A bunch of tricks

Pierri Damian, Martínez Julián

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Abstract

The purpose of this note is to show that, in certain environments, it is possible to obtain a recursive representation of a non-optimal general equilibrium model with a finite number of exogenous shocks that has an ergodic invariant measure and a compact and stationary state space. The seminal paper of Duffie, et. al. (1994) shows that in non-optimal economies with a finite number of exogenous shocks there is a trade off between the generality of a recursive representation and a well behaved steady state, which is defined by an ergodic invariant measure of an stationary Markov Process. The authors "convexified" the state space using "sunspots" in order to prove the ergodicity of the measure. By enlarging the number of variables in the state space, this paper shows that it is possible to obtain a continuous markovian representation which allows deriving an ergodic invariant measure using a state space with no sunspots. It also shows that, contrarily to what is claimed in Blume (1982), it is possible to obtain an economy with multiple equilibria and a continuous markovian representation.

1 Introduction

Since the seminal paper of [6] macroeconomists have been concerned with the recursive representation of sequential equilibria in general equilibrium models. This representation is relevant not only for computational purposes but also for theoretical ones. As regards the former, it is easier to numerically approximate a first order stationary dynamic process rather than the sequential representation originally defined. Furthermore, it is possible to accurately simulate a Markov process with a well defined steady state (see [8] for a detailed discussion). As regards the latter, a markovian structure allows to define a well behaved long term equilibria (i.e. a steady state) using a recursive equilibrium notion (see for instance [1]).

Unfortunately, there is no free lunch. In order to improve the emprical performance of general equilibrium models, macroeconomists have turned to the incomplete markets framework. [3] showed that in this enviroment there is a trade off between the generality of a recursive representation and a well behaved steady state. The authors showed that in order to obtain an ergodic invariant measure, which is the natural representation of a steady state under the most general markovian environment, it is necessary to artificially convexify an appropriately enlarged state space; thus affecting the predictive performance of the model as each time period has an arbitrarily large number of possible continuations which are called "sunspots". [1] showed that it is possible to prove the existence of an invariant measure using a state space with no Sunspots as long as there is an uncountable number of exogenous shocks. This last fact is inconvinient from a numerical point of view as the computed policy functions must be evaluated in an arbitrarily large number of different shocks in order to satisfy the required assumption.

The purpose of this note is to show that, in certain environments, it is possible to obtain a recursive representation with an ergodic invariant measure, a finite number of exogenous shocks and a well behaved state space (i.e. a state space with no sunspots).

By enlarging the number of variables in the state space, it is possible to obtain a continuous markovian representations which allows to derive an ergodic invariant measure by applying standard results. [5] argued that the lack of a minimal state space recursive representation is associated with the presence of multiple equilibria; this fact justifies the necessity of an enlarged state space in an incomplete markets general equilibrium framework [1]. This note shows that, in certain cases, an appropriate enlargement is enough to: i) derive at least 1, possible 2, continuous markovian process which represents a subset of all (possibly multiple) sequential equilibria, ii) obtain a sunspots free state space with a finite number of exogenous shocks and, iii) determine a suitable steady state.

Fact i) above is relevant from a theoretical point of view as it provides a counterexample for the equivalence between a continuous markovian representation and the uniqueness of the sequential equilibrium. In words of [1]:

"the existence of a continuos selection - tantamount to the uniqueness of equilibrium in each state - is not often satisfied".

The results are based on a RBC model due to [7] which is well known because it illustrates the implications of looking for a computationally efficient recursive representation. In particular, Santos presents an example of an economy with a discontinuous markovian representation and minimal state space. In this note, it is shown that an enlargement similar to the one used in [3] is enough to derive a continuous markovian representation with well behaved steady state and state space.

2 A Continuous Recursive Representation

2.1 Setting of the Model

The model is a stochastic version of [7] (section 3.2). Consider a representative agent economy with discrete time, $t = 0, 1, 2....$ Exogenous shocks are markovian and will be denoted z. For the sake of simplicity let us assume that the state space for these shocks is $\{0, 1\}$. An element of the transition matrix will be denoted $p(.,.)$. Let $\{z_t\}$ be a sequence of shocks and Z^t the set of histories up to time t, being a typical element z^t. Using

[1] [2] provided conditions to guarantee the uniqueness of equilibria in an infinite horizon economy with complete markets. There is no analogous result for incomplete markets

standard results (see [10], Ch. 8) it is possible to define, for any $z_0 \in \{0,1\}$, a stochastic process $(\Omega, \sigma_\Omega, \mu_{z_0})$ on Z^∞.

There is a unique decreasing return to scale firm which only uses capital as input and its technology is characterized by $y_t = A(z_t)f(k_t)$ with $f' > 0$, $f'' < 0$ and $f(0) = 0$ as usual. The firm is owned by the consumer as she is endowed with $k_0 > 0$ units of capital. Thus, the agent has two sources of current income derived from her endowment: benefits, denoted by π_t, and rents from capital, denoted by $r_t k_t$. Besides, the flow of taxes paid and transfers received is $\tau(k_t)r_t k_t$ and T_t respectively. Note that the tax rate depends on the stock of capital. In particular, it is given by a piecewise linear continuous function (see [7], page 87 for details).

The problem faced by the consumer is to choose a pair of functions $c : Z^\infty \longrightarrow \mathbb{R}$ and $x : Z^\infty \longrightarrow \mathbb{R}$ that solves the following problem:

$$\max_{\{c,x\}} \sum_t \sum_{z^t \in Z^t} \gamma^t u(c(z^t))\mu_{z_0}(z^t) \tag{1}$$

s.t.

$$k(z^t) = x(z^t) + (1-\delta)k(z^{t-1}) \tag{2}$$
$$c(z^t) + x(z^t) \leqslant \pi(z^{t-1}) - (1-\tau(z^{t-1}))r(z^t)k(z^{t-1}) + T(z^t) \tag{3}$$

$c(z^t) \geqslant 0, x(z^t) \geqslant 0$ for any $z^t \in Z^t$, z_0 and $k_0 > 0$ given, $\delta \in [0,1]$ is the depreciation rate and $\gamma \in (0,1)$ the discount factor.

In what follows $\tau(z^{t-1})$ stands for $\tau(k(z^{t-1}))$ or abusing notation $\tau(k_t(z^{t-1}))$.

That is, the tax rate affects the rents obtained from capital holdings at time t, which is in turn affected by the information contained in z^{t-1} because $k_t(z^{t-1}) = x_{t-1}(z^{t-1}) + (1-\delta)k_{t-1}(z^{t-2})$. A similar argument can be used to understand $r(z^t)$ because the agent knows the clearing condition for the market of factors and the optimality condition for the firm to be described below.

The problem of the firm is standard. Taking r_t as given it solves:

$$\max_{K_t} A(z_t)f(K_t) - r_t K_t, \qquad \text{for any } z_t \in \{0,1\}. \tag{4}$$

Observe that the optimality of the firm implies $r_t = A(z_t)f'(K_t)$. The Government simply transfers to the consumer the tax revenues:

$$T = \tau(z^{t-1})r(z^t)k(z^{t-1}). \tag{5}$$

Finally, goods and factor markets clear:

$$c(z^t) + x(z^t) = A(z_t)f(K_t) \qquad \text{Goods Market}$$
$$k(z^t) = K_{t+1} \qquad \text{Capital Market}$$

where both equations hold for any $z^t \in Z^t$.

Note that in equilibrium, the optimality condition of the firm and the market clearing equation for capital holdings implies $r_t = A(z_t)f'(k(z^{t-1}))$ which in turn implies $r_t = r(z^t)$ as claimed. Further, both market clearing conditions imply $c(z^t) + x(z^t) = A(z_t)f(k(z^{t-1})) = y(z^t)$ as expected.

2.2 Equilibrium Equation

In this case, the solution to the model can be characterized by the equilibrium Euler equation, which can be obtained by putting the optimality condition for the firm, the budget constraint for the Government and the market clearing conditions into the optimality condition for the consumer.

Assume that $u(c) = \ln(c)$ and $\delta = 1$. Then, the equilibrium equation is given by:

$$\tfrac{1}{C_t} = \gamma \sum_{z_{t+1}=0,1} \frac{A(z_{t+1})p(z_t, z_{t+1})(1-\tau(K_{t+1}))f'(K_{t+1})}{C_{t+1}}, \tag{6}$$

with constrains given by

$$K_{t+1} = A(z_t)f(K_t) - C_t. \tag{7}$$

Note that the market clearing condition for capital implies that *given* z^t the demand for capital K_{t+1} does not depend on the realizations of the exogenous shock at $t+1$. Hence, by replacing C_{t+1} in (6) with its expression obtained from (7) and after some algebra we can rewrite (6) in the following way:

$$\underbrace{\frac{1}{\gamma(A(z_t)f(K_t) - K_{t+1})(1-\tau(K_{t+1}))f'(K_{t+1})}}_{c} = \frac{\overbrace{A(0)p(z_t,0)}^{c_1}}{\underbrace{A(0)f(K_{t+1})}_{d_1} - K_{t+2}} + \frac{\overbrace{A(1)p(z_t,1)}^{c_2}}{\underbrace{A(1)f(K_{t+1})}_{d_1} - K_{t+2}}. \tag{8}$$

The purpose of this note is to find an equation $\Psi : X \longrightarrow X$, where X is an appropriately defined state space and Ψ is a function that maps $x_t \longmapsto x_{t+1}$ with (x_t, x_{t+1}) satisfying equation (8) for any t.

Notice that by standard arguments, by fixing $\delta = 1$ and $f(0) = 0$, K_t stays in $[0, K^{UB}]$ (see [10], Ch. 5) for any t.

Let $X = [0, K^{UB}] \times [0, K^{UB}] \times \{0, 1\}$. With this state space Ψ becomes a vector valued function of the form $x_t \longmapsto (\Psi_1(x_t), \Psi_2(x_t), \Psi_3(x_t))$ with $x_t = (K_t, U_t, z_t)$.

Let $\{z_n\}$ be a realization of $(\Omega, \sigma_\Omega, \mu_{z_0})$. Then, it is possible to define each coordinate in the image of Ψ as follows:

$$\begin{aligned} K_{t+1} &= \Psi_1(x_t) = U_t \\ z_{t+1} &= \Psi_3(x_t) = \{z_n\}(t+1). \end{aligned}$$

In order define Ψ_2 we could use (8). Notice that (8) takes the form

$$c = \frac{c_1}{d_1 - U_{t+1}} + \frac{c_2}{d_2 - U_{t+1}}, \tag{9}$$

or equivalently,

$$c(d_1 - U_{t+1})(d_2 - U_{t+1}) = c_1(d_2 - U_{t+1}) + c_2(d_1 - U_{t+1}). \tag{10}$$

Due to the fact that this is just a quadratic equation we can get U_{t+1} as a *continuous function* of the parameters, namely:

$$U_{t+1} = \frac{\pm\sqrt{(-d_1c - d_2c + c_1 + c_2)^2 - 4c(d_1d_2c - c_1d_2 - c_2d_1)} + (d_1 + d_2)c - c_1 - c_2}{c}. \tag{11}$$

Equivalently:

$$U_{t+1} \equiv g(d_1, c, d_2, c_1, c_2)$$

It is important to observe that (11) gives at most *2 different mechanisms* [2] ,each of them characterized by a different root of (11). Furthermore, note that $c(K_t, U_t, z_t)$, $d_1(U_t)$, $d_2(U_t)$ and the rest of the parameters in (11) depend on z_t . Thus, Ψ_2 is given by:

$$U_{t+1} = g(d_1, c, d_2, c_1, c_2) =: \Psi_2(x_t).$$

In order to show the continuity of Ψ (on K_t and U_t), provided that the discriminant in g is positive, it suffices to verify the continuity of C, d_1, d_2 (on K_t and U_t).

2.3 Discussion

Translating this, we just wrote K_{t+2} in terms of (K_t, K_{t+1}, z_t); ie,

$$K_{t+2} = g(K_t, K_{t+1}, z_t).$$

Though g might be awful, it is still *explicit* and, even more, continuous (of course, this representation has economic content if we can assure that the discriminant in g is positive under reasonable parameterizations for any $x \in X$).

The (numerical) cost of this representation is the enlargement of the state space with respect to the natural one (i.e. (K_t, z_t)). As discussed in [5], enlarging the state space might provide a recursive representation. Unfortunately, the results in that paper does not address the continuity of the mechanism; an aspect that has severe consequences for the steady state of the model as discussed in [3]. This paper shows that it is possible to obtain a continuous selection from a correspondence based on a recursive representation. Moreover, this example shows that the restrictions implied by [9] or [1] may not always be necessary as it is possible to have multiple equilibria and continuous mechanisms.

In particular, as $U_t := K_{t+1}$, we have now the following iterative system: Take first an arbitrary initial condition (K_0, U_0, z_0) and a drawn $\{z_n\}$, then

$$\begin{aligned} K_{t+1} &= U_t \\ U_{t+1} &= g(K_t, U_t, z_t), \end{aligned}$$

[2]Note that (8) implies that this model does not have a trivial solution at $K_t = 0$ as $u = \ln$ and investment is not allowed to be negative. This fact in turn implies that the parameters in (8) are all bounded away from 0. Of course, in order to have two non-trivial solutions it suffice to impose conditions on the discriminant of (11)

provides a sequence $\{X_n\}$. Such a sequence defines a Feller mechanism, with compact state space X. Thus, it has an ergodic invariant measure (see [4]), which guarantees that the process K_t has an invariant measure as well.

Moreover, using standard results on laws of large numbers for markov processes (see [11]), it can be shown that choosing an appropriate initial condition it suffices to guarantee that:

$$\frac{\sum_{t\in 0,\ldots,T} h(X_t)}{T} \quad \text{converges almost surely to} \quad E_\mu(h),$$

where h is a σ_X-measurable function and μ is one of the possibly many ergodic invariant measures described above.

It is worth mentioning that this "trick" can also be done in the case of discrete shocks if the number of total states is 3 or 4. This relies on the fact that there are explicit expressions for the roots of a polynomial in terms of its coefficients whenever the degree of the equation is smaller or equal than 4. Thus, it is possible to obtain an expression similar to (11) even if we allow for a more realistic exogenous state space.

3 Conclusions

This note presents an example of an economy with multiple equilibria and continuous policy functions (i.e. Ψ is not unique). This type of equilibrium is useful for accurately assessing the predictions of the model as it allows to generate reliable simulations. These simulations can be used to generate counterfactuals which are useful to evaluate alternative economic policies.

The paper also connects two branches of the recursive literature: the one concerned with the existence of a steady state (see for instance [8]) and the one concerned with the existence of a recursive representation of the sequential equilibria ([5]).

It is clear that the results in this note have to be generalized. In particular, it is necessary to understand the connection between the number of possible exogenous states and the number of distinct economically meaningful recursive equilibria. That is, as the degree of the polynomial in g is increasing in the number of exogenous states and each root of the polynomial defines a different mechanism (provided that the root is real and positive), there is a trade off between a realistic shock process and the predictive performance of the model as more than one possible mechanism generates a less conclusive model.

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