direction and heads towards point Z, a point 12 km away on a bearing of 090°. Once at Z, the crew want to sail back to X. Calculate:

a the distance ZX,
b the bearing of X from Z.

3 An aeroplane sets off from G on a bearing of 024° towards H, a point 250 km away. At H, it changes course and heads towards J, which is 180 km away on a bearing of 055°.

a How far is H to the north of G?
b How far is H to the east of G?
c How far is J to the north of H?



d How far is J to the east of H?

e What is the shortest distance between G and J?

f What is the bearing of G from J?

4 $PQRS$ is a quadrilateral. The sides RS and QR are the same length. The sides QP and RS are parallel. Calculate:

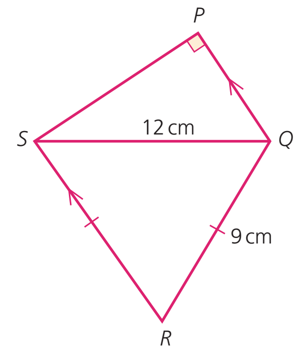
a angle SQR,

b angle PSQ,

c length PQ,

d length PS,

e the area of $PQRS$.



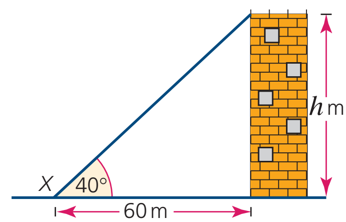
Angles of elevation and depression

The **angle of elevation** is the angle above the horizontal through which a line of view is raised. The **angle of depression** is the angle below the horizontal through which a line of view is lowered.

Worked examples

a The base of a tower is 60 m away from a point X on the ground. If the angle of elevation of the top of the tower from X is 40°, calculate the height of the tower.

Give your answer to the nearest metre.



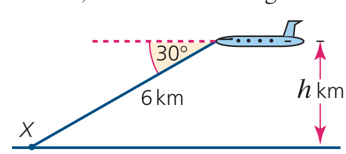
$$\tan 40° = \frac{h}{60}$$

$$h = 60 \times \tan 40° = 50.3$$

The height is 50 m to the nearest metre.

b An aeroplane receives a signal from a point X on the ground. If the angle of depression of point X from the aeroplane is 30°, calculate the height at which the aeroplane is flying.

Give your answer to the nearest 0.1 km.



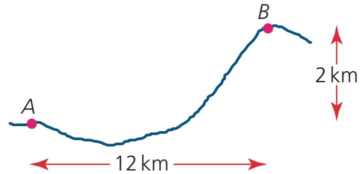
$$\sin 30° = \frac{h}{6}$$

$$h = 6 \times \sin 30° = 3$$

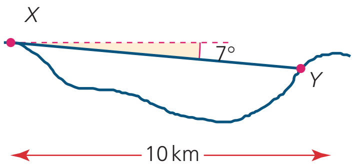
The height is 3 km.

Exercise 29.6

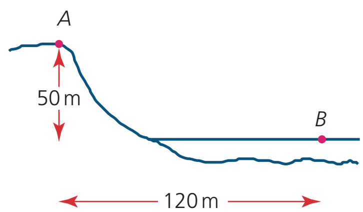
1 A and B are two villages. If the horizontal distance between them is 12 km and the vertical distance between them is 2 km, calculate:
  a the shortest distance between the two villages,
  b the angle of elevation of B from A.



2 X and Y are two towns. If the horizontal distance between them is 10 km and the angle of depression of Y from X is 7°, calculate:
  a the shortest distance between the two towns,
  b the vertical height between the two towns.

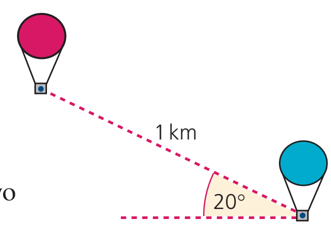


3 A girl standing on a hill at A can see a small boat at a point B on a lake.

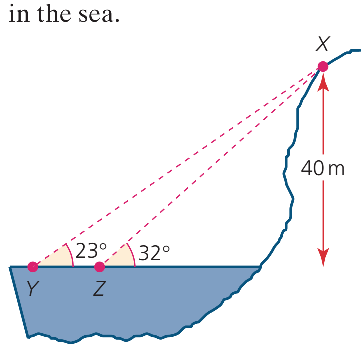


If the girl is at a height of 50 m above B and at a horizontal distance of 120 m away from B, calculate:
  a the angle of depression of the boat from the girl,
  b the shortest distance between the girl and the boat.

4 Two hot air balloons are 1 km apart in the air. If the angle of elevation of the higher from the lower balloon is 20°, calculate, giving your answers to the nearest metre:
  a the vertical height between the two balloons,
  b the horizontal distance between the two balloons.



5 A boy X can be seen by two of his friends, Y and Z, who are swimming in the sea.



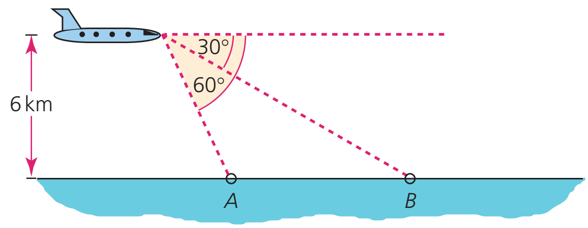
If the angle of elevation of X from Y is 23° and from Z is 32°, and the height of X above Y and Z is 40 m, calculate:

a the horizontal distance between X and Z,

b the horizontal distance between Y and Z.

Note: XYZ is a vertical plane

6 A plane is flying at an altitude of 6 km directly over the line AB. It spots two boats, A and B, on the sea. If the angles of depression of A and B from the plane are 60° and 30° respectively, calculate the horizontal distance between A and B.



Note

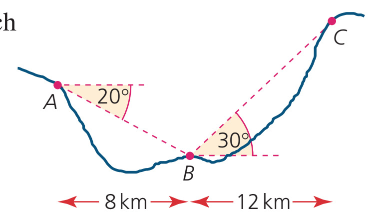
You may find a sketch of this information will help you solve the problem.

7 Two planes are flying directly above each other. A person standing at P can see both of them. The horizontal distance between the two planes and the person is 2 km. If the angles of elevation of the planes from the person are 65° and 75°, calculate:

a the altitude at which the higher plane is flying,

b the vertical distance between the two planes.

8 Three villages, A, B and C, can see each other across a valley. The horizontal distance between A and B is 8 km, and the horizontal distance between B and C is 12 km. The angle of depression of B from A is 20° and the angle of elevation of C from B is 30°.



Calculate, giving all answers to 1 d.p.:

a the vertical height between A and B,

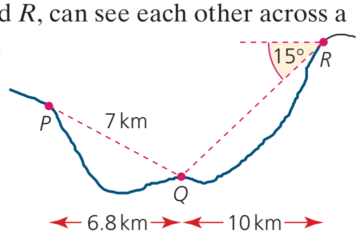
b the vertical height between B and C,

c the angle of elevation of C from A,

d the shortest distance between A and C.

Note: A, B and C are in the same vertical plane.

9 Using binoculars, three people, P, Q and R, can see each other across a valley. The horizontal distance between P and Q is 6.8 km and the horizontal distance between Q and R is 10 km. If the shortest distance between P and Q is 7 km and the angle of depression of Q from R is 15°, calculate, giving appropriate answers:



a the vertical height between Q and R,

b the vertical height between P and R,

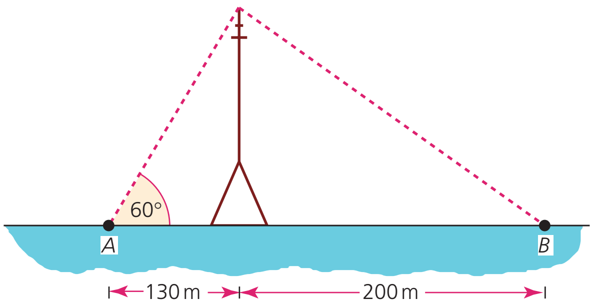
c the angle of elevation of R from P,

d the shortest distance between P and R.

Note: P, Q and R are in the same vertical plane.

Exercise 29.6 (cont)

10 Two people, A and B, are standing either side of a transmission mast. A is 130 m away from the mast and B is 200 m away.

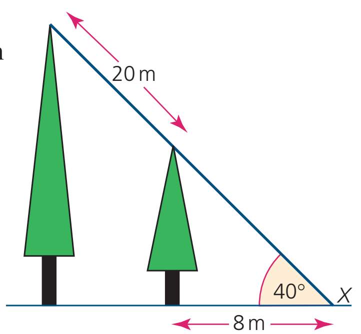


If the angle of elevation of the top of the mast from A is 60°, calculate:

a the height of the mast to the nearest metre,

b the angle of elevation of the top of the mast from B.

11 Two trees are standing on flat ground. The angle of elevation of their tops from a point X on the ground is 40°. If the horizontal distance between X and the small tree is 8 m and the distance between the tops of the two trees is 20 m, calculate:



a the height of the small tree,

b the height of the tall tree,

c the horizontal distance between the trees.

Special angles and their trigonometric ratios

So far, most of the angles you have worked with have required the use of a calculator in order to calculate their sine, cosine or tangent. However, some angles produce exact values and a calculator is both unnecessary and unhelpful when exact solutions are required.

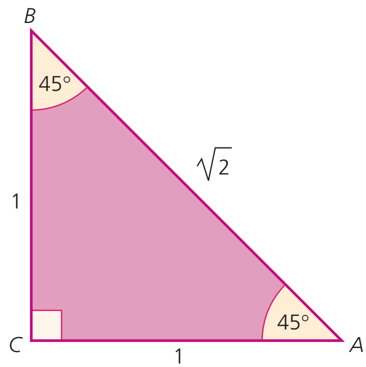
There are a number of angles which have 'neat' trigonometric ratios, for example 0°, 30°, 45°, 60° and 90°. Their trigonometric ratios are derived below.

Consider the right-angled isosceles triangle ABC.

Let the perpendicular sides AC and BC each have a length of 1 unit.

As triangle ABC is isosceles, angle ABC = angle CAB = 45°.

Using Pythagoras' theorem, AB can also be calculated:



$$(AB)^2 = (AC)^2 + (BC)^2$$
$$(AB)^2 = 1^2 + 1^2 = 2$$
$$AB = \sqrt{2}$$

From the triangle, it can be deduced that $\sin 45° = \frac{1}{\sqrt{2}}$.

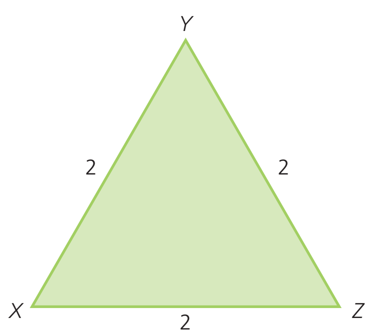
When rationalised, this can be written as $\sin 45° = \frac{\sqrt{2}}{2}$.

Similarly, $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

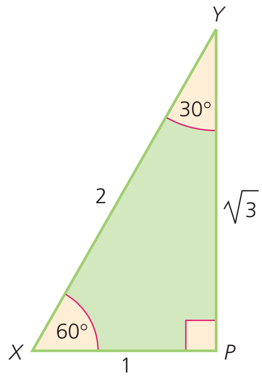
Therefore $\sin 45° = \cos 45°$

$\tan 45° = \frac{1}{1} = 1$

Consider also the equilateral triangle XYZ (below) in which each of its sides has a length of 2 units.



If a vertical line is dropped from the vertex Y until it meets the base XZ at P, the triangle is bisected. Consider now the right-angled triangle XYP.



Angle $XYP = 30°$ as it is half of angle XYZ.

$XP = 1$ unit length as it is half of XZ.

The length YP can be calculated using Pythagoras' theorem:

$$(XY)^2 = (XP)^2 + (YP)^2$$

$$(YP)^2 = (XY)^2 - (XP)^2 = 2^2 - 1^2 = 3$$

$$YP = \sqrt{3}$$

Therefore from this triangle the following trigonometric ratios can be deduced:

$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

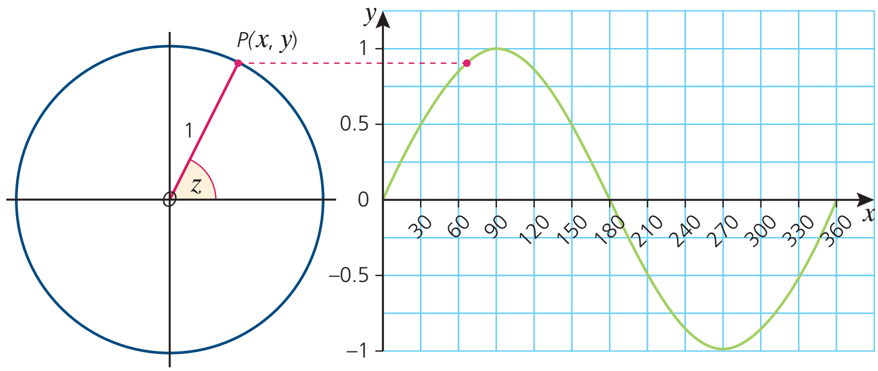
$\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

These results and those obtained from the trigonometric graphs shown on the next page are summarised in the table below:

Angle (°)	**sin (°)**	**cos (°)**	**tan (°)**
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
45°	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	–

There are other angles which have the same trigonometric ratios as those shown in the table. The following section explains why, using a unit circle, i.e. a circle with a radius of 1 unit.

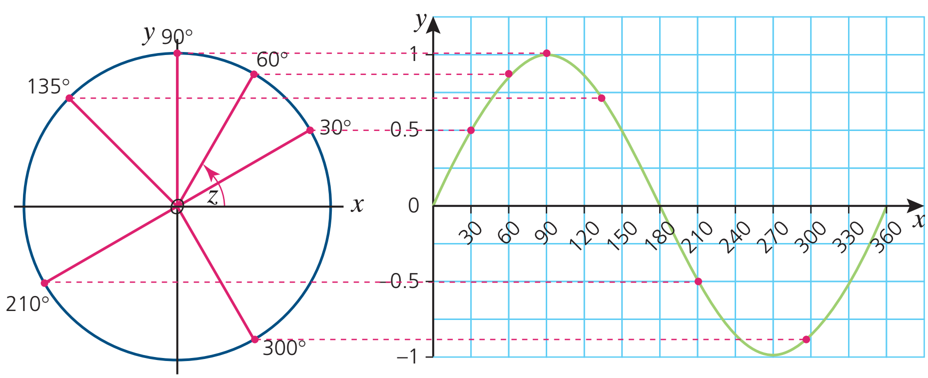
Graphs of trigonometric functions



In the diagram above, P is a point on the circumference of a circle with centre at O and a radius of 1 unit. P has coordinates (x, y). As the position of P changes, then so does the angle z.

$\sin z = \frac{y}{1} = y$, i.e. the sine of the angle z is represented by the y-coordinate of P.

The graph therefore shows the different values of $\sin z$ as z varies. A more complete diagram is shown below. Note that the angle z is measured anticlockwise from the positive x-axis.



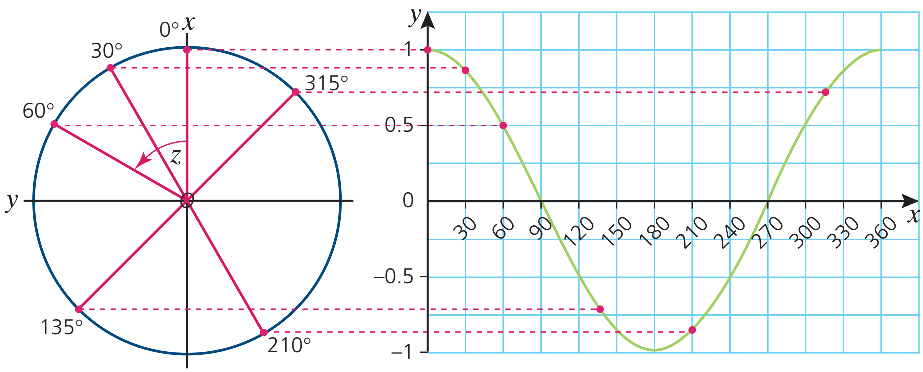
The graph of $y = \sin x$ has:

- a period of 360° (i.e. it repeats itself every 360°)
- a maximum value of +1
- a minimum value of –1
- symmetry, e.g. $\sin z = \sin (180 - z)$.

Similar diagrams and graphs can be constructed for $\cos z$ and $\tan z$.

From the unit circle, it can be deduced that $\cos z = \frac{x}{1} = x$,

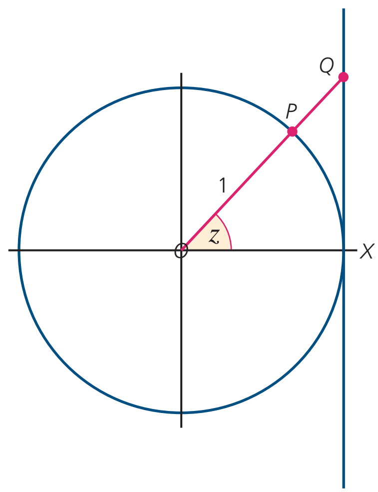
i.e. the cosine of the angle z is represented by the x-coordinate of P. Since $\cos z = x$, to be able to compare the graphs, the axes should be rotated through 90° as shown.



The properties of the cosine curve are similar to those of the sine curve. It has:

- a period of 360°
- a maximum value of +1
- a minimum value of –1
- symmetry, e.g. $\cos z = \cos(360 - z)$.

The cosine curve is a translation of the sine curve of –90°, i.e. $\cos z = \sin(z + 90)$.



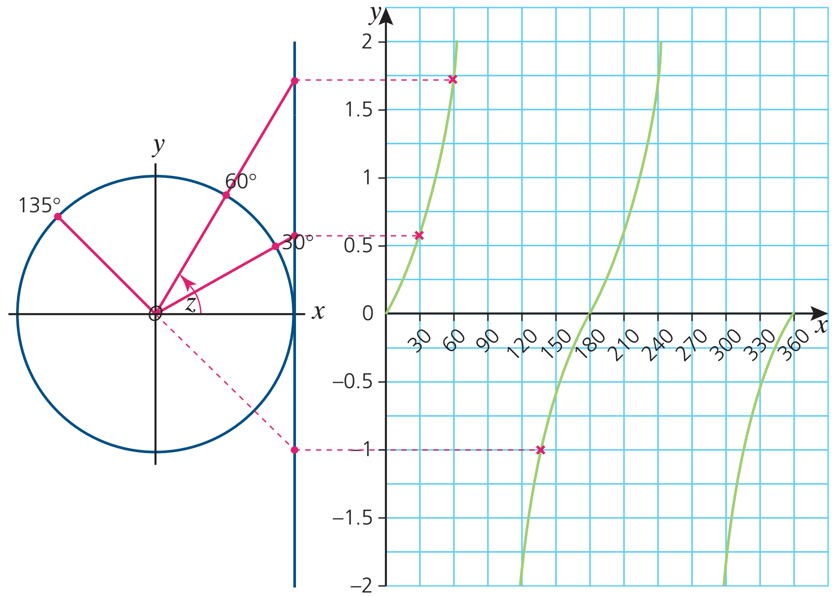
From the unit circle it can be deduced that $\tan z = \frac{y}{x}$.

In order to compare all the graphs, a tangent to the unit circle is drawn at (1, 0). OP is extended to meet the tangent at Q as shown.

As $OX = 1$ (radius of the unit circle), $\tan z = \frac{QX}{OX} = QX$.

i.e. $\tan z$ is equal to the y-coordinate of Q.

The graph of tan z is therefore shown below:

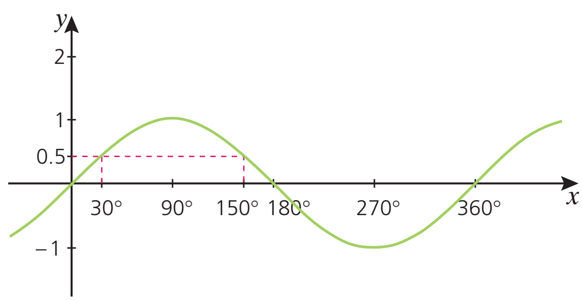


The graph of tan z has:

- a period of 180°
- no maximum or minimum value
- symmetry
- asymptotes at 90° and 270°.

Worked examples

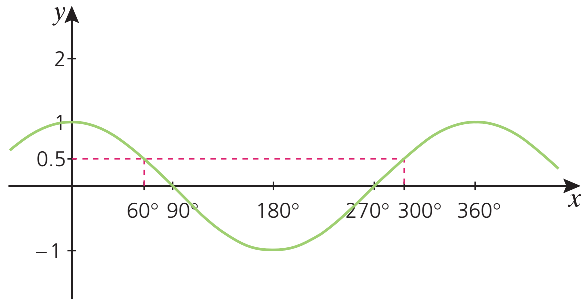
a sin 30° = 0.5. Which other angle between 0° and 360° has a sine of 0.5?



From the graph above it can be seen that sin 150° = 0.5.

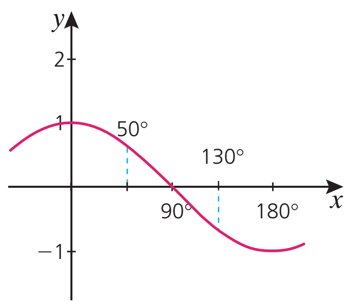
Also sin x = sin (180° – x); therefore sin 30° = sin (180° – 30) = sin 150°.

b cos 60° = 0.5. Which other angle between 0° and 360° has a cosine of 0.5?



From the graph above it can be seen that cos 300° = 0.5.

c The cosine of which angle between 0° and 180° is equal to the negative of cos 50°?
Cos 50° has the same magnitude but different sign to cos 130° because of the symmetrical properties of the cosine curve.
Therefore cos 130° = −cos 50°



Exercise 29.7

1 Write each of the following in terms of the sine of another angle between 0° and 360°.

a sin 60°	**b** sin 80°	**c** sin 115°
d sin 200°	**e** sin 300°	**f** sin 265°

2 Write each of the following in terms of the sine of another angle between 0° and 360°.

a sin 35°	**b** sin 50°	**c** sin 30°
d sin 248°	**e** sin 304°	**f** sin 327°

3 Find the two angles between 0° and 360° which have the following sine. Give each angle to the nearest degree.

a 0.33	**b** 0.99	**c** 0.09
d $-\frac{1}{2}$	**e** $-\frac{\sqrt{3}}{2}$	**f** $-\frac{1}{\sqrt{2}}$

4 Find the two angles between 0° and 360° which have the following sine. Give each angle to the nearest degree.

a 0.94	**b** 0.16	**c** 0.80
d −0.56	**e** −0.28	**f** −0.33

Exercise 29.8

1 Write each of the following in terms of the cosine of another angle between 0° and 360°.
  a cos 20° b cos 85° c cos 32°
  d cos 95° e cos 147° f cos 106°

2 Write each of the following in terms of the cosine of another angle between 0° and 360°.
  a cos 98° b cos 144° c cos 160°
  d cos 183° e cos 211° f cos 234°

3 Write each of the following in terms of the cosine of another angle between 0° and 180°.
  a −cos 100° b −cos 90° c −cos 110°
  d −cos 45° e −cos 122° f −cos 25°

4 The cosine of which acute angle has the same value as:
  a cos 125° b cos 107° c −cos 120°
  d −cos 98° e −cos 92° f −cos 110°?

5 Explain with reference to a right-angled triangle why the tangent of 90° is undefined.

Solving trigonometric equations

Knowledge of the graphs of the trigonometric functions enables us to solve trigonometric equations.

Worked examples

a Angle A is an obtuse angle. If $\sin A = \frac{\sqrt{3}}{2}$, calculate the size of A.

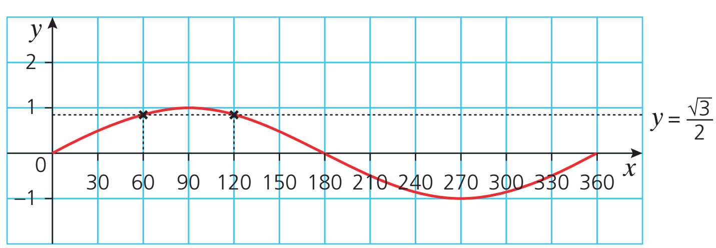
This ratio is one of the special angles covered earlier. Therefore, a calculator is not really needed to work out the size of A.

$$\sin A = \frac{\sqrt{3}}{2}$$

$$A = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$A = 60°.$$

However, the question states that A is an obtuse angle (i.e. $90° < A < 180°$), therefore $A \neq 60°$. Because of the symmetry properties of the sine curve, it can be deduced that $\sin 60° = \sin 120°$ as shown.



Therefore $A = 120°$.

This ratio is also of a special angle. A calculator is therefore not needed.

b If $\tan x = \frac{1}{\sqrt{3}}$, calculate the possible values for x in the range $0 \leqslant x \leqslant 360$.

$\tan x = \frac{1}{\sqrt{3}}$

$x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$

But the graph of $y = \tan x$ has a period of 180°. Therefore another solution in the range would be $30° + 180° = 210°$.

Therefore $x = 30°, 210°$.

Exercise 29.9

Try and do parts **b**, **c** and **d** without a calculator.

1 Solve each of the following equations, giving all the solutions in the range $0 \leqslant x \leqslant 360$.

a $\sin x = \frac{1}{4}$ **b** $\cos x = \frac{1}{\sqrt{2}}$ **c** $\sin x = -\frac{1}{2}$

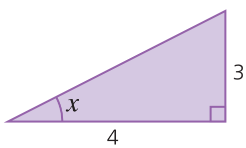
d $\tan x = -\sqrt{3}$ **e** $5\cos x + 1 = 2$ **f** $\frac{1}{2}\tan x + 2 = 1$

2 In the triangle below, $\tan x = \frac{3}{4}$

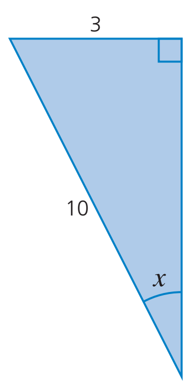
Deduce, without a calculator, the value of:

a $\sin x$

b $\cos x$



3 In the triangle below, $\sin x = \frac{3}{10}$

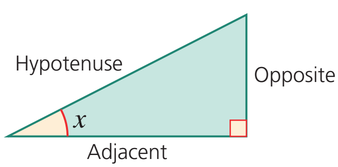


Deduce, without a calculator, the exact value of:

a $\cos x$

b $\tan x$

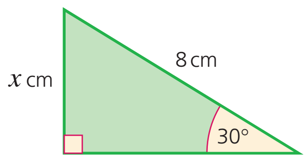
4 By using the triangle below as an aid, explain why the solution to the equation $\sin x = \cos x$ occurs when $x = 45°$.



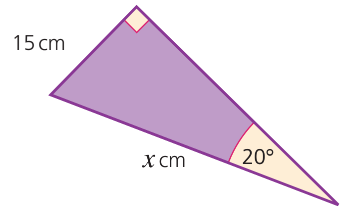
Student assessment 1

1 Calculate the length of the side marked x cm in each of the following.

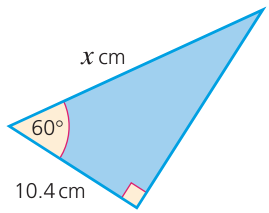
a



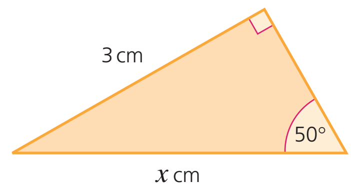
b



c

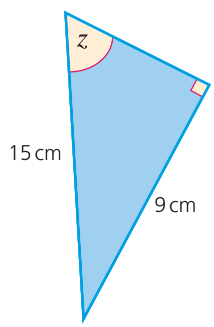


d

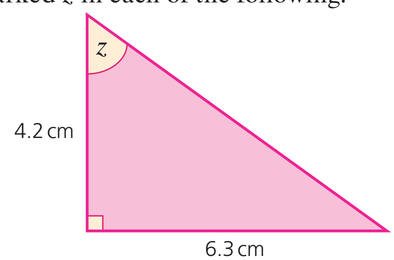


2 Calculate the size of the angle marked z in each of the following.

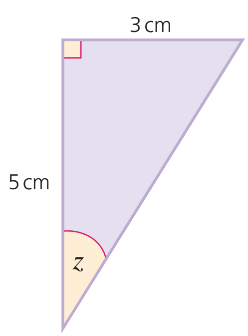
a



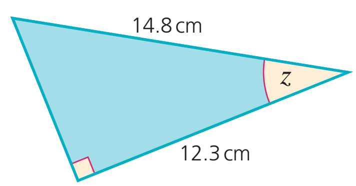
b



c

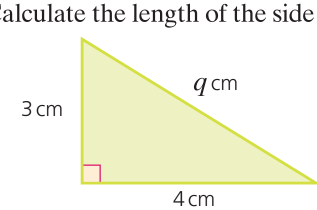


d

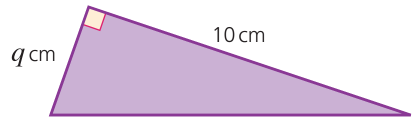


3 Calculate the length of the side marked q cm in each of the following.

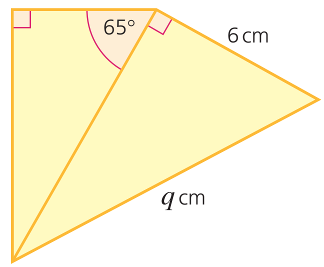
a



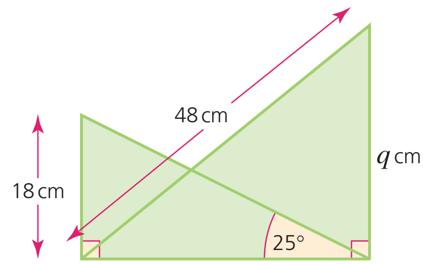
b



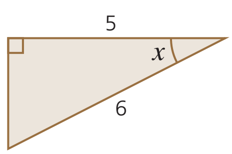
c



d



4 In the triangle below, $\cos x = \frac{5}{6}$.



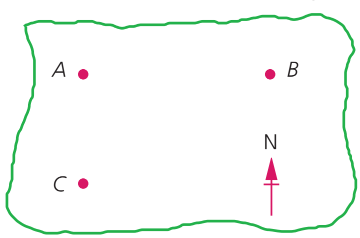
Showing your working clearly, deduce the exact value of:

a $\sin x$

b $\tan x$

Student assessment 2

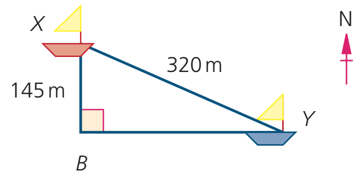
1 A map shows three towns A, B and C. Town A is due north of C. Town B is due east of A. The distance AC is 75 km and the bearing of C from B is 245°. Calculate, giving your answers to the nearest 100 m:



a the distance AB,

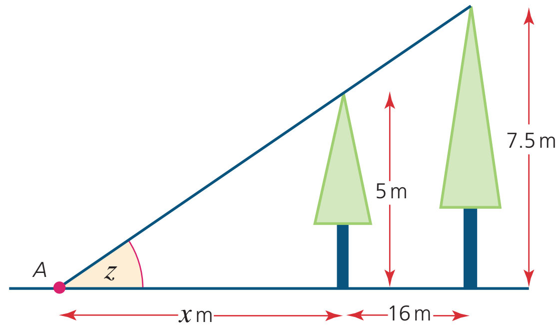
b the distance BC.

2 Two boats X and Y, sailing in a race, are shown in the diagram. Boat X is 145 m due north of a buoy B. Boat Y is due east of buoy B. Boats X and Y are 320 m apart. Calculate:



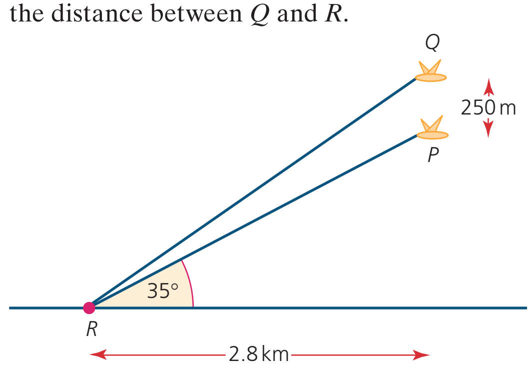
a the distance BY,
b the bearing of Y from X,
c the bearing of X from Y.

3 Two trees stand 16 m apart. Their tops make an angle z at point A on the ground.



a Express z in terms of the height of the shorter tree and its distance x metres from point A.
b Express z in terms of the height of the taller tree and its distance from A.
c Form an equation in terms of x.
d Calculate the value of x.
e Calculate the value z.

4 Two hawks P and Q are flying vertically above one another. Hawk Q is 250 m above hawk P. They both spot a snake at R.
Using the information given, calculate:
a the height of P above the ground,
b the distance between P and R,
c the distance between Q and R.



5 Solve the following trigonometric equations, giving all the solutions in the range $0 \leqslant x \leqslant 360$.

a $\sin x = \frac{2}{5}$ **b** $\tan x = -\frac{\sqrt{3}}{3}$
c $\cos x = -0.1$ **d** $\sin x = 1$

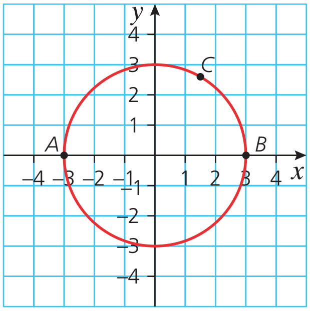
Student assessment 3

1 Explain, with the aid of a graph, why the equation $\cos x = \frac{3}{2}$ has no solutions.

2 The cosine of which other angle between 0 and 180° has the same value as:

a cos 128° **b** −cos 80°?

3 A circle of radius 3 cm, centre at *O*, is shown on the axes below.

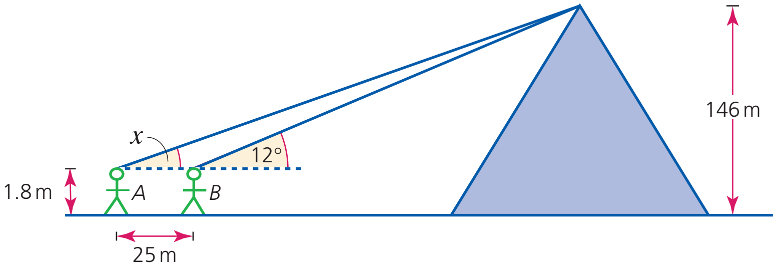


The points *A* and *B* lie where the circumference of the circle intersects the *x*-axis. Point *C* is free to move on the circumference of the circle.

a Deduce, justifying your answer, the size of angle *ACB*.

b If *BC* = 3 cm, calculate the possible coordinates of point *C*, giving your answers in exact form.

4 The Great Pyramid at Giza is 146 m high. Two people *A* and *B* are looking at the top of the pyramid. The angle of elevation of the top of the pyramid from *B* is 12°. The distance between *A* and *B* is 25 m.



If both *A* and *B* are 1.8 m tall, calculate:

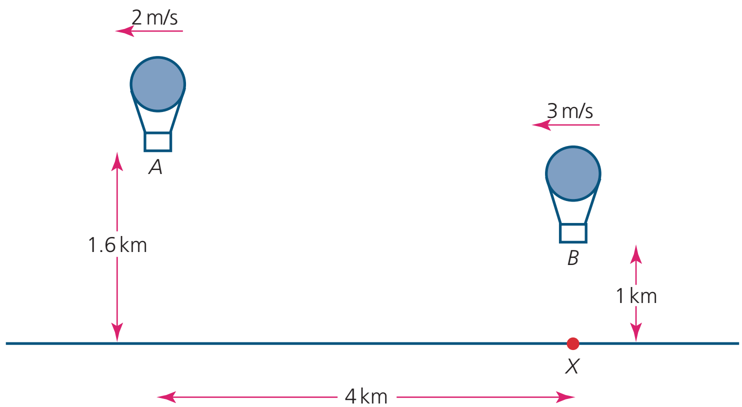
a the distance on the ground from *B* to the centre of the base of the pyramid,

b the angle of elevation *x* of the top of the pyramid from *A*,

c the distance between *A* and the top of the pyramid.

Note: *A*, *B* and the top of the pyramid are in the same vertical plane.

5 Two hot air balloons A and B are travelling in the same horizontal direction as shown in the diagram below. A is travelling at 2 m/s and B at 3 m/s. Their heights above the ground are 1.6 km and 1 km, respectively.



At midday, their horizontal distance apart is 4 km and balloon B is directly above a point X on the ground.
Calculate:

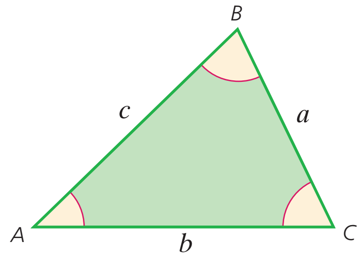
a the angle of elevation of A from X at midday,
b the angle of depression of B from A at midday,
c their horizontal distance apart at 1230,
d the angle of elevation of B from X at 1230,
e the angle of elevation of A from B at 1230,
f how much closer A and B are at 1230 compared with midday.

6 **a** On one diagram, plot the graph of $y = \sin x°$ and the graph of $y = \cos x°$, for $0° \leqslant x° \leqslant 180°$.
b Use your graph to find the angles for which $\sin x° = \cos x°$.

30 Further trigonometry

The sine rule

With right-angled triangles, we can use the basic trigonometric ratios of sine, cosine and tangent. The **sine rule** is a relationship which can be used with non-right-angled triangles.



The sine rule states that:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or alternatively

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

You should know this formula, but you do not need to memorise it.

Worked examples

a Calculate the length of side BC.

Using the sine rule:

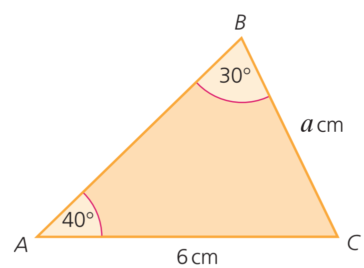
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 40°} = \frac{6}{\sin 30°}$$

$$a = \frac{6 \times \sin 40°}{\sin 30°}$$

$a = 7.7$ (1 d.p.)

$BC = 7.7$ cm



b Calculate the size of angle C.

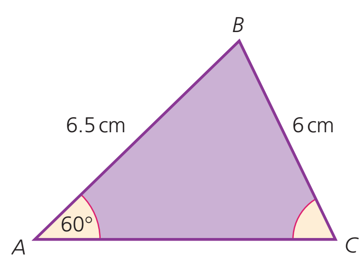
Using the sine rule:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\sin C = \frac{6.5 \times \sin 60°}{6}$$

$C = \sin^{-1}(0.94)$

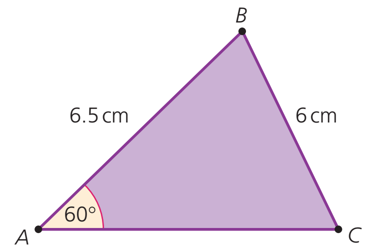
$C = 69.8°$ (1 d.p.)



In the worked example **b** above, $C = 69.8°$ is not the only possible solution from the information given, as the question does not state that the angle is acute and often, diagrams are not drawn to scale.

The triangle ABC_1 is as shown in **b** above and therefore, angle $AC_1B = 69.8°$ as calculated.

However, if a circle of radius 6 cm and centre at B is drawn, then it can be seen to intersect the base of the triangle in another place, C_2.



Therefore, triangle ABC_2 is another possible triangle.

But angle AC_2B is not the same size as angle AC_1B.

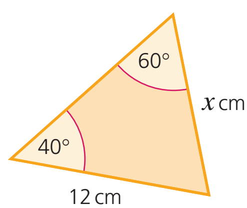
It can be calculated in the same way as before, however. Using our knowledge of the sine curve, it can be calculated that angle AC_2B is $180 - 69.8 = 110.2°$

This is known as the **ambiguous case** of the sine rule as there are two possible answers.

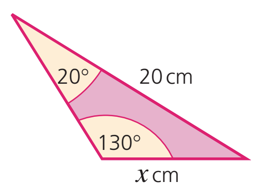
Exercise 30.1

1 Calculate the length of the side marked x in each of the following.

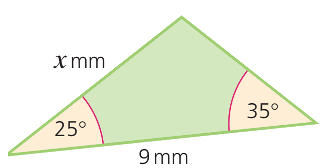
a



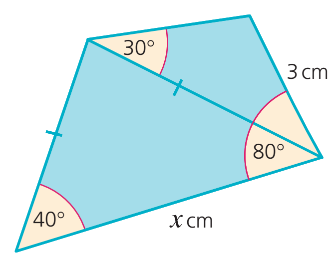
b



c



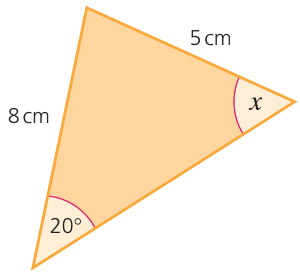
d



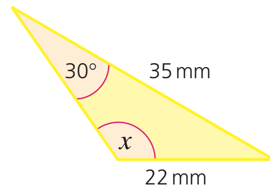
Exercise 30.1 (cont)

2 Calculate the size of the angle marked x in each of the following. If two values for the angle are possible, give both values.

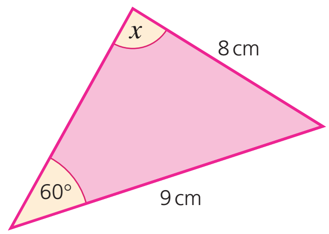
a



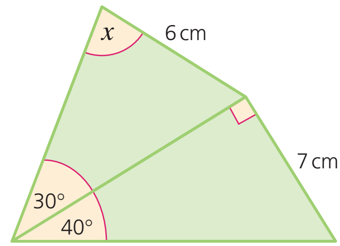
b



c



d

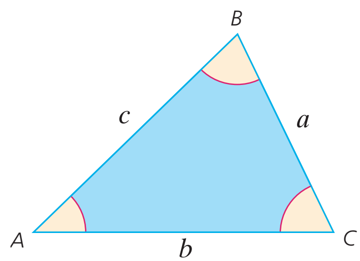


3 Triangle ABC has the following dimensions:
$AC = 10$ cm, $AB = 8$ cm and angle $ACB = 20°$.
  a Calculate the two possible values for angle CBA.
  b Sketch and label the two possible shapes for triangle ABC.

4 Triangle PQR has the following dimensions:
$PQ = 6$ cm, $PR = 4$ cm and angle $PQR = 40°$.
  a Calculate the two possible values for angle QRP.
  b Sketch and label the two possible shapes for triangle PQR.

The cosine rule

The **cosine rule** is another relationship which can be used with non-right-angled triangles.



The cosine rule states that:

You should know this formula, but you do not need to memorise it.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Worked examples

a Calculate the length of the side BC.

Using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

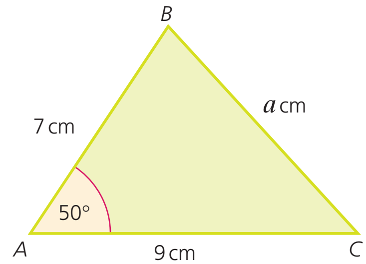
$$a^2 = 9^2 + 7^2 - (2 \times 9 \times 7 \times \cos 50°)$$

$$= 81 + 49 - (126 \times \cos 50°) = 49.0$$

$$a = \sqrt{49.0}$$

$$a = 7.00 \text{ (3 s.f.)}$$

$$BC = 7.00\text{ cm (3 s.f.)}$$



b Calculate the size of angle A.

Using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

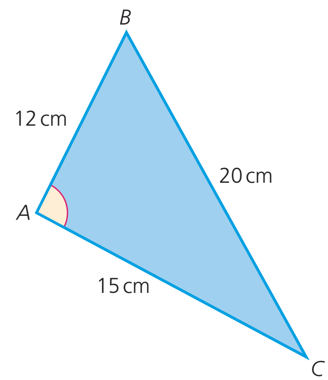
Rearranging the equation gives:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{15^2 + 12^2 - 20^2}{2 \times 15 \times 12} = -0.086$$

$$A = \cos^{-1}(-0.086)$$

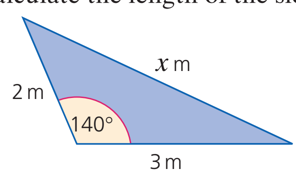
$$A = 94.9° \text{ (1 d.p.)}$$



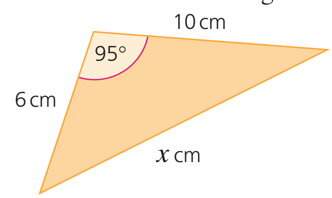
Exercise 30.2

1 Calculate the length of the side marked x in each of the following.

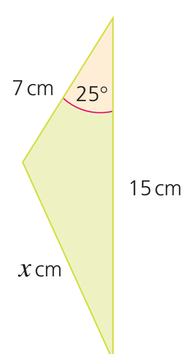
a



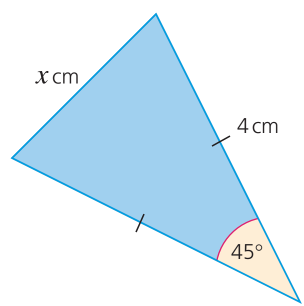
b



c

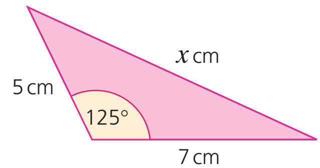


d



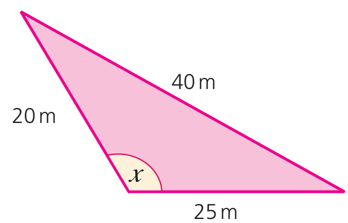
Exercise 30.2 (cont)

e

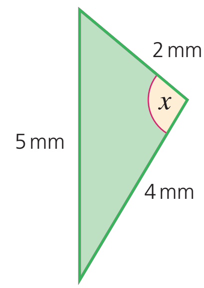


2 Calculate the angle marked x in each of the following.

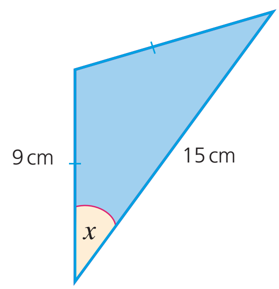
a



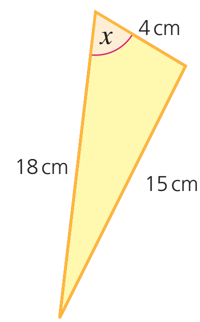
b



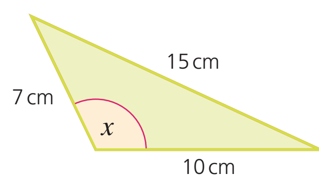
c



d



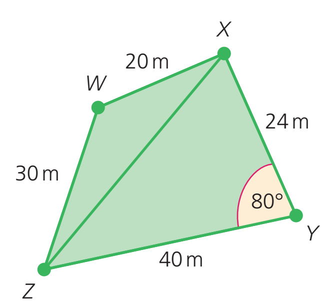
e



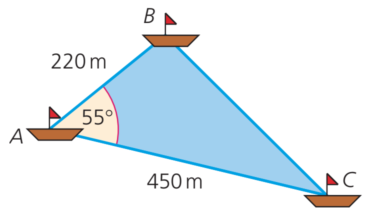
Exercise 30.3

1 Four players, W, X, Y and Z, are on a rugby pitch. The diagram shows a plan view of their relative positions.
Calculate:

- **a** the distance between players X and Z,
- **b** angle ZWX,
- **c** angle WZX,
- **d** angle YZX,
- **e** the distance between players W and Y.

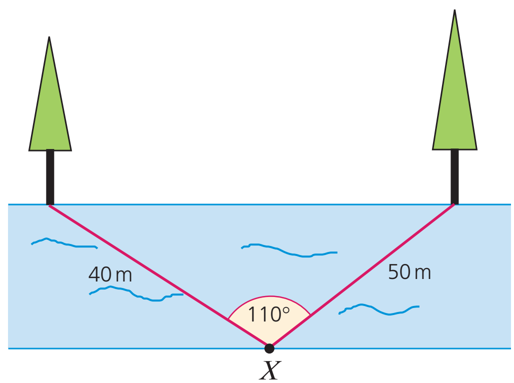


2 Three yachts, A, B and C, are racing off Cape Comorin in India. Their relative positions are shown (below).



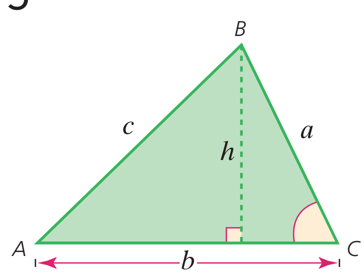
Calculate the distance between B and C to the nearest 10 m.

3 There are two trees standing on one side of a river bank. On the opposite side, a boy is standing at X.



Using the information given, calculate the distance between the two trees.

The area of a triangle



$$\text{Area} = \tfrac{1}{2}bh$$

Also:

$$\sin C = \frac{h}{a}$$

Rearranging:

$$h = a \sin C$$

Therefore

$$\text{Area} = \tfrac{1}{2}ab\sin C$$

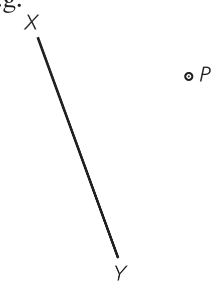
You should know this formula, but you do not need to memorise it.

Shortest distance from a point to a line

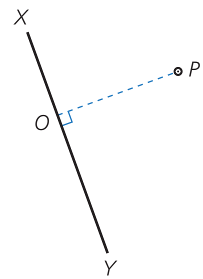
The height of a triangle is measured perpendicular to the base of the triangle, as shown above.

In general, the shortest distance from a point to a line is the distance measured perpendicular to the line and passing through the point.

e.g.



To calculate the shortest distance from point P to the line XY, draw a line perpendicular to XY passing through P.

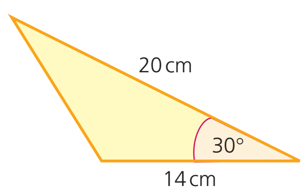


The shortest distance from P to the line XY is therefore the distance OP.

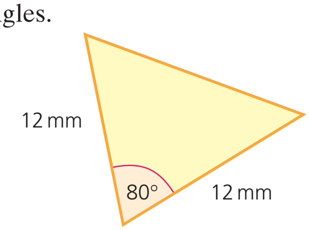
Exercise 30.4

1 Calculate the area of the following triangles.

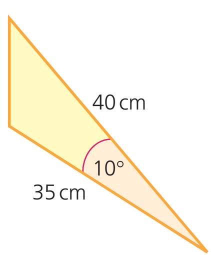
a



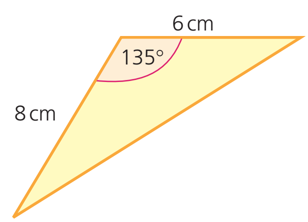
b



c

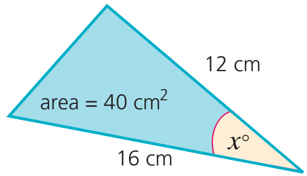


d

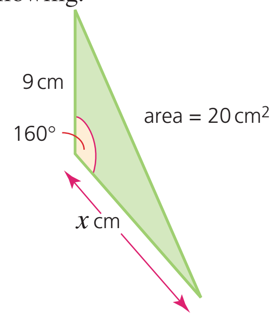


2 Calculate the value of x in each of the following.

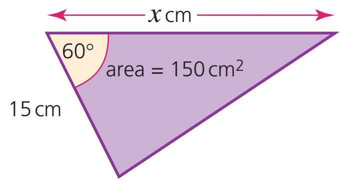
a



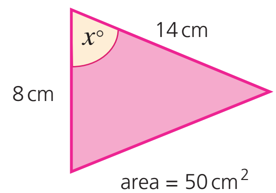
b



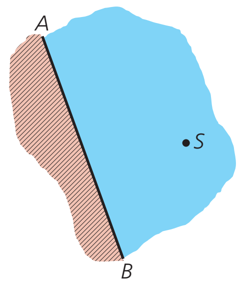
c



d

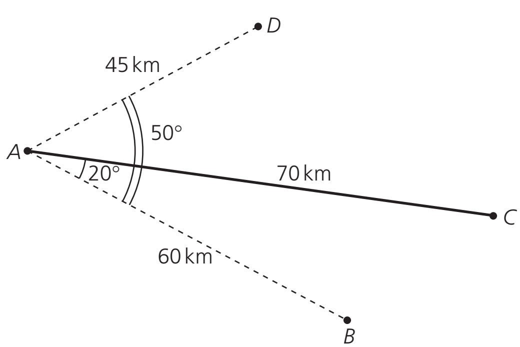


3 A straight stretch of coast AB is shown below. A ship S at sea is 1.1 km from A.



If angle $SAB = 25°$, calculate the shortest distance between the ship and the coast, giving your answer to the nearest metre.

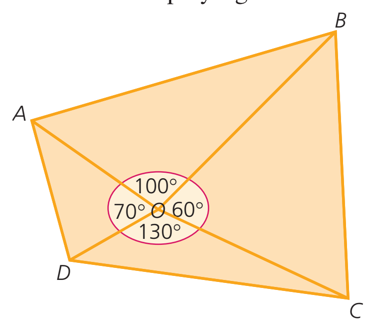
4 The four corners, A, B, C and D, of a large nature reserve form a quadrilateral and are shown below.
$AC = 70$ km, $AD = 45$ km and $AB = 60$ km.



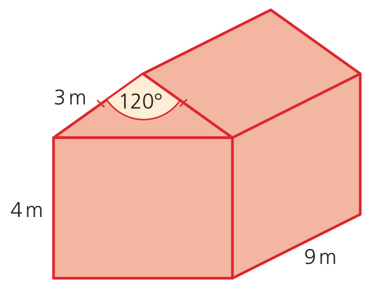
If angle $BAC = 20°$ and angle $BAD = 50°$, calculate the area of the nature reserve, giving your answer to the nearest square kilometre.

Exercise 30.4 (cont)

5 $ABCD$ is a school playing field. The following lengths are known:
$OA = 83\,\text{m}$,
$OB = 122\,\text{m}$,
$OC = 106\,\text{m}$,
$OD = 78\,\text{m}$.
Calculate the area of the school playing field to the nearest $100\,\text{m}^2$.



6 The roof of a garage has a slanting length of 3 m and makes an angle of 120° at its vertex. The height of the walls of the garage is 4 m and its depth is 9 m.

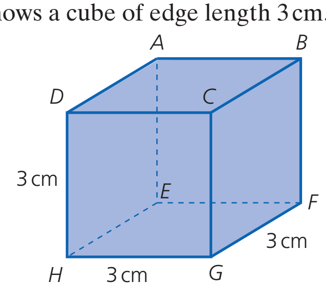


Calculate:
a the cross-sectional area of the roof,
b the volume occupied by the whole garage.

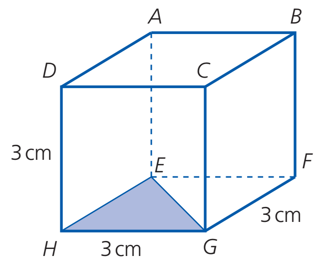
Trigonometry in three dimensions

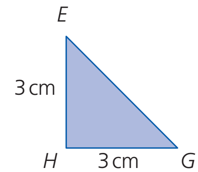
Worked examples

The diagram (below) shows a cube of edge length 3 cm.



a Calculate the length EG.



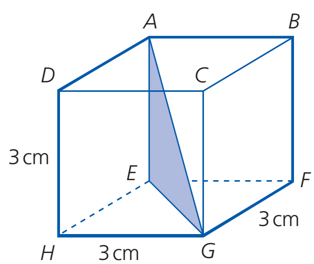


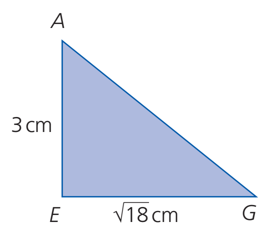
Triangle EHG (above) is right-angled. Use Pythagoras' theorem to calculate the length EG.

$$(EG)^2 = (EH)^2 + (HG)^2$$
$$(EG)^2 = 3^2 + 3^2 = 18$$
$$EG = \sqrt{18}\text{ cm} = 4.24\text{ cm (3 s.f.)}$$

b Calculate the length AG.

Triangle AEG (below) is right-angled. Use Pythagoras' theorem to calculate the length AG.



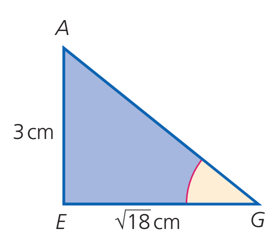


$$(AG)^2 = (AE)^2 + (EG)^2$$
$$(AG)^2 = 3^2 + (\sqrt{18})^2$$
$$(AG)^2 = 9 + 18$$
$$AG = \sqrt{27}\text{ cm} = 5.20\text{ cm (3 s.f.)}$$

c Calculate the angle EGA.

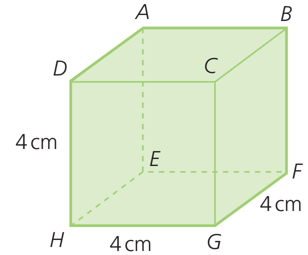
To calculate angle EGA we use the triangle EGA:

$$\tan G = \frac{3}{\sqrt{18}}$$
$$G = 35.3° \text{ (1 d.p.)}$$

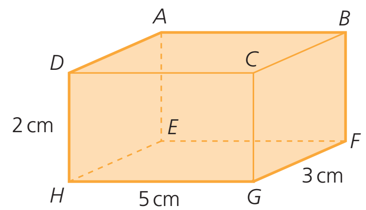


Exercise 30.5

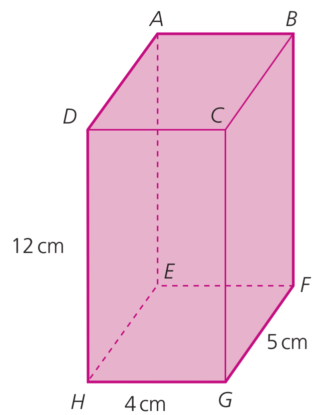
1 a Calculate the length *HF*.
  b Calculate the length *HB*.
  c Calculate the angle *BHG*.



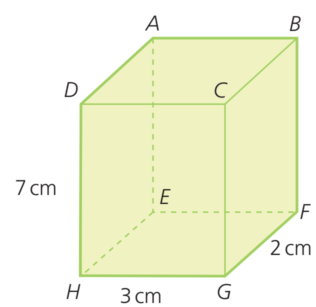
2 a Calculate the length *CA*.
  b Calculate the length *CE*.
  c Calculate the angle *ACE*.



3 a Calculate the length *EG*.
  b Calculate the length *AG*.
  c Calculate the angle *AGE*.



4 a Calculate the angle *BCE*.
  b Calculate the angle *GFH*.



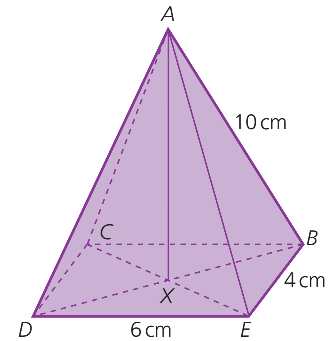
5 The diagram shows a right pyramid where A is vertically above X.

a **i** Calculate the length DB.

ii Calculate the angle DAX.

b **i** Calculate the angle CED.

ii Calculate the angle DBA.



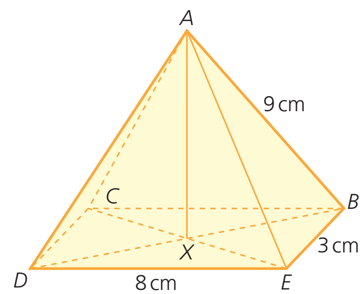
6 The diagram shows a right pyramid where A is vertically above X.

a **i** Calculate the length CE.

ii Calculate the angle CAX.

b **i** Calculate the angle BDE.

ii Calculate the angle ADB.

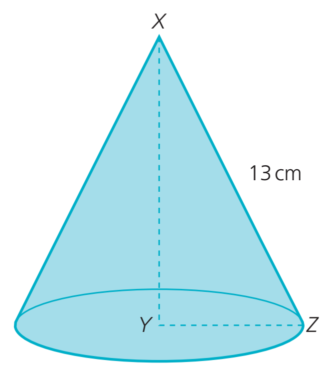


7 In this cone the angle $YXZ = 60°$. Calculate:

a the length XY,

b the length YZ,

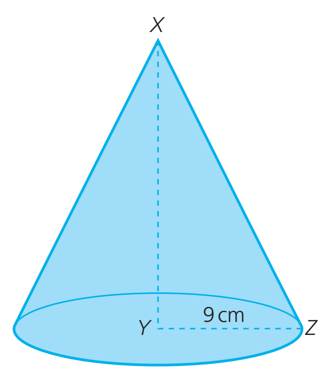
c the circumference of the base.



8 In this cone the angle $XZY = 40°$. Calculate:

a the length XZ,

b the length XY.



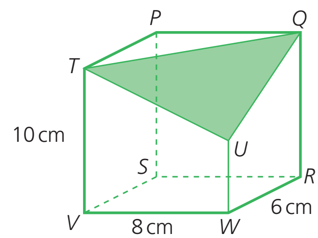
Exercise 30.5 (cont)

9 One corner of this cuboid has been sliced off along the plane QTU. $WU = 4$ cm.

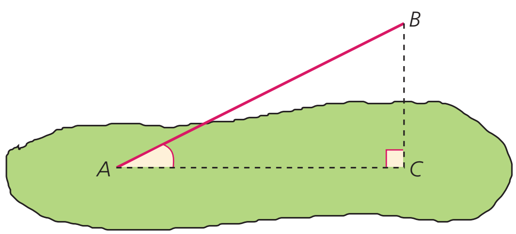
a Calculate the length of the three sides of the triangle QTU.

b Calculate the three angles Q, T and U in triangle QTU.

c Calculate the area of triangle QTU.



The angle between a line and a plane



To calculate the size of the angle between the line AB and the shaded plane, drop a perpendicular from B. It meets the shaded plane at C. Then join AC.

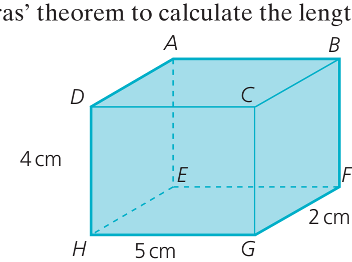
The angle between the lines AB and AC represents the angle between the line AB and the shaded plane.

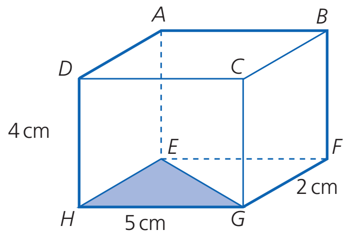
The line AC is the projection of the line AB on the shaded plane.

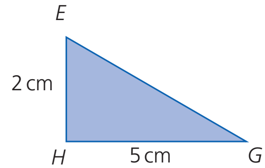
Worked examples

a Calculate the length EC.

First use Pythagoras' theorem to calculate the length EG:







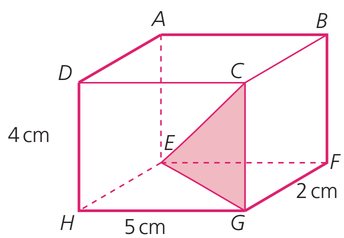
$(EG)^2 = (EH)^2 + (HG)^2$

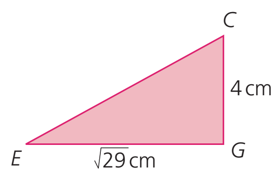
$(EG)^2 = 2^2 + 5^2$

$(EG)^2 = 29$

$EG = \sqrt{29}\,\text{cm}$

Now use Pythagoras' theorem to calculate *CE*:





$(EC)^2 = (EG)^2 + (CG)^2$

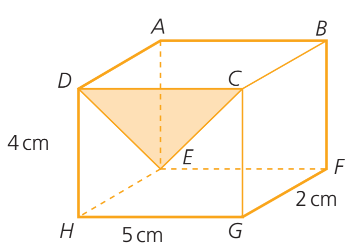
$(EC)^2 = (\sqrt{29})^2 + 4^2$

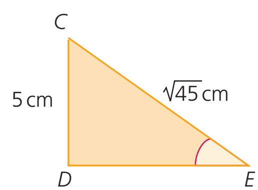
$(EC)^2 = 29 + 16$

$EC = \sqrt{45}\,\text{cm} = 6.71\,\text{cm (3 s.f.)}$

b Calculate the angle between the line *CE* and the plane *ADHE*.

To calculate the angle between the line *CE* and the plane *ADHE* use the right-angled triangle *CED* and calculate the angle *CED*.





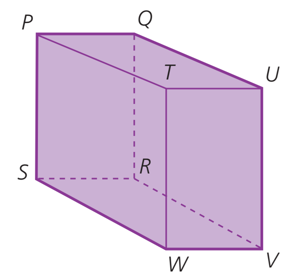
$\sin E = \frac{CD}{CE} = \frac{5}{\sqrt{45}}$

$E = \sin^{-1}\frac{5}{\sqrt{45}} = 48.2^\circ \text{ (1 d.p.)}$

Exercise 30.6

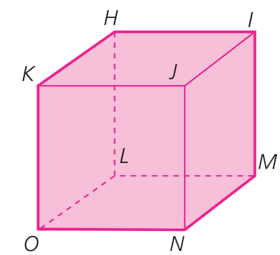
1 Name the projection of each line onto the given plane:

a TR onto $RSWV$
b TR onto $PQUT$
c SU onto $PQRS$
d SU onto $TUVW$
e PV onto $QRVU$
f PV onto $RSWV$



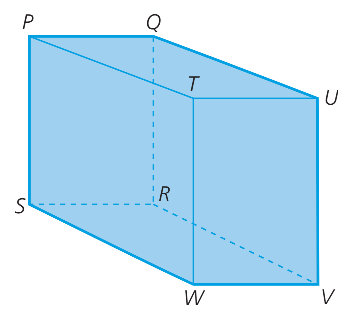
2 Name the projection of each line onto the given plane:

a KM onto $IJNM$
b KM onto $JKON$
c KM onto $HIML$
d IO onto $HLOK$
e IO onto $JKON$
f IO onto $LMNO$

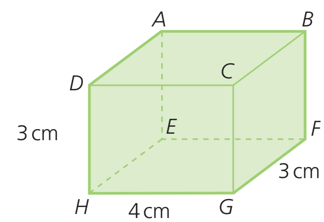


3 Name the angle between the given line and plane:

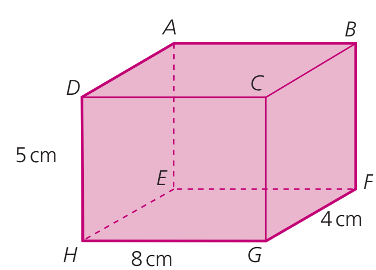
a PT and $PQRS$
b PU and $PQRS$
c SV and $PSWT$
d RT and $TUVW$
e SU and $QRVU$
f PV and $PSWT$



4 a Calculate the length BH.
b Calculate the angle between the line BH and the plane $EFGH$.



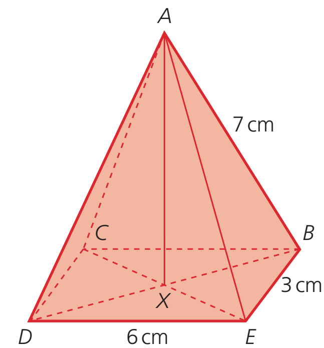
5 a Calculate the length AG.
b Calculate the angle between the line AG and the plane $EFGH$.
c Calculate the angle between the line AG and the plane $ADHE$.



6 The diagram shows a right pyramid where A is vertically above X.

a Calculate the length BD.

b Calculate the angle between AB and $CBED$.

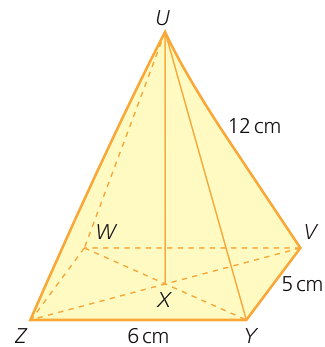


7 The diagram shows a right pyramid where U is vertically above X.

a Calculate the length WY.

b Calculate the length UX.

c Calculate the angle between UX and UZY.



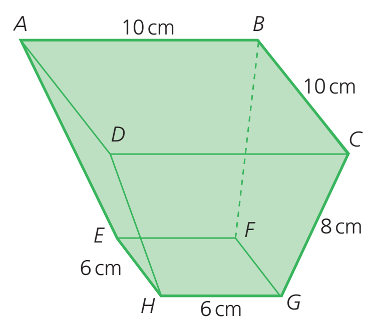
8 $ABCD$ and $EFGH$ are square faces lying parallel to each other. Calculate:

a the length DB,

b the length HF,

c the vertical height of the object,

d the angle DH makes with the plane $ABCD$.



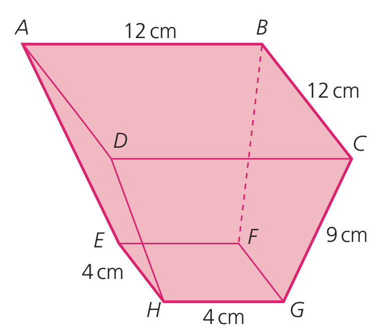
9 $ABCD$ and $EFGH$ are square faces lying parallel to each other. Calculate:

a the length AC,

b the length EG,

c the vertical height of the object,

d the angle CG makes with the plane $EFGH$.

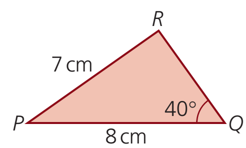


Student assessment 1

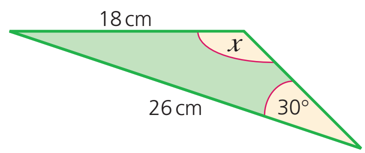
1 The triangle PQR is shown right.

a Calculate the two possible values for angle PRQ.

b Calculate the shortest distance possible of R from side PQ.



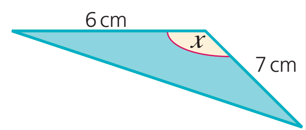
2 Calculate the size of the obtuse angle marked x in the triangle (right).



3 The area of the triangle is 10.5cm^2.

a Calculate the value of $\sin x$.

b If x is an obtuse angle, calculate the value of x.

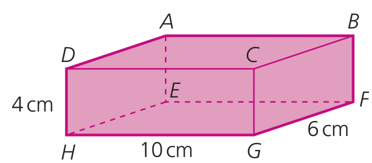


4 For the cuboid, calculate:

a the length EG,

b the length EC,

c angle BEC.



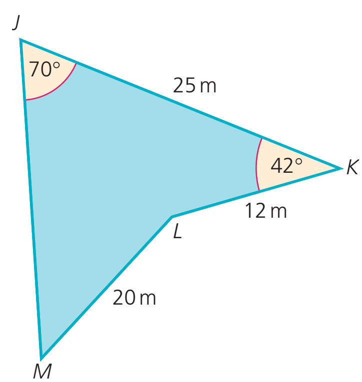
5 For the quadrilateral (right), calculate:

a the length JL,

b angle KJL,

c the length JM,

d the area of $JKLM$.



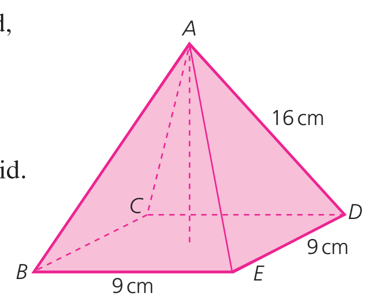
6 For the square-based right pyramid, calculate:

a the length BD,

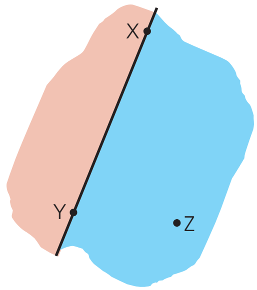
b angle ABD,

c the area of triangle ABD,

d the vertical height of the pyramid.



Student assessment 2

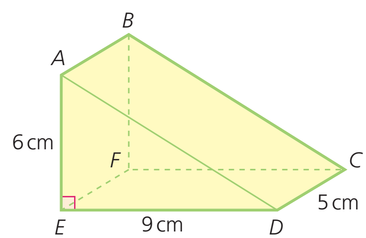
1 

Two points X and Y are 1000 m apart on a straight piece of coastline. A buoy Z is out at sea. $XZ = 850$ m and $YZ = 625$ m. Imani wishes to swim from the coast to the buoy.

- **a** What is the shortest distance she will have to swim? Give your answer to 3 s.f.
- **b** How far along the coast from Y will she have to set off in order to swim the least distance?

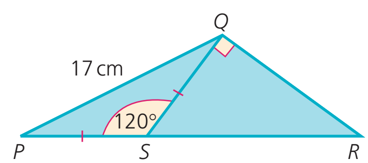
2 Using the triangular prism, calculate:

- **a** the length AD,
- **b** the length AC,
- **c** the angle AC makes with the plane $CDEF$,
- **d** the angle AC makes with the plane $ABFE$.



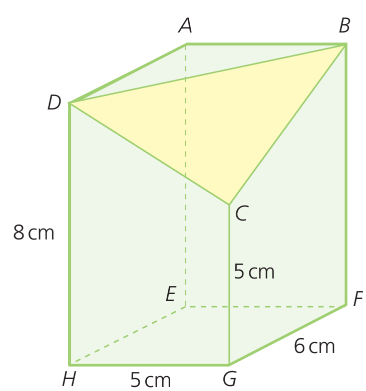
3 For the triangle, calculate:

- **a** the length PS,
- **b** angle QRS,
- **c** the length SR.



4 The cuboid (right) has one of its corners removed to leave a flat triangle BDC.
Calculate:

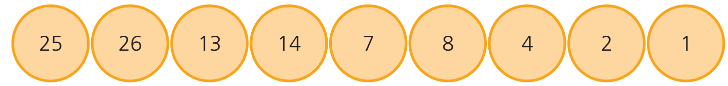
- **a** length DC,
- **b** length BC,
- **c** length DB,
- **d** angle CBD,
- **e** the area of triangle BDC,
- **f** the angle AC makes with the plane $AEHD$.



6 Mathematical investigations and ICT 6

Numbered balls

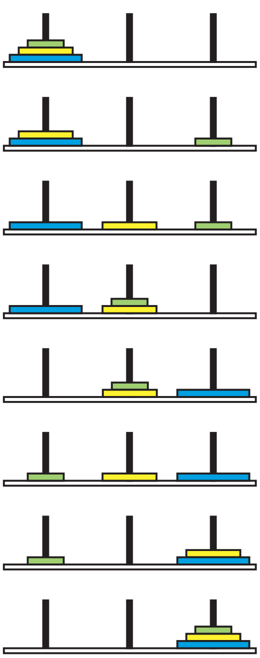
The balls below start with the number 25 and then subsequent numbered balls are added according to a rule. The process stops when ball number 1 is added.



1. Express in words the rule for generating the sequence of numbered balls.
2. What is the longest sequence of balls starting with a number less than 100?
3. Is there a strategy for generating a long sequence?
4. Use your rule to state the longest sequence of balls starting with a number less than 1000.
5. Extend the investigation by having a different term-to-term rule.

Towers of Hanoi

This investigation is based on an old Vietnamese legend. The legend is as follows:



At the beginning of time a temple was created by the Gods. Inside the temple stood three giant rods. On one of these rods, 64 gold discs, all of different diameters, were stacked in descending order of size, i.e. the largest at the bottom rising to the smallest at the top. Priests at the temple were responsible for moving the discs onto the remaining two rods until all 64 discs were stacked in the same order on one of the other rods. When this task was completed, time would cease and the world would come to an end.

The discs, however, could only be moved according to certain rules. These were:

- Only one disc could be moved at a time.
- A disc could only be placed on top of a larger one.

The diagram (left) shows the smallest number of moves required to transfer three discs from the rod on the left to the rod on the right.

With three discs, the smallest number of moves is seven.

1. What is the smallest number of moves needed for two discs?
2. What is the smallest number of moves needed for four discs?
3. Investigate the smallest number of moves needed to move different numbers of discs.

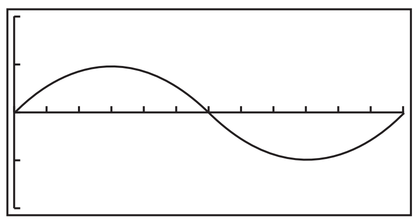
4 Display the results of your investigation in an ordered table.
5 Describe any patterns you see in your results.
6 Predict, from your results, the smallest number of moves needed to move ten discs.
7 Determine a formula for the smallest number of moves for n discs.
8 Assume the priests have been transferring the discs at the rate of one per second and assume the Earth is approximately 4.54 billion years old (4.54×10^9 years).

According to the legend, is the world coming to an end soon? Justify your answer with relevant calculations.

ICT activity

In this activity you will need to use a graphics calculator to investigate the relationship between different trigonometric ratios.

1 a Using the calculator, plot the graph of $y = \sin x$ for $0° \leqslant x \leqslant 360°$. The graph should look similar to the one shown below:



b Using the equation solving facility, evaluate the following:
i sin 70°
ii sin 125°
iii sin 300°

c Referring to the graph, explain why $\sin x = 0.7$ has two solutions between 0° and 360°.

d Use the graph to solve the equation $\sin x = 0.5$.

2 a On the same axes as before, plot $y = \cos x$.

b How many solutions are there to the equation $\sin x = \cos x$ between 0° and 360°?

c What is the solution to the equation $\sin x = \cos x$ between 180° and 270°?

3 By plotting appropriate graphs, solve the following for $0° \leqslant x \leqslant 360°$.

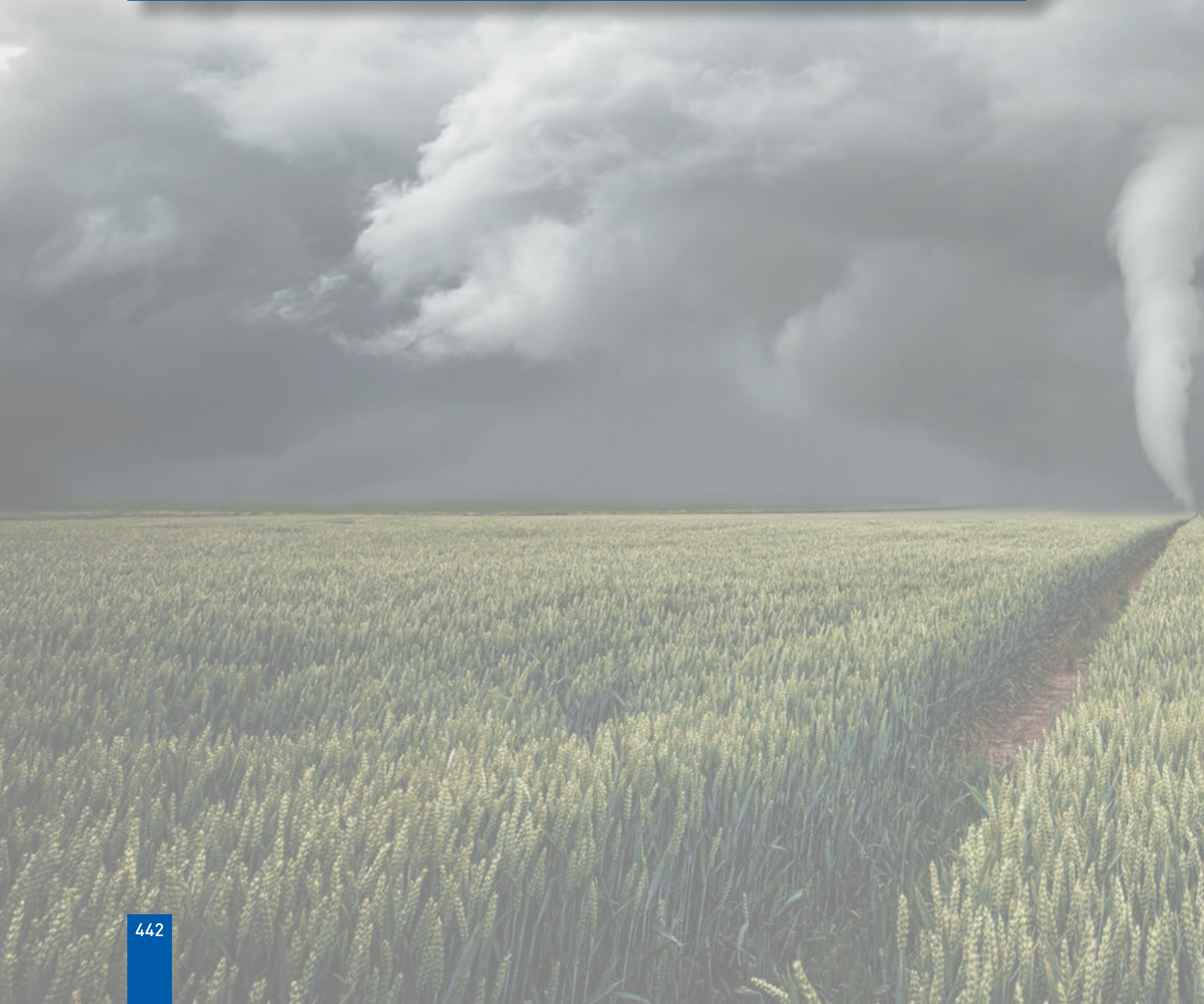
a $\sin x = \tan x$

b $\cos x = \tan x$

TOPIC 7

Vectors and transformations

Contents



Learning objectives

E7.1 Transformations

Recognise, describe and draw the following transformations:

1 Reflection of a shape in a straight line.
2 Rotation of a shape about a centre through multiples of 90°.
3 Enlargement of a shape from a centre by a scale factor.
4 Translation of a shape by a vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

E7.2 Vectors in two dimensions

1 Describe a translation using a vector represented by $\begin{pmatrix} x \\ y \end{pmatrix}$, $\overrightarrow{AB}$ or **a**.
2 Add and subtract vectors.
3 Multiply a vector by a scalar.

E7.3 Magnitude of a vector

Calculate the magnitude of a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ as $\sqrt{x^2 + y^2}$.

E7.4 Vector geometry

1 Represent vectors by directed line segments.
2 Use position vectors.
3 Use the sum and difference of two or more vectors to express given vectors in terms of two coplanar vectors.
4 Use vectors to reason and to solve geometric problems.

The Italians

Leonardo Pisano (known today as Fibonacci) introduced new methods of arithmetic to Europe from the Hindus, Persians and Arabs. He discovered the sequence 1, 1, 2, 3, 5, 8, 13, … , which is now called the Fibonacci sequence, and some of its occurrences in nature. He also brought the decimal system, algebra and the 'lattice' method of multiplication to Europe. Fibonacci has been called the 'most talented mathematician of the middle ages'. Many books say that he brought Islamic mathematics to Europe, but in Fibonacci's own introduction to *Liber Abaci*, he credits the Hindus.

Fibonacci (1170–1250)

The Renaissance began in Italy. Art, architecture, music and the sciences flourished. Girolamo Cardano (1501–1576) wrote his great mathematical book *Ars Magna* (Great Art) in which he showed, among much algebra that was new, calculations involving the solutions to cubic equations. He published this book, the first algebra book in Latin, to great acclaim. However, although he continued to study mathematics, no other work of his was ever published.

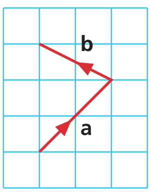
31 Vectors

Translations

A **translation** (a sliding movement) can be described using **column vectors**. A column vector describes the movement of the object in both the *x* direction and the *y* direction.

Worked examples

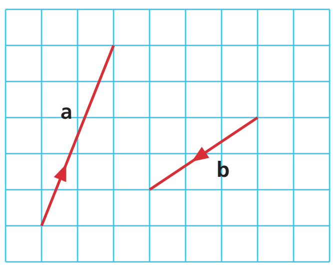
Define **a** and **b** in the diagram (left) using column vectors.



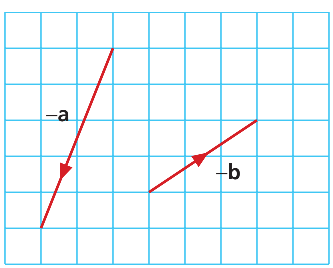
$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Note: When you represent **vectors** by single letters, i.e. **a**, in handwritten work, you should write them as $\underset{\sim}{a}$.

If $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$, they can be represented diagrammatically.



The diagrammatic representation of –**a** and –**b** is shown below.



It can be seen from the diagram above that $-\mathbf{a} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ and $-\mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Translations can also be named using letters to represent the start and end point, with an arrow above the letters showing the direction of the translation.

Worked examples

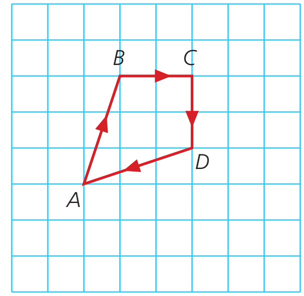
Note

The notation $\overrightarrow{AB}$ or **a** for vectors is only required for the Extended syllabus.

a Describe the translation from A to B in the diagram in terms of a column vector.

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

i.e. 1 unit in the x direction, 3 units in the y direction.



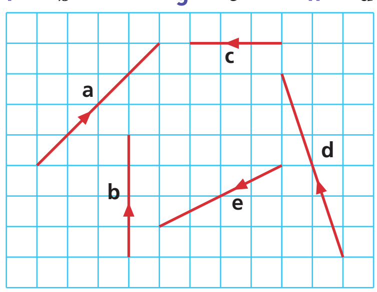
b Describe $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$ in terms of column vectors.

$\overrightarrow{BC} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\overrightarrow{CD} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ $\overrightarrow{DA} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

Exercise 31.1

1 Describe each translation using a column vector.

a **a** b **b** c **c** d **d** e **e**
f **–b** g **–c** h **–d** i **–a**



2 Draw and label the following vectors on a square grid.

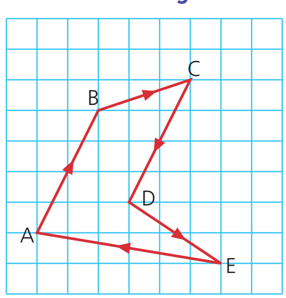
a $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ b $\mathbf{b} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$ c $\mathbf{c} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

d $\mathbf{d} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ e $\mathbf{e} = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$ f $\mathbf{f} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$

g $\mathbf{g} = -\mathbf{c}$ h $\mathbf{h} = -\mathbf{b}$ i $\mathbf{i} = -\mathbf{f}$

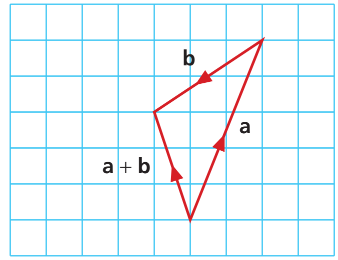
3 Describe each translation using a column vector.

a $\overrightarrow{AB}$ b $\overrightarrow{BC}$ c $\overrightarrow{CD}$ d $\overrightarrow{DE}$ e $\overrightarrow{EA}$
f $\overrightarrow{AE}$ g $\overrightarrow{DA}$ h $\overrightarrow{CA}$ i $\overrightarrow{DB}$



Addition and subtraction of vectors

Vectors can be added together and represented diagrammatically as shown.

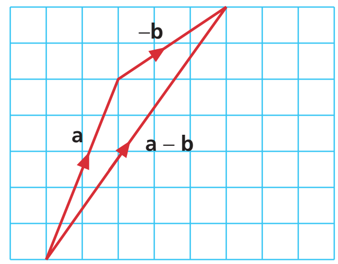


The translation represented by **a** followed by **b** can be written as a single transformation **a** + **b**:

i.e. $\begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Worked examples

$\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$



a Draw a diagram to represent **a** – **b**, where **a** – **b** = (**a**) + (–**b**).

b Calculate the vector represented by **a** – **b**.

$\begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

Exercise 31.2

In the following questions,

$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad \mathbf{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

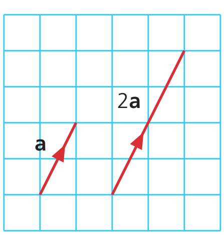
1 Draw vector diagrams to represent the following.
   a **a** + **b** b **b** + **a** c **a** + **d**
   d **d** + **a** e **b** + **c** f **c** + **b**

2 What conclusions can you draw from your answers to Question 1 above?

3 Draw vector diagrams to represent the following.
   a **b** – **c** b **d** – **a** c –**a** – **c**
   d **a** + **c** – **b** e **d** – **c** – **b** f –**c** + **b** + **d**

4 Represent each of the vectors in Question 3 by a single column vector.

Multiplying a vector by a scalar

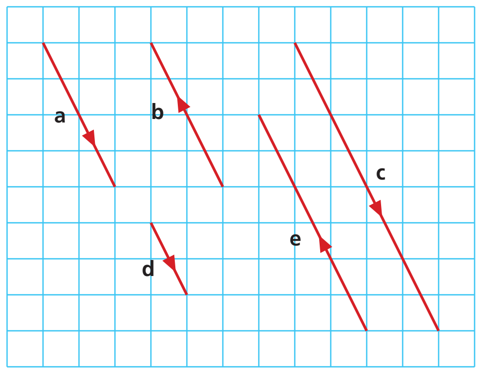


Look at the two vectors in the diagram.

$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{2a} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Worked example

If $\mathbf{a} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$, express the vectors **b**, **c**, **d** and **e** in terms of **a**.

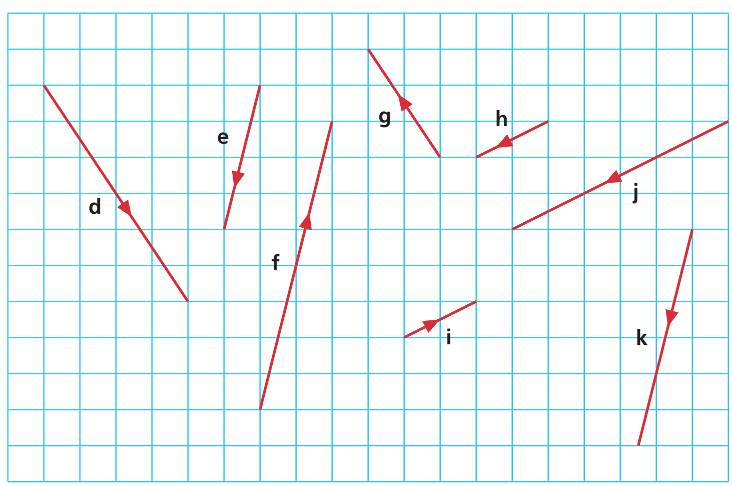


$\mathbf{b} = -\mathbf{a}$ $\quad \mathbf{c} = 2\mathbf{a}$ $\quad \mathbf{d} = \frac{1}{2}\mathbf{a}$ $\quad \mathbf{e} = -\frac{3}{2}\mathbf{a}$

Exercise 31.3

1 $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\quad \mathbf{b} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$ $\quad \mathbf{c} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

Express the following vectors in terms of either **a**, **b** or **c**.



2 $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\quad \mathbf{b} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ $\quad \mathbf{c} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

Represent each of the following as a single column vector.

a $2\mathbf{a}$ b $3\mathbf{b}$ c $-\mathbf{c}$ d $\mathbf{a} + \mathbf{b}$ e $\mathbf{b} - \mathbf{c}$

f $3\mathbf{c} - \mathbf{a}$ g $2\mathbf{b} - \mathbf{a}$ h $\frac{1}{2}(\mathbf{a} - \mathbf{b})$ i $2\mathbf{a} - 3\mathbf{c}$

Exercise 31.3 (cont)

3 $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

Express each of the following vectors in terms of **a**, **b** and **c**.

a $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ b $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ c $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$

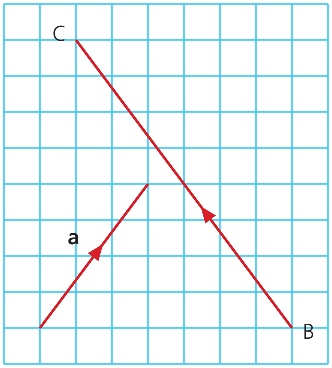
d $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$ e $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$ f $\begin{pmatrix} 10 \\ -5 \end{pmatrix}$

The magnitude of a vector

The **magnitude** or size of a vector is represented by its length, i.e. the longer the length, the greater the magnitude. The magnitude of a vector **a** or $\overrightarrow{AB}$ is denoted by $|\mathbf{a}|$ or $|\overrightarrow{AB}|$ respectively and is calculated using Pythagoras' theorem.

Worked examples

$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$



a Represent both of the above vectors diagrammatically.

b i Calculate $|\mathbf{a}|$.

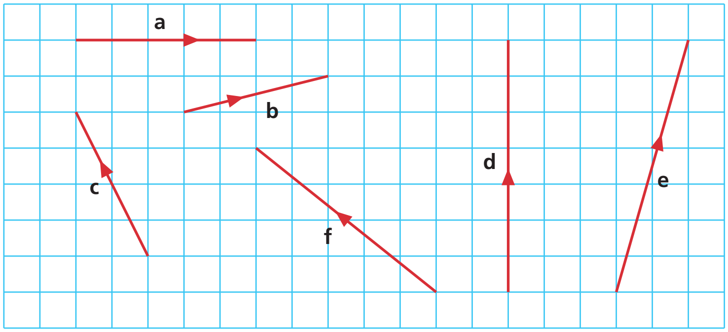
$$|\mathbf{a}| = \sqrt{(3^2 + 4^2)}$$
$$= \sqrt{25} = 5$$

ii Calculate $|\overrightarrow{BC}|$.

$$|\overrightarrow{BC}| = \sqrt{(-6)^2 + 8^2}$$
$$= \sqrt{100} = 10$$

Exercise 31.4

1 Calculate the magnitude of the vectors shown below. Give your answers correct to 1 d.p. where appropriate.



2 Calculate the magnitude of the following vectors, giving your answers to 1 d.p. where appropriate.

a $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ b $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ c $\overrightarrow{CD} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$

d $\overrightarrow{DE} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$ e $2\overrightarrow{AB}$ f $2\overrightarrow{CD}$

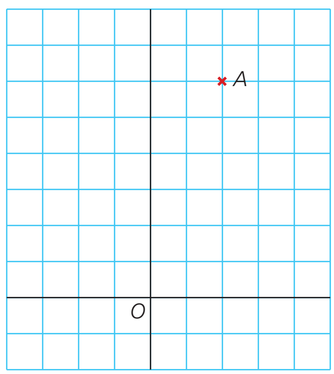
3 $\mathbf{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} -1 \\ -8 \end{pmatrix}$

Calculate the magnitude of the following, giving your answers to 1 d.p.

a $\mathbf{a} + \mathbf{b}$ b $2\mathbf{a} - \mathbf{b}$ c $\mathbf{b} - \mathbf{c}$

d $2\mathbf{c} + 3\mathbf{b}$ e $2\mathbf{b} - 3\mathbf{a}$ f $\mathbf{a} + 2\mathbf{b} - \mathbf{c}$

Position vectors

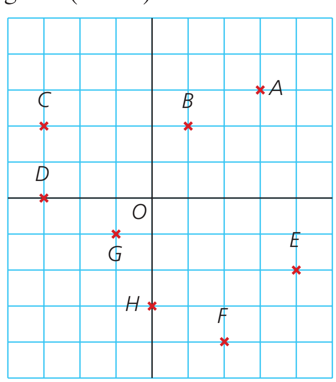


Sometimes a vector is fixed in position relative to a specific point.

In the diagram, the position vector of A relative to O is $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$.

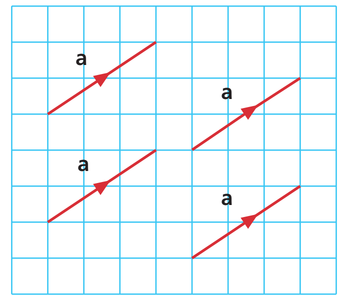
Exercise 31.5

Give the position vectors of A, B, C, D, E, F, G and H relative to O in the diagram (below).



Vector geometry

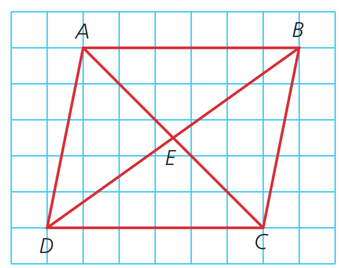
In general, vectors are not fixed in position. If a vector **a** has a specific magnitude and direction, then any other vector with the same magnitude and direction as **a** can also be labelled **a**.



If $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ then all the vectors shown in the diagram can also be labelled **a**, as they all have the same magnitude and direction.

This property of vectors can be used to solve problems in vector geometry.

Worked examples



a Name a vector equal to $\overrightarrow{AD}$.

$\overrightarrow{BC} = \overrightarrow{AD}$

b Write $\overrightarrow{BD}$ in terms of $\overrightarrow{BE}$.

$\overrightarrow{BD} = 2\overrightarrow{BE}$

c Express $\overrightarrow{CD}$ in terms of $\overrightarrow{AB}$.

$\overrightarrow{CD} = \overrightarrow{BA} = -\overrightarrow{AB}$

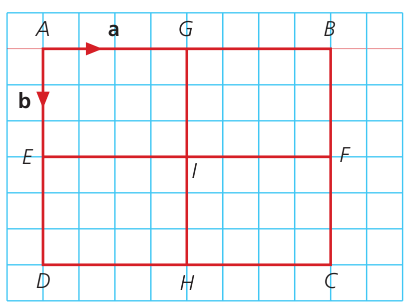
Exercise 31.6

1 If $\overrightarrow{AG} = \mathbf{a}$ and $\overrightarrow{AE} = \mathbf{b}$, express the following in terms of **a** and **b**.

a $\overrightarrow{EI}$ b $\overrightarrow{HC}$ c $\overrightarrow{FC}$

d $\overrightarrow{DE}$ e $\overrightarrow{GH}$ f $\overrightarrow{CD}$

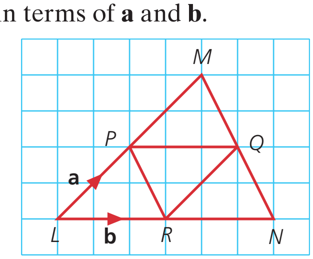
g $\overrightarrow{AI}$ h $\overrightarrow{GE}$ i $\overrightarrow{FD}$



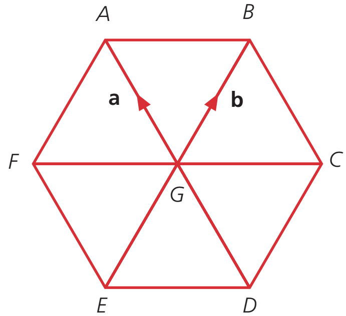
2 If $\overrightarrow{LP} = \mathbf{a}$ and $\overrightarrow{LR} = \mathbf{b}$, express the following in terms of **a** and **b**.

a $\overrightarrow{LM}$ b $\overrightarrow{PQ}$ c $\overrightarrow{PR}$

d $\overrightarrow{MQ}$ e $\overrightarrow{MP}$ f $\overrightarrow{NP}$



3 *ABCDEF* is a regular hexagon.

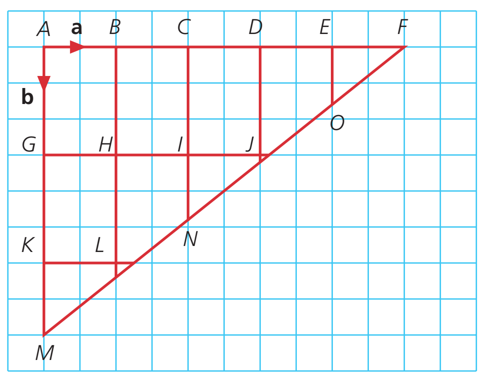


If $\overrightarrow{GA} = \mathbf{a}$ and $\overrightarrow{GB} = \mathbf{b}$, express the following in terms of **a** and **b**.

a $\overrightarrow{AD}$	b $\overrightarrow{FE}$	c $\overrightarrow{DC}$
d $\overrightarrow{AB}$	e $\overrightarrow{FC}$	f $\overrightarrow{EC}$
g $\overrightarrow{BE}$	h $\overrightarrow{FD}$	i $\overrightarrow{AE}$

4 If $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AG} = \mathbf{b}$, express the following in terms of **a** and **b**.

a $\overrightarrow{AF}$	b $\overrightarrow{AM}$	c $\overrightarrow{FM}$
d $\overrightarrow{FO}$	e $\overrightarrow{EI}$	f $\overrightarrow{KF}$
g $\overrightarrow{CN}$	h $\overrightarrow{AN}$	i $\overrightarrow{DN}$



Exercise 31.7

1 *T* is the midpoint of the line *PS* and *R* divides the line *QS* in the ratio 1 : 3.

$\overrightarrow{PT} = \mathbf{a}$ and $\overrightarrow{PQ} = \mathbf{b}$.

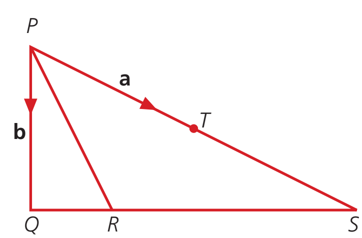
a Express each of the following in terms of **a** and **b**.

i $\overrightarrow{PS}$

ii $\overrightarrow{QS}$

iii $\overrightarrow{PR}$

b Show that $\overrightarrow{RT} = \frac{1}{4}(2\mathbf{a} - 3\mathbf{b})$.



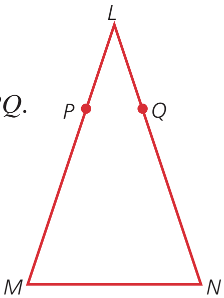
Exercise 31.7 (cont)

2 $\overrightarrow{PM} = 3\overrightarrow{LP}$ and $\overrightarrow{QN} = 3\overrightarrow{LQ}$

Prove that:

a the line PQ is parallel to the line MN,

b the line MN is four times the length of the line PQ.



3 $PQRS$ is a parallelogram. The point T divides the line PQ in the ratio 1 : 3, and U, V and W are the midpoints of SR, PS and QR respectively.

$\overrightarrow{PT} = \mathbf{a}$ and $\overrightarrow{PV} = \mathbf{b}$.

a Express each of the following in terms of **a** and **b**.

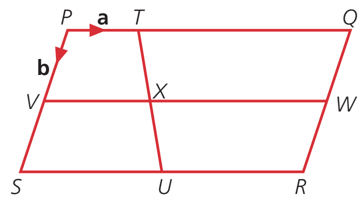
i $\overrightarrow{PQ}$

ii $\overrightarrow{SU}$

iii $\overrightarrow{PU}$

iv $\overrightarrow{VX}$

b Show that $\overrightarrow{XR} = \frac{1}{2}(5\mathbf{a} + 2\mathbf{b})$.

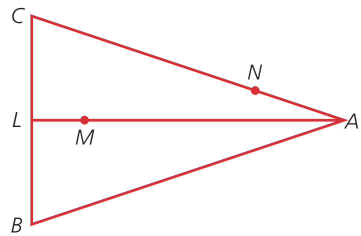


4 ABC is an isosceles triangle. L is the midpoint of BC. M divides the line LA in the ratio 1 : 5, and N divides AC in the ratio 2 : 5.

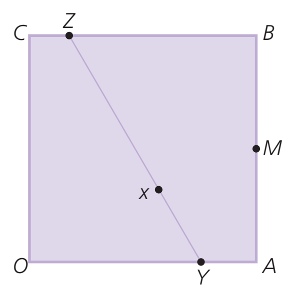
a $\overrightarrow{BC} = \mathbf{p}$ and $\overrightarrow{BA} = \mathbf{q}$. Express the following in terms of **p** and **q**.

i $\overrightarrow{LA}$ ii $\overrightarrow{AN}$

b Show that $\overrightarrow{MN} = \frac{1}{84}(46\mathbf{q} - 11\mathbf{p})$.



5 A square $OABC$ is shown opposite.
Point Y divides OA in the ratio 5:3.
Point M is the midpoint of AB.
Point Z divides CB in the ratio 1:7.
If $\overrightarrow{OA} = \boldsymbol{a}$ and if $\overrightarrow{OC} = \boldsymbol{c}$, prove that O, X and M are collinear.



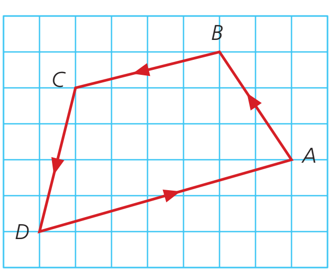
Note

Points which lie on the same line are collinear.

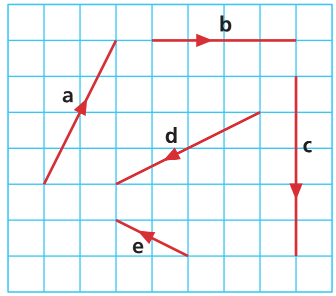
Student assessment 1

1 Using the diagram, describe the following translations using column vectors.

a $\overrightarrow{AB}$ b $\overrightarrow{DA}$ c $\overrightarrow{CA}$



2 Describe each of the translations shown using column vectors.



3 Using the vectors in Question 2, draw diagrams to represent:

a $\mathbf{a} + \mathbf{b}$ b $\mathbf{e} - \mathbf{d}$ c $\mathbf{c} - \mathbf{e}$ d $2\mathbf{e} + \mathbf{b}$

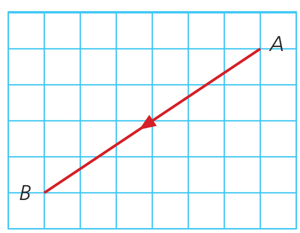
4 In the following, $\mathbf{a} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

Calculate:

a $\mathbf{a} + \mathbf{b}$ b $\mathbf{c} - \mathbf{b}$ c $2\mathbf{a} + \mathbf{b}$ d $3\mathbf{c} - 2\mathbf{b}$

Student assessment 2

1 a Calculate the magnitude of the vector $\overrightarrow{AB}$ shown in the diagram.



b Calculate the magnitude of the following vectors.

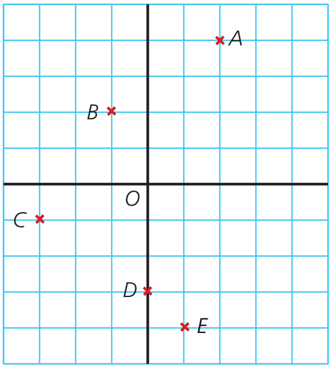
$\mathbf{a} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -7 \\ -4 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$

2 $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

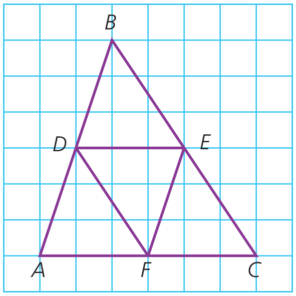
Calculate the magnitude of the following, giving your answers to 3 s.f.

a $3\mathbf{p} - 2\mathbf{q}$ b $\frac{1}{2}\mathbf{r} + \mathbf{q}$

3 Give the position vectors of A, B, C, D and E relative to O for the diagram below.

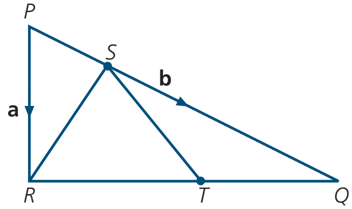


4 a Name another vector equal to $\overrightarrow{DE}$ in the diagram.
b Express $\overrightarrow{DF}$ in terms of $\overrightarrow{BC}$.
c Express $\overrightarrow{CF}$ in terms of $\overrightarrow{DE}$.



Student assessment 3

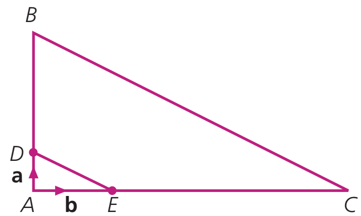
1 In the triangle PQR, the point S divides the line PQ in the ratio 1:3, and T divides the line RQ in the ratio 3:2.
$\overrightarrow{PR} = \mathbf{a}$ and $\overrightarrow{PQ} = \mathbf{b}$.



a Express the following in terms of **a** and **b**.
i $\overrightarrow{PS}$ ii $\overrightarrow{SR}$ iii $\overrightarrow{TQ}$

b Show that $\overrightarrow{ST} = \frac{1}{20}(8\mathbf{a} + 7\mathbf{b})$.

2 In the triangle ABC, the point D divides the line AB in the ratio 1 : 3, and E divides the line AC also in the ratio 1 : 3.

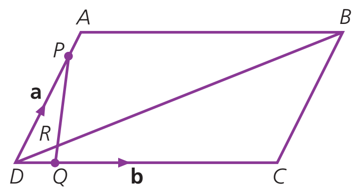


If $\overrightarrow{AD} = \mathbf{a}$ and $\overrightarrow{AE} = \mathbf{b}$, prove that:

a $\overrightarrow{BC} = 4\overrightarrow{DE}$,

b $BCED$ is a trapezium.

3 The parallelogram $ABCD$ shows the points P and Q dividing each of the lines AD and DC in the ratio 1 : 4 respectively.



a If $\overrightarrow{DA} = \mathbf{a}$ and $\overrightarrow{DC} = \mathbf{b}$, express the following in terms of **a** and **b**.

i $\overrightarrow{AC}$ **ii** $\overrightarrow{CB}$ **iii** $\overrightarrow{DB}$

b **i** Find the ratio in which R divides DB.

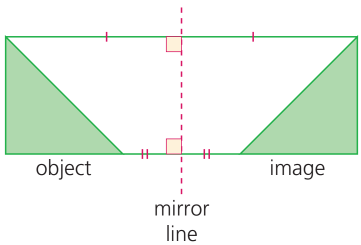
ii Find the ratio in which R divides PQ.

32 Transformations

An object undergoing a transformation changes in either position or shape. In its simplest form this change can occur because of a **reflection**, **rotation**, **translation** or **enlargement**. When an object undergoes a transformation, then its new position or shape is known as the **image**.

Reflection

When an object is reflected, it undergoes a 'flip' movement about a dashed (broken) line known as the **mirror line**, as shown in the diagram:

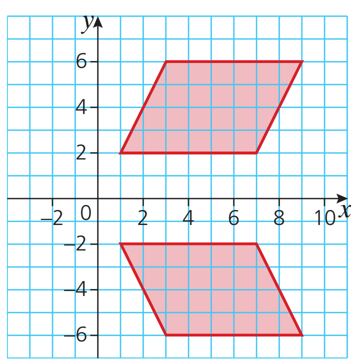


A point on the object and its equivalent point on the image are equidistant from the mirror line. This distance is measured at right angles to the mirror line. The line joining the point to its image is perpendicular to the mirror line.

The position of the mirror line is essential when describing a reflection. Sometimes, its equation as well as its position will be required.

Worked examples

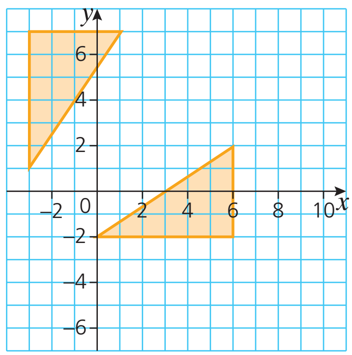
a Find the equation of the mirror line in the reflection given in the diagram (below).

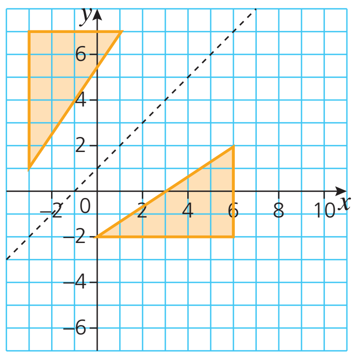


Here, the mirror line is the x-axis. The equation of the mirror line is therefore $y = 0$.

b A **reflection** is shown below.

i Draw the position of the mirror line.





ii Give the equation of the mirror line.
Equation of mirror line: $y = x + 1$.

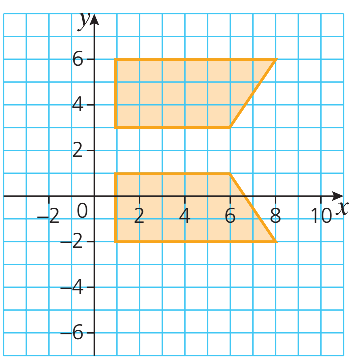
Exercise 32.1

Copy each of the following diagrams, then:

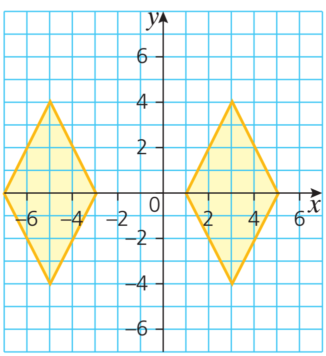
a draw the position of the mirror line(s),

b give the equation of the mirror line(s).

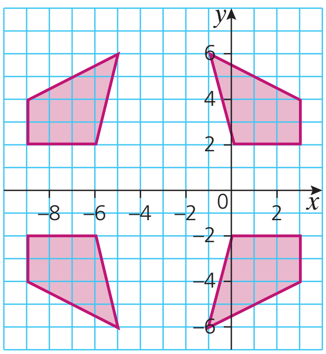
1



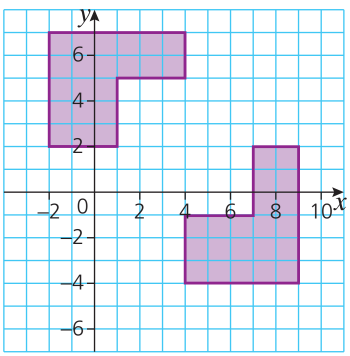
2



3

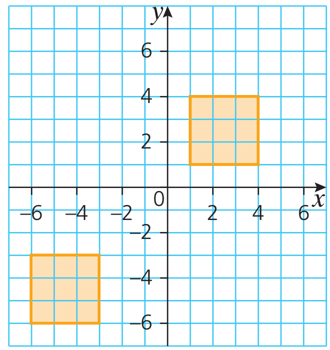


4

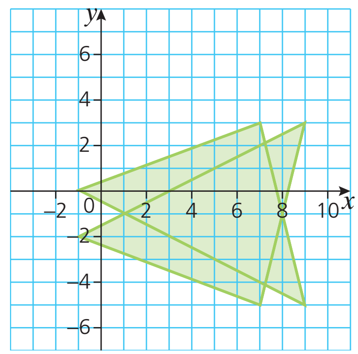


Exercise 32.1 (cont)

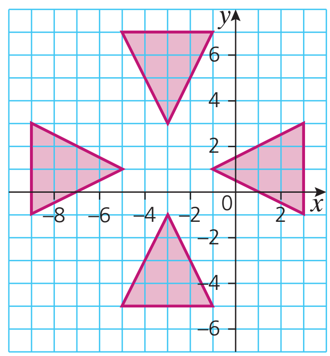
5



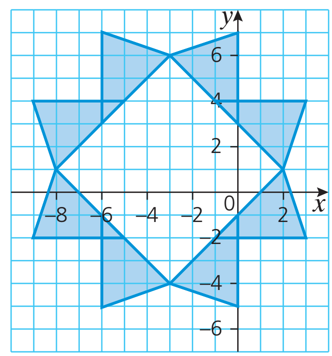
6



7



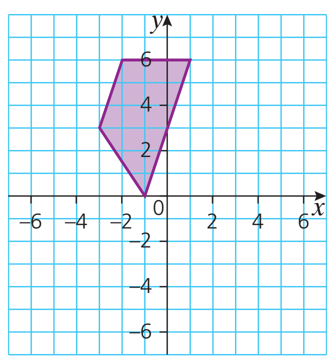
8



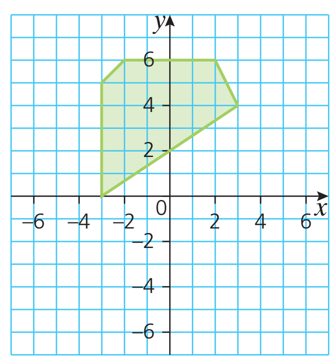
Exercise 32.2

In Questions 1 and 2, copy each diagram four times and reflect the object in each of the lines given.

1 **a** $x = 2$
b $y = 0$
c $y = x$
d $y = -x$

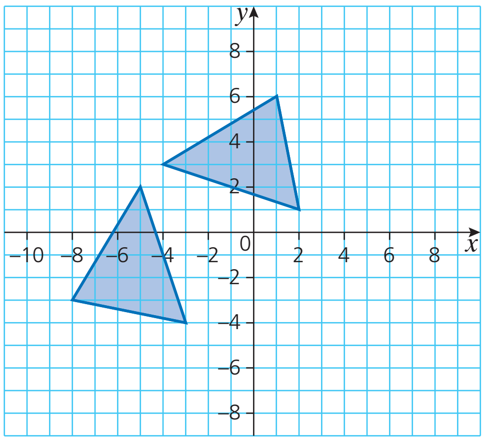


2 **a** $x = -1$
b $y = -x - 1$
c $y = x + 2$
d $x = 0$



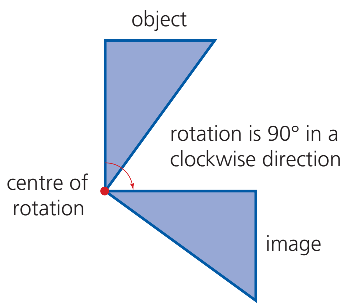
3 Copy the diagram (below), and reflect the triangles in the following lines:

$x = 1$ and $y = -3$.



Rotation

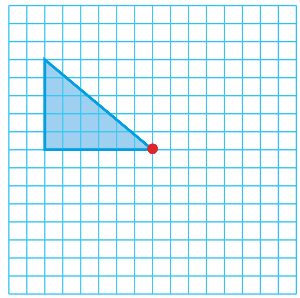
When an object is rotated, it undergoes a 'turning' movement about a specific point known as the **centre of rotation**. When describing a rotation, it is necessary to identify not only the position of the centre of rotation, but also the angle and direction of the turn, as shown in the diagram:



Exercise 32.3

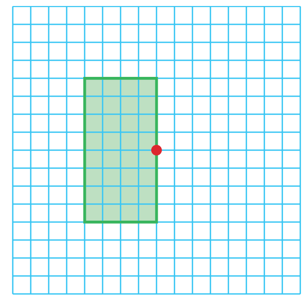
In the following, the object and centre of rotation have both been given. Copy each diagram and draw the object's image under the stated rotation about the marked point.

1



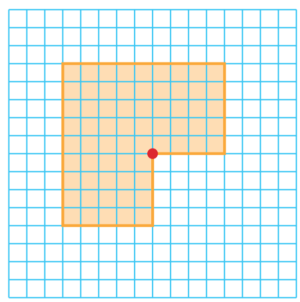
rotation 180°

2



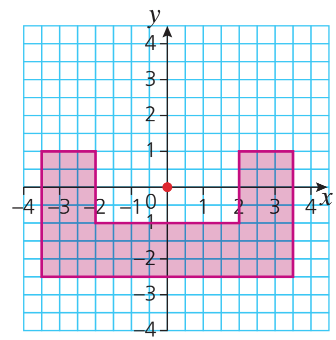
rotation 90° clockwise

3



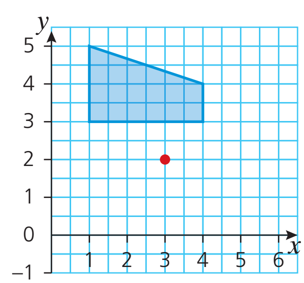
rotation 180°

4



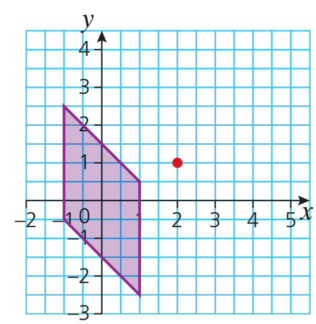
rotation 90° anticlockwise about (0, 0)

5



rotation 90° clockwise about (3, 2)

6



rotation 90° clockwise about (2, 1)

Exercise 32.4

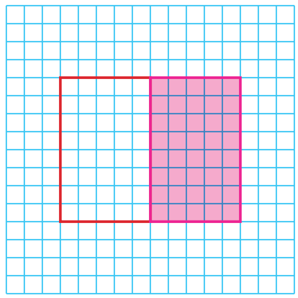
In the following, the object (unshaded) and image (shaded) have been drawn.

Copy each diagram and:

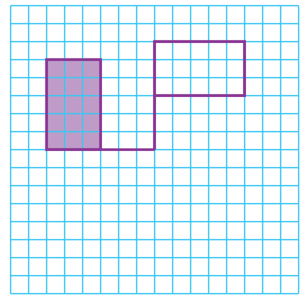
a mark the centre of rotation,

b calculate the angle and direction of rotation.

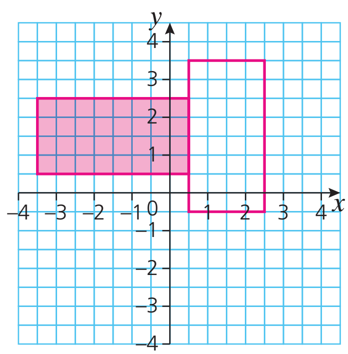
1



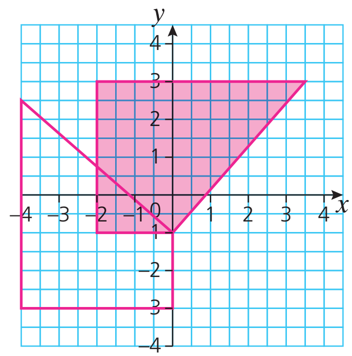
2



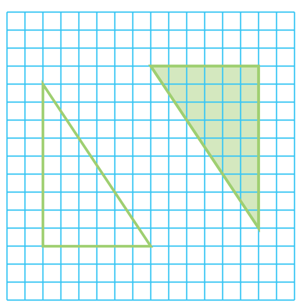
3



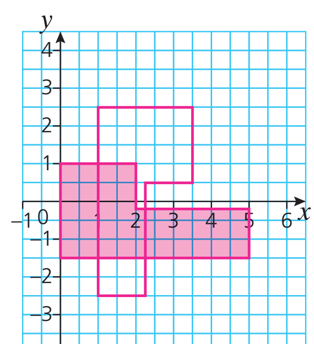
4



5



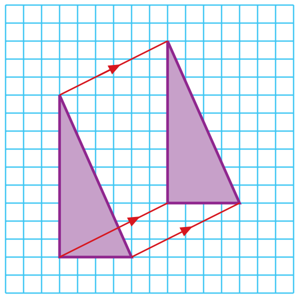
6



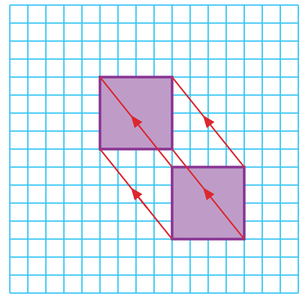
> **Note**
>
> To describe a rotation, three pieces of information need to be given. These are the centre of rotation, the angle of rotation and the direction of rotation.

Translation

When an object is translated, it undergoes a 'straight sliding' movement. When describing a translation, it is necessary to give the translation vector. As no rotation is involved, each point on the object moves in the same way to its corresponding point on the image, e.g.



Vector $= \begin{pmatrix} 6 \\ 3 \end{pmatrix}$



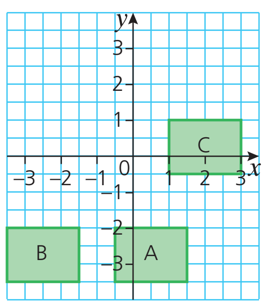
Vector $= \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

Exercise 32.5

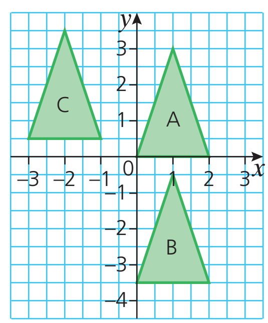
In the following diagrams, object A has been translated to each of images B and C.

Give the translation vectors in each case.

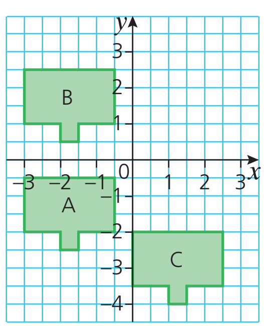
1



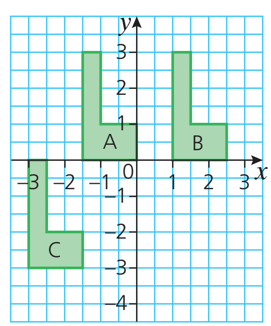
2



3



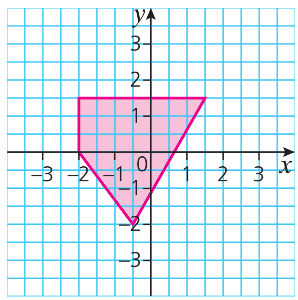
4



Exercise 32.6

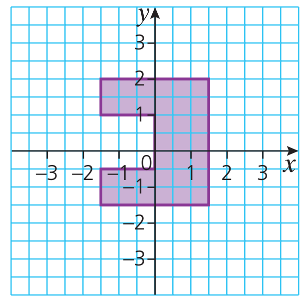
a Copy each of the following diagrams and draw the object.
b Translate the object by the vector given in each case and draw the image in its position.
(Note that a bigger grid than the one shown may be needed.)

1



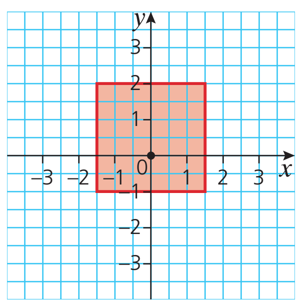
Vector = $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

2



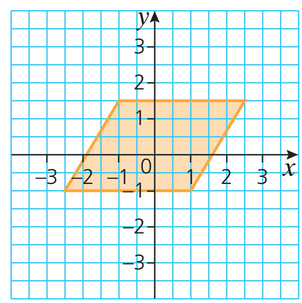
Vector = $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$

3



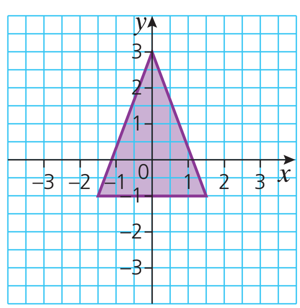
Vector = $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$

4



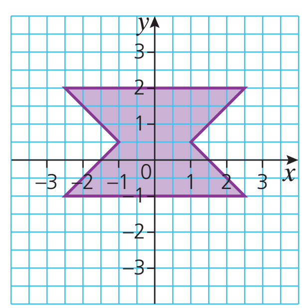
Vector = $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$

5



Vector = $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$

6



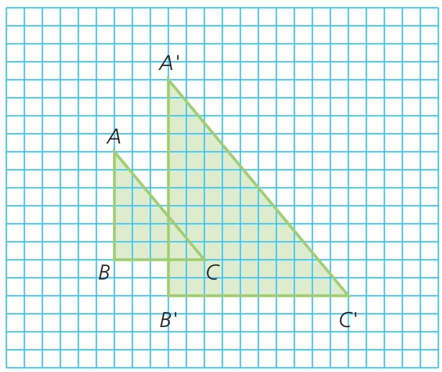
Vector = $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Enlargement

When an object is enlarged, the result is an image which is mathematically similar to the object but of a different size. The image can be either larger or smaller than the original object. When describing an enlargement, two pieces of information need to be given, the position of the **centre of enlargement** and the **scale factor of enlargement**.

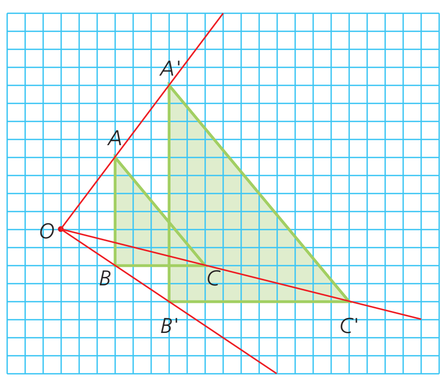
Worked examples

a In the diagram below, triangle ABC is enlarged to form triangle $A'B'C'$.



i Find the centre of enlargement.

The centre of enlargement is found by joining corresponding points on the object and image with a straight line. These lines are then extended until they meet. The point at which they meet is the centre of enlargement O.



ii Calculate the scale factor of enlargement.

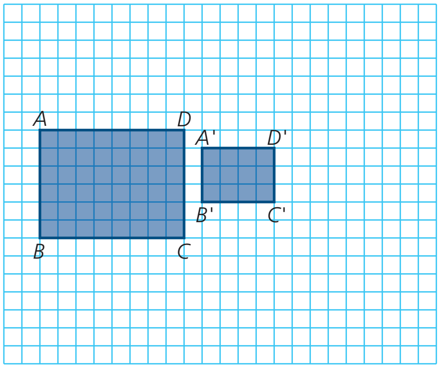
The scale factor of enlargement can be calculated in one of two ways. From the diagram above it can be seen that the distance OA' is twice the distance OA. Similarly OC' and OB' are both twice OC and OB respectively, hence the scale factor of enlargement is 2.

Alternatively, the scale factor can be found by considering the ratio of the length of a side on the image to the length of the corresponding side on the object, i.e.

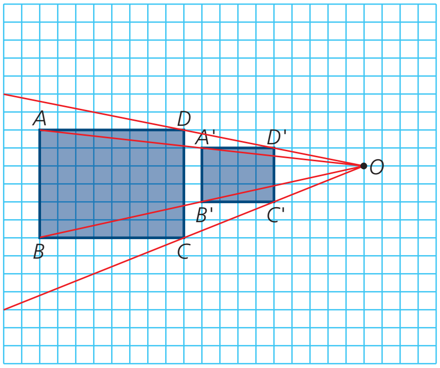
$$\frac{A'B'}{AB} = \frac{12}{6} = 2$$

Hence the scale factor of enlargement is 2.

b In the diagram below, the rectangle $ABCD$ undergoes a transformation to form rectangle $A'B'C'D'$.



i Find the centre of enlargement.
By joining corresponding points on both the object and the image, the centre of enlargement is found at O.



ii Calculate the scale factor of enlargement.

The scale factor of enlargement $= \frac{A'B'}{AB} = \frac{3}{6} = \frac{1}{2}$

Note

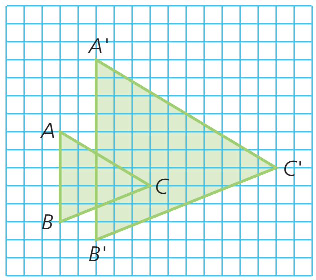
If the scale factor of enlargement is greater than 1, then the image is larger than the object. If the scale factor lies between 0 and 1, then the resulting image is smaller than the object. In these cases, although the image is smaller than the object, the transformation is still known as an enlargement.

Exercise 32.7

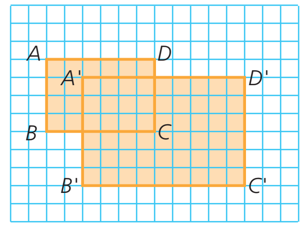
Copy the following diagrams and find:

a the centre of enlargement,
b the scale factor of enlargement.

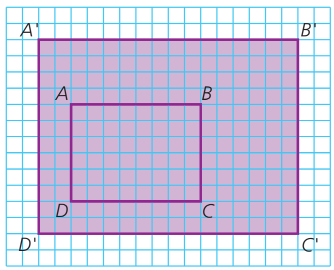
1



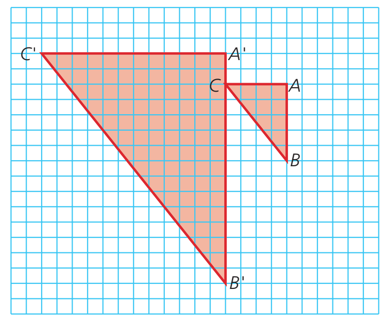
2



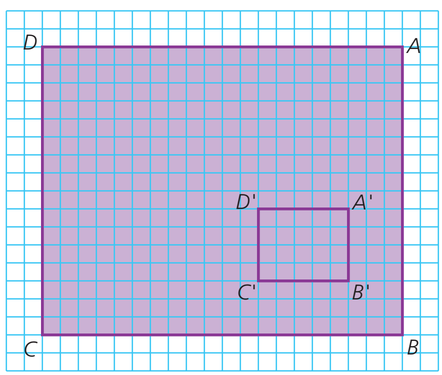
3



4



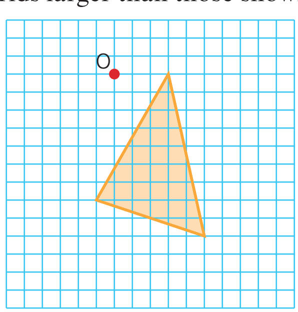
5



Exercise 32.8

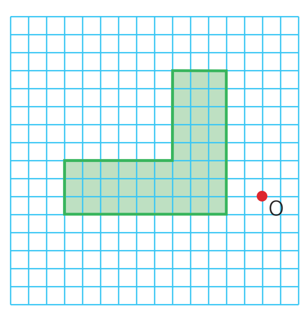
Copy the following diagrams and enlarge the objects by the scale factor given and from the centre of enlargement shown.
(Grids larger than those shown may be needed.)

1



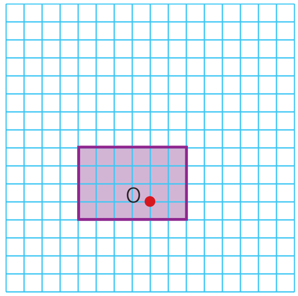
scale factor 2

2



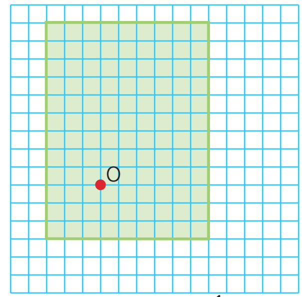
scale factor 2

3



scale factor 3

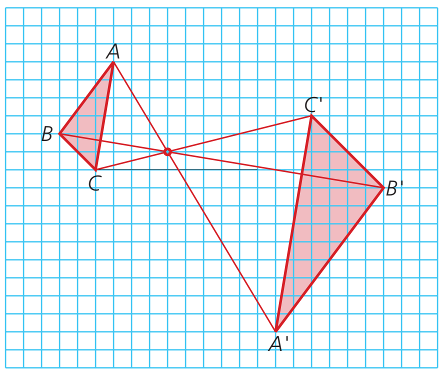
4



scale factor $\frac{1}{3}$

Negative enlargement

The diagram below shows an example of **negative enlargement**.

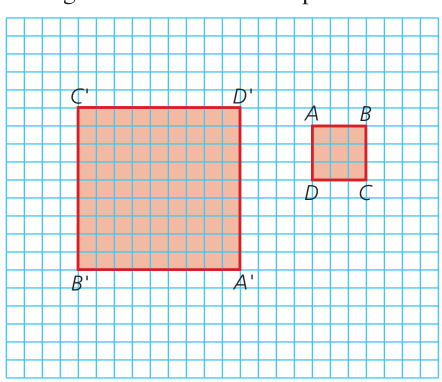


scale factor of enlargement is –2

With negative enlargement each point and its image are on opposite sides of the centre of enlargement. The scale factor of enlargement is calculated in the same way, remembering, however, to write a '–' sign before the number.

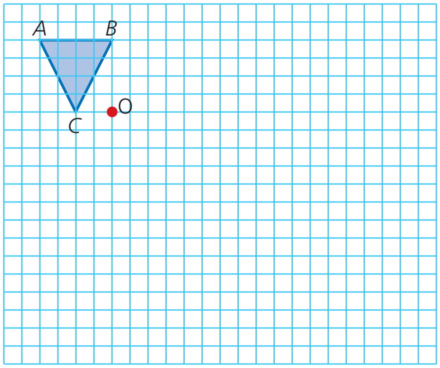
Exercise 32.9

1 Copy the following diagram and then calculate the scale factor of enlargement and show the position of the centre of enlargement.



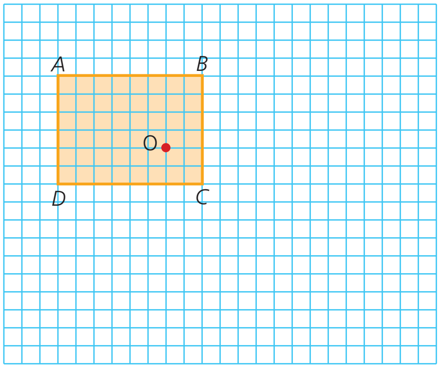
Exercise 32.9 (cont)

2 The scale factor of enlargement and centre of enlargement are both given. Copy and complete the diagram.



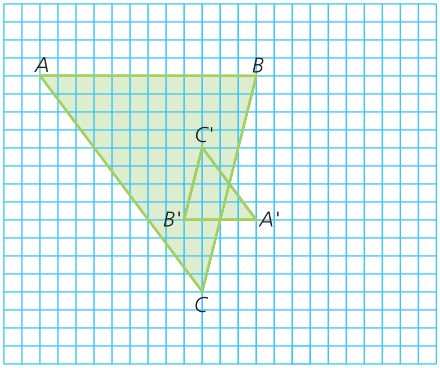
scale factor of enlargement is –2.5

3 The scale factor of enlargement and centre of enlargement are both given. Copy and complete the diagram.

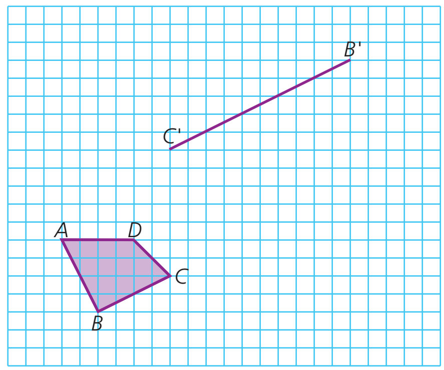


scale factor of enlargement is –2

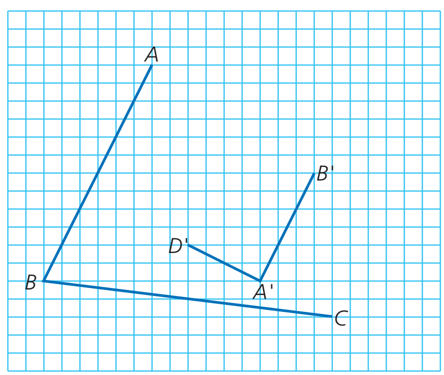
4 Copy the following diagram and then calculate the scale factor of enlargement and show the position of the centre of enlargement.



5 An object and part of its image under enlargement are given in the diagram below. Copy the diagram and complete the image. Also find the centre of enlargement and calculate the scale factor of enlargement.



6 In the diagram below, part of an object in the shape of a quadrilateral and its image under enlargement are drawn. Copy and complete the diagram. Also find the centre of enlargement and calculate the scale factor of enlargement.



Combinations of transformations

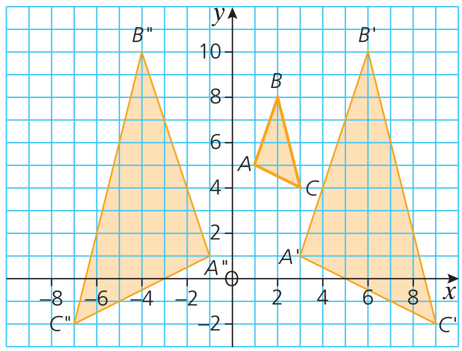
An object may not just undergo one type of transformation. It can undergo a succession of different transformations.

Worked example

A triangle ABC maps onto $A'B'C'$ after an enlargement of scale factor 3 from the centre of enlargement (0, 7). $A'B'C'$ is then mapped onto $A''B''C''$ by a reflection in the line $x = 1$.

a Draw and label the image $A'B'C'$.

b Draw and label the image $A''B''C''$.

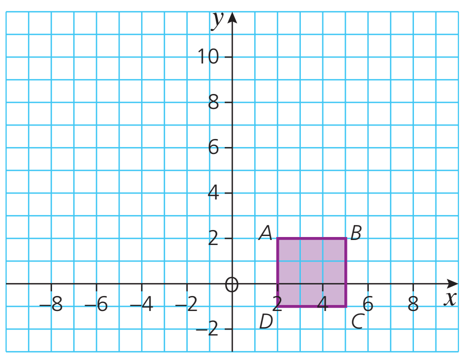


Exercise 32.10

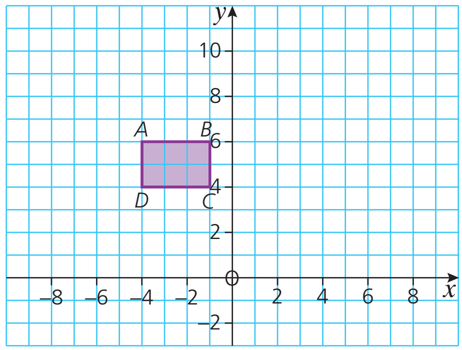
For each of the following questions, copy the diagram.

After each transformation, draw the image on the same grid and label it clearly.

1 The square $ABCD$ is mapped onto $A'B'C'D'$ by a reflection in the line $y = 3$. $A'B'C'D'$ then maps onto $A''B''C''D''$ as a result of a 90° rotation in a clockwise direction about the point (–2, 5).



2 The rectangle $ABCD$ is mapped onto $A'B'C'D'$ by an enlargement of scale factor –2 with its centre at (0, 5). $A'B'C'D'$ then maps onto $A''B''C''D''$ as a result of a reflection in the line $y = -x + 7$.

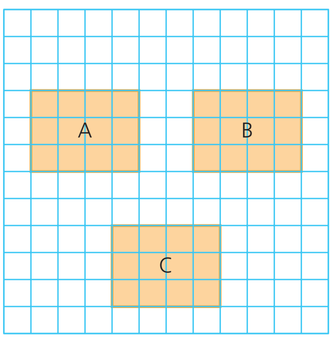


Student assessment 1

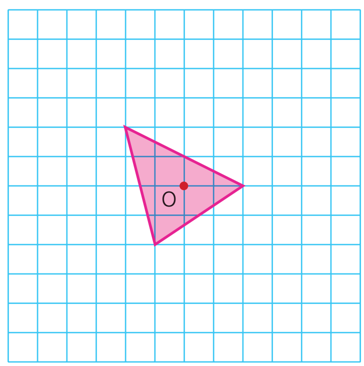
1 Write down the column vector of the translation which maps:

a rectangle A to rectangle B,

b rectangle B to rectangle C.



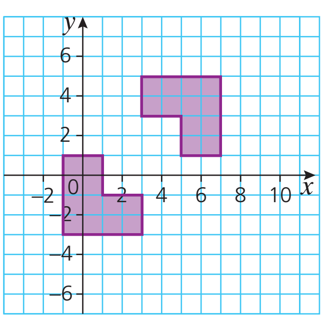
2 Enlarge the triangle below by scale factor 2 and from the centre of enlargement O.



3 Copy the diagram below, which shows an object and its reflected image.

a Draw on your diagram the position of the mirror line.

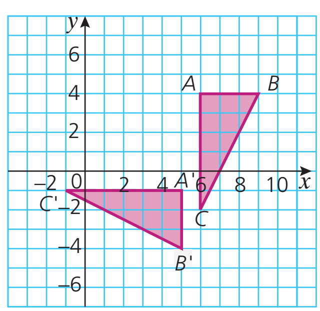
b Find the equation of the mirror line.



4 The triangle ABC is mapped onto triangle $A'B'C'$ by a rotation (below).

a Find the coordinates of the centre of rotation.

b Give the angle and direction of rotation.

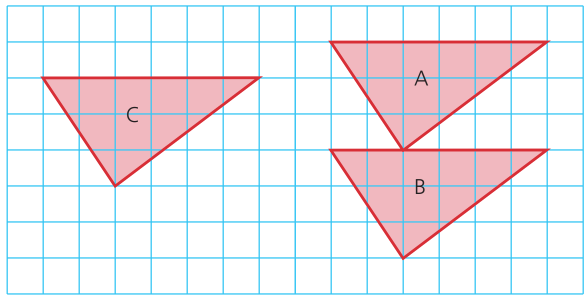


Student assessment 2

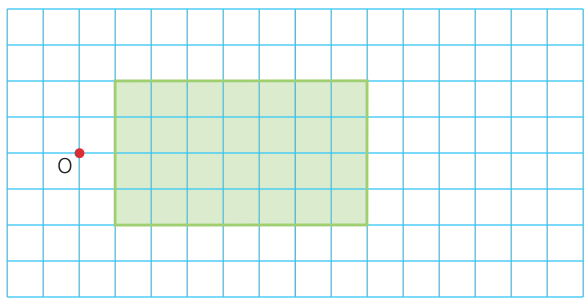
1 Write down the column vector of the translation which maps:

a triangle A to triangle B,

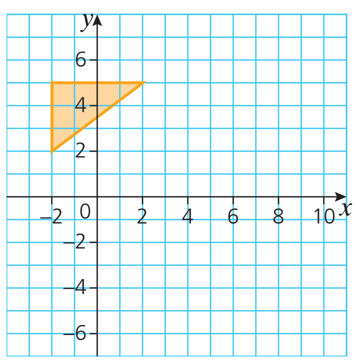
b triangle B to triangle C.



2 Enlarge the rectangle below by scale factor 1.5 and from the centre of enlargement *O*.



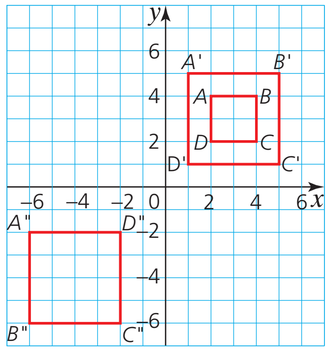
3 Copy the diagram below.



a Draw in the mirror line with equation $y = x - 1$.

b Reflect the object in the mirror line.

4 Square $ABCD$ is mapped onto square $A'B'C'D'$. Square $A'B'C'D'$ is subsequently mapped onto square $A''B''C''D''$.



a Describe fully the transformation which maps $ABCD$ onto $A'B'C'D'$.

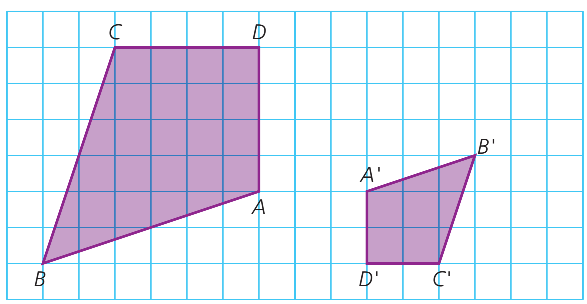
b Describe fully the transformation which maps $A'B'C'D'$ onto $A''B''C''D''$.

Student assessment 3

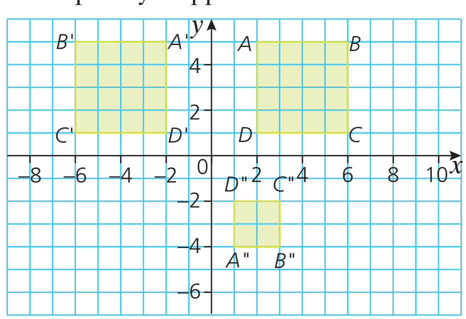
1 An object $ABCD$ and its image $A'B'C'D'$ are shown below.

a Find the position of the centre of enlargement.

b Calculate the scale factor of enlargement.



2 The square $ABCD$ is mapped onto $A'B'C'D'$. $A'B'C'D'$ is subsequently mapped onto $A''B''C''D''$.



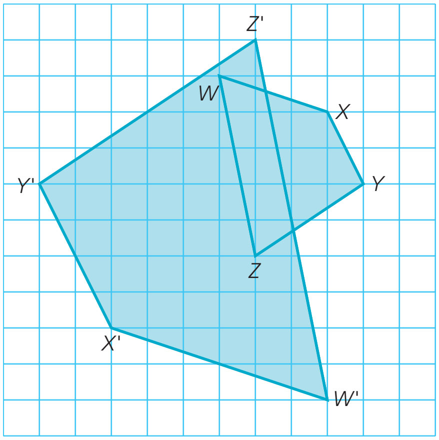
a Describe in full the transformation which maps $ABCD$ onto $A'B'C'D'$.

b Describe in full the transformation which maps $A'B'C'D'$ onto $A''B''C''D''$.

3 An object $WXYZ$ and its image $W'X'Y'Z'$ are shown below.

a Find the position of the centre of enlargement.

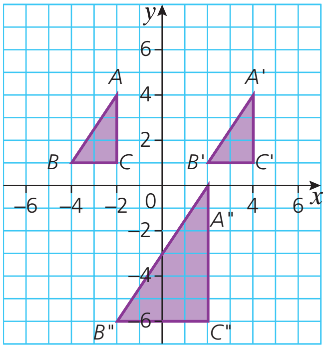
b Calculate the scale factor of enlargement.



4 Triangle ABC is mapped onto $A'B'C'$. $A'B'C'$ is subsequently mapped onto $A''B''C''$.

a Describe in full the transformation which maps ABC onto $A'B'C'$.

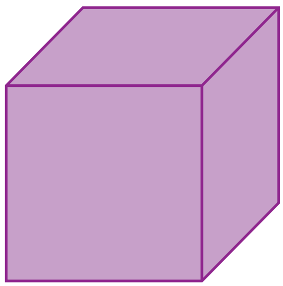
b Describe in full the transformation which maps $A'B'C'$ onto $A''B''C''$.

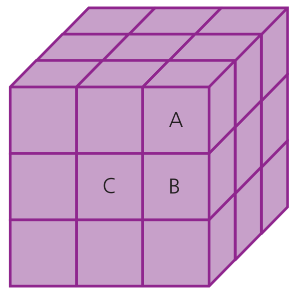


7 Mathematical investigations and ICT 7

A painted cube

A $3 \times 3 \times 3$ cm cube is painted on the outside as shown in the left-hand diagram below:





The large cube is then cut up into 27 smaller cubes, each $1\,\text{cm} \times 1\,\text{cm} \times 1\,\text{cm}$ as shown on the right.

$1 \times 1 \times 1$ cm cubes with 3 painted faces are labelled type A.

$1 \times 1 \times 1$ cm cubes with 2 painted faces are labelled type B.

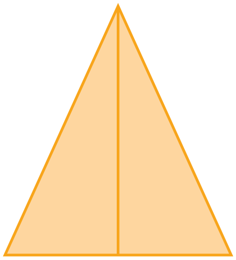
$1 \times 1 \times 1$ cm cubes with 1 face painted are labelled type C.

$1 \times 1 \times 1$ cm cubes with no faces painted are labelled type D.

1 a How many of the 27 cubes are type A?
  b How many of the 27 cubes are type B?
  c How many of the 27 cubes are type C?
  d How many of the 27 cubes are type D?
2 Consider a $4 \times 4 \times 4$ cm cube cut into $1 \times 1 \times 1$ cm cubes. How many of the cubes are type A, B, C and D?
3 How many type A, B, C and D cubes are there when a $10 \times 10 \times 10$ cm cube is cut into $1 \times 1 \times 1$ cm cubes?
4 Generalise for the number of type A, B, C and D cubes in an $n \times n \times n$ cube.
5 Generalise for the number of type A, B, C and D cubes in a cuboid l cm long, w cm wide and h cm high.

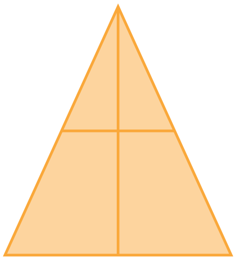
Triangle count

The diagram below shows an isosceles triangle with a vertical line drawn from its apex to its base.



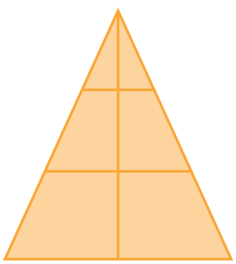
There is a total of 3 triangles in this diagram.

If a horizontal line is drawn across the triangle, it will look as shown:

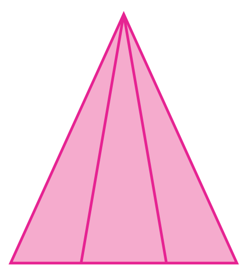


There is a total of 6 triangles in this diagram.

When one more horizontal line is added, the number of triangles increases further:



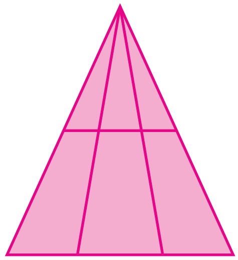
1 Calculate the total number of triangles in the diagram above with the two inner horizontal lines.
2 Investigate the relationship between the total number of triangles (t) and the number of inner horizontal lines (h). Enter your results in an ordered table.
3 Write an algebraic rule linking the total number of triangles and the number of inner horizontal lines.



The triangle (right) has two lines drawn from the apex to the base.

There is a total of six triangles in this diagram.

If a horizontal line is drawn through this triangle, the number of triangles increases as shown:



4 Calculate the total number of triangles in the diagram above with two lines from the vertex and one inner horizontal line.

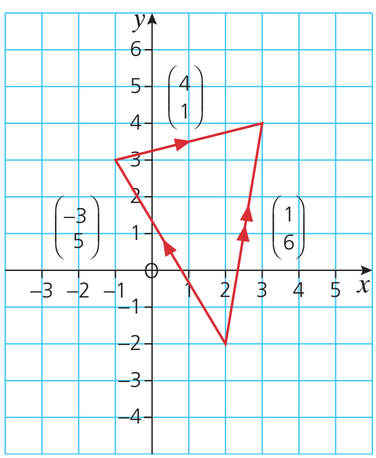
5 Investigate the relationship between the total number of triangles (t) and the number of inner horizontal lines (h) when two lines are drawn from the apex. Enter your results in an ordered table.

6 Write an algebraic rule linking the total number of triangles and the number of inner horizontal lines.

ICT activity

Using Autograph or another appropriate software package, prepare a help sheet for your revision that demonstrates the addition, subtraction and multiplication of vectors.

An example is shown below:



Vector addition:

$$\begin{pmatrix}-3\\5\end{pmatrix}+\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}1\\6\end{pmatrix}$$

TOPIC 8

Probability

Contents

Learning objectives

E8.1 Introduction to probability

1. Understand and use the probability scale from 0 to 1.
2. Understand and use probability notation.
3. Calculate the probability of a single event.
4. Understand that the probability of an event not occurring = 1 – the probability of the event occurring.

E8.2 Relative and expected frequencies

1. Understand relative frequency as an estimate of probability.
2. Calculate expected frequencies.

E8.3 Probability of combined events

Calculate the probability of combined events using, where appropriate:

- Sample space diagrams
- Venn diagrams
- Tree diagrams.

E8.4 Conditional probability

Calculate conditional probability using Venn diagrams, tree diagrams and tables.

Order and chaos

Blaise Pascal and Pierre de Fermat (known for his last theorem) corresponded about problems connected to games of chance.

Although Newton and Galileo had had some thoughts on the subject, this is accepted as the beginning of the study of what is now called probability. Later, in 1657, Christiaan Huygens wrote the first book on the subject, entitled *The Value of all Chances in Games of Fortune*.

In 1821, Carl Friedrich Gauss (1777–1855) worked on normal distribution.

Blaise Pascal (1623–1662)

At the start of the nineteenth century, the French mathematician Pierre-Simon Laplace was convinced of the existence of a Newtonian universe. In other words, if you knew the position and velocities of all the particles in the universe, you would be able to predict the future because their movement would be predetermined by scientific laws. However, quantum mechanics has since shown that this is not true. Chaos theory is at the centre of understanding these limits.

33 Probability

Probability is the study of chance, or the likelihood of an event happening. However, because probability is based on chance, what theory predicts does not necessarily happen in practice.

A **favourable outcome** refers to the event in question actually happening. The **total number of possible outcomes** refers to all the different types of outcome one can get in a particular situation. In general:

$$\text{Probability of an event} = \frac{\text{number of favourable outcomes}}{\text{total number of equally likely outcomes}}$$

If the probability = 0, the event is impossible.

If the probability = 1, the event is certain to happen.

If an event can either happen or not happen then:

Probability of the event not occurring
= 1 – the probability of the event occurring.

Worked examples

Although probabilities are written here as fractions, they could also be expressed as decimals or percentages.

a A fair spinner numbered 1 - 6 is spun. Calculate the probability of getting a six.

Number of favourable outcomes = 1 (i.e. getting a 6)

Total number of possible outcomes = 6 (i.e. getting a 1, 2, 3, 4, 5 or 6)

Probability of getting a six = $\frac{1}{6}$

Probability of not getting a six = $1 - \frac{1}{6} = \frac{5}{6}$

b A fair spinner numbered 1 - 6 is spun. Calculate the probability of getting an even number.

Number of favourable outcomes = 3 (i.e. getting a 2, 4 or 6)

Total number of possible outcomes = 6 (i.e. getting a 1, 2, 3, 4, 5 or 6)

Probability of getting an even number = $\frac{3}{6} = \frac{1}{2}$

c Thirty students are asked to choose their favourite subject out of Maths, English and Art. The results are shown in the table below.

	Maths	English	Art
Girls	7	4	5
Boys	5	3	6

A student is chosen at random.

i What is the probability that it is a girl?

Total number of girls is 16.
Probability of choosing a girl is $\frac{16}{30} = \frac{8}{15}$.

ii What is the probability that it is a boy whose favourite subject is Art?

Number of boys whose favourite subject is Art is 6.

Probability is therefore $\frac{6}{30} = \frac{1}{5}$.

iii What is the probability of **not** choosing a girl whose favourite subject is English?

There are two ways of approaching this:

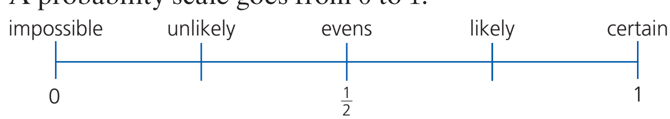
Method 1:
Total number of students who are not girls whose favourite subject is English is $7 + 5 + 5 + 3 + 6 = 26$.
Therefore probability is $\frac{26}{30} = \frac{13}{15}$.
Method 2:
Total number of girls whose favourite subject is English is 4.
Probability of choosing a girl whose favourite subject is English is $\frac{4}{30}$.

Therefore the probability of **not** choosing a girl whose favourite subject is English is:

$1 - \frac{4}{30} = \frac{26}{30} = \frac{13}{15}$

The likelihood of an event such as 'you will play sport tomorrow' will vary from person to person. Therefore, the probability of the event is not constant. However, the probability of some events, such as the result of spinning a coin or spinner, can be found by experiment or calculation.

A probability scale goes from 0 to 1.



Exercise 33.1

1 Copy the probability scale above.
Mark on the probability scale the probability that:
- a a day chosen at random is a Saturday,
- b a coin will show tails when spun,
- c the sun will rise tomorrow,
- d a woman will run 100 metres in under 10 seconds,
- e the next car you see will be silver.

2 Express your answers to Question 1 as fractions, decimals and percentages.

Exercise 33.2

1 Calculate the theoretical probability, when spinning a fair 1–6 spinner, of getting each of the following:
  - a a score of 1
  - b a score of 2, 3, 4, 5 or 6
  - c an odd number
  - d a score less than 6
  - e a score of 7
  - f a score less than 7

2 a Calculate the probability of:
  - i being born on a Wednesday,
  - ii not being born on a Wednesday.

  b Explain the result of adding the answers to **a i** and **ii** together.

3 250 balls are numbered from 1 to 250 and placed in a box. A ball is picked at random. Find the probability of picking a ball with:
  - a the number 1
  - b an even number
  - c a three-digit number
  - d a number less than 300

4 In a class there are 25 girls and 15 boys. The teacher takes in all of their books in a random order. Calculate the probability that the teacher will:
  - a mark a book belonging to a girl first,
  - b mark a book belonging to a boy first.

5 Twenty-six tiles, each printed with one different letter of the alphabet, are put into a bag. If one tile is taken out at random, calculate the probability that it is:
  - a an A or P
  - b a vowel
  - c a consonant
  - d an X, Y or Z
  - e a letter in your first name.

6 A boy was late for school 5 times in the previous 30 school days. If tomorrow is a school day, calculate the probability that he will arrive late.

7 a Three red, 10 white, 5 blue and 2 green counters are put into a bag. If one is picked at random, calculate the probability that it is:
  - i a green counter
  - ii a blue counter.

  b If the first counter taken out is green and it is not put back into the bag, calculate the probability that the second counter picked is:
  - i a green counter
  - ii a red counter.

8 A circular spinner with an arrow has the numbers 0 to 36 equally spaced around its edge. Assuming that it is unbiased, calculate the probability on spinning the arrow of getting:
  - a the number 5
  - b not 5
  - c an odd number
  - d zero
  - e a number greater than 15
  - f a multiple of 3
  - g a multiple of 3 or 5
  - h a prime number.

9 The letters R, C and A can be combined in several different ways.
  - a Write the letters in as many different orders as possible.

  If a computer writes these three letters at random, calculate the probability that:
  - b the letters will be written in alphabetical order,
  - c the letter R is written before both the letters A and C,
  - d the letter C is written after the letter A,
  - e the computer will spell the word CART if the letter T is added.

10 A normal pack of playing cards contains 52 cards. These are made up of four suits (hearts, diamonds, clubs and spades). Each suit consists of 13 cards. These are labelled Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King. The hearts and diamonds are red; the clubs and spades are black.

If a card is picked at random from a normal pack of cards, calculate the probability of picking:

a a heart
b not a heart
c a 4
d a red King
e a Jack, Queen or King
f the Ace of spades
g an even-numbered card
h a 7 or a club.

Exercise 33.3

1 Zuri conducts a survey on the types of vehicle that pass her house. The results are shown below.

Vehicle type	Car	Lorry	Van	Bicycle	Motorbike	Other
Frequency	28	6	20	48	32	6

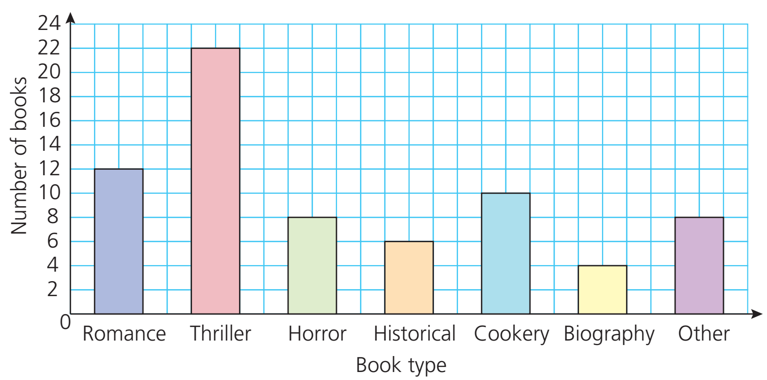
a How many vehicles passed Zuri's house?
b A vehicle is chosen at random from the results. Calculate the probability that it is:
i a car **ii** a lorry **iii** not a van.

2 In a class, data is collected about whether each student is right-handed or left-handed. The results are shown below.

	Left-handed	**Right-handed**
Boys	2	12
Girls	3	15

a How many students are in the class?
b A student is chosen at random. Calculate the probability that the student is:
i a girl
ii left-handed
iii a right-handed boy
iv not a right-handed boy.

3 A library keeps a record of the books that are borrowed during one day. The results are shown in the chart below.



a How many books were borrowed that day?
b A book is chosen at random from the ones borrowed. Calculate the probability that it is:
i a thriller
ii a horror or a romance
iii not a horror or a romance
iv not a biography.

Venn diagrams

In Chapter 10, we saw how Venn diagrams are used to display information written as sets. Venn diagrams are also a good way of representing information when carrying out probability calculations.

Set notation symbols include:

ξ universal set

{ } set

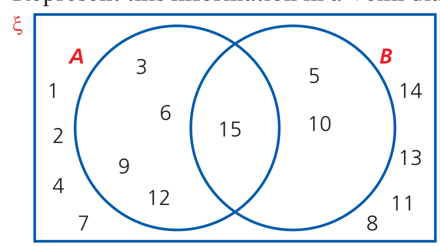
∪ union of

∩ intersection of

Worked examples

Cards numbered 1–15 are arranged in two sets:
A = {multiples of 3} and B = {multiples of 5}.

a Represent this information in a Venn diagram.



Probability notation is only required for the Extended syllabus.

Note

A' means not A

b A card is picked at random. Calculate the probability that it is from set A. (i.e. $P(A)$).

From the diagram it can be seen that the number of elements in set A is five.

(i.e $n(A) = 5$)

Therefore $P(A) = \frac{5}{15} = \frac{1}{3}$

c If a card is picked at random, calculate the probability that it is not from set A. (i.e. $P(A')$).

As $P(A) = \frac{1}{3}$,

then $P(A') = 1 - \frac{1}{3} = \frac{2}{3}$

d Calculate $P(A \cap B)$.

This represents the probability of picking a card belonging to both set A and set B.

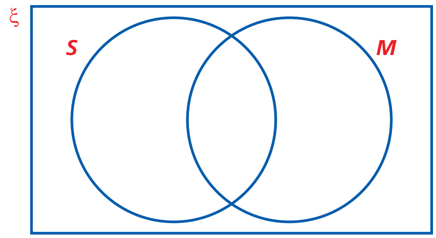
$n(A \cap B) = 1$

Therefore $P(A \cap B) = \frac{1}{15}$

Exercise 33.4

1 In a class of 30 students, all students study at least one language: Spanish (S) or Mandarin Chinese (M). 16 students study only Spanish and 8 students study only Mandarin.

a Copy and complete the Venn diagram for the number of students studying each language.



b If a student is chosen at random, calculate:

i $P(S)$ ii $P(M)$

iii $P(S \cap M)$ iv $P(S \cup M)$

2 One hundred students take part in a school sports event. 70 students take part in track events (T), 62 students take part in field events (F) and 8 students take part in neither.

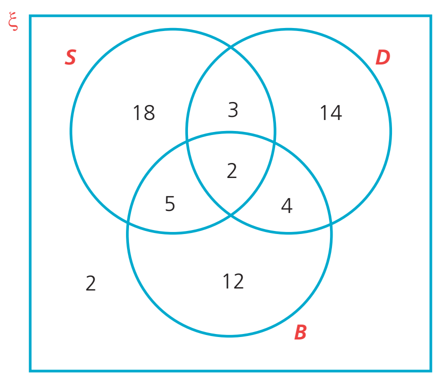
a Construct a Venn diagram to show this information.

b If a student is chosen at random, calculate:

i $P(T)$ ii $P(T \cap F)$

iii $P(F')$ iv $P(T \cup F)'$

3 Sixty students were asked where they had travelled to in the last twelve months. The three most popular destinations were Singapore (S), Dubai (D) and Great Britain (B). The number of students travelling to each destination is shown in the Venn diagram.



If a student is chosen at random, calculate the probability of the following:

a $P(S)$ b $P(B \cap D)$

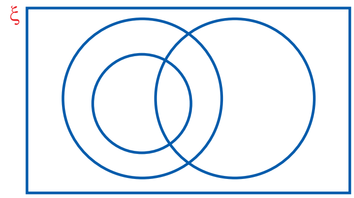
c $P(S \cup D)$ d $P(S \cap D \cap B)$

e $P(B')$ f $P(S \cap B \cap D')$

Exercise 33.4 (cont)

4 Cards numbered 1–20 are arranged in three sets as follows:
A = {multiples of 2}
B = {multiples of 3}
C = {multiples of 4}

a Copy and complete the Venn diagram:



b If a number is chosen at random, calculate each of the following probabilities:

i $P(A)$ **ii** $P(A \cap B)$ **iii** $P(B \cap C)$
iv $P(B \cup C)$ **v** $P(A \cup B)'$ **vi** $P(A \cup B \cup C')$

Relative frequency

A football referee always used a special coin to toss for ends. She noticed that out of the last twenty matches the coin had come down heads far more often than tails. She wanted to know if the coin was fair, that is, if it was as likely to come down heads as tails.

She decided to do a simple experiment by spinning the coin lots of times. Her results are shown below.

Number of trials	Number of heads	Relative frequency
100	40	0.4
200	90	0.45
300	142	0.47…
400	210	0.525
500	260	0.52
600	290	0.48…
700	345	0.49…
800	404	0.505
900	451	0.50…
1000	499	0.499

The **relative frequency** $= \dfrac{\text{number of successful trials}}{\text{total number of trials}}$

In the 'long run', that is after many trials, did the coin appear to be fair?

Notice that the greater the number of trials, the better the estimated probability or relative frequency is likely to be. The key idea is that increasing the number of trials gives a better estimate of the probability and the closer the result obtained by experiment will be to that obtained by calculation.

Worked examples

a There is a group of 250 people in a hall. A girl calculates that the probability of randomly picking someone that she knows from the group is 0.032. Calculate the number of people in the group that the girl knows.

$$\text{Probability} = \frac{\text{number of favourable results } (F)}{\text{number of possible results}}$$

$$0.032 = \frac{F}{250}$$

$250 \times 0.032 = F$ so $8 = F$

The girl knows 8 people in the group.

b A boy enters 8 cakes into a baking competition. His mother knows how many cakes have been entered into the competition in total and tells her son that he has a probability of 0.016 of winning the first prize (assuming all the cakes have an equal chance). How many cakes were entered into the competition?

$$\text{Probability} = \frac{\text{number of favourable results}}{\text{number of possible results } (T)}$$

$$0.016 = \frac{8}{T}$$

$$T = \frac{8}{0.016} = 500$$

So, 500 cakes were entered into the competition.

Exercise 33.5

1. Mikki calculates that she has a probability of 0.004 of winning the first prize in a photography competition if the selection is made at random. If 500 photographs are entered into the competition, how many photographs did Mikki enter?
2. The probability of getting any particular number on a spinner game is given as 0.04. How many numbers are there on the spinner?
3. A bag contains 7 red counters, 5 blue, 3 green and 1 yellow. If one counter is drawn, what is the probability that it is:
   a yellow **b** red **c** blue or green
   d red, blue or green **e** not blue?
4. Luca collects marbles. He has the following colours in a bag: 28 red, 14 blue, 25 yellow, 17 green and 6 purple. If he draws one marble from the bag, what is the probability that it is:
   a red **b** blue **c** yellow or blue
   d purple **e** not purple?
5. The probability of Hanane drawing a marble of one of the following colours from another bag of marbles is:
   blue 0.25 red 0.2 yellow 0.15 green 0.35 white 0.05
   If there are 49 green marbles, how many of each other colour does she have in her bag?
6. There are six red sweets in a bag. If the probability of randomly picking a red sweet is 0.02, calculate the number of sweets in the bag.
7. The probability of getting a bad egg in a batch of 400 is 0.035. How many bad eggs are there likely to be in a batch?
8. A sports arena has 25 000 seats, some of which are VIP seats. For a charity event, all the seats are allocated randomly. The probability of getting a VIP seat is 0.008. How many VIP seats are there?
9. The probability of Harts Utd winning 4–0 is 0.05. How many times are they likely to win by this score in a season of 40 matches?

Student assessment 1

1 What is the probability of spinning the following numbers with a fair 1–6 spinner?
 a a 2 b not a 2
 c less than 5 d a 7

2 If you have a normal pack of 52 cards, what is the probability of drawing:
 a a diamond b a 6 c a black card
 d a picture card e a card less than 5?

3 250 coins, one of which is gold, are placed in a bag. What is the probability of getting the gold coin if I take, without looking, the following numbers of coins?
 a 1 b 5 c 20
 d 75 e 250

4 A bag contains 11 blue, 8 red, 6 white, 5 green and 10 yellow counters. If one counter is taken from the bag, what is the probability that it is:
 a blue b green c yellow d not red?

5 The probability of drawing a red, blue or green marble from a bag containing 320 marbles is:
 red 0.5 blue 0.3 green 0.2
 How many marbles of each colour are there?

6 In a small town there are a number of sports clubs. The clubs have 750 members in total. The table below shows the types of sports club and the number of members each has.

	Tennis	Football	Golf	Hockey	Athletics
Men	30	110	40	15	10
Women	15	25	20	45	30
Boys	10	200	5	10	40
Girls	20	35	0	30	60

 A sports club member is chosen at random from the town. Calculate the probability that the member is:
 a a man b a girl
 c a woman who does athletics d a boy who plays football
 e not a boy who plays football f not a golf player
 g a male who plays hockey.

7 A 1–6 spinner is thought to be biased. In order to test it, Monique spins it 12 times and gets the following results:

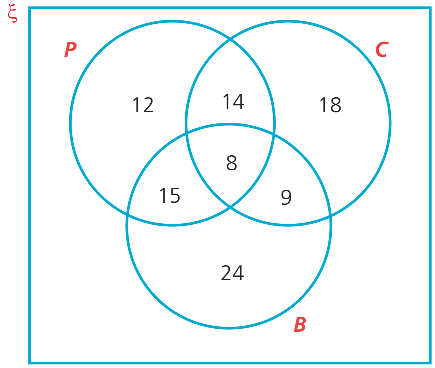
Number	1	2	3	4	5	6
Frequency	2	2	2	2	2	2

 Jas decides to test the same spinner and spins it 60 times. The table below shows her results:

Number	1	2	3	4	5	6
Frequency	3	3	47	3	2	2

 a Which results are likely to be more reliable? Justify your answer.
 b What conclusion can you make about whether the spinner is biased?

8 In a school, students can study science as the individual subjects of Physics, Chemistry and Biology. Each student must study at least one of the subjects. The following Venn diagram gives the number of students studying each subject:

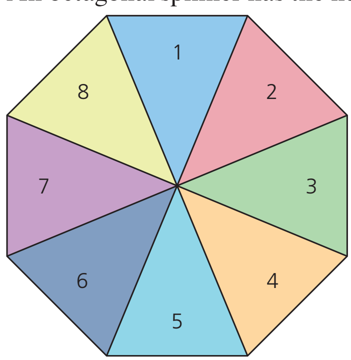


If a student is chosen at random, calculate each of the following probabilities:

a $P(C)$ **b** $P(P \cap B)$ **c** $P(P \cap C \cap B)$
d $P(B \cap C \cap P')$ **e** $P(P \cup C)$

Student assessment 2

1 An octagonal spinner has the numbers 1 to 8 on it as shown.



What is the probability of spinning:
a a 7
b not a 7
c a factor of 12
d a 9?

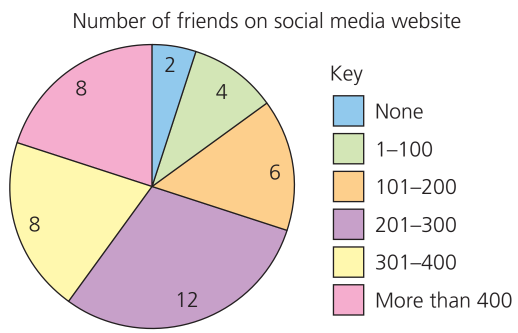
2 A game requires the use of all the playing cards in a normal pack from 6 to King inclusive.
a How many cards are used in the game?
b What is the probability of choosing:
i a 6 **ii** a picture
iii a club **iv** a prime number
v an 8 or a spade?

3 180 students in a school are offered a chance to attend a football match for free. If the students are chosen at random, what is the chance of being picked to go if the following numbers of tickets are available?
a 1 **b** 9 **c** 15 **d** 40 **e** 180

4 A bag contains 11 white, 9 blue, 7 green and 5 red counters. What is the probability that a single counter drawn will be:
a blue **b** red or green **c** not white?

5 The probability of drawing a red, blue or green marble from a bag containing 320 marbles is:
red 0.4 blue 0.25 green 0.35
If there are no other colours in the bag, how many marbles of each colour are there?

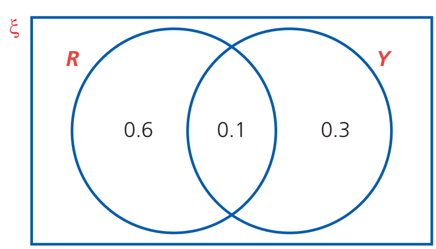
6 Students in a class conduct a survey to see how many friends they have on a social media website. The results were grouped and are shown in the pie chart below.



A student is chosen at random. What is the probability that she:
a has 101–200 friends on the social media website
b uses the social media website
c has more than 200 friends on the website?

7 **a** If I enter a competition and have a 0.00002 probability of winning, how many people entered the competition?
b What assumption do you have to make in order to answer part **a**?

8 A large bag contains coloured discs. The discs are completely red (R), completely yellow (Y) or half red and half yellow.
The Venn diagram below shows the probability of picking each type of disc:



If there are 120 discs that are coloured yellow (either fully or partly), calculate:
a the number of discs coloured completely red,
b the total number of discs in the bag.

34 Further probability

Combined events

Combined events look at the probability of two or more events.

Worked examples

a Two coins are tossed. Show in a **two-way table** all the possible outcomes.

A two-way table is a simple form of a sample space diagram.

		Coin 1	
		Head	Tail
Coin 2	Head	HH	TH
	Tail	HT	TT

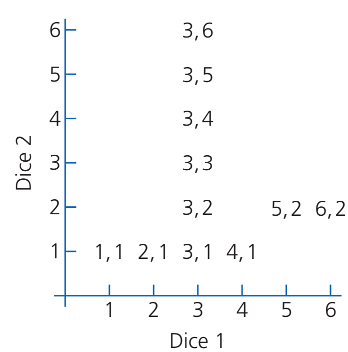
b Calculate the probability of getting two heads.
All four outcomes are equally likely: therefore, the probability of getting HH is $\frac{1}{4}$.

c Calculate the probability of getting a head and a tail in any order.
The probability of getting a head and a tail in any order, i.e. HT or TH, is $\frac{2}{4} = \frac{1}{2}$.

Exercise 34.1

1 **a** Two fair, tetrahedral dice are rolled. If each is numbered 1–4, draw a two-way table to show all the possible outcomes.
b What is the probability that both dice show the same number?
c What is the probability that the number on one dice is double the number on the other?
d What is the probability that the sum of both numbers is prime?

2 Two fair dice are rolled. Copy and complete the diagram to show all the possible combinations.
What is the probability of getting:
a a double 3,
b any double,
c a total score of 11,
d a total score of 7,
e an even number on both dice,
f an even number on at least one dice,
g a total of 6 or a double,
h scores which differ by 3,
i a total which is either a multiple of 2 or 5?



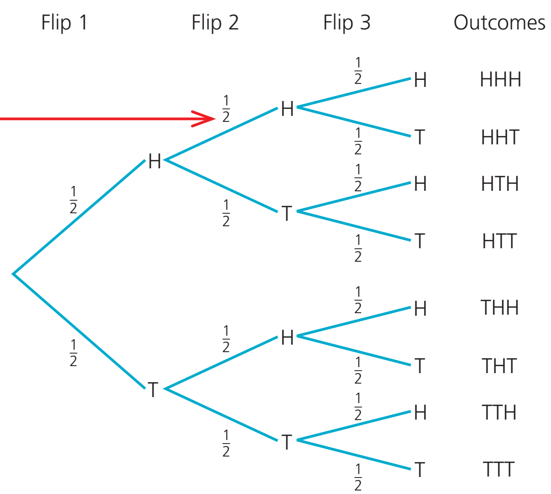
Tree diagrams

When more than two combined events are being considered, then two-way tables cannot be used and another method of representing information diagrammatically is needed. Tree diagrams are a good way of doing this.

Worked examples

a If a coin is flipped three times, show all the possible outcomes on a tree diagram, writing each of the probabilities at the side of the branches.

On tree diagrams, outcomes should be written at the end of the branches and the probabilities by the side of each branch.



b What is the probability of getting three heads?

There are eight equally likely outcomes, therefore the probability of getting HHH is $\frac{1}{8}$.

c What is the probability of getting two heads and one tail in any order?
The successful outcomes are HHT, HTH, THH.

Therefore the probability is $\frac{3}{8}$.

d What is the probability of getting at least one head?
This refers to any outcome with either one, two or three heads, i.e. all of them *except* TTT.

Therefore the probability is $\frac{7}{8}$.

e What is the probability of getting no heads?
The only successful outcome for this event is TTT.

Therefore the probability is $\frac{1}{8}$.

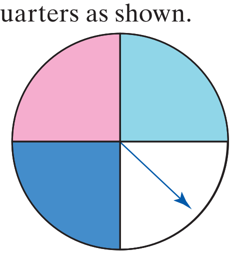
Exercise 34.2

1 a A computer uses the numbers 1, 2 or 3 at random to make three-digit numbers. Assuming that a number can be repeated, show on a tree diagram all the possible combinations that the computer can print.
  b Calculate the probability of getting:
   i the number 131,
   ii an even number,
   iii a multiple of 11,
   iv a multiple of 3,
   v a multiple of 2 or 3.

2 a A cat has four kittens. Draw a tree diagram to show all the possible combinations of males and females. [assume P (male) = P (female)]
  b Calculate the probability of getting:
   i all female,
   ii two females and two males,
   iii at least one female,
   iv more females than males.

3 a A netball team plays three matches. In each match the team is equally likely to win, lose or draw. Draw a tree diagram to show all the possible outcomes over the three matches.
  b Calculate the probability that the team:
   i wins all three matches,
   ii wins more times than loses,
   iii loses at least one match,
   iv either draws or loses all three matches.
  c Explain why it is not very realistic to assume that the outcomes are equally likely in this case.

4 A spinner is split into quarters as shown.



  a If it is spun twice, draw a probability tree showing all the possible outcomes.
  b Calculate the probability of getting:
   i two dark blues,
   ii two blues of either shade,
   iii a pink and a white in any order.

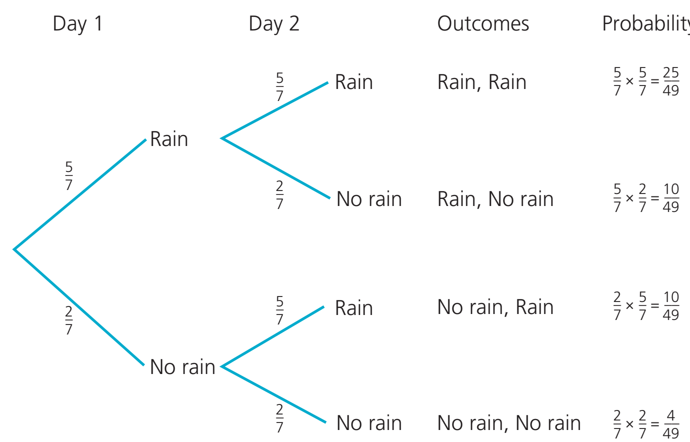
In each of the cases considered so far, all of the outcomes have been assumed to be equally likely. However, this need not be the case.

Worked example

In winter, the probability that it rains on any one day is $\frac{5}{7}$.

a Using a tree diagram, show all the possible combinations for two consecutive days.

b Write each of the probabilities by the sides of the branches.



Note how the probability of each outcome is arrived at by multiplying the probabilities of the branches.

c Calculate the probability that it will rain on both days.

$P(\text{R,R}) = \frac{5}{7} \times \frac{5}{7} = \frac{25}{49}$

d Calculate the probability that it will rain on the first but not the second day.

$P(\text{R,NR}) = \frac{5}{7} \times \frac{2}{7} = \frac{10}{49}$

e Calculate the probability that it will rain on at least one day.

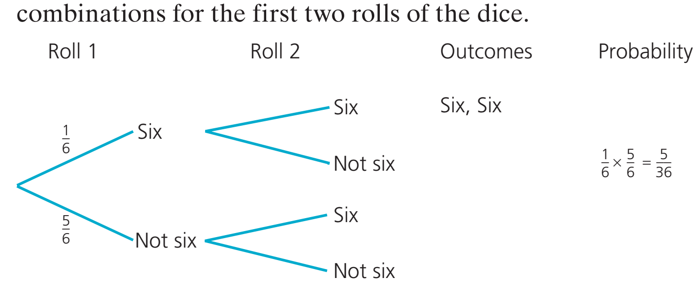
The outcomes which satisfy this event are (R, R) (R, NR) and (NR, R).

Therefore the probability is $\frac{25}{49} + \frac{10}{49} + \frac{10}{49} = \frac{45}{49}$.

Exercise 34.3

1 A board game involves players rolling a dice. However, before a player can start, each player needs to roll a 6.

a Copy and complete the tree diagram below showing all the possible combinations for the first two rolls of the dice.



Exercise 34.3 (cont)

b Calculate the probability of the following:
  i getting a 6 on the first roll,
  ii starting within the first two rolls,
  iii starting on the second roll,
  iv not starting within the first three rolls,
  v starting within the first three rolls.

c If you add the answers to **b iv** and **v** what do you notice? Explain.

2 In Italy $\frac{3}{5}$ of the cars are foreign made. By drawing a tree diagram and writing the probabilities next to each of the branches, calculate the following probabilities:
  a the next two cars to pass a particular spot are both Italian,
  b two of the next three cars are foreign,
  c at least one of the next three cars is Italian.

3 The probability that a morning bus arrives on time is 65%.
  a Draw a tree diagram showing all the possible outcomes for three consecutive mornings.
  b Label your tree diagram and use it to calculate the probability that:
    i the bus is on time on all three mornings,
    ii the bus is late the first two mornings,
    iii the bus is on time two out of the three mornings,
    iv the bus is on time at least twice.

4 A normal pack of 52 cards is shuffled and three cards are picked at random. Draw a tree diagram to help calculate the probability of picking:
  a two clubs first,
  b three clubs,
  c no clubs,
  d at least one club.

5 Light bulbs are packaged in cartons of three. 10% of the bulbs are found to be faulty. Calculate the probability of finding two faulty bulbs in a single carton.

6 A volleyball team has a 0.25 chance of losing a game. Calculate the probability of the team achieving:
  a two consecutive wins,
  b three consecutive wins,
  c 10 consecutive wins.

7 A bowl of fruit contains one kiwi fruit, one banana, two mangos and two lychees. Two pieces of fruit are chosen at random and eaten.
  a Draw a probability tree showing all the possible combinations of the two pieces of fruit eaten.
  b Use your tree diagram to calculate the probability that:
    i both the pieces of fruit eaten are mangos,
    ii a kiwi fruit and a banana are eaten,
    iii at least one lychee is eaten.

8 A class has n number of girls and n number of boys.
Two students are chosen at random.
  a Draw a tree diagram to show all the possible outcomes, labelling the probability of each branch in terms of n, where appropriate.
  b Show that the probability of two girls being chosen is $\frac{n-1}{2(2n-1)}$.

9 A bag of candies contains n red candies and $n + 3$ yellow candies.
A child takes two candies from the bag at random.
  a Draw a tree diagram to show all the possible outcomes and label the probability of each branch in terms of n.
  b Calculate the probability that the child picks two yellow candies.

Conditional probability

So far, all the probability considered has been based on random events with no other information given. However, sometimes more information is known and, as a result, the probability of the event happening changes. This is an example of **conditional probability**.

> **Note**
>
> Although the notation $P(A|B)$ is not explicitly required for this course, it has been included here as a more efficient way of describing conditional probability.

Worked examples

a The table shows the number of boys and girls studying Maths and Art in a school.

	Maths	Art	Total
Boys	26	14	40
Girls	34	12	46
Total	60	26	86

i A student is chosen at random. Calculate the probability that they study Maths.

60 students out of a total of 86 study Maths.

Therefore, the probability that they study Maths is $\frac{60}{86} = \frac{30}{43}$.

ii A student is chosen at random. Calculate the probability that they study Maths **given** that the student is a girl.

Here more information has been given than in part i. We already know that the student chosen is a girl, therefore, the student studying Maths is only being chosen from a group of 46 students (i.e. the total number of girls).

Therefore, the probability of choosing a student who studies Maths who is a girl is $\frac{34}{46} = \frac{17}{23}$.

In general, the notation used for conditional probability is $P(A|B)$.

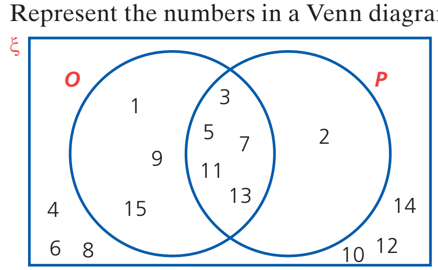
This states the probability of event A happening given that event B has happened.

In the example ii above, if the event of a student studying Maths is M and the event of choosing a girl is G, then the probability of choosing a student who studies Maths who is a girl is written as follows:

$P(M \mid G) = \frac{34}{46}$

b The numbers 1–15 are arranged into two sets of numbers where O = {odd numbers} and P = {prime numbers}.

i Represent the numbers in a Venn diagram.



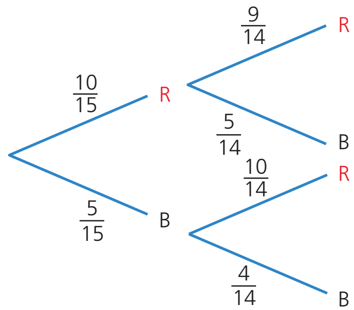
ii A number is chosen at random. Calculate the probability that it is prime.

$P(P) = \frac{6}{15} = \frac{2}{5}$

iii A number is chosen at random. Calculate the probability that it is prime **given** that it is odd, i.e. calculate $P(P|O)$.
We already know that the number is odd, therefore the total number of possible outcomes is only eight.
Therefore $\boldsymbol{P}(P|O) = \frac{5}{8}$.

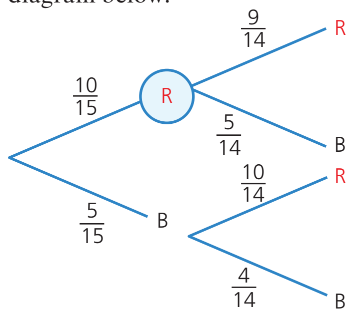
c A bag contains ten red beads (R) and five black beads (B).
A bead is taken out of the bag at random, its colour is noted and then it is **not** placed back in the bag.

i Draw a tree diagram to show the possible outcomes and the probabilities of the first two beads removed from the bag.



ii Calculate the probability that the first two beads are red.
$\boldsymbol{P}(RR) = \frac{10}{15} \times \frac{9}{14} = \frac{3}{7}$

iii Calculate the probability that the second bead is black given that the first bead chosen is red.
As the first bead is already known to be red, then the bottom half of the tree diagram can be ignored as it involves a black bead having been chosen first.
The top half of the tree diagram has a total probability of $\frac{2}{3}$
$\left(\text{i.e. } \frac{10}{15} \times \frac{9}{14} + \frac{10}{15} \times \frac{5}{14} = \frac{2}{3}\right)$
The probability of getting a red bead followed by a black bead when considering only the top half of the tree diagram is
$\frac{\frac{10}{15} \times \frac{5}{14}}{\frac{2}{3}} = \frac{5}{14}$.
Another way to consider this is to note that, as it is already known that the first bead is red, we are already at the stage indicated in the diagram below:



The probability of picking a black bead after this stage is therefore $\frac{5}{14}$.

Exercise 34.4

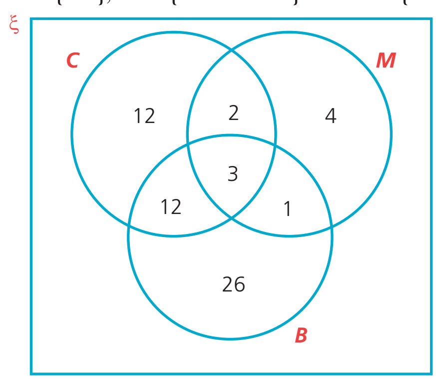
1 Each of the boys and girls in a class are asked how many brothers and sisters they have. The results are shown in the table:

	Number of brothers and sisters					
	0	1	2	3	⩾4	Total
Boys	2	3	3	2	0	10
Girls	3	1	6	3	1	14
Total	5	4	9	5	1	24

If a student is picked at random, calculate the probability:

a that it is a girl,
b that it is a boy with two or more brothers or sisters,
c that they have no brothers or sisters given that it is a girl,
d that it is a boy given that they have four or more brothers or sisters.

2 The Venn diagram shows the type of vehicle owned by a group of 60 people, where:
C = {car}, M = {motorbike} and B = {bicycle}.



If a person is picked at random, calculate the probability that they own the following vehicle type.

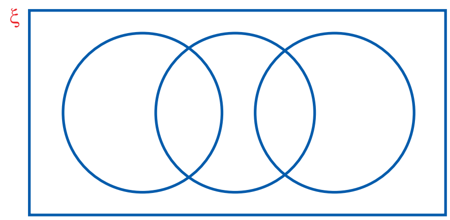
a $P(C)$
b $P(M \cup B)$
c $P(C \cap B)$
d $P(B|C)$
e $P(C|B)$
f $P(C \cap M|B)$
g $P(M|C')$
h $P(M'|B)$

3 Over the course of a season, a volleyball player records how often he is selected to play for his team and whether or not his team wins. At the end of the season, he analyses the results and finds that the probability of being selected was 0.8. If he was selected, the probability of the team winning was 0.65; if he wasn't selected, the probability of the team winning was 0.45.

a Draw a tree diagram to represent this information.
b If the team played 100 matches during the season, calculate how many they won.
c Given that he played in a match, calculate the probability that the team won.
d Given that the team won a match, calculate the probability that he played.

4 The numbers 1–15 are arranged into three sets as follows:
A = {odd numbers}, B = {prime numbers} and C = {multiples of two}.

a Copy and complete the Venn diagram, labelling each circle and then placing each number in the correct region.



b If a number is picked at random, calculate the following probabilities:

i $P(B)$

ii $P(B \cap C)$

iii $P(A \cap C)$

iv $P(A|B)$

v $P(B|C)$

vi $P(C|B)$

vii $P(B|B \cap C)$

c Explain why two of the circles do not overlap each other.

Student assessment 1

1 Two normal and fair dice are rolled and their scores added together.

a Using a two-way table, show all the possible scores that can be achieved.

b Using your two-way table, calculate the probability of getting:

i a score of 12,

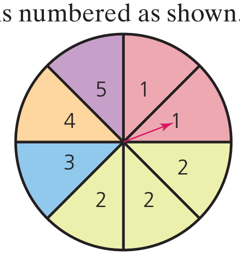
ii a score of 7,

iii a score less than 4,

iv a score of 7 or more.

c Two dice are rolled 180 times. In theory, how many times would you expect to get a total score of 6?

2 A spinner is numbered as shown.



a If it is spun once, calculate the probability of getting:

i a 1,

ii a 2.

b If it is spun twice, calculate the probability of getting:

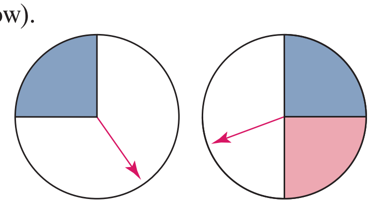
i a 2 followed by a 4,

ii a 2 and a 4 in any order,

iii at least one 1,

iv at least one 2.

3 Two spinners are coloured as shown (below).



a They are both spun. Draw and label a tree diagram showing all the possible outcomes.

b Using your tree diagram, calculate the probability of getting:

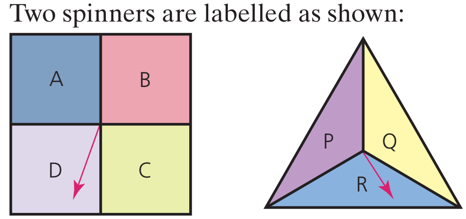
i two blues,

ii two whites,

iii a white and a pink,

iv at least one white.

4 Two spinners are labelled as shown:



Calculate the probability of getting:

a A and P,

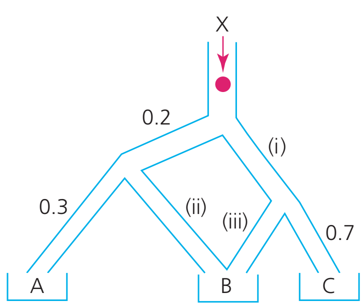
b A or B and R,

c C but not Q.

5 A vending machine accepts $1 and $2 coins. The probability of a $2 coin being rejected is 0.2. The probability of a $1 coin being rejected is 0.1.
A sandwich costing $3 is bought. Calculate the probability of getting a sandwich first time if:
- **a** one of each coin is used,
- **b** three $1 coins are used.

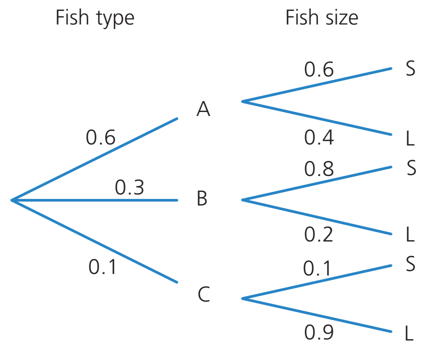
6 A biased coin is flipped three times. On each occasion, the probability of getting a head is 0.6.
- **a** Draw a tree diagram to show all the possible outcomes after three flips. Label each branch clearly with the probability of each outcome.
- **b** Using your tree diagram, calculate the probability of getting:
  - **i** three heads,
  - **ii** three tails,
  - **iii** at least two heads.

7 A ball enters a chute at X.



- **a** What are the probabilities of the ball going down each of the chutes labelled (i), (ii) and (iii)?
- **b** Calculate the probability of the ball landing in:
  - **i** tray A,
  - **ii** tray C,
  - **iii** tray B.
- **c** Given that the ball goes down chute (i), calculate the probability of it landing in tray B.
- **d** Given that the ball lands in tray B, calculate the probability that it came down chute (iii).

8 A fish breeder keeps three types of fish: A, B and C. Each type can be categorised into two sizes, small (S) and large (L). The probabilities of each are given in the tree diagram:



- **a** A fish is chosen at random. Calculate the probability that:
  - **i** it is a large fish of type B,
  - **ii** it is a large fish,
  - **iii** it is a large fish given that it is of type B.
- **b** If a fish chosen at random is small, calculate the probability of it being of type:
  - **i** A,
  - **ii** B,
  - **iii** C.

9 A football team decide to analyse their match results. They look at their probability of winning depending on whether they scored first in the game or not. The results are presented in the table:

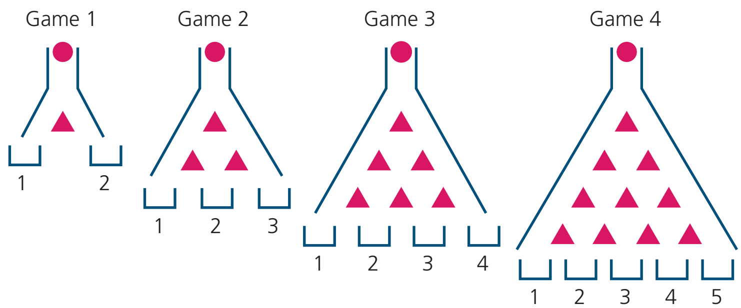
	Win	**Lose**	**Draw**
Score first	0.38	0.12	0.18
Don't score first	0.02	0.22	0.08

- **a** Calculate the probability that the team score first.
- **b** Calculate the probability that the team draw.
- **c** Given that the team score first, calculate the probability that they draw.
- **d** Given that the team lose, calculate the probability that they didn't score first.

8 Mathematical investigations and ICT 8

Probability drop

A game involves dropping a red marble down a chute. On hitting a triangle divider, the marble can bounce either left or right. On completing the drop, the marble lands in one of the trays along the bottom. The trays are numbered from left to right. Different sizes of game exist, the four smallest versions are shown below:



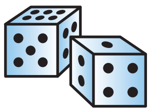
To land in tray 2 in the second game above, the ball can travel in one of two ways. These are: Left – Right or Right – Left.

This can be abbreviated to LR or RL.

1 State the different routes the marble can take to land in each of the trays in the third game.
2 State the different routes the marble can take to land in each of the trays in the fourth game.
3 State, giving reasons, the probability of a marble landing in tray 1 in the fourth game.
4 State, giving reasons, the probability of a marble landing in each of the other trays in the fourth game.
5 Investigate the probability of the marble landing in each of the different trays in larger games.
6 Using your findings from your investigation, predict the probability of a marble landing in tray 7 in the tenth game (11 trays at the bottom).
7 Investigate the links between this game and the sequence of numbers generated in Pascal's triangle.

Dice sum

Two ordinary dice are rolled and their scores added together. Below is an incomplete table showing the possible outcomes:



		Dice 1					
		1	**2**	**3**	**4**	**5**	**6**
Dice 2	**1**	2			5		
	2						
	3				7		
	4				8		
	5				9	10	11
	6						12

1 Copy and complete the table to show all possible outcomes.
2 How many possible outcomes are there?
3 What is the most likely total when two dice are rolled?
4 What is the probability of getting a total score of 4?
5 What is the probability of getting the most likely total?
6 How many times more likely is a total score of 5 compared with a total score of 2?

Now consider rolling two four-sided dice, each numbered 1–4. Their scores are also added together.

7 Draw a table to show all the possible outcomes when the two four-sided dice are rolled and their scores added together.
8 How many possible outcomes are there?
9 What is the most likely total?
10 What is the probability of getting the most likely total?
11 Investigate the number of possible outcomes, the most likely total and its probability when two identical dice are rolled together and their scores are added, i.e. consider eight-sided dice, ten-sided dice, etc.
12 Consider two m-sided dice rolled together and their scores added.
   a What is the total number of outcomes in terms of m?
   b What is the most likely total, in terms of m?
   c What, in terms of m, is the probability of the most likely total?
13 Consider an m-sided and n-sided dice rolled together, where $m > n$.
   a In terms of m and n, deduce the total number of outcomes.
   b In terms of m and/or n, deduce the most likely total(s).
   c In terms of m and/or n, deduce the probability of getting the most likely total.

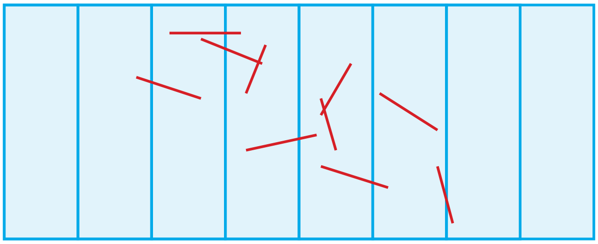
ICT activity: Buffon's needle experiment

You will need to use a spreadsheet for this activity.

The French count Le Comte de Buffon devised the following probability experiment.

1 Measure the length of a match (with the head cut off) as accurately as possible.
2 On a sheet of paper, draw a series of straight lines parallel to each other. The distance between each line should be the same as the length of the match.
3 Take ten identical matches and drop them randomly on the paper. Count the number of matches that cross or touch any of the lines.

For example, in the diagram below, the number of matches crossing or touching lines is six.



4 Repeat the experiment a further nine times, making a note of your results, so that altogether you have dropped 100 matches.
5 Set up a spreadsheet similar to the one shown below and enter your results in cell B2.

	A	B	C	D	E	F	G	H	I	J	K
1	Number of drops (N)	100	200	300	400	500	600	700	800	900	1000
2	Number of matches crossing/touching lines (n)										
3	Probability of crossing a line (p = n/N)										
4	$2/p$										

6 Repeat 100 match drops again, making a total of 200 drops, and enter cumulative results in cell C2.
7 By collating the results of your fellow students, enter the cumulative results of dropping a match 300–1000 times in cells D2–K2 respectively.
8 Using an appropriate formula, get the spreadsheet to complete the calculations in Rows 3 and 4.
9 Use the spreadsheet to plot a line graph of N against $\frac{2}{p}$.
10 What value does $\frac{2}{p}$ appear to get closer to?

TOPIC 9

Statistics

Contents

Learning objectives

E9.1 Classifying statistical data
Classify and tabulate statistical data.

E9.2 Interpreting statistical data

1. Read, interpret and draw inferences from tables and statistical diagrams.
2. Compare sets of data using tables, graphs and statistical measures.
3. Appreciate restrictions on drawing conclusions from given data.

E9.3 Averages and measures of spread

1. Calculate the mean, median, mode, quartiles, range and interquartile range for discrete data and distinguish between the purposes for which these are used.
2. Calculate an estimate of the mean for grouped discrete or continuous data.
3. Identify the modal class from a grouped frequency distribution.

E9.4 Statistical charts and diagrams
Draw and interpret:

(a) bar charts
(b) pie charts
(c) pictograms
(d) stem-and-leaf diagrams
(e) simple frequency distributions.

E9.5 Scatter diagrams

1. Draw and interpret scatter diagrams.
2. Understand what is meant by positive, negative and zero correlation.
3. Draw by eye, interpret and use a straight line of best fit.

E9.6 Cumulative frequency diagrams

1. Draw and interpret cumulative frequency tables and diagrams.
2. Estimate and interpret the median, percentiles, quartiles and interquartile range from cumulative frequency diagrams.

E9.7 Histograms

1. Draw and interpret histograms.
2. Calculate with frequency density.

Statistics in history

The earliest writing on statistics was found in a ninth-century book entitled *Manuscript on Deciphering Cryptographic Messages*, written by the Arab philosopher Al-Kindi (801–873), who lived in Baghdad. In his book, he gave a detailed description of how to use statistics to unlock coded messages.

The *Nuova Cronica*, a fourteenth-century history of Florence by the Italian banker Giovanni Villani, includes much statistical information on population, commerce, trade and education.

Florence Nightingale (1820–1910) was a famous British nurse who treated casualties in the Crimean War (1853–1856). By using statistics she realised that most of the deaths that occurred were not as a result of battle injuries but from preventable illnesses afterwards, such as cholera and typhoid. By understanding these statistics, Florence Nightingale was able to improve the sanitary conditions of the injured soldiers and therefore reduce their mortality rates.

Early statistics served the needs of states, state-istics. By the early nineteenth century, statistics included the collection and analysis of data in general. Today, statistics are widely employed in government, business, and natural and social sciences. The use of modern computers has enabled large-scale statistical computation and has also made possible new methods that are impractical to perform manually.

35 Mean, median, mode and range

Average

'Average' is a word which in general use is taken to mean somewhere in the middle. For example, a woman may describe herself as being of average height. A student may think they are of average ability in Maths. Mathematics is more exact and uses three principal methods to measure average.

- The **mode** is the value occurring the most often.
- The **median** is the middle value when all the data is arranged in order of size.
- The **mean** is found by adding together all the values of the data and then dividing that total by the number of data values.

Spread

It is often useful to know how spread out the data is. It is possible for two sets of data to have the same mean and median but very different spreads.

The simplest measure of spread is the **range**. The range is simply the difference between the largest and smallest values in the data.

Another measure of spread is known as the interquartile range. This is covered in more detail in Chapter 37.

Note

Interquartile range is not part of the Core syllabus.

Advantages and disadvantages of different averages		
	Advantage	**Disadvantage**
Mean	It includes all data values.	It can be affected by extreme values.
Median	It is not affected by extreme values. It gives a sense of where the 'middle' value is.	As it only considers the middle value, it does not take into account all data values.
Mode	It is not affected by extreme values. The most common result is used.	It does not take into account all data values. It is not useful if there are lots of different data values.

Worked examples

a **i** Find the mean, median and mode of the data listed below.
1, 0, 2, 4, 1, 2, 1, 1, 2, 5, 5, 0, 1, 2, 3

Mean = $\frac{1+0+2+4+1+2+1+1+2+5+5+0+1+2+3}{15} = 2$

Arranging all the data in order of magnitude and then picking out the middle number gives the median:
0, 0, 1, 1, 1, 1, 1, (2), 2, 2, 2, 3, 4, 5, 5
The mode is the number which appears most often.
Therefore the mode is 1.

ii Calculate the range of the data.
Largest value = 5
Smallest value = 0
Therefore the range is 5 – 0 = 5

b **i** The bar chart (below) shows the score out of 10 achieved by a class in a maths test.
Calculate the mean, median and mode for this data.

Transferring the results to a **frequency distribution table** gives:

This frequency table also shows the **frequency distribution** of the test scores. →

Test score	0	1	2	3	4	5	6	7	8	9	10	Total
Frequency	1	2	3	2	3	5	4	6	4	1	1	32
Frequency × score	0	2	6	6	12	25	24	42	32	9	10	168

In the total column we can see the number of students taking the test, i.e. 32, and also the total number of marks obtained by all the students, i.e. 168.

Therefore, the mean score = $\frac{168}{32} = 5.25$

Arranging all the scores in order gives:

0, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 5, (5, 6), 6, 6, 6, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 9, 10

Because there is an even number of students there isn't one middle number. There is, however, a middle pair. The median is $\frac{(5+6)}{2} = 5.5$.

The mode is 7 as it is the score which occurs most often.

ii Calculate the range of the data.
Largest value = 10   Smallest value = 0
Therefore the range is 10 – 0 = 10.

Exercise 35.1

In Questions 1–5, find the mean, median, mode and range for each set of data.

1 A hockey team plays 15 matches. Below is a list of the numbers of goals scored in these matches.
1, 0, 2, 4, 0, 1, 1, 1, 2, 5, 3, 0, 1, 2, 2

2 The total scores when two dice are thrown 20 times are:
7, 4, 5, 7, 3, 2, 8, 6, 8, 7, 6, 5, 11, 9, 7, 3, 8, 7, 6, 5

3 The ages of a group of girls are:
14 years 3 months, 14 years 5 months,
13 years 11 months, 14 years 3 months,
14 years 7 months, 14 years 3 months,
14 years 1 month

4 The numbers of students present in a class over a three-week period are:
28, 24, 25, 28, 23, 28, 27, 26, 27, 25, 28, 28, 28, 26, 25

5 An athlete keeps a record of her training times, in seconds, for the 100 m race:
14.0, 14.3, 14.1, 14.3, 14.2, 14.0, 13.9, 13.8, 13.9, 13.8, 13.8, 13.7, 13.8, 13.8, 13.8

6 The mean mass of the 11 players in a football team is 80.3 kg. The mean mass of the team plus a substitute is 81.2 kg. Calculate the mass of the substitute.

7 After eight matches, a basketball player had scored a mean of 27 points. After three more matches his mean was 29. Calculate the total number of points he scored in the last three games.

Exercise 35.2

1 An ordinary dice was rolled 60 times. The frequency distribution is shown in the table below. Calculate the mean, median, mode and range of the scores.

Score	1	2	3	4	5	6
Frequency	12	11	8	12	7	10

2 Two dice were thrown 100 times. Each time their combined score was recorded. Below is a frequency distribution of the results. Calculate the mean score.

Score	2	3	4	5	6	7	8	9	10	11	12
Frequency	5	6	7	9	14	16	13	11	9	7	3

3 Sixty flowering shrubs of the same variety are planted. At their flowering peak, the number of flowers per shrub is counted and recorded. The frequency distribution is shown in the table below.

Flowers per shrub	0	1	2	3	4	5	6	7	8
Frequency	0	0	0	6	4	6	10	16	18

a Calculate the mean, median, mode and range of the number of flowers per shrub.

b Which of the mean, median and mode would be most useful when advertising the shrub to potential buyers?

The mean for grouped data

The mean for grouped data can only be an estimate as the position of the data within a group is not known. An estimate is made by calculating the mid-interval value for a group and then assigning that mid-interval value to all of the data within the group.

Worked example

The history test scores for a group of 40 students are shown in the grouped frequency table below.

Score, S	**Frequency**	**Mid-interval value**	**Frequency × mid-interval value**
$0 \leqslant S < 20$	2	10	20
$20 \leqslant S < 40$	4	30	120
$40 \leqslant S < 60$	14	50	700
$60 \leqslant S < 80$	16	70	1120
$80 \leqslant S < 100$	4	90	360

a Calculate an estimate for the mean test result.

$$\text{Mean} = \frac{20 + 120 + 700 + 1120 + 360}{40} = 58$$

b What is the modal class?
This refers to the class with the greatest frequency, if the class width is constant. Therefore the modal class is $60 \leqslant S < 80$.

Exercise 35.3

1 The heights of 50 basketball players attending a tournament are recorded in the grouped frequency table.

Height (m)	$1.8 < H \leqslant 1.9$	$1.9 < H \leqslant 2.0$	$2.0 < H \leqslant 2.1$	$2.1 < H \leqslant 2.2$	$2.2 < H \leqslant 2.3$	$2.3 < H \leqslant 2.4$
Frequency	2	5	10	22	7	4

a Copy the table and complete it to include the necessary data with which to calculate the mean height of the players.
b Estimate the mean height of the players.
c What is the modal class height of the players?

2 The number of hours of overtime worked by employees at a factory over a period of a month is given in the table (below).

Hours of overtime	0–9	10–19	20–29	30–39	40–49	50–59
Frequency	12	18	22	64	32	20

a Calculate an estimate for the mean number of hours of overtime worked by the employees that month.
b What is the modal class?

Exercise 35.3 (cont)

3 The length of the index finger of 30 students in a class was measured. The results were recorded and are shown in the table below.

Length (cm)	$5.0 < L \leqslant 5.5$	$5.5 < L \leqslant 6.0$	$6.0 < L \leqslant 6.5$	$6.5 < L \leqslant 7.0$	$7.0 < L \leqslant 7.5$
Frequency	3	8	10	7	2

a Calculate an estimate for the mean index finger length of the students.
b What is the modal class?

Student assessment 1

1 A javelin thrower keeps a record of her best throws over ten competitions. These are shown in the table below.

Competition	1	2	3	4	5	6	7	8	9	10
Distance (m)	77	75	78	86	92	93	93	93	92	89

Find the mean, median, mode and range of her throws.

2 The bar chart shows the marks out of 10 for a chemistry test taken by a class of students.

a Calculate the number of students who took the test.
b Calculate for the class:
i the mean test result,
ii the median test result,
iii the modal test result,
iv the range of the test results.
c The teacher is happy with these results as she says that the average result was 10/10. Another teacher says that the average is only 7.5/10. Which teacher is correct? Give a reason for your answer.

3 The range, mode, median and mean of five positive integers are all equal to 12. Work out one possible set of these five integers.

4 A hundred sacks of coffee with a stated mass of 10 kg are unloaded from a train. The mass of each sack is checked and the results are presented in the table.

Mass (kg)	Frequency
$9.8 \leqslant M < 9.9$	14
$9.9 \leqslant M < 10.0$	22
$10.0 \leqslant M < 10.1$	36
$10.1 \leqslant M < 10.2$	20
$10.2 \leqslant M < 10.3$	8

a Calculate an estimate for the mean mass.
b What is the modal class?

36 Collecting, displaying and interpreting data

Tally charts and frequency tables

The number of chocolate buttons in each of twenty packets is:

35 36 38 37 35 36 38 36 37 35

36 36 38 36 35 38 37 38 36 38

The figures can be shown on a tally chart:

Number	Tally	Frequency
35	IIII	4
36	卌 II	7
37	III	3
38	卌 I	6

When the tallies are added up to find the frequency, the chart is usually called a **frequency table**. The information can then be displayed in a variety of ways.

Pictograms

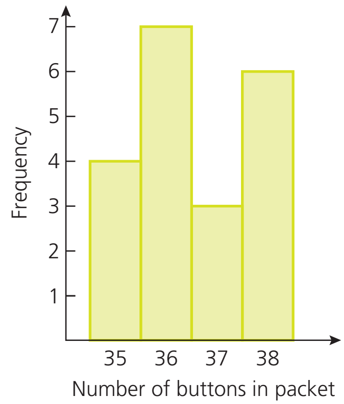
● = 4 packets, ◕ = 3 packets, ◑ = 2 packets, ◔ = 1 packet

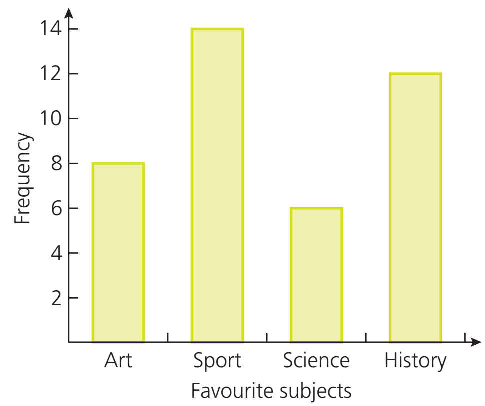
Buttons per packet	
35	●
36	● ◕
37	◕
38	● ◑

Note

When a bar chart is showing categorical data (such as favourite subjects) rather than numerical data, there are gaps between the bars.

Bar charts





Stem-and-leaf diagrams

Discrete data is data that has a specific, fixed value. A stem-and-leaf diagram can be used to display discrete data in a clear and organised way. It has an advantage over bar charts as the original data can easily be recovered from the diagram.

The ages of people on a coach transferring them from an airport to a ski resort are as follows:

22	24	25	31	33	23	24	26	37	42
40	36	33	24	25	18	20	27	25	33
28	33	35	39	40	48	27	25	24	29

Displaying the data on a stem-and-leaf diagram produces the following graph.

1	8															
2	0	2	3	4	4	4	4	5	5	5	5	6	7	7	8	9
3	1	3	3	3	3	5	6	7	9							
4	0	0	2	8												

Key 2 | 5 represents 25

In this form the data can be analysed quite quickly:

- The youngest person is 18
- The oldest is 48
- The modal ages are 24, 25 and 33

As the data is arranged in order, the median age can also be calculated quickly. The middle people out of 30 will be the 15th and 16th people. In this case the 15th person is 27 years old and the 16th person 28 years old, therefore the median age is 27.5.

Back-to-back stem-and-leaf diagrams

Stem-and-leaf diagrams are often used as an easy way to compare two sets of data. The leaves are usually put 'back-to-back' on either side of the stem.

Continuing from the example given above, consider a second coach from the airport taking people to a golfing holiday. The ages of these people are shown below:

43	46	52	61	65	38	36	28	37	45
69	72	63	55	46	34	35	37	43	48
54	53	47	36	58	63	70	55	63	64

Displaying the two sets of data on a back-to-back stem-and-leaf diagram is shown below:

Golf	Stem	Skiing
	1	8
8	2	0 2 3 4 4 4 4 5 5 5 5 6 7 7 8 9
8 7 7 6 6 5 4	3	1 3 3 3 3 5 6 7 9
8 7 6 6 5 3 3	4	0 0 2 8
8 5 5 4 3 2	5	
9 5 4 3 3 3 1	6	
2 0	7	

Key: 5 | 3 | 6 represents 35 to the left and 36 to the right

From the back-to-back diagram it is easier to compare the two sets of data. This data shows that the people on the coach going to the golf resort tend to be older than those on the coach to the ski resort.

Grouped frequency tables

If there is a big range in the data, it is easier to group the data in a **grouped frequency table**.

The groups are arranged so that no score can appear in two groups.

The scores for the first round of a golf competition are:

71	75	82	96	83	75	76	82	103	85	79	77	83	85	88
104	76	77	79	83	84	86	88	102	95	96	99	102	75	72

This data can be grouped as shown:

Score	Frequency
71–75	5
76–80	6
81–85	8
86–90	3
91–95	1
96–100	3
101–105	4
Total	30

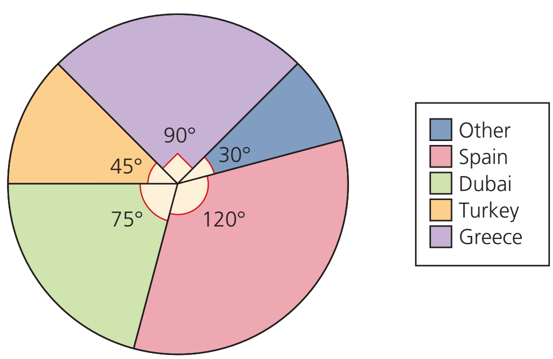
Note: It is not possible to score 70.5 or 77.3 at golf. The scores are **discrete**. If the data is **continuous**, for example when measuring time, the intervals can be shown as $0 \leqslant t < 10$, $10 \leqslant t < 20$, $20 \leqslant t < 30$ and so on.

Pie charts

Data can be displayed on a **pie chart** – a circle divided into sectors. The size of the sector is in direct proportion to the frequency of the data. The sector size does not show the actual frequency. The actual frequency can be calculated easily from the size of the sector.

Worked examples

a



In a survey, 240 people were asked to vote for their favourite holiday destination. The results are shown on the pie chart above. Calculate the actual number of votes for each destination.

The total 240 votes are represented by 360°.

It follows that if 360° represents 240 votes:

There were $240 \times \frac{120}{360}$ votes for Spain so, 80 votes for Spain.

There were $240 \times \frac{75}{360}$ votes for Dubai so, 50 votes for Dubai.

There were $240 \times \frac{45}{360}$ votes for Turkey so, 30 votes for Turkey.

There were $240 \times \frac{90}{360}$ votes for Greece so, 60 votes for Greece.

Other destinations received $240 \times \frac{30}{360}$ votes so, 20 votes for other destinations.

Note: It is worth checking your result by adding them:

$80 + 50 + 30 + 60 + 20 = 240$ total votes

b The table below shows what percentage of the money raised during a fundraising campaign that charities, which support different causes, received. If a total of \$5 million was raised, how much money did each charitable cause receive?

Charitable cause	Percentage of money
Children and young adults	45%
Disability and mental health	36%
Animal welfare	15%
Others	4%

Children and young adults received $\frac{45}{100} \times \$5$ million
so, \$2.25 million.

Disability and mental health received $\frac{36}{100} \times \$5$ million
so, \$1.8 million.

Animal welfare received $\frac{15}{100} \times \$5$ million
so, \$750 000.

Other charitable causes received $\frac{4}{100} \times \$5$ million
so, \$200 000.

Check total:

$2.25 + 1.8 + 0.75 + 0.2 = 5$ (million dollars)

c The table shows the results of a survey among 72 students to find their favourite sport. Display this data on a pie chart.

Sport	Frequency
Football	35
Tennis	14
Volleyball	10
Hockey	6
Basketball	5
Other	2

72 students are represented by 360°, so 1 student is represented by $\frac{360}{72}$ degrees. Therefore, the size of each sector can be calculated:

Football $35 \times \frac{360}{72}$ degrees i.e. 175°

Tennis $14 \times \frac{360}{72}$ degrees i.e. 70°

Volleyball $10 \times \frac{360}{72}$ degrees i.e. 50°

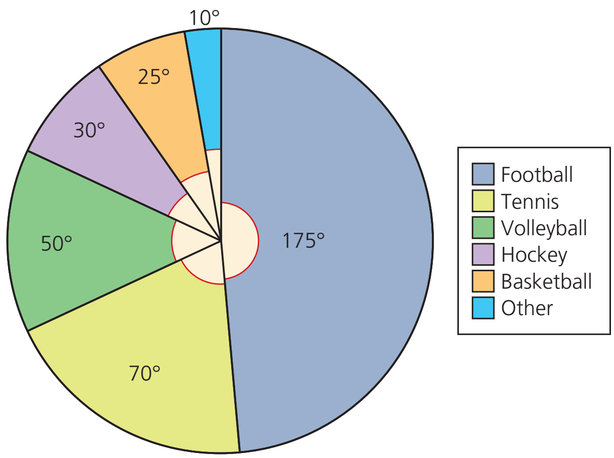
Hockey $6 \times \frac{360}{72}$ degrees i.e. 30°

Basketball $5 \times \frac{360}{72}$ degrees i.e. 25°

Other sports $2 \times \frac{360}{72}$ degrees i.e. 10°

Check total:

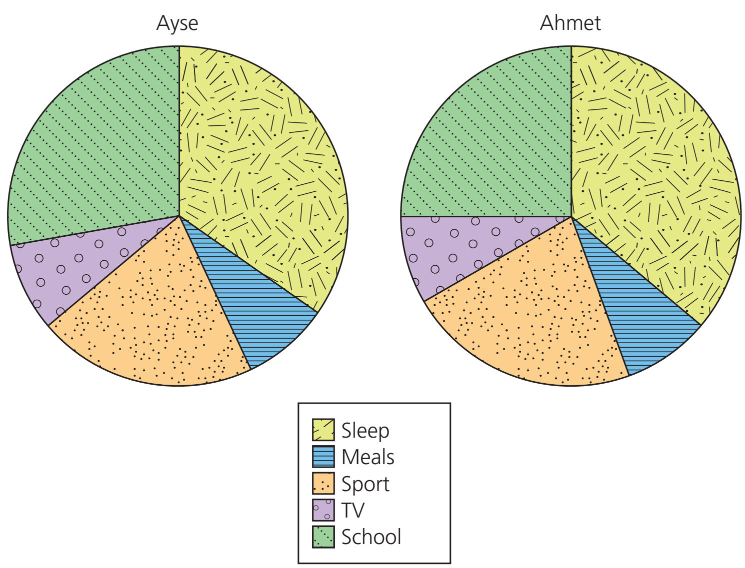
$175 + 70 + 50 + 30 + 25 + 10 = 360$



Exercise 36.1

You will need an angle measurer or protractor for this question.

1 The unlabelled pie charts below show how Ayse and her brother, Ahmet, spent one day. Calculate how many hours they spent on each activity. The diagrams are to scale.



Exercise 36.1 (cont)

2 A survey was carried out among a class of 40 students. The question asked was, 'How would you spend a gift of $15?' The results are shown below:

Choice	Frequency
Music	14
Books	6
Clothes	18
Cinema	2

Illustrate these results on a pie chart.

3 A student works during the holidays. He earns a total of $2400. He estimates that the money has been spent as follows: clothes, $\frac{1}{3}$; transport, $\frac{1}{5}$; entertainment, $\frac{1}{4}$. He has saved the rest.

Calculate how much he has spent on each category, and illustrate this information on a pie chart.

4 Two universities in central Asia compared the percentages of people who enrolled on different engineering courses in 2022. The results are shown in the table below.

Engineering course	University A (percentage)	University B (percentage)
Civil	22	38
Mechanical	16	8
Chemical	24	16
Electrical	12	24
Industrial	8	10
Aerospace	18	4

a Illustrate this information on two pie charts, and make two statements that could be supported by the data.

b If 3000 people enrolled on engineering courses at University B in 2022, calculate the number who enrolled on either mechanical or aerospace engineering courses.

5 A village has two sports clubs.
The ages of people in each club are listed below:

Ages in Club 1									
38	8	16	15	18	8	59	12	14	55
14	15	24	67	71	21	23	27	12	48
31	14	70	15	32	9	44	11	46	62

Ages in Club 2									
42	62	10	62	74	18	77	35	38	66
43	71	68	64	66	66	22	48	50	57
60	59	44	57	12					

a Draw a back-to-back stem-and-leaf diagram for the ages of the members of each club.

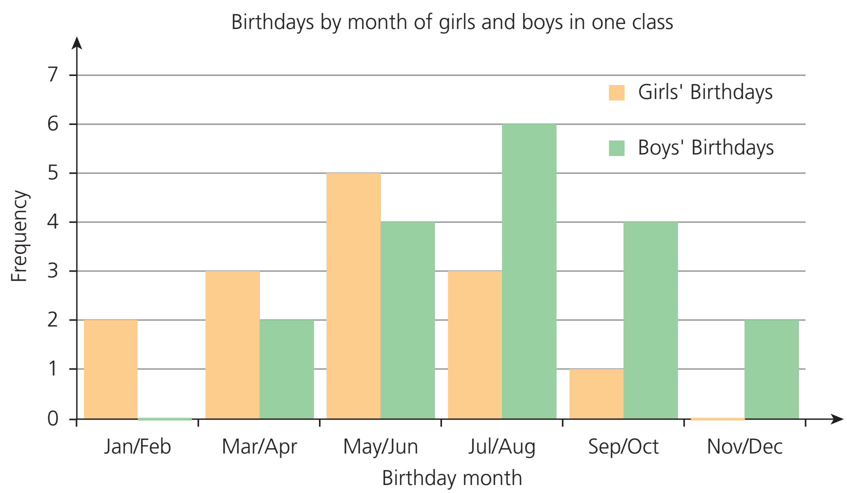
b For each club, calculate:

i the age range of their members,

ii the median age.

c One of the clubs is the golf club, the other is the athletics club. Which club is **likely** to be which? Give a reason for your answer.

6 The birthday months of boys and girls in one class are plotted as a dual bar chart below:



a How many girls are there in the class?

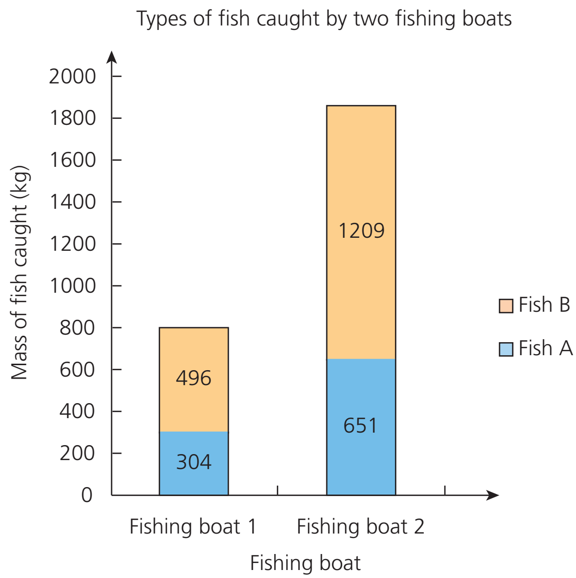
b How many more boys than girls have their birthdays in either July or August?

c Describe the differences in the birthdays of boys and girls in this class.

d Construct a dual bar chart for the birthday months of boys and girls in your own class.

Exercise 36.1 (cont)

7 Two fishing boats return to port and the mass of two types of fish caught by each boat is recorded. The masses are shown in the composite (stacked) bar chart below.



a Which boat caught the greater mass of fish type A?
b Assuming only the two types of fish were caught, which boat's catch had a higher percentage of fish type A? Show your working.
c The above composite bar chart shows the mass of fish in kg on the vertical axis.
Construct a composite bar chart comparing the catches of both boats, but with percentages on the vertical axis.

Surveys

A survey requires data to be collected, organised, analysed and presented.

A survey may be carried out for interest's sake, for example, to find out how many cars pass your school in an hour. A survey could be carried out to help future planning – information about traffic flow could lead to the building of new roads, or the placing of traffic lights or a pedestrian crossing.

Exercise 36.2

1 Below are ten statements, some of which you may have heard or read before.
Conduct a survey to collect data which will support or disprove one of the statements. Where possible, use pie charts to illustrate your results.
a Magazines are full of adverts.
b If you go to a football match you are lucky to see more than one goal scored.
c Every other car on the road is white.
d Most retired people use public transport.
e Children today do nothing but watch TV.
f Newspapers have more sport than news in them.

- **g** More people prefer to drink coffee than tea.
- **h** Nobody walks to school any more.
- **i** Nearly everybody has a computer at home.
- **j** Most of what is on TV comes from America.

2 Below are some instructions relating to a washing machine, written in English, French, German, Dutch and Italian.
Analyse the data and write a report. You may wish to comment upon:
- **a** the length of words in each language,
- **b** the frequency of letters of the alphabet in different languages.

ENGLISH

ATTENTION

Do not interrupt drying during the programme.

This machine incorporates a temperature safety thermostat which will cut out the heating element in the event of a water blockage or power failure. In the event of this happening, reset the programme before selecting a further drying time. For further instructions, consult the user manual.

FRENCH

ATTENTION

N'interrompez pas le séchage en cours de programme.

Une panne d'électricité ou un manque d'eau momentanés peuvent annuler le programme de séchage en cours. Dans ces cas arrêtez l'appareil, affichez de nouveau le programme et après remettez l'appareil en marche.

Pour d'ultérieures informations, rapportez-vous à la notice d'utilisation.

GERMAN

ACHTUNG

Die Trocknung soll nicht nach Anlaufen des Programms unterbrochen werden.

Ein kurzer Stromausfall bzw. Wassermangel kann das laufende Trocknungsprogramm annullieren. In diesem Falle Gerät ausschalten, Programm wieder einstellen und Gerät wieder einschalten.

Für nähere Angaben beziehen Sie sich auf die Bedienungsanleitung.

ESTONIAN

TÄHELEPANU

Ärge katkestage kuivatamist programmi ajal.

Sellel masinal on temperatuuri turvatermostaat, mis lõikab veeummistuse või voolukatkestuse korral kütteelemendi välja. Juhul kui see peaks juhtuma, lähtestage programm uuesti enne uue kuivamisaja valimist. Täiendavate juhiste saamiseks vaadake kasutusjuhendit.

MALAY

PERHATIAN

Jangan ganggu pengeringan semasa program.

Mesin ini menggabungkan termostat keselamatan suhu yang akan dipotong keluarkan elemen pemanas sekiranya berlaku penyumbatan air atau kegagalan kuasa. Sekiranya ini berlaku, tetapkan semula atur cara sebelum memilih a masa pengeringan selanjutnya. Untuk arahan lanjut, rujuk manual pengguna.

Scatter diagrams

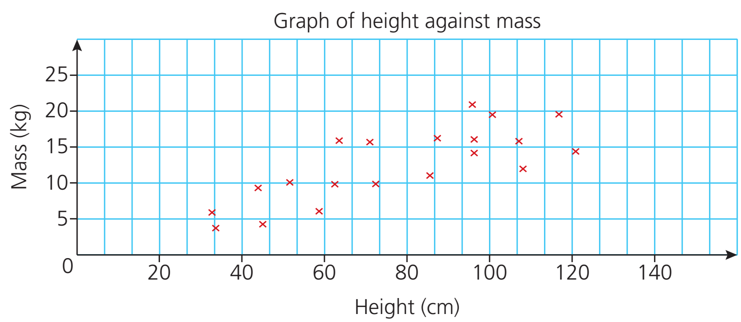
Scatter diagrams are particularly useful if we wish to see if there is a **correlation** (relationship) between two sets of data. The two values of data collected represent the coordinates of each point plotted. How the points lie when plotted, indicates the type of relationship between the two sets of data.

Worked example

The heights and masses of 20 children under the age of five were recorded. The heights were recorded in centimetres and the masses in kilograms. The data is shown in a table:

Height	32	34	45	46	52
Mass	5.8	3.8	9.0	4.2	10.1
Height	59	63	64	71	73
Mass	6.2	9.9	16.0	15.8	9.9
Height	86	87	95	96	96
Mass	11.1	16.4	20.9	16.2	14.0
Height	101	108	109	117	121
Mass	19.5	15.9	12.0	19.4	14.3

a Plot a scatter diagram of the above data.

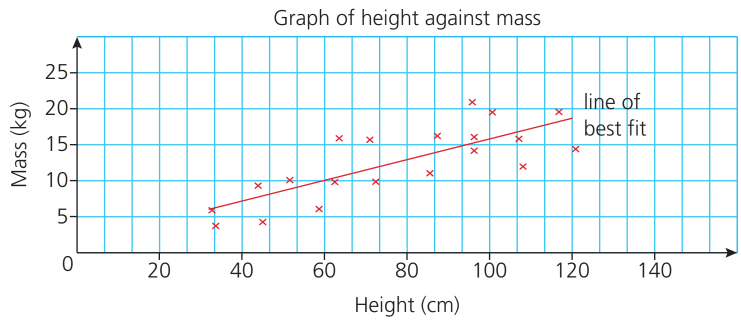


b Comment on any relationship you see.

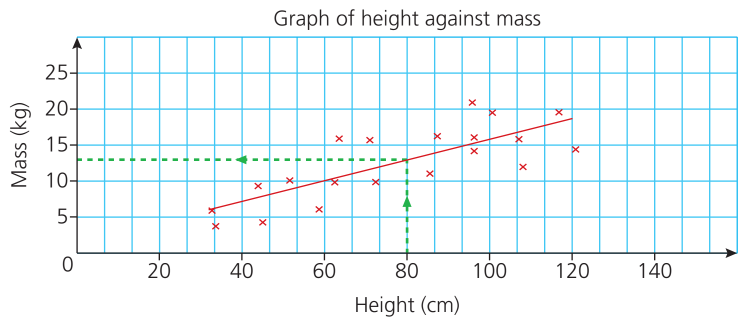
The points tend to lie in a diagonal direction from bottom left to top right. This suggests that as height increases then, in general, mass increases too. Therefore there is a **positive correlation** between height and mass.

c If another child was measured as having a height of 80 cm, approximately what mass would you expect them to be?

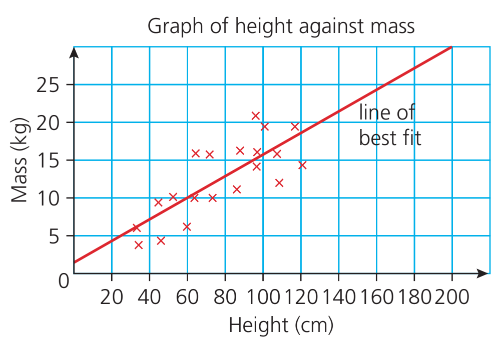
We assume that this child will follow the trend set by the other 20 children. To deduce an approximate value for the mass, we draw a **line of best fit**. This is done by eye and is a solid straight line which passes through the points as closely as possible, as shown.



The line of best fit can now be used to give an approximate solution to the question. If a child has a height of 80 cm, you would expect their mass to be in the region of 13 kg.



d Someone decides to extend the line of best fit in both directions because they want to make predictions for heights and masses beyond those of the data collected. The graph is shown below.



Explain why this should not be done.

It should not be done ecause we cannot assume that the relationship between mass and height continues in the same pattern beyond the collected data.

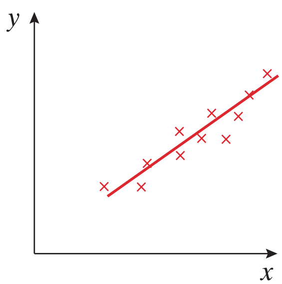
Types of correlation

There are several types of correlation, depending on the arrangement of the points plotted on the scatter diagram.

A **strong positive correlation** between the variables x and y.

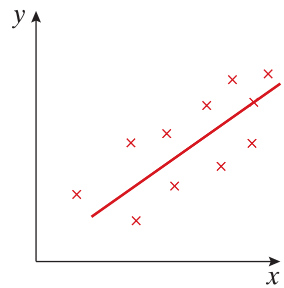
The points lie very close to the line of best fit.

As x increases, so does y.

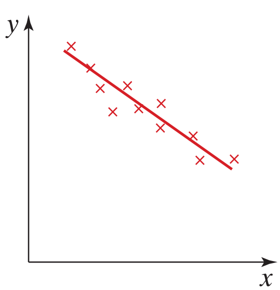


A **weak positive correlation**. Although there is direction to the way the points are lying, they are not tightly packed around the line of best fit.

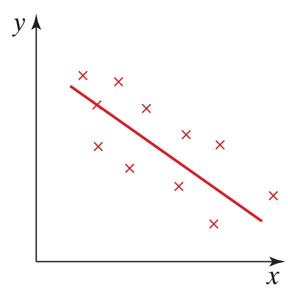
As x increases, y tends to increase too.



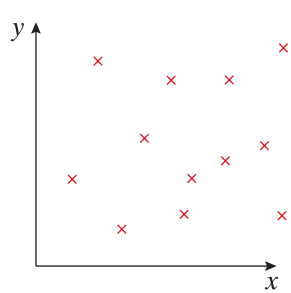
A **strong negative correlation**. The points lie close to the line of best fit. As x increases, y decreases.



A **weak negative correlation**. The points are not tightly packed around the line of best fit. As x increases, y tends to decrease.



Zero or no correlation. As there is no pattern to the way in which the points are lying, there is no correlation between the variables x and y. As a result there can be no line of best fit.



Exercise 36.3

1 State what type of correlation you might expect, if any, if the following data was collected and plotted on a scatter diagram. Give reasons for your answer.
- **a** A student's score in a maths exam and their score in a science exam.
- **b** A student's hair colour and the distance they have to travel to school.
- **c** The outdoor temperature and the number of cold drinks sold by a shop.
- **d** The age of a motorcycle and its second-hand selling price.
- **e** The number of people living in a house and the number of rooms the house has.
- **f** The number of goals your opponents score and the number of times you win.
- **g** A child's height and the child's age.
- **h** A car's engine size and its fuel consumption.

2 A website gives average monthly readings for the number of hours of sunshine and the amount of rainfall in millimetres for several cities in Europe. The table below is a summary for July.

Place	Hours of sunshine	Rainfall (mm)
Athens	12	6
Belgrade	10	61
Copenhagen	8	71
Dubrovnik	12	26
Edinburgh	5	83
Frankfurt	7	70
Geneva	10	64
Helsinki	9	68
Innsbruck	7	134
Krakow	7	111
Lisbon	12	3
Marseilles	11	11
Naples	10	19
Oslo	7	82
Plovdiv	11	37
Reykjavik	6	50
Sofia	10	68
Tallinn	10	68
Valletta	12	0
York	6	62
Zurich	8	136

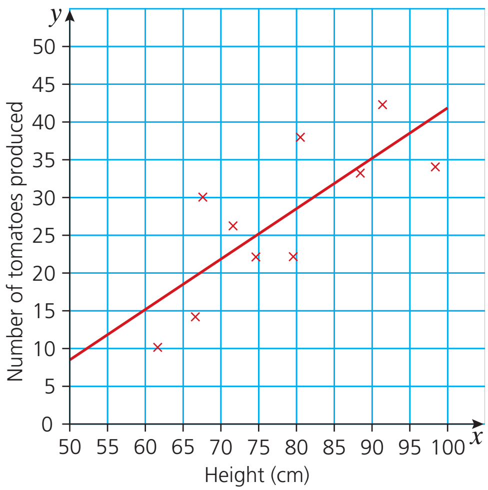
- **a** Plot a scatter diagram of the number of hours of sunshine against the amount of rainfall. Use a spreadsheet if possible.
- **b** What type of correlation, if any, is there between the two variables? Comment on whether this is what you would expect.

Exercise 36.3 (cont)

3 The United Nations keeps an up-to-date database of statistical information on its member countries. The table below shows some of the information available.

Country	Life expectancy at birth (years, 2005–2010)		Adult illiteracy rate (%, 2009)	Infant mortality rate (per 1000 births, 2005–2010)
	Female	Male		
Australia	84	79	1	5
Barbados	80	74	0.3	10
Brazil	76	69	10	24
Chad	50	47	68.2	130
China	75	71	6.7	23
Colombia	77	69	7.2	19
Cuba	81	77	0.2	5
Democratic Republic of the Congo	55	53	18.9	79
Egypt	72	68	33	35
France	85	78	1	4
Germany	82	77	1	4
India	65	62	34	55
Israel	83	79	2.9	5
Japan	86	79	1	3
Kenya	55	54	26.4	64
Mexico	79	74	7.2	17
Nepal	67	66	43.5	42
Portugal	82	75	5.1	4
Russian Federation	73	60	0.5	12
Saudi Arabia	75	71	15	19
South Africa	53	50	12	49
United Kingdom	82	77	1	5
United States of America	81	77	1	6

a By plotting a scatter diagram, decide if there is a correlation between the adult illiteracy rate and the infant mortality rate.
b Are your findings in part **a** what you expected? Explain your answer.
c Without plotting a graph, decide if you think there is likely to be a correlation between male and female life expectancy at birth. Explain your reasons.
d Plot a scatter diagram to test if your predictions for part **c** were correct.



4 Kris plants 10 tomato plants. He wants to see if there is a relationship between the number of tomatoes the plant produces and its height in centimetres.
The results are presented in the scatter diagram left. The line of best fit is also drawn.

a Describe the correlation (if any) between the height of a plant and the number of tomatoes it produced.

b Kris has another plant grown in the same conditions as the others. If the height is 85 cm, estimate from the graph the number of tomatoes he can expect it to produce.

c Another plant only produces 15 tomatoes. Deduce its height from the graph.

d Kris thinks he will be able to make more predictions if the height axis starts at 0 cm rather than 50 cm and if the line of best fit is then extended.
By re-plotting the data on a new scatter graph and extending the line of best fit, explain whether Kris's idea is correct.

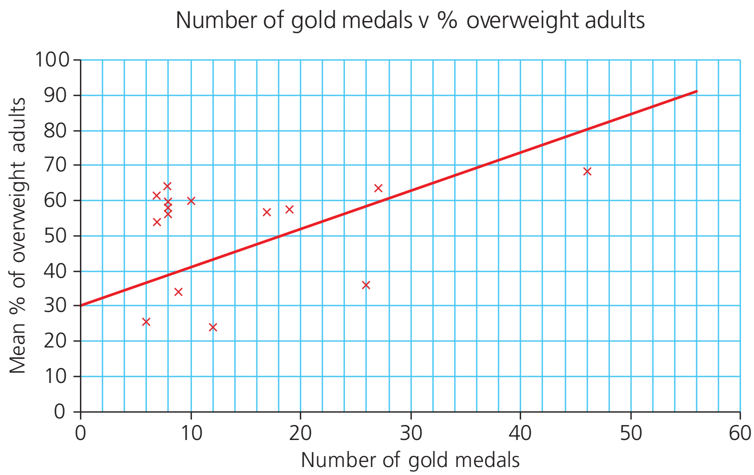
5 The table shows the 15 countries that won the most medals at the 2016 Rio Olympics. In addition, statistics relating to each of those countries' population, wealth, percentage of people with higher education and percentage who are overweight are also given.

Country	Olympic medals			Population (million)	Average wealth per person in 1000's $	% with a Higher Education Qualification	% Adult population that are overweight	
	Gold	Silver	Bronze				Male	Female
USA	46	37	38	322	345	45	73	63
UK	27	23	17	65	289	44	68	59
China	26	18	26	1376	23	10	39	33
Russia	19	18	19	143	10	54	60	55
Germany	17	10	15	81	185	28	64	49
Japan	12	8	21	127	231	50	29	19
France	10	18	14	664	244	34	67	52
S. Korea	9	3	9	50	160	45	38	30
Italy	8	12	8	60	202	18	66	53
Australia	8	11	10	24	376	43	70	58
Holland	8	7	4	17	184	35	63	49
Hungary	8	3	4	10	34	24	67	49
Brazil	7	6	6	208	18	14	55	53
Spain	7	4	6	46	116	35	67	55
Kenya	6	6	1	46	2	11	17	34

Exercise 36.3 (cont)

A sports scientist wants to see if there is a correlation between the number of medals a country won and the percentage of overweight people in that country.

To obtain a simple scatter graph, she plots the number of gold medals against the mean percentage of overweight people and adds the line of best fit:



a Describe the type of correlation implied by the above graph.
b The sports scientist states that the graph shows that the more overweight you are the more likely you are to win a gold medal. Give two reasons why this conclusion may not be accurate.
c Analyse the correlation between two other sets of data and comment on whether the results are expected or not. Justify your answer.

Histograms

A **histogram** displays the frequency of either continuous or grouped discrete data in the form of bars. There are several important features of a histogram which distinguish it from a bar chart.

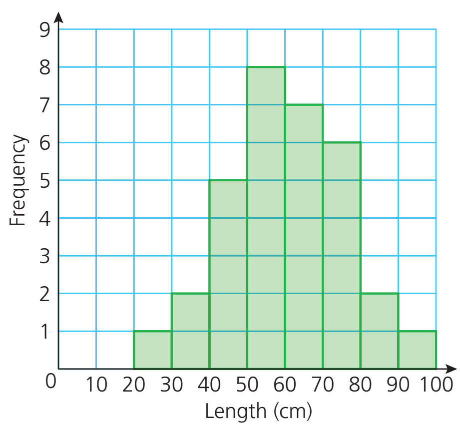
- The bars are joined together.
- The bars can be of varying width.
- The frequency of the data is represented by the area of the bar and not the height (though in the case of bars of equal width, the area is directly proportional to the height of the bar and so the height is usually used as the measure of frequency).

Worked example

A zoo keeper measures the length (L cm) of 32 lizards kept in the reptile section of the zoo. Draw a histogram of the data.

Length (cm)	Frequency
$0 < L \leqslant 10$	0
$10 < L \leqslant 20$	0
$20 < L \leqslant 30$	1
$30 < L \leqslant 40$	2
$40 < L \leqslant 50$	5
$50 < L \leqslant 60$	8
$60 < L \leqslant 70$	7
$70 < L \leqslant 80$	6
$80 < L \leqslant 90$	2
$90 < L \leqslant 100$	1

All the class intervals are the same. As a result, the bars of the histogram will all be of equal width and the frequency can be plotted on the vertical axis. The histogram is shown below.



Exercise 36.4

1 The table (below) shows the distances travelled to school by a class of 30 students. Represent this information on a histogram.

Distance (km)	Frequency
$0 \leqslant d < 1$	8
$1 \leqslant d < 2$	5
$2 \leqslant d < 3$	6
$3 \leqslant d < 4$	3
$4 \leqslant d < 5$	4
$5 \leqslant d < 6$	2
$6 \leqslant d < 7$	1
$7 \leqslant d < 8$	1

2 The heights of students in a class were measured. The results are shown in the table (below). Draw a histogram to represent this data.

Height (cm)	Frequency
$145 \leqslant h < 150$	1
$150 \leqslant h < 155$	2
$155 \leqslant h < 160$	4
$160 \leqslant h < 165$	7
$165 \leqslant h < 170$	6
$170 \leqslant h < 175$	3
$175 \leqslant h < 180$	2
$180 \leqslant h < 185$	1

Note that both questions in Exercise 36.4 deal with **continuous data**.

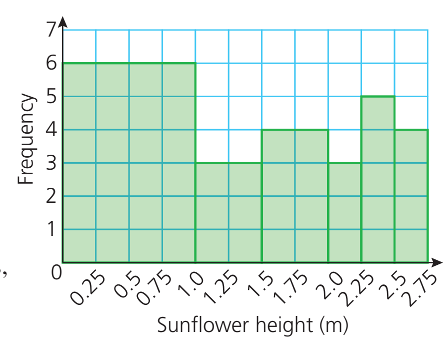
So far the work on histograms has only dealt with problems in which the class intervals are of the same width. This, however, need not be the case.

Worked example

The heights of 25 sunflowers were measured and the results recorded in the table (below).

Height (m)	Frequency
$0 \leqslant h < 1.0$	6
$1.0 \leqslant h < 1.5$	3
$1.5 \leqslant h < 2.0$	4
$2.0 \leqslant h < 2.25$	3
$2.25 \leqslant h < 2.50$	5
$2.50 \leqslant h < 2.75$	4

If a histogram were drawn with frequency plotted on the vertical axis, then it could look like the one drawn opposite.



This graph is misleading because it leads people to the conclusion that most of the sunflowers were under 1 m, simply because the area of the bar is so great. In fact, only approximately one quarter of the sunflowers were under 1 m.

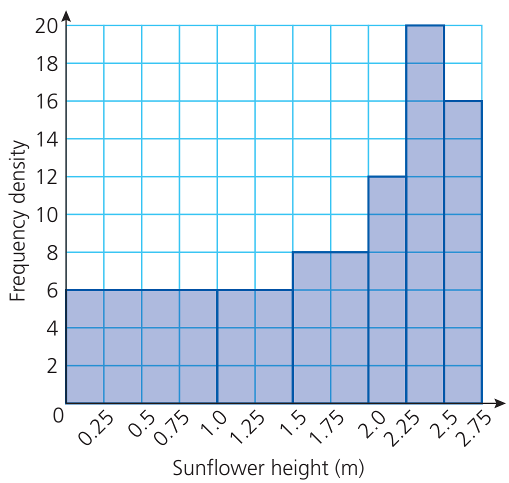
When class intervals are different it is the area of the bar which represents the frequency, not the height. Instead of frequency being plotted on the vertical axis, **frequency density** is plotted.

$$\text{Frequency density} = \frac{\text{frequency}}{\text{class width}}$$

The results of the sunflower measurements in the example above can therefore be written as:

Height (m)	Frequency	Frequency density
$0 \leqslant h < 1.0$	6	$6 \div 1 = 6$
$1.0 \leqslant h < 1.5$	3	$3 \div 0.5 = 6$
$1.5 \leqslant h < 2.0$	4	$4 \div 0.5 = 8$
$2.0 \leqslant h < 2.25$	3	$3 \div 0.25 = 12$
$2.25 \leqslant h < 2.50$	5	$5 \div 0.25 = 20$
$2.50 \leqslant h < 2.75$	4	$4 \div 0.25 = 16$

The histogram can therefore be redrawn giving a more accurate representation of the data:



Exercise 36.5

1 The table below shows the time taken, in minutes, by 40 students to travel to school.

Time (min)	Frequency	Frequency density
$0 \leqslant t < 10$	6	
$10 \leqslant t < 15$	3	
$15 \leqslant t < 20$	13	
$20 \leqslant t < 25$	7	
$25 \leqslant t < 30$	3	
$30 \leqslant t < 40$	4	
$40 \leqslant t < 60$	4	

a Copy the table and complete it by calculating the frequency density.
b Represent the information on a histogram.

2 On Sundays, Maria helps her father feed their chickens. Over a period of one year, she kept a record of how long it took. Her results are shown in the table below.

Time (min)	Frequency	Frequency density
$0 \leqslant t < 30$	8	
$30 \leqslant t < 45$	5	
$45 \leqslant t < 60$	8	
$60 \leqslant t < 75$	9	
$75 \leqslant t < 90$	10	
$90 \leqslant t < 120$	12	

a Copy the table and complete it by calculating the frequency density. Give the answers correct to 1 d.p.
b Represent the information on a histogram.

3 Frances and Ali did a survey of the ages of the people living in their village. Part of their results are set out in the table below.

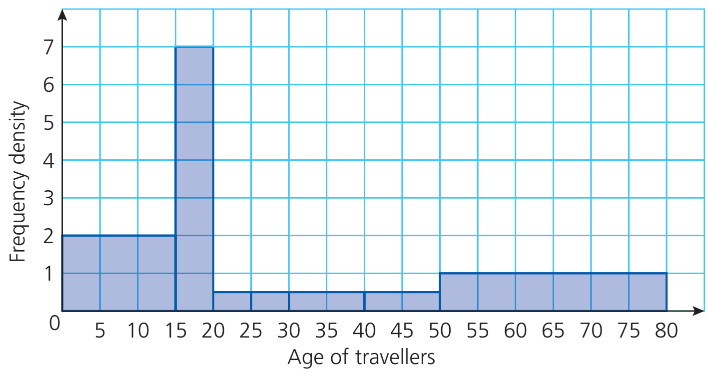
Age (years)	Frequency	Frequency density
$0 \leqslant y < 1$	35	
$1 \leqslant y < 5$		12
$5 \leqslant y < 10$		28
$10 \leqslant y < 20$	180	
$20 \leqslant y < 40$	260	
$40 \leqslant y < 60$		14
$60 \leqslant y < 90$	150	

a Copy the table and complete it by calculating either the frequency or the frequency density.
b Represent the information on a histogram.

4 The table below shows the ages of 150 people, chosen randomly, taking the 06 00 train into a city.

Age (years)	Frequency
$0 \leqslant y < 15$	3
$15 \leqslant y < 20$	25
$20 \leqslant y < 25$	20
$25 \leqslant y < 30$	30
$30 \leqslant y < 40$	32
$40 \leqslant y < 50$	30
$50 \leqslant y < 80$	10

The histogram below shows the results obtained when the same survey was carried out on the 11 00 train.



a Draw a histogram for the 06 00 train.
b Compare the two sets of data and give two possible reasons for the differences.

Student assessment 1

1 The table below shows the population (in millions) of the continents. Display this information on a pie chart.

Continent	Asia	Europe	America	Africa	Oceania
Population (millions)	4140	750	920	995	35

2 A department store decides to investigate whether there is a correlation between the number of pairs of gloves it sells and the outside temperature. Over a one-year period the store records, every two weeks, how many pairs of gloves are sold and the mean daytime temperature during the same period. The results are given in the table:

Mean temperature (°C)	3	6	8	10	10	11	12	14	16	16	17	18	18
Number of pairs of gloves	61	52	49	54	52	48	44	40	51	39	31	43	35
Mean temperature (°C)	19	19	20	21	22	22	24	25	25	26	26	27	28
Number of pairs of gloves	26	17	36	26	46	40	30	25	11	7	3	2	0

a Plot a scatter diagram of mean temperature against number of pairs of gloves.
b What type of correlation is there between the two variables?
c How might this information be useful for the department store in the future?
d The mean daytime temperature during the next two-week period is predicted to be 20°C. Draw a line of best fit on your graph and use it to estimate the number of pairs of gloves the department store can expect to sell.

3 A test in physics is marked out of 40. The scores of a class of 32 students are shown below.

24	27	30	33	26	27	28	39
21	18	16	33	22	38	33	11
16	11	14	23	37	36	38	22
28	11	9	17	28	11	36	34

a Display the data on a stem-and-leaf diagram.
b The teacher says that any student getting less than the average score will have to sit a re-test. How many students will sit the re-test? Justify your answer fully.

4 The grouped frequency table below shows the number of points scored by a school basketball player.

Points	Number of games	Frequency density
$0 \leqslant p < 5$	2	
$5 \leqslant p < 10$	3	
$10 \leqslant p < 15$	8	
$15 \leqslant p < 25$	9	
$25 \leqslant p < 35$	12	
$35 \leqslant p < 50$	3	

a Copy and complete the table by calculating the frequency densities. Give your answers to 1 d.p.
b Draw a histogram to illustrate the data.

37 Cumulative frequency

Cumulative frequency

Calculating the **cumulative frequency** is done by adding up the frequencies as we go along. A cumulative frequency diagram is particularly useful when trying to calculate the median of a large set of data, grouped or continuous data, or when trying to establish how consistent the results in a set of data are.

Worked example

The duration of two different brands of battery, A and B, is tested. 50 batteries of each type are randomly selected and tested in the same way. The duration of each battery is then recorded. The results of the tests are shown in the tables below.

Brand A		
Duration (h)	**Frequency**	**Cumulative frequency**
$0 \leqslant t < 5$	3	3
$5 \leqslant t < 10$	5	8
$10 \leqslant t < 15$	8	16
$15 \leqslant t < 20$	10	26
$20 \leqslant t < 25$	12	38
$25 \leqslant t < 30$	7	45
$30 \leqslant t < 35$	5	50

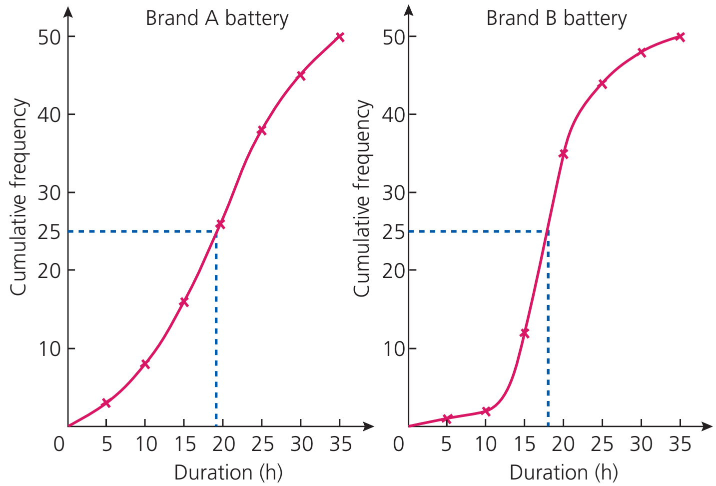
Brand B		
Duration (h)	**Frequency**	**Cumulative frequency**
$0 \leqslant t < 5$	1	1
$5 \leqslant t < 10$	1	2
$10 \leqslant t < 15$	10	12
$15 \leqslant t < 20$	23	35
$20 \leqslant t < 25$	9	44
$25 \leqslant t < 30$	4	48
$30 \leqslant t < 35$	2	50

Note that a smooth curve is drawn through the points.

Note

For large data sets the median is at the mid-point of the cumulative frequency.

a Plot a cumulative frequency diagram for each brand of battery.



Both cumulative frequency diagrams are plotted above.

Notice how the points are plotted at the upper boundary of each class interval and *not* at the middle of the interval.

b Calculate the median duration for each brand.

The median value is the value which occurs halfway up the cumulative frequency axis. Therefore:

Median for brand A batteries ≈ 19 h

Median for brand B batteries ≈ 18 h

This tells us that the same number of batteries are still working as have stopped working after approximately 19 h for brand A and approximately 18 h for brand B.

Exercise 37.1

1 Sixty athletes enter a long-distance run. Their finishing times are recorded and are shown in the table below.

Finishing time (h hours)	Frequency	Cumulative freq.
$0 \leqslant h < 0.5$	0	
$0.5 \leqslant h < 1.0$	0	
$1.0 \leqslant h < 1.5$	6	
$1.5 \leqslant h < 2.0$	34	
$2.0 \leqslant h < 2.5$	16	
$2.5 \leqslant h < 3.0$	3	
$3.0 \leqslant h < 3.5$	1	

a Copy the table and calculate the values for the cumulative frequency.
b Draw a cumulative frequency diagram of the results.
c Show how your diagram could be used to find the approximate median finishing time.
d What does the median value tell us?

Exercise 37.1 (cont)

2 Three physics classes take the same test in preparation for their final exam. Their raw scores are shown in the table below.

Class A	12, 21, 24, 30, 33, 36, 42, 45, 53, 53, 57, 59, 61, 62, 74, 88, 92, 93
Class B	48, 53, 54, 59, 61, 62, 67, 78, 85, 96, 98, 99
Class C	10, 22, 36, 42, 44, 68, 72, 74, 75, 83, 86, 89, 93, 96, 97, 99, 99

a Using the class intervals $0 \leqslant x < 20$, $20 \leqslant x < 40$ etc., draw up a grouped frequency and cumulative frequency table for each class.
b Draw a cumulative frequency diagram for each class.
c Show how your diagrams could be used to find the median score for each class.
d What does the median value tell us?

3 The table below shows the heights of students in a class over a three-year period.

Height (cm)	Frequency 2020	Frequency 2020	Frequency 2022
$150 \leqslant h < 155$	6	2	2
$155 \leqslant h < 160$	8	9	6
$160 \leqslant h < 165$	11	10	9
$165 \leqslant h < 170$	4	4	8
$170 \leqslant h < 175$	1	3	2
$175 \leqslant h < 180$	0	2	2
$180 \leqslant h < 185$	0	0	1

a Construct a cumulative frequency table for each year.
b Draw a cumulative frequency diagram for each year.
c Show how your diagrams could be used to find the median height for each year.
d What does the median value tell us?

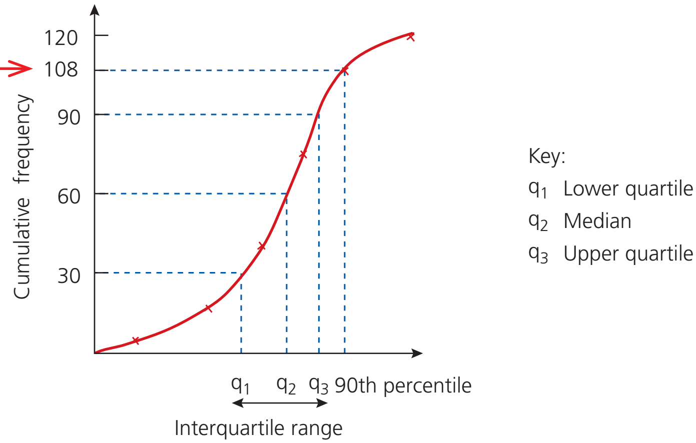
Quartiles, percentiles and the interquartile range

The cumulative frequency axis can also be represented in terms of **percentiles**. A percentile scale divides the cumulative frequency scale into hundredths. The maximum value of cumulative frequency is found at the 100th percentile. Similarly, the median, being the middle value, is called the 50th percentile. The 25th percentile is known as the **lower quartile**, and the 75th percentile is called the **upper quartile**.

The **range** of a distribution is found by subtracting the lowest value from the highest value. Sometimes this will give a useful result, but often it will not. A better measure of spread is given by looking at the spread of the middle half of the results, i.e. the difference between the upper and lower quartiles. This result is known as the **interquartile range**.

The diagram (below) shows the terms mentioned above.

As the cumulative frequency shows a total of 120, the 90th percentile is calculated from the cumulative frequency as 108, because $120 \times 90 \div 100 = 108$.



Advantages and disadvantages of different types of range		
	Advantage	**Disadvantage**
Range	The spread takes into account all data values.	It can be affected by extreme values.
Interquartile range	As it only considers the middle 50% of the data, it is not affected by extreme results.	It doesn't take into account all the data values, i.e. it disregards the top and bottom 25%.

Worked examples

Consider again the two brands of batteries, A and B, discussed earlier on page 535.

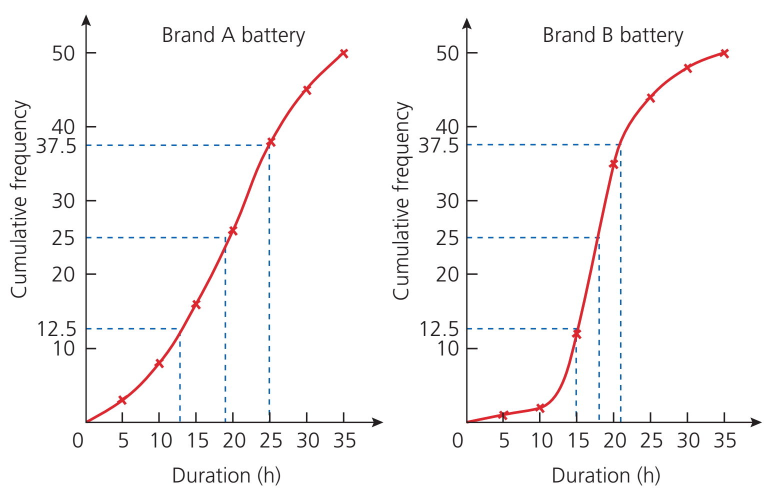
a Using the diagrams, estimate the upper and lower quartiles for each battery.

Lower quartile of brand A ≈ 13 h

Upper quartile of brand A ≈ 25 h

Lower quartile of brand B ≈ 15 h

Upper quartile of brand B ≈ 21 h



b Calculate the interquartile range for each brand of battery.

Interquartile range of brand A ≈ 12 h

Interquartile range of brand B ≈ 6 h

c Based on these results, how might the manufacturers advertise the two brands of battery?

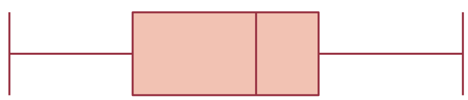
Brand A: on 'average' the longer-lasting battery.

Brand B: the more reliable battery.

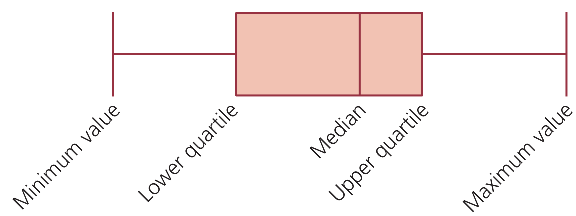
Box-and-whisker plots

The information calculated from a cumulative frequency diagram can be represented in a diagram called a **box-and-whisker plot**.

The typical shape of a box-and-whisker plot is:



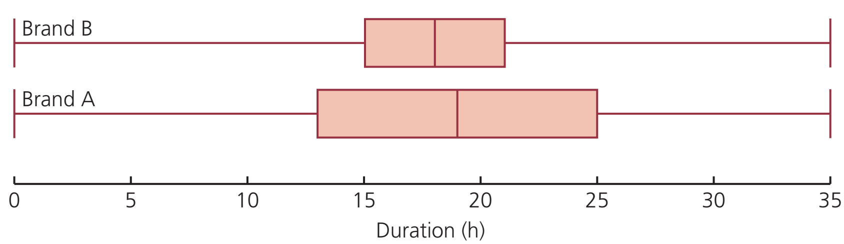
Its different components represent different aspects of the data.



When looking at two sets of data, box-and-whisker plots are an efficient way of comparing them.

Worked example

Consider the two battery brands A and B discussed in the previous worked example.
Draw box-and-whisker plots for both brands on the same scale and comment on any similarities and differences.



- Both battery brands have the same overall range.
- The median of brand A is slightly bigger than that of brand B.

- The spread of the middle 50% of the data, represented by the rectangular block in the centre and known as the interquartile range, is much smaller for brand B batteries. This implies that the middle 50% are more consistent for brand B than for brand A batteries.

Note: From the tables on page 535, the first group was for a duration of $0 \leqslant t < 5$ hours. For brand A batteries there were three results in that group. It is not known exactly where in the group the three results lie, therefore the minimum value is taken as the lower bound of the group. Similarly, the last group for the data was for a duration of $30 \leqslant t < 35$ hours. There were five brand A batteries in that group. It is not known where those five batteries lie within the group, therefore the maximum value is taken to be the upper bound of the group.

Exercise 37.2

1 Using the results obtained from Question 2 in Exercise 37.1:
  a find the interquartile range of each of the classes taking the mathematics test,
  b analyse your results and write a brief summary comparing the three classes.

2 Using the results obtained from Question 3 in Exercise 37.1:
  a find the interquartile range of the students' heights each year,
  b analyse your results and write a brief summary comparing the three years.

3 Forty students enter a school javelin competition. The distances thrown are recorded below.

Distance thrown (m)	$0 \leqslant d < 20$	$20 \leqslant d < 40$	$40 \leqslant d < 60$	$60 \leqslant d < 80$	$80 \leqslant d < 100$
Frequency	4	9	15	10	2

  a Construct a cumulative frequency table for the above results.
  b Draw a cumulative frequency diagram.
  c If the top 20% of students are considered for the final, estimate (using the cumulative frequency diagram) the qualifying distance.
  d Calculate the interquartile range of the throws.
  e Calculate the median distance thrown.

4 The masses of two different types of orange are compared. Eighty oranges are randomly selected from each type and weighed. The results are shown in the table.

Type A			Type B	
Mass (g)	Frequency		Mass (g)	Frequency
$75 \leqslant m < 100$	4		$75 \leqslant m < 100$	0
$100 \leqslant m < 125$	7		$100 \leqslant m < 125$	16
$125 \leqslant m < 150$	15		$125 \leqslant m < 150$	43
$150 \leqslant m < 175$	32		$150 \leqslant m < 175$	10
$175 \leqslant m < 200$	14		$175 \leqslant m < 200$	7
$200 \leqslant m < 225$	6		$200 \leqslant m < 225$	4
$225 \leqslant m < 250$	2		$225 \leqslant m < 250$	0

  a Construct a cumulative frequency table for each type of orange.
  b Draw a cumulative frequency diagram for each type of orange.

c Calculate the median mass for each type of orange.

d Using your diagrams, estimate for each type of orange:

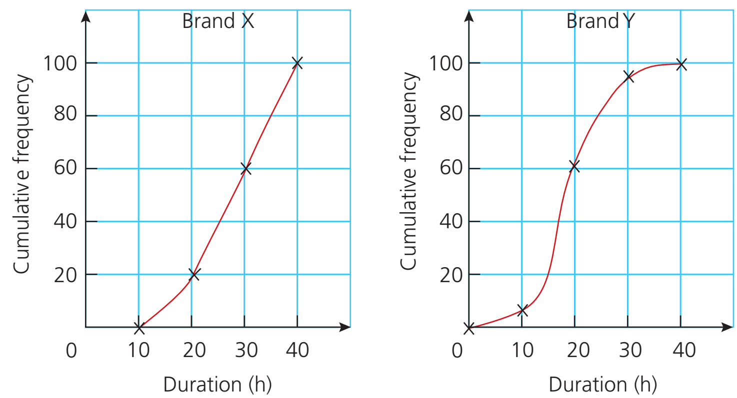
  i the lower quartile,

  ii the upper quartile,

  iii the interquartile range.

e Write a brief report, comparing the two types of orange. You may wish to draw a box-and-whisker plot to help you.

5 Two competing brands of battery, X and Y, are compared. One hundred batteries of each brand are tested and the duration of each battery is recorded. The results of the tests are shown in the cumulative frequency diagrams below.



a The manufacturers of brand X claim that on average their batteries will last at least 40% longer than those of brand Y. Showing your method clearly, decide whether this claim is true.

b The manufacturers of brand X also claim that their batteries are more reliable than those of brand Y. Is this claim true? Show your working clearly.

Student assessment 1

1 Thirty students sit a maths exam. Their marks are given as percentages and are shown in the table below.

Mark (%)	$20 \leqslant m < 30$	$30 \leqslant m < 40$	$40 \leqslant m < 50$	$50 \leqslant m < 60$	$60 \leqslant m < 70$	$70 \leqslant m < 80$	$80 \leqslant m < 90$	$90 \leqslant m < 100$
Frequency	2	3	5	7	6	4	2	1

a Construct a cumulative frequency table of the above results.

b Draw a cumulative frequency diagram of the results.

c Using the diagram, estimate a value for:

  i the median,

  ii the upper and lower quartiles,

  iii the interquartile range.

2 Four hundred students sit an exam. Their marks (as percentages) are shown in the table below.

Mark (%)	Frequency	Cumulative frequency
$31 \leq m < 40$	21	
$41 \leq m < 50$	55	
$51 \leq m < 60$	125	
$61 \leq m < 70$	74	
$71 \leq m < 80$	52	
$81 \leq m < 90$	45	
$91 \leq m < 100$	28	

a Copy and complete the above table by calculating the cumulative frequency.
b Draw a cumulative frequency diagram of the results.
c Using the diagram, estimate a value for:
  i the median result,
  ii the upper and lower quartiles,
  iii the interquartile range.

3 Eight hundred students sit an exam. Their marks (as percentages) are shown in the table below.

Mark (%)	Frequency	Cumulative frequency
$1 \leq m < 10$	10	
$11 \leq m < 20$	30	
$21 \leq m < 30$	40	
$31 \leq m < 40$	50	
$41 \leq m < 50$	70	
$51 \leq m < 60$	100	
$61 \leq m < 70$	240	
$71 \leq m < 80$	160	
$81 \leq m < 90$	70	
$91 \leq m < 100$	30	

a Copy and complete the above table by calculating the cumulative frequency.
b Draw a cumulative frequency diagram of the results.
c An A grade is awarded to a student at or above the 80th percentile. What mark is the minimum requirement for an A grade?
d A C grade is awarded to any student between and including the 55th and the 70th percentile. What marks form the lower and upper boundaries of a C grade?
e Calculate the interquartile range for this exam.

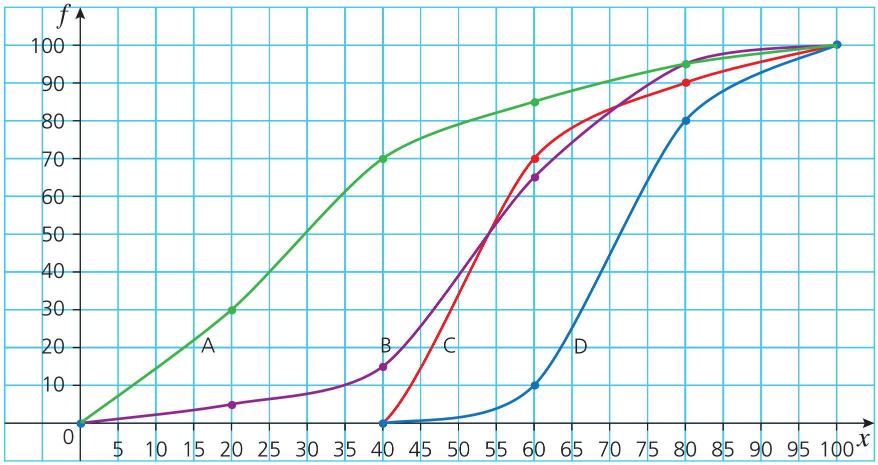
4 Identify which graph(s) correspond with each of the following statements.

a The graph with the highest lower quartile.

b The graph with the largest interquartile range. Justify your answer.

c The graphs with the same 84th percentile. Justify your answer.

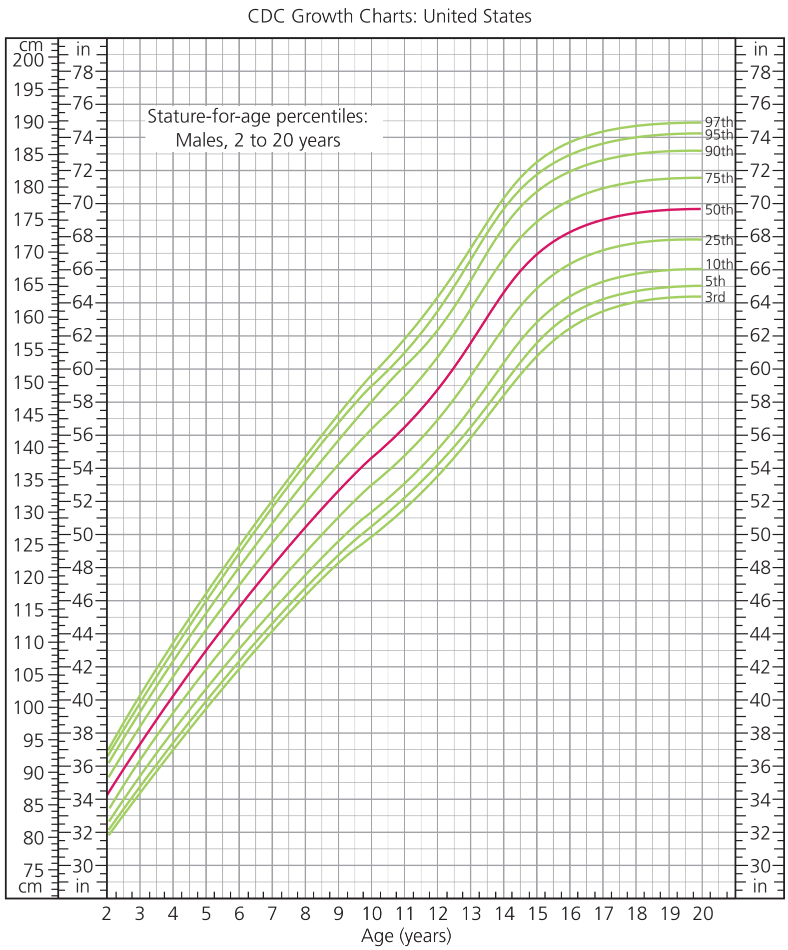
d The graphs with the greatest range.



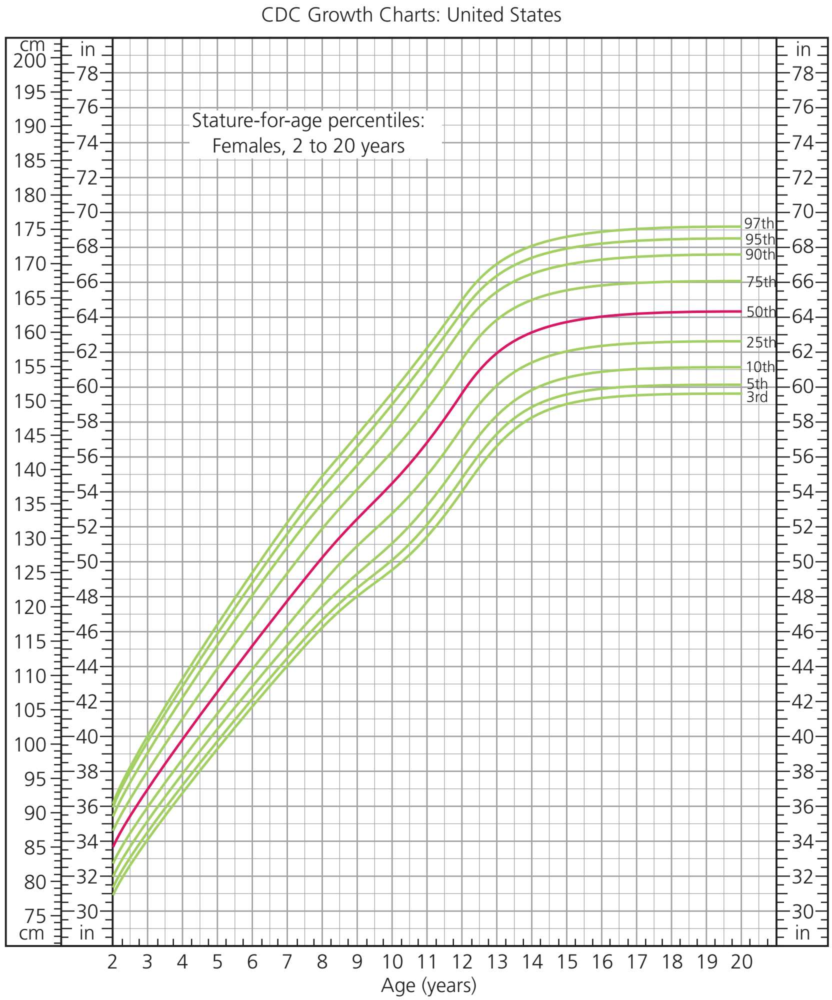
9 Mathematical investigations and ICT 9

Heights and percentiles

The graphs below show the height charts for males and females from the age of 2 to 20 years in the United States.



Note: Heights have been given in both centimetres and inches.



1 From the graph, find the height corresponding to the 75th percentile for 16-year-old females.

2 Find the height which 75% of 16-year-old males exceed.

3 What is the median height for 12-year-old females?

4 Measure the heights of students in your class. By carrying out appropriate statistical calculations, write a report comparing your data to that shown in the graphs.

5 Would all cultures use the same height charts? Explain your answer.

Reading ages

Depending on their target audience, newspapers, magazines and books have different levels of readability. Some are easy to read and others more difficult.

1 Decide on some factors that you think would affect the readability of a text.
2 Write down the names of two newspapers which you think would have different reading ages. Give reasons for your answer.

There are established formulas for calculating the reading age of different texts. One of these is the Gunning Fog Index. It calculates the reading age as follows:

Reading age $= \frac{2}{5}\left(\frac{A}{n} + \frac{100L}{A}\right)$ where

A = number of words

n = number of sentences

L = number of words with three or more syllables.

3 Select one article from each of the two newspapers you chose in Question 2. Use the Gunning Fog Index to calculate the reading ages for the articles. Do the results support your predictions?
4 Write down some factors which you think may affect the reliability of your results.

ICT activity

In this activity you will be collecting the height data of all the students in your class and then plotting a cumulative frequency diagram of the results.

1 Measure the heights of all the students in your class.
2 Group your data appropriately.
3 Enter your data into graphing software such as Excel or Autograph.
4 Produce a cumulative frequency diagram of the results.
5 From your graph, find:
  a the median height of the students in your class,
  b the interquartile range of the heights.
6 Compare the cumulative frequency diagram from your class with one produced from data collected from another class in a different year group. Comment on any differences or similarities between the two diagrams.

Glossary

= = means is equal to. For example, $3 + 4 = 7$.

≠ ≠ means is not equal to. For example, $3 + 4 \neq 8$.

> > means is greater than. For example, $8 > 3 + 4$.

< < means is less than. For example, $3 + 4 < 8$.

⩾ ⩾ means is greater than or equal to. For example, $x \geqslant 5$ means x is any number greater than or equal to 5.

⩽ ⩽ means is less than or equal to. For example, $x \leqslant 5$ means x is any number less than or equal to 5.

∈ ∈ means is an element of. So $e \in S$ means the element e belongs to the set S.

∉ ∉ means is NOT an element of. So $e \notin S$ means the element e does not belong to the set S.

⊆ ⊆ means is a subset of. So $X \subseteq Y$ means X is a subset of Y.

⊈ ⊈ means is NOT a subset of. So $X \nsubseteq Y$ means X is not a subset of Y.

$A \cap B$ $A \cap B$ means all the elements that belong to BOTH set A and set B. $A \cap B$ denotes the elements that are in the intersection of A and B on a Venn diagram.

$A \cup B$ The union of sets A and B, $A \cup B$, is all the elements that belong to EITHER set A OR set B OR both sets.

n(A) The number of elements in set A.

A' The complement of set A.

$A \subseteq B$ A is a subset of B.

$A \nsubseteq B$ A is not a subset of B.

ℰ The universal set, ℰ, for any particular problem is the set which contains all the possible elements for that problem.

∅ The empty set is a set with no elements. It is written as ∅ .

12-hour clock 12-hour clock is when the day is split into two halves 'am' and 'pm. The times before 12 noon are written using am and times after 12 noon are written as pm.

24-hour clock 24-hour clock is when the time is given as the number of hours that have passed since midnight. The hours part of the time is given two digits. For example, 01 30 is 1.30 am and 13 30 is 1.30 pm.

accuracy The accuracy of a measurement tells you how close the measurement is to the true value. For example, if you measure a pencil correct to the nearest centimetre, your measurement will be within 0.5 cm of the true measurement.

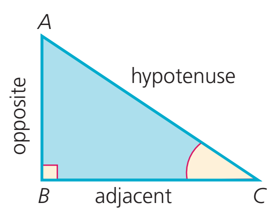
acute angle An acute angle lies between 0° and 90°.

acute-angled isosceles An acute-angled isosceles triangle has two equal angles and two sides of equal length, and all three angles are less than 90°.

acute-angled triangle In an acute-angled triangle, all three angles are less than 90°.

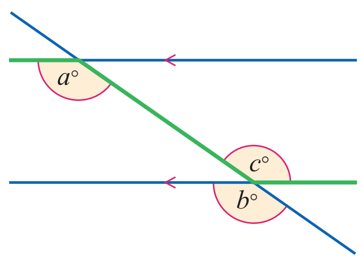
addition Addition is one of the four operations: addition, subtraction, multiplication and division. It means to find the total or sum of two or more numbers or quantities.

adjacent In a right-angled triangle, the adjacent is the side which is next to the angle.



algebraic fraction An algebraic fraction is a fraction with a denominator that is an algebraic expression. For example, $\frac{3}{x}$ or $\frac{1}{x+3}$.

alternate angles Alternate angles are formed when a line crosses a pair of parallel lines. Alternate angles are equal. Look for a Z shape.



altitude The altitude of a triangle is the perpendicular height.

angle of depression The angle of depression is the angle below the horizontal through which a line of view is lowered.

angle of elevation The angle of elevation is the angle above the horizontal through which a line of view is raised.

angles at a point The angles at a point add up to 360°.

angles on a straight line The angles on a straight line add up to 180°.

apex The apex of a pyramid is the point where the triangular faces of the pyramid meet.

arc An arc is part of the circumference of a circle between two radii. When the angle between the two radii of length r is x, then: arc length $= \frac{x}{360} \times 2\pi r$

area The area of a shape is the amount of surface that it covers. Area is measured in mm^2, cm^2, m^2, km^2, etc.

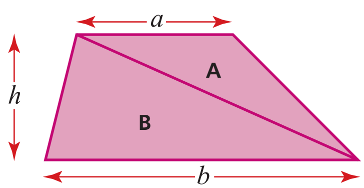
area factor When shape A is an enlargement by scale factor k of shape B, then the area factor is k2.

area of a circle The area, A, of a circle of radius r is: $A = \pi r^2$

area of a parallelogram The area, A, of a parallelogram of base length b and perpendicular height h is: $A = bh$

area of a rectangle The area, A, of a rectangle of length l and breadth b is: $A = lb$

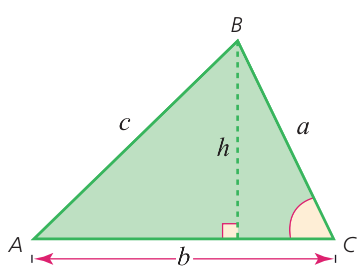
area of a trapezium The area, A, of a trapezium is: $A = \frac{1}{2}h(a+b)$



area of a sector The area of a sector is given by: $\frac{\varnothing}{360} \times \pi r^2$

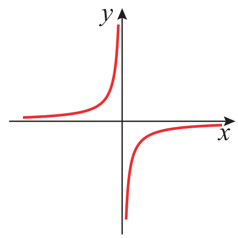
area of a triangle The area, A, of a triangle of base b and perpendicular height h is: $A = \frac{1}{2}bh$

area of any triangle The area of any triangle is given by: area $= \frac{1}{2}ab\sin C$



arithmetic sequence An arithmetic sequence is a sequence where the difference between any two terms is a constant.

asymptote An asymptote is a line that a graph tends towards but does not meet. Here the x-axis and y-axis are both asymptotes:



average An average is a measure of the typical value in a data set. There are three measures: mean, mode and median.

average speed average speed $= \frac{\text{total distance}}{\text{total time}}$

back bearing If the bearing of B from A is given, then the back bearing is the bearing of A from B. It is the bearing that takes you from B back to A. The back bearing is in the reverse direction to the original bearing – it represents the direction of the return journey. (*See* bearing)

bar chart A bar chart is a chart that uses rectangular bars to display data. The height of each bar represents the frequency.

base The base of a triangle is one of its sides. Any side can be the base, but the height must be measured perpendicular to the chosen base.

basic pay Basic pay is the fixed pay that an employee is given for working a certain number of hours.

basic week A basic week is the fixed number of hours that an employee is expected to work each week.

bearing A bearing is a direction. It is the angle measured clockwise from North. Bearings are given as 3 figures so, for example, for an angle of 45° the three-figure bearing is 045°.

bisect Bisect means to divide in half.

bonus A bonus is an extra payment that is sometimes added to an employee's basic pay.

breadth The breadth of a rectangle is the measure of its shortest side.

centre of enlargement The centre of enlargement is a specific point about which an object is enlarged.

centre of rotation The centre of rotation is a specific 'pivot' point about which an object is rotated.

chord A chord is a straight line that joins two points on the circumference of a circle.

circumference The circumference is the perimeter of a circle.

circumference of a circle The circumference, C, of a circle of radius r is: $C = 2\pi r$

column vector A column vector describes the movement of an object in both the x direction and the y direction.

common difference The common difference, d, is the difference between one term and the next in an arithmetic sequence.

common ratio The common ratio, r, is the ratio between one term and the next in a geometric sequence.

complement The complement of set A is the set of elements which are in $\mathscr{E}$ but not in A. The complement of A is written as A'.

complementary angle Two angles which add together to total 90° are called complementary angles.

composite bar chart A composite bar chart shows the bars stacked on top of each other.

composite function A composite function is when one function is applied to the results of another function. fg(x) means apply function g first, and then apply function f to the result.

compound interest Compound interest is interest that is paid not only on the principal amount, but also on the interest itself. So the amount of interest earned each year increases.

compound measure A compound measure is one made up of two or more other measures.

compound shape A compound shape is a shape that can be split into simpler shapes.

cone A cone is a like a pyramid, but with a circular base.

congruent Congruent shapes are exactly the same shape and size – they are identical.

constant of proportionality When two quantities, x and y, are in direct proportion, $\frac{y}{x} = k$ (or $y = kx$), where k is the constant of proportionality.

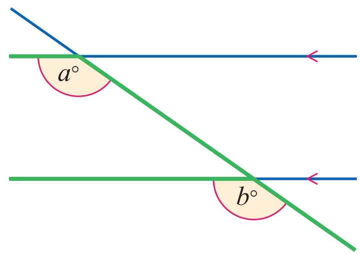
construction A construction is an accurate drawing made using a ruler and a pair of compasses.

continuous data Continuous data is numerical data that can take on any value in a certain range. For example, height and weight (mass) are continuous data.

conversion graph A conversion graph is a straight-line graph used to convert one set of units to another.

correlation Correlation is the relationship between two sets of data.

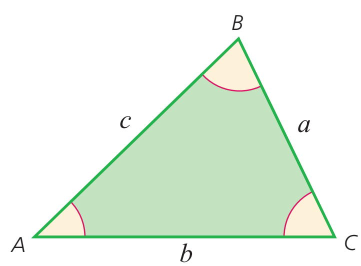
corresponding angles Corresponding angles are formed when a line crosses a pair of parallel lines. Corresponding angles are equal. Look for an F shape.



cosine The cosine of an angle, cos x, in a right-angled triangle is the ratio of the adjacent side to the hypotenuse: $\cos x = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$

cosine rule The cosine rule is:

$a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.



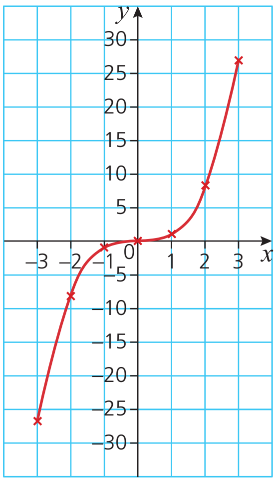
cost price The cost price is the total amount of money that it costs to produce a good or service, before any profit is made.

cube number A cube number is the result when an integer is multiplied by itself twice. The cube numbers are 1, 8, 27, 64, 125, …

cube root The cube root of a number is the number which when multiplied by itself twice gives the original number. The inverse of cubing is cube rooting. For example, the cube root of 27 is 3 (as $3 \times 3 \times 3 = 27$). The symbol $\sqrt[3]{\ }$ is used for the cube root of a number, so $\sqrt[3]{27} = 3$.

cubic A cubic function has the form $f(x) = ax^3$

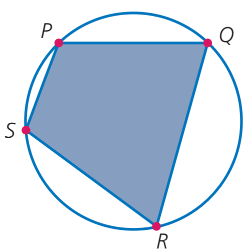
cubic curve A cubic curve is the graph of a cubic function.



cuboid A cuboid is a prism with a rectangular cross-section.

cumulative frequency The cumulative frequency is the running total of the frequencies in a data set.

cyclic quadrilateral A cyclic quadrilateral is a quadrilateral whose vertices all lie on the circumference of a circle.



cylinder A cylinder is three-dimensional shape with a constant circular cross-section.

decagon A decagon is a 10-sided polygon.

decimal A decimal is a number with digits after the decimal point. It is a number which is not an integer.

decimal fraction A decimal fraction is a fraction between 0 and 1 in which the denominator is a power of 10 and the numerator is an integer.

decimal place The decimal place is the number of digits after the decimal point. For example, 3.2 has 1 decimal place and 5.678 has 3 decimal places.

denominator The denominator is the bottom line of a fraction; it tells you how many equal parts the whole is divided into. For example, $\frac{3}{8}$ has a denominator of 8, so the 'whole' has been divided into 8 equal parts.

density Density is a measure of the mass of a substance per unit of its volume. It is calculated using the formula: density $= \frac{\text{mass}}{\text{volume}}$

depreciate When the value of something decreases over a period of time, it is said to depreciate.

derivative Differentiating a function produces the derivative (or gradient function).

diameter A diameter is a straight line which passes through the centre of a circle and joins two points on the circumference.

difference of two squares The difference of two squares is an expression in the form $x^2 - y^2$. Note, $x^2 - y^2$ factorises to give $(x + y)(x - y)$.

differentiation Differentiation is the process of finding the derivative or gradient function.

direct proportion Two quantities, x and y, are in direct proportion when the ratio $\frac{y}{x}$ is a constant. So $y = kx$ and the graph of y against x is a straight line passing through the origin. An increase in one quantity causes an increase in the other. For example, when the amount of ingredients is doubled for some cakes, the number of cakes made also doubles.

directed number A directed number is a number that is positive or negative. A number has size (magnitude) and its sign (+ or −) tells you which *direction* to move along a number line from 0 in order to reach that number.

discount An item sold at 10% discount is 10% cheaper than the full selling price.

discrete data Discrete data is numerical data that can only take on certain values, usually whole numbers. For example, the number of peas in a pod is discrete data.

distance between two points The distance, d, between two points (x_1, y_1) and (x_2, y_2) is:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

division Division is one of the four operations: addition, subtraction, multiplication and division. To divide one number by another means to find how many times one number goes into another number. For example,

$20 \div 5 = 4$

$5 \div 20 = 0.25$

double time Overtime is often paid at a higher rate. When overtime is paid at twice the basic pay, it is called double time.

dual bar chart A dual bar chart shows the bars side by side.

element An object or symbol in a set is called an element.

elevation An elevation is a two-dimensional view of a three-dimensional object. A side elevation is the view from one side of the object and the front elevation is the view from the front.

elimination method The elimination method is a method for solving simultaneous equations. One of the unknowns is eliminated by either adding or subtracting the pair of equations.

enlargement An enlargement changes the size of an object. When a shape is enlarged, the image is mathematically similar to the object but is a different size. Note: the image may be larger or smaller than the original object.

equation An equation says that one expression is equal to another. For example, $6 + 4 = 16 - 6$. When an expression contains an unknown, it can be solved. For example, the solution to the equation $x + 4 = 16 - x$ is $x = 6$.

equation of a straight line The equation of a straight line can be written in the form $y = mx + c$, where m is the gradient of the line and the y-intercept is at $(0, c)$.

equilateral triangle An equilateral triangle has three equal angles (all 60°) and three sides of equal length.

equivalent fraction Equivalent fractions have the same decimal value. For example, $\frac{3}{5} = 0.6$ and $\frac{9}{15} = 0.6$, so $\frac{3}{5}$ and $\frac{9}{15}$ are equivalent.

estimate / estimation Estimation is a way of working out the approximate answer or estimate to a calculation. The numbers in the calculation are rounded (usually to 1 significant figure) so that the calculation is easier to work out without a calculator. Estimation is useful for checking a calculation.

evaluate Evaluate means to work out the value of something.

expand Expand means to multiply out or remove the brackets.

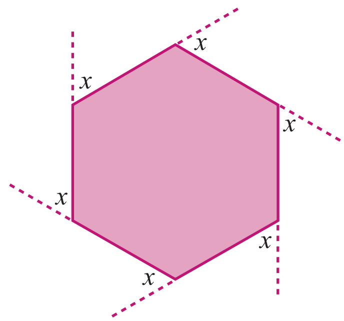
exponential An exponential expression is in the form a^x where a is a positive constant and x is a variable.

exponential equation An exponential equation has a variable (unknown) as the index. For example, $y = 2^x$ is an exponential equation.

exponential function An exponential function is in the form $f(x) = a^x$, where a is a positive constant.

exponential sequence An exponential sequence is a sequence where there is a common ratio (r) between successive terms. The nth term can be written as $T_n = ar^{n-1}$, where a is the first term.

exterior angle The exterior angle of an n-sided regular polygon $= \frac{360°}{n}$.



factor A factor of a number divides into that number exactly. For example, the factors of 18 are 1, 2, 3, 6, 9 and 18.

factorise Factorise means to remove common factors and write an equivalent expression using brackets. For example, $2x - 6$ factorises to give $2(x - 3)$.

favourable outcome A favourable outcome refers to the event in question (for example, getting a 6 when a dice is thrown) actually happening.

fraction A fraction represents a part of a whole.

frequency Frequency is the number of times a particular outcome happens.

frequency density frequency density $= \frac{\text{frequency}}{\text{class width}}$

frequency table A frequency table shows the frequency of each data value in a data set.

frustrum A frustrum is the base part of a cone or pyramid when the top of the cone or pyramid is removed.

geometric sequence A geometric sequence is a sequence where the ratio between any two terms is a constant.

gradient Gradient is a measure of how steep a line is. The gradient of the line joining the points (x_1, y_1) and (x_2, y_2) is given by: gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

gradient-intercept form The gradient-intercept form of the equation of a straight line is the form $y = mx + c$, where m is the gradient and c is the y-intercept.

gross earnings Gross earnings are the total earnings *before* all the deductions such as tax, insurance and pension contributions are made.

grouped frequency table A grouped frequency table is a method of displaying a large data set so that it is easier to handle.

height The height of a triangle is the perpendicular distance from its base to its third vertex.

hemisphere A hemisphere is made when a sphere is cut into two congruent halves. It is a half a sphere.

hexagon A hexagon is a 6-sided polygon.

highest common factor The highest common factor (HCF) of two numbers is the greatest integer that divides exactly into both numbers. For example, the highest common factor of 6 and 15 is 3.

histogram A histogram is a chart used to display grouped continuous data as bars. Both axes have continuous scales, and the vertical axis shows the frequency density of each bar.

hyperbola The graph of a reciprocal function in the form $y = \frac{k}{x}$, where k is a constant, is a hyperbola.

hypotenuse The hypotenuse is the longest side of a right-angled triangle.

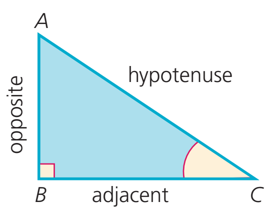


image When an object undergoes a transformation, the resulting position or shape is the image.

improper fraction (or vulgar fraction) In an improper fraction, the numerator is more than the denominator. For example, $\frac{8}{3}$ is an improper fraction.

index The index is the power to which a number is raised. For example, in 4^3 the power (or index) is 3 and so $4^3 = 4 \times 4 \times 4$.

inequality An inequality says that one expression is not equal to a second expression. For example, $x + 2 < 8$ or $7 > 6$

integer An integer is a positive or negative whole number (including zero). The set of integers is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

interest Interest is the money added by a bank to a sum deposited by a customer. Interest is also the money charged by a bank for a loan to a customer. It can be either simple interest or compound interest (*see* simple interest *and* compound interest).

interior angle The sum of the interior angles of an n-sided polygon is $180(n - 2)°$.

interquartile range interquartile range = upper quartile – lower quartile

intersection The intersection of two sets is the elements that are common to both sets. It is represented by the symbol $\cap$.

inverse of a function The inverse of a function is its reverse, i.e. it 'undoes' the function's effects. The inverse of the function f(x) is written $f^{-1}(x)$.

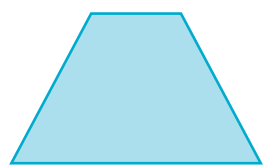
inverse proportion Two quantities, x and y, are in inverse proportion when the product of the two quantities xy is constant, i.e. when an increase in one quantity causes a decrease in the second quantity.

irrational number An irrational number is any number (positive or negative) that cannot be written as a fraction. Any decimal which neither terminates nor recurs is irrational. The square root of any number other than square numbers is also irrational. Some examples of irrational numbers are π, $\sqrt{2}$ and $\sqrt{10}$.

irregular polygon An irregular polygon is a polygon that does not have equal sides or equal angles.

isosceles trapezium An isosceles trapezium is a quadrilateral with the following properties:

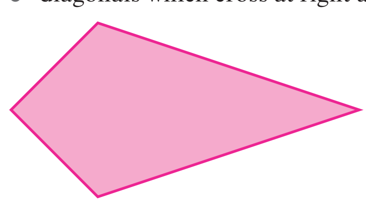
- one pair of parallel sides
- the other pair of sides are equal in length
- two pairs of equal base angles
- opposite base angles add up to 180°.



isosceles triangle An isosceles triangle has two equal angles and two sides of equal length.

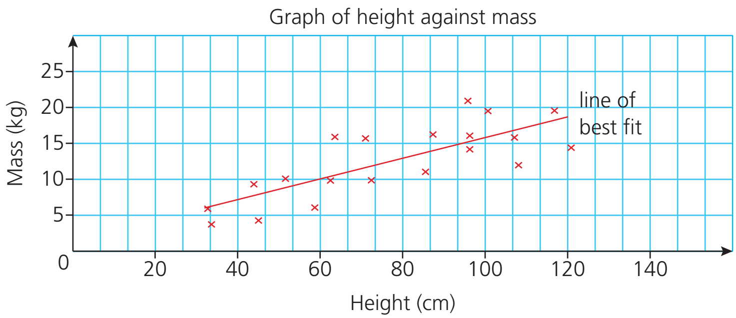
kite A kite is a quadrilateral with the following properties:

- two pairs of equal sides
- one pair of equal angles
- diagonals which cross at right angles.



laws of indices The laws of indices are:

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n$ or $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^1 = a$
- $a^0 = 1$
- $a^{-m} = \frac{1}{a^m}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \sqrt[n]{(a^m)}$ or $\left(\sqrt[n]{(a)}\right)^m$



line of symmetry A line of symmetry divides a two-dimensional shape into two congruent (identical) shapes.

line segment A line segment is part of a line.

linear A linear function is a function whose graph is a straight line. It is of the form $f(x) = ax$.

linear equation The graph of a linear equation is a straight line. The highest power of the variable is 1.

linear function The graph of a linear function is a straight line.

linear inequality An inequality says that one expression is not equal to a second expression. For example, $x + 2 < 8$. In a linear inequality, the highest power of the variable is 1.

local maxima A local maxima of the function f is a stationary point where $f(x)$ reaches a maximum within a given range.

local minima A local minima of the function f is a stationary point where $f(x)$ reaches a minimum within a given range.

loss When an item is sold for less than it cost to make, it is sold at a loss: loss = cost price – selling price

lower bound Measurement is only approximate; the actual value of a measurement could be half the rounded unit above or below the given value. The lower bound is the least possible value that the true measurement could be. For example, the length of a pencil is 15.5 cm to the nearest millimetre. So the lower bound is 15.5 cm – 0.5 mm = 15.45 cm. The actual length, l, of the pencil is greater than or equal to 15.45 cm, so $15.45 \leqslant l$.

length The length of a rectangle is the measure of its longest side.

line A line is a one-dimensional object with length but no width. It has infinite length.

line of best fit A line of best fit is a straight line that passes as close as possible to as many points as possible on a scatter diagram.

lower quartile The lower quartile is the 25th percentile.

lowest common multiple The lowest common multiple (LCM) of two numbers is the lowest integer that is a multiple of both numbers. For example, the lowest common multiple of 6 and 15 is 30.

lowest terms (or simplest form) A fraction is in its lowest terms when the highest common factor of the numerator and denominator is 1. In other words, the fraction cannot be cancelled down any further. For example, $\frac{30}{45} = \frac{6}{9} = \frac{2}{3}$, so $\frac{2}{3}$ is a fraction in its lowest terms.

magnitude Magnitude means size.

mean The mean is found by adding together all of the data values and then dividing this total by the number of data values. The mean is one of the three ways to measure an average.

median The median is the middle value when the data set is organised in order of size. The median is one of the three ways to measure an average.

metric units of capacity
1 litre (l) = 1000 millilitres (ml)   1 ml = 1 cm^3

metric units of length
1 kilometre (km) = 1000 metres (m)
1 metre (m) = 100 centimetres (cm)
1 centimetre (cm) = 10 millimetres (mm)

metric units of mass 1 tonne (t) = 1000 kilograms (kg) 1 kilogram (kg) = 1000 grams (g) 1 gram (g) = 1000 milligrams (mg)

midpoint The midpoint of a line segment AB, where $A(x_1, y_1)$ and $B(x_2, y_2)$ is: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

mirror line The mirror line is the line in which an object is reflected.

mixed number A mixed number is made up of a whole number and a proper fraction. For example, $2\frac{3}{8}$ is a mixed number.

modal class The modal class is the class or group in a grouped frequency table with highest frequency.

mode The mode is the value occurring most often in a data set. The mode is one of the three ways to measure an average.

multiple The multiple of a number is the result when you multiply that number by a positive integer. For example, the multiples of 6 are 6, 12, 18, 24, 30, …

multiplication Multiplication is one of the four operations: addition, subtraction, multiplication and division. Multiplication is repeated addition, so 3 multiplied by 4 means 3 + 3 + 3 + 3.

natural number A natural number is a whole number (integer) that is used in counting. Natural numbers start at zero and continue 0, 1, 2, 3, … The set of natural numbers is {…, –3, –2, –1, 0, 1, 2, 3, …}.

negative correlation Two quantities have negative correlation if, in general, one decreases as the other increases.

negative enlargement In a negative enlargement, the object and image are on opposite sides of the centre of enlargement. The scale factor of enlargement is negative.

negative number A negative number is any number less than 0.

net A net is a two-dimensional shape which can be folded up to form a three-dimensional shape.

net pay Net pay is sometimes called ‘take-home’ pay. It is the money left *after* all the deductions such as tax, insurance and pension contributions are made.

no correlation Correlation is the relationship between two sets of data. If there is no correlation, then there is no relationship between the two data sets. In a scatter graph showing no correlation, there is no pattern in the plotted points.

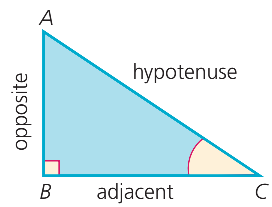
numerator The numerator is the top line of a fraction. It represents the number of equal parts of the whole. For example, $\frac{3}{8}$ has a numerator of 3, so there are 3 equal parts and each part is equal to $\frac{1}{8}$ of the ‘whole’.

obtuse angle An obtuse angle lies between 90° and 180°.

obtuse-angled triangle In an obtuse-angled triangle, one angle is greater than 90°.

octagon An octagon is an 8-sided polygon.

opposite In a right-angled triangle, the opposite side is the one which is opposite the angle.



order of operations When a calculation contains a mixture of brackets and/or the operations (×, ÷, + and –), the order that the operations should be carried out in is:

- First work out any … Brackets and Indices
- … then carry out any … Multiplication and Division
- … finally Addition and Subtraction

When a calculation contains operations of equal priority (e.g. + and –, or × and ÷), work from left to right. For example, 10 – 7 + 2 = 3 + 2 = 5.

order of rotational symmetry The order of rotational symmetry is the number of times a shape, when rotated about a central point, fits its outline during a complete revolution of 360°.

origin The origin is the point at which the x-axis and the y-axis meet.

overtime Overtime is any hours worked in excess of the basic week.

parabola The graph of a quadratic function is a parabola.

parallel A pair of parallel lines can be continued to infinity in either direction without meeting. Parallel lines have the same gradient.

parallelogram A parallelogram is a quadrilateral with the following properties:

- two pairs of parallel sides
- opposite sides are equal
- opposite angles are equal.

pentagon A pentagon is a 5-sided polygon.

per cent (%) Per cent means parts per 100.

percentage A percentage is the number of parts per 100.

percentage interest (or interest rate) Interest is earned on a fixed percentage of the principal. The interest rate gives the percentage interest earned.

percentage loss percentage loss $= \frac{\text{loss}}{\text{cost price}} \times 100\%$

percentage profit percentage profit $= \frac{\text{profit}}{\text{cost price}} \times 100\%$

percentile The cumulative frequency can be divided into percentiles. The maximum value of the cumulative frequency is the 100th percentile.

perfect square A quadratic equation is called a perfect square if it is in the form $y = x^2 + 2ax + a^2$, where a is a constant. This factorises to give $y = (x + a)^2$.

perimeter The perimeter of a shape is the distance around the outside edge of the shape. Perimeter is measured in mm, cm, m, km, etc.

perimeter of a rectangle The perimeter of a rectangle of length l and breadth b is: $2l + 2b$

perpendicular Two lines are perpendicular if they meet at right angles. The product of the gradients of two perpendicular lines is –1. So if the gradient of a line is m_1, then the gradient of a line perpendicular to it is: $m_2 = -\frac{1}{m_1}$

perpendicular bisector The perpendicular bisector of a line AB is another line which meets AB at right angles and cuts AB exactly in half.

pictogram A pictogram is a chart that uses pictures or symbols to display data.

pie chart A pie chart is a circular chart divided into sectors that is used to display data. The area of each sector is proportional to the frequency.

piece work Piece work is when an employee is paid for the number of articles made (rather than the time spent working).

plan A plan of an object is a scale diagram of the view from above the object, looking directly down on the object.

plane of symmetry A plane of symmetry divides a three-dimensional shape into two congruent (identical) three-dimensional shapes.

point A point is an exact location or position.

polygon A polygon is a two-dimensional shape made up of straight lines.

polynomials A polynomial function is a function such as a quadratic, cubic, etc. that includes only non-negative powers of x.

population density Population density is a measure of the population per unit of area. It is calculated using the formula:

population density $= \frac{\text{population}}{\text{area}}$

positive correlation Two quantities have positive correlation if, in general, one increases as the other increases.

positive number A positive number is any number greater than 0.

power For example, in 4^3 the power is 3 and so $4^3 = 4 \times 4 \times 4$.

pressure Pressure is a compound measurement and is measured in Pascals (Pa) or N/m^2:

Pressure $= \frac{\text{Force}}{\text{Area}}$

prime factor A prime factor of a number is any factor of that number that is also a prime. For example, the prime factors of 60 are 2, 3 and 5.

prime number A prime number is a number with exactly two factors: one and itself. The prime numbers are 2, 3, 5, 7, 11, … Note: 1 is not a prime number as it only has one factor.

principal The principal is the amount of money deposited by a customer in a bank account.

prism A prism is a three-dimensional object with a constant cross-sectional area.

probability Probability is the study of chance. The probability of an event happening is a measure of how likely that event is to happen. Probability is given on a scale of 0 (an impossible event) to 1 (a certain event):

probability $= \frac{\text{number of favourable outcomes}}{\text{total number of equally likely outcomes}}$

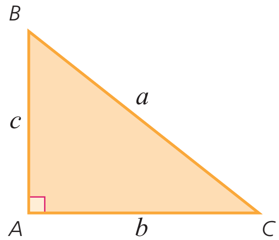
probability scale A probability scale is a scale that indicates how likely an event is, ranging from impossible to certain.

profit When an item is sold for more than it cost to make, it is sold at a profit: profit = selling price – cost price

proper fraction In a proper fraction, the numerator is less than its denominator. For example, $\frac{3}{8}$ is a proper fraction.

pyramid A pyramid is a three-dimensional shape. It has a polygon for a base and the other faces are triangles which meet at a common vertex, called the apex.

Pythagoras' theorem Pythagoras' theorem states the relationship between the lengths of the three sides of a right-angled triangle. Pythagoras' theorem is: $a^2 = b^2 + c^2$



quadratic equation A quadratic equation can be written in the form $y = ax^2 + bx + c$, where a, b and c are constants.

quadratic expression In a quadratic expression, the highest power of any of the terms is 2. A quadratic expression can be written in the form $ax^2 + bx + c$, where a, b and c are constants.

quadratic formula The quadratic formula is used to solve a quadratic equation in the form $ax^2 + bx + c = 0$. The formula is: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

quadratic function A quadratic function is in the form $y = ax^2 + bx + c$, where a, b and c are constants.

quadrilateral A quadrilateral is a 4-sided polygon.

radius A radius is a straight line which joins the centre of a circle to a point on the circumference.

range Range is a measure of the spread of a data set. The range is the difference between the largest and smallest data values.

rate Rate is a ratio of two measurements, usually the second measurement is time. For example, water flows through a pipe at a rate of 1 litre per second or a computer programmer types at a rate of 30 words per minute.

ratio A ratio is the comparison of one quantity with another.

ratio method The ratio method is used to solve problems involving direct proportion by comparing the ratios. For example, a bottling machine fills 500 bottles in 15 minutes. How many bottles will it fill in 90 minutes?

$\frac{x}{90} = \frac{500}{15}$ so $x = \frac{500 \times 90}{15} = 3000$

3000 bottles are filled in 90 minutes.

rational number A rational number is any number (positive or negative) that can be written as a fraction. All integers and all terminating and recurring decimals are rational numbers.

real number The real numbers are the all the rational and irrational numbers. So any integer, fraction or decimal is a real number.

reciprocal The reciprocal of a number is 1 divided by that number. So the reciprocal of 4 is $1 \div 4 = \frac{1}{4} = 0.25$ and the reciprocal of $\frac{1}{5}$ is $1 \div \frac{1}{5} = 5$.

reciprocal function A reciprocal function is in the form $y = \frac{k}{x}$, where k is a constant.

rectangle A rectangle is a quadrilateral with the following properties:

- two pairs of parallel sides
- opposite sides are equal
- four equal angles (each 90°).

recurring decimal A recurring decimal has digits that repeat forever. For example, $\frac{2}{9} = 0.2222\ldots = 0.\dot{2}$ and $\frac{415}{999} = 0.415415415415\ldots = 0.\dot{4}\dot{1}\dot{5}$

reflection A reflection is a 'flip' movement in a mirror line. The mirror line is the line of symmetry between the object and its image.

reflective symmetry A shape has reflective symmetry if it has one or more lines or planes of symmetry.

reflex angle A reflex angle lies between 180° and 360°.

region A region is a part of a graph, shape or Venn diagram.

regular polygon A regular polygon has all sides of equal length and all angles of equal size.

relative frequency

relative frequency = $\frac{\text{number of successful trials}}{\text{total number of trials}}$

rhombus A rhombus is a quadrilateral with the following properties:

- two pairs of parallel sides
- four equal sides
- opposite angles are equal
- diagonals which cross at right angles.

right angle A right angle is 90°.

right-angled triangle In a right-angled triangle, one angle is 90°.

roots (of an equation) The root(s) of an equation are the value(s) of x when $y = 0$. On a graph, these are the values of x where the curve crosses the x-axis.

rotation A rotation is a 'turning' movement about a specific point known as the centre of rotation.

rotational symmetry A shape has rotational symmetry if, when rotated about a central point, it fits its outline more than once in a complete turn.

round (or rounding) Rounding is a way of rewriting a number so it is simpler than the original number. A rounded number should be approximately equal to the unrounded (exact) number and be of the same order of magnitude (size). Rounded numbers are often given to 2 decimal places (2 d.p.) or 3 significant figures (3 s.f.), for example.

sample space diagram A sample space diagram shows all the possible outcomes of an experiment.

scalar A scalar is a quantity with magnitude (size) only.

scale A scale on a drawing shows the ratio of a length on the drawing to the length on the actual object.

scale factor of enlargement The scale factor of enlargement is the ratio between corresponding sides on an object and its image.

scalene triangle In a scalene triangle, none of the angles are of equal size and none of the sides are of equal length.

scatter diagram A scatter diagram is a graph of plotted points which shows the relationship between two variables.

sector A sector is the region of a circle enclosed by two radii and an arc.

segment A segment is an area of a circle formed by a line (chord) and an arc.

selling price The selling price is the total amount of money that an item is sold for.

semicircle A semicircle is made when a circle is cut into two congruent halves. A semicircle is half a circle.

sequence A sequence is a collection of terms arranged in a specific order, where each term is obtained according to a rule.

set A set is a well-defined group of objects or symbols.

significant figures The first significant figure of a number is the first non-zero digit in the number. The second significant figure is the next digit in the number, and so on. For example, in the numbers 78 046 and 0.0078 046 the first significant figure is 7, the second significant figure is 8 and the third significant figure is 0.

similar Two shapes are similar if the corresponding angles are equal and the corresponding sides are in proportion to each other.

simple interest Simple interest is calculated only on the principal (initial) amount deposited in an account. When simple interest is earned, the amount of interest paid is the same each year.

$$\text{simple interest} = \frac{\text{principal} \times \text{time in years} \times \text{rate per cent}}{100}$$

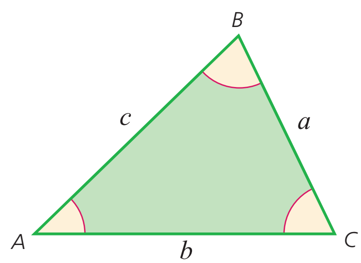
simplest form (or lowest terms) A fraction is in its simplest form when the highest common factor of the numerator and denominator is 1. In other words, the fraction cannot be cancelled down any further. For example, $\frac{30}{45} = \frac{6}{9} = \frac{2}{3}$, so $\frac{2}{3}$ is a fraction in its simplest form.

simultaneous equations Simultaneous equations are a pair of equations involving two unknowns.

sine The sine of an angle, sin x, in a right-angled triangle is the ratio of the side opposite the angle and the hypotenuse.

$$\sin x = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$

sine rule The sine rule is: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



speed $\text{speed} = \frac{\text{distance}}{\text{time}}$ When the speed is not constant: $\text{average speed} = \frac{\text{total distance}}{\text{total time}}$

sphere A sphere is a three-dimensional shape which is a ball.

square A square is a quadrilateral with the following properties:

- two pairs of parallel sides
- four equal sides
- four equal angles (90°)
- diagonals which cross at right angles.

square number A square number is the result when an integer is multiplied by itself. The square numbers are 1, 4, 9, 16, 25, …

square root The square root of a number is the number which when multiplied by itself gives the original number. The inverse of squaring is square rooting. Every number has two square roots, for example, the square root of 9 is 3 (as $3 \times 3 = 9$) and -3 (as $-3 \times -3 = 9$). The symbol $\sqrt{\ }$ is used for the positive square root of a number, so $\sqrt{9} = 3$.

standard form Standard form is a way of writing very large or very small numbers. A number in standard form is written as $A \times 10^n$, where $1 \leq A < 10$ and n is a positive or negative integer.

Examples of numbers in standard from are 5×10^3 and 2.7×10^{-18}.

stationary point A stationary point is a point on a curve where the gradient is zero.

stem-and-leaf diagram A stem and leaf diagram is a diagram where each date value is split into two parts – the 'stem' and the 'leaf' (usually the last digit). The data is then grouped so that data values with the same stem appear on the same line.

straight line A straight line is the shortest distance between two points.

subject The subject of a formula is the single variable (often on the left-hand side of a formula) that the rest of the formula is equal to. For example, in $C = 2\pi r$, C is the subject and in $a^2 + b^2 = c^2$, c2 is the subject.

subset When all the elements of set X are also elements of set Y, then X is a subset of Y. Every set has itself and the empty set as subsets.

substitute / substitution Substitution is replacing the variables (letter symbols) in an expression or formula with numbers.

substitution method The substitution method is a method for solving simultaneous equations, where one unknown is made the subject of one of the equations, and then this expression is substituted into the second equation.

subtraction Subtraction is one of the four operations: addition, subtraction, multiplication and division. It means to take one number away from another.

supplementary angle Two angles that add together to total 180° are called supplementary angles.

surd A surd is a square root or cube root of a number which cannot be simplified by removing the root. A surd is an irrational number. For example, $\sqrt{4}$ is not a surd as $\sqrt{4} = 2$. $\sqrt{5} = 2.2360679...$ is a surd.

surface area of a cuboid The surface area of a cuboid of length l, width w and height h is:
surface area $= 2(wl + lh + wh)$

surface area of a cylinder The surface area of a cylinder of radius r and height h is:
surface area $= 2\pi r(r + h)$

surface area of a sphere The surface area of a sphere is $4\pi r^2$.

tally table A tally table is table where the frequencies of each outcome are recorded using marks like ||| for 3 or ~~||||~~ for 5.

tangent The tangent to a curve at a point is a straight line that just touches the curve at that point. The gradient of the tangent is the same as the gradient of the curve at that point.

tangent (*x*) The tangent of an angle, tan *x*, in a right-angled triangle is the ratio of the sides opposite and adjacent to the angle.

$$\tan x = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$

term Each number in a sequence is a called a term.

terminating decimal A terminating decimal has digits after the decimal point that do not continue forever. For example, 0.123 and 0.987654321.

term-to-term rule A term-to-term rule describes how to use one term in a sequence to find the next term.

three-figure bearing A three-figure bearing is a measure of the direction in which an object is travelling. North is 000° and South is 180°.

time and a half Overtime is often paid at a higher rate. When overtime is paid at 1.5 × basic pay, it is called time and a half.

total number of possible outcomes The total number of possible outcomes refers to all the different types of outcomes one can get in a particular situation.

transformation A transformation changes either the position or size of an object, such as translation, rotation, reflection and enlargement.

translation A translation is a sliding movement. Each point on the object moves in the same way to its corresponding point on the image, as described by its translation vector.

translation vector A translation vector describes a translation in terms of its horizontal and vertical movement. For example, the translation vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ describes a translation of 2 units right and 3 units down.

trapezium A trapezium is a quadrilateral with one pair of parallel sides.

travel graph A travel graph is a diagram showing the journey of one or more objects on the same pair of axes. The vertical axis is distance and the horizontal axis is time.

triangle A triangle is a 3-sided polygon.

triangular numbers The triangular numbers are the numbers in the sequence 1, 3, 6, 10, 15, etc., where the difference between terms increases by 1 each time. The formula for the n^{th} triangular number is: $\frac{1}{2}n(n+1)$

turning point The turning point of a quadratic graph is its highest or lowest point. If the x^2 term is positive, the graph will have a lowest point. If the x^2 term is negative, it will have a highest point.

union The union of two sets is everything that belongs to EITHER or BOTH sets. It is represented by the symbol $\cup$.

unitary method The unitary method is used to solve problems involving direct proportion by first finding the value of a single unit. For example, 5 pens cost \$8. Work out the cost of 7 pens.

5 pens cost \$8

1 pen costs \$8 ÷ 5 = \$1.60 (the cost of 1 unit)

So 7 pens cost \$1.60 × 7 = \$11.20

universal set The universal set for any particular problem is the set which contains all the possible elements for that problem. It is represented by the symbol $\mathscr{E}$.

upper bound Measurement is only approximate; the actual value of a measurement could be half the rounded unit above or below the given value. The upper bound is the greatest value up to which the true measurement can be. For example, the length of a pencil is 15.5 cm to the nearest millimetre. So, the upper bound is the measurement up to, but not equalling, 15.5 cm + 0.5 mm = 15.55 cm. Therefore, the actual length, l, of the pencil is less than 15.55 cm, so $l < 15.55$.

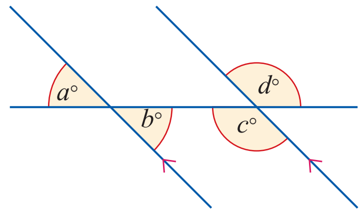
upper quartile The upper quartile is the 75th percentile.

vector A vector is a quantity with both magnitude (size) and direction. A vector can be used to describe the position of one point in space relative to another.

Venn diagram A Venn diagram is a diagram comprising of overlapping circles, which is used to display sets.

vertex (plural: vertices) A vertex of a shape is a point where two sides meet.

vertically opposite angles Vertically opposite angles are formed when two lines cross. Vertically opposite angles are equal.



volume (capacity) The volume of a 3D solid is the amount of space the solid fills.

volume factor When shape A is an enlargement by scale factor k of shape B, the volume factor is k^3.

volume of a cylinder The volume of a cylinder of radius r and height h is given by: $volume = \pi r^2 h$

volume of a cone The volume of a cone with height h and a base of radius r is given by: $volume = \frac{1}{3}\pi r^2 h$

volume of a prism The volume of a prism is given by: volume = area of cross-section × length

volume of a pyramid The volume of pyramid is given by: volume = $\frac{1}{3}$ × area of base × perpendicular height

volume of a sphere The volume of a sphere is given by: $volume = \frac{4}{3}\pi r^3$

***x*-axis** The x-axis is the horizontal axis on a graph.

***y*-axis** The y-axis is the vertical axis on a graph.

Index

D

E

K

L

M

N

O

P

Q

R

S

T

U

V

X

Y

Z

Confidently deliver and ensure full coverage of the latest Cambridge IGCSE™ Mathematics syllabuses (0580/0980) using a tried-and-tested approach to improve mathematical skills with an emphasis on problem-solving and interrogating findings in new topics.

- Put theory into practice with worked examples that show full solutions, plus plenty of exercise questions covering both Core and Extended content, all put together by an experienced team of authors.
- Encourage full understanding of mathematical principles with commentaries and additional explanations.
- Develop problem-solving skills with guidance on techniques to help complete open-ended investigations and justify reasoning for solutions.
- Check understanding with end-of-chapter student assessment questions to consolidate learning and test skills.
- Numerical answers to all questions are available free on **hoddereducation.com/cambridgeextras**. Worked solutions for the student assessments are available in the *Cambridge IGCSE Core and Extended Mathematics Teacher's Guide with Boost subscription.*

This resource is endorsed by Cambridge Assessment International Education

- ✓ Supports the full Cambridge IGCSE™ and IGCSE™ (9–1) Mathematics syllabuses (0580/0980) for examination from 2025
- ✓ Has passed Cambridge International's rigorous quality-assurance process
- ✓ Developed by subject experts
- ✓ For Cambridge schools worldwide

For over 30 years we have been trusted by Cambridge schools around the world to provide quality support for teaching and learning. For this reason we have been selected by Cambridge Assessment International Education as an official publisher of endorsed material for their syllabuses.

Working for over 30 YEARS WITH Cambridge Assessment International Education

Registered Cambridge International Schools benefit from high-quality programmes, qualifications, assessments and a wide range of support so that teachers can effectively deliver in the classroom. Visit ***www.cambridgeinternational.org*** *to find out more.*

HODDER EDUCATION

e: education@hachette.co.uk

w: hoddereducation.com

ISBN 978-1-3983-7391-4

FSC