Some Useful Results for the Simulation of Non Optimal General Equilibrium Economies

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May, 2017

PRELIMINARY DRAFT, WORK IN PROGRESS.

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Abstract

This paper investigates the empirical evaluation of infinite horizon non-optimal economies by means of numerical simulations. In particular, the paper answers the following question: is it possible to derive a general framework which guarantees that numerical simulations truly reflect the behavior of endogenous variables in the model?. Under mild assumptions, this paper provides an affirmative answer to this question for endowment economies with incomplete markets and infinitely many exogenous states. For this type of models, the paper presents an accurate calibration method. For economies with finitely many shocks, even under stronger assumptions, it is only possible to show that a numerically computable, time independent and recursive representation of the sequential equilibrium generates a stationary Markov process, which is a necessary condition to answer the above mentioned question.

Keywords: non-optimal economies, Markov equilibrium, numerical methods, simulations.

JEL Classification: C63, C68, D52, D58.

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1 Introduction

The empirical performance of infinite horizon economies with complete markets or with a representative agent has been repeatedly questioned. In particular, in models were both assumptions are supposed to hold, Singleton (1990) and Mehra and Prescott (1985), using different methodologies, reported parameters and predicted values out of a reasonable range. For economies with a representative agent and incomplete markets, the literature has shown that Deaton and Paxson's (1994) result[2], which is the most relevant empirical finding in the field, does not hold when is tested using different samples (see Guvenen, 2011, page 20). Further, the empirical relevance of economies with heterogeneous agents and complete markets has been challenged by Hayashi et. al. (1996), Cochrane (1991) and Attanasio and Davis (1996). Because of the inability of these economies to match observed behavior, the literature has moved in different directions. In this sense, the relevance of financial market incompleteness, market failures and agent heterogeneity for economic analysis has been recognized by the empirical evidence in several fields (see for example Pijoan – Mas (2007), Heathcote (2005), Krueger (1999) and Akyol (2004).

Generally, these economies, typically referred as non-optimal, investigate the long and short run effects of alternative economic policies under market frictions. To achieve this purpose, frequently, the observable variables in the model are *computed, simulated* and *compare with its empirical counterpart*. This is done since empirically meaningful general equilibrium models often do not have a closed form solution. Further, simulating an economy is the most immediate way to explore its quantitative properties. It allows going beyond a comparative statics analysis and obtaining information about the dynamic behavior of the economy.

Unfortunately, there is *no general method* to compute non-optimal economies. Further, the commonly used procedures generate surprisingly different outcomes (see Hatchondo, et. al., 2010, De Groot, et. al., 2013, among others) and the simulations obtained from them may not provide accurate representations of the economies depicted by the models (see for instance

[2] Deaton and Paxson's (1994) matched the evolution of cross sectional income and consumption variance using a simple finite horizon model with a representative agent who is only allowed to hold a riskless real asset. These economies are called "self-insurance models" and have been recently picked up by the literature (see for instance Guvenen, 2009) as the Krusell – Smith (1998) "approximate aggregation theorem" suggests that this type of models could empirically perform as well as a Bewley – economy.

Feng, et.al., 2013). *This is the gap that this paper tries to bridge by answering the following question:*

Is it possible to approximate, simulate, and empirically evaluate an infinite horizon non-optimal economy in a general framework which guarantees that numerical simulations truly reflect the behavior of endogenous variables in the model?.

This paper provides conditions that allow answering this question positively for endowment economies with incomplete markets and infinitely many exogenous states as in Mas-Colell and Zame (1996). In particular, a *recursive equilibrium* notion due to Feng, et. al. (2013) is used to derive a *stationary Markov process* under mild assumptions. This is the first step to compute and simulate an infinite horizon non-optimal economy as it endows the sequential equilibrium with a dynamically simple, and thus computable, representation which generates a well behaved stochastic process. Then, it is shown that: a) under minor additional assumptions on the transition functions, the process is shown to be ergodic. b) Assuming uniform convergence of numerical approximations together with a restriction on the computed transition functions, the simulations obtained from them asymptotically replicate the actual long run behavior of the model, c) it is possible to derive an accurate calibration procedure based on a) and b).

For non-optimal economies with finitely many shocks, even after a significant increase in the strength of the assumptions, it is only possible to show that the recursive equilibrium in Feng, et. al. generates a stationary Markov process. This results in not enough to prove the accuracy of simulations even if the algorithm is assumed to approximate the model uniformly. Technically, it is not possible to show that the process is ergodic, which in turn affects the long run behavior of the computed and actual simulations.

Thus, this paper provides a *general setting for the computation and simulation* of non-optimal economies and gives conditions which guarantee that the parameters obtained from a calibrated model are consistent with the stationary behavior of the model.

The necessity of a general framework that allows evaluating quantitatively non-optimal economies comes from the failure of methods frequently used (i.e. Kydland and Prescott, 1982, Krusell and Smith, 1998, Cooley and Quadrini, 2001, Chari, Kehoe and Mcgrattan, 2002, among others) in providing *simultaneously*: an adequate representation of the steady state, a well defined stationary law of motion for the endogenous variables, a result which guarantee the accuracy of simulations even if an appropriate algorithm is available .

Inadequacy of the steady state: one of the most popular procedures in the literature (see for instance Allub and Erosa, 2013) can be summarized in 4 steps: i) set up of the model, ii) calibration around a non-stochastic steady state, iii) computation using a local algorithm and iv) simulation (see ch. 6 in DeJong and Dave, 2007, for a detailed discussion). As there are no sufficient conditions for the uniqueness of equilibria in non-optimal economies, there may be multiple stationary solutions. Further, even if the equilibrium is unique, the presence of non degenerate exogenous shocks implies that the endogenous variables may not rest at a point with probability 1. Thus, the assumed representation of the steady state is not realistic as the economy will fluctuate randomly around a potentially large number of points; implying that the set of parameters obtained in ii) may not be relevant.

Lack of an appropriate transition function: it is possible to compute the model without using a local algorithm, thus avoiding the dependence of the procedure on a non-stochastic steady state. However, these methods (like the projection algorithm in Judd, Kubler and Schmedders, 2002 or the Bellman equation methods in Arellano, 2008) depend on the existence of a continuous policy function mapping the natural state space (i.e. exogenous shocks and the distribution of wealth) to the rest of the payoff relevant variables. While this procedure may be suitable for efficient equilibria, it is not appropriate for non-optimal economies as such policy function may not exist in the presence of multiple equilibria (see for instance, Kubler and Schmedders, 2002, Santos, 2002 and Kubler and Polemarchakis, 2004).

Inaccuracy of simulations: Santos and Peralta Alva (2005) or Feng, et. al. (2013) showed that simulations may not replicate the actual behavior of the model as numerical errors could accumulate over time. That is, unless the approximated simulations can be guaranteed to reach the actual steady state of the model, they may contain severe biases even if convergence to some stationary value is attained.

This paper shows that one way to obtain accurate numerical simulations is to solve all the above mentioned problems at the same time. In particular, in order to deal with multiplicity and computability, this paper borrows a correspondence based recursive equilibrium notion from Feng, et. al. Then, it is proved that this notion not only generates a computable time invariant transition function but also a Markov process with an appropriate steady state, called *invariant measure,* which guarantees the stationarity of the process after appropriately selecting its initial conditions. Then, a set of additional assumptions are used to prove that this measure is *ergodic*, a property that assures that (the time averages of) computed simulations converge accurately if the algorithm used is appropriately chosen.

Technically, the challenge is to obtain a (numerically) computable representation of a non optimal economy that generates convergent and empirically meaningful simulations. The literature, with the notable exception of Santos and Peralta Alva (2013), has not addressed these problems at the same time. Duffie, et. al. (1994) and Blume (1982) showed the existence of an ergodic invariant measure for some non optimal economies but they did not take care of numerical part of the problem. Feng, et. al. derived a time invariant recursive representation of (possibly multiple) equilibria that depends on a low dimensional observable state space but they were not able to prove that this representation generates an ergodic stochastic process.

This paper fills the gap in the literature by refining some of the results in Santos and Peralta Alva (2013). In particular, a modified version of the assumptions made by these authors is derived from primitive conditions of the economy: the number of possible exogenous shocks and a set of restrictions on the discontinuity of the transition function. While Araujo, et. al. (1996) showed that the former is a plausible assumption in non optimal infinite horizon endowment economies, the conditions which guarantee the latter has still to be shown.

Also, the strategies used for the proofs differ from previous results. One of the consequences of allowing for multiple equilibria is that the selected transitions may not be continuous. This fact causes a serious problem for the existence of an invariant measure, as can be seen in Stokey, Lucas and Prescott (1989, page 376), because it affects the continuity of the generated Markov process. The literature has circumvented this problem by using a fixed point theorem for correspondences defined in functional spaces, like Fan – Glicksberg. Unfortunately, this approach requires conditions which affect the computability of transitions (like the convexification technique used in Duffie, et. al. or the impossibility to identify an appropriate selection in Blume). The strategy in this paper is to derive *verifiable conditions on each computable transition* that restore the continuity of the Markov process. The proofs are based on a seminal paper by Ito (1964), a recent characterization of Portmanteu's theorem due to Molchanov and Zuyev (2011) and requirement slightly (weaker) stronger than the absolute continuity of the Markov operator which allows to prove the (stationarity) ergodicity of the process. Once the existence of an ergodic invariant measure is established, the paper shows the accuracy of simulations by using Birkhoff's ergodic theorem and adapting theorem 2 in Santos and Peralta Alva (2005).

Finally, the paper also extends the literature in infinite horizon general equilibrium models with incomplete markets (see Maguill and Quinzii 1996, among others[3]) beyond an existence result for the sequential competitive equilibrium by deriving a theoretical structure composed by a computable Markovian representation of a subset of all possible sequential equilibria, a well behaved steady state (i.e. an invariant measure) and a law of large numbers. From a purely theoretical point of view, these results are important as the conditions that guarantee the existence of this subset of sequential equilibria follows (almost) directly from the existence of the recursive structure (see for instance Duffie, et. al. section 3 or Miao 2006).

Finally, the paper is organized as follows: Section 2 provides a brief review of the most relevant function based recursive equilibrium concepts and discusses their limitations for the purpose of this paper by means of an illustrative example. Then, the necessity of a correspondence based recursive structure is discussed together with 3 simple facts that allow understanding how the theoretical results in this paper are going to be used to answer the motivating question. Section 3 proves the existence of an invariant measure and establishes sufficient conditions for its ergodicity. Section 4 proves the implications of having an ergodic recursive representation for the purposes of this paper. Section 5 presents an infinite horizon economy that satisfies the assumptions made in section 3. Section 6 concludes and estates directions for future research.

2. Motivation and Relation with the Literature

This section uses a standard infinite horizon general equilibrium model with incomplete markets to introduce some recursive equilibrium concepts, discussed its existence and several properties which are useful for the purposes of this paper.

[3] Magill and Quinzii (2008) provided an exhaustive review of this branch of the literature.

A part of this section is devoted to keep the paper self-contained. The reader who is familiar with the concepts of sequential competitive equilibrium, recursive and wealth recursive equilibrium, Duffie, et. al.'s time homogeneous markov equilibrium and Feng, et. al.'s recursive equilibrium is invited to go directly to sections 2.4, which discusses the existence of "standard" recursive equilibrium concepts, and 2.6 that addresses the applicability of "modern" recursive equilibrium notions for the purposes in this paper.

Sections 2.3 to 2.6 are devoted to justify the use of Feng, et. al.'s correspondence based recursive equilibrium to derive an ergodic Markov process that allows answering the question at hand. Further, these sections explain why standard function based recursive equilibrium notions are unsuitable in the presence of multiple equilibria and how an ergodic Markov process can be used to accurately calibrate the model using numerical simulations.

The only directly related paper is Santos and Peralta Alva (2013). These authors also study the accuracy of numerical simulations for non-optimal economies. This paper refines and extends some of the results in Santos and Peralta Alva. A detailed discussion of the connection between these 2 papers is postponed to section 3.1 as it requires some investment in notation.

2.1 Structure of the Economy

A standard infinite horizon discrete time pure exchange economy is considered. An exogenous Markov chain defines the law of motion for the exogenous state variable[4]. For every period t, an exogenous shock s_t occurs; $s_t \in S$ and $S = (1,2,\dots,\mathrm{S})$. To model the evolution of uncertainty, an event tree approach is assumed. Each tree $\mathfrak{T}$ has a unique root, $\sigma_0 = s_0$. A typical element will be denoted $\sigma_t = (s_0, s_1, \dots, s_t)$. Each σ_t has a unique predecessor $\sigma_t^* = (s_0, s_1, \dots, s_{t-1})$ and S sucessors, $\sigma_t s$, for each $s \in \mathrm{S}$.

Since the exogenous shocks follow a first order Markov process and S is finite, the evolution of $\{s_t\}_{t=0}^{\infty}$ can be characterized by a transition matrix, $p = [p(s_i, s_j)]$. For any given s_0, the probability of σ_t will be denoted $\mu_t(\sigma_t) = \prod_{t=1}^{t} p(s_{t-1}, s_t)$ and $\mu_0(\sigma_0) = \delta_{s_0}$, where δ_{s_0} is the Dirac measure at s_0.

[4] The set of exogenous shocks is assumed to be finite in all the equilibrium concepts mentioned in this section. This is done because the conditions to guarantee the existence of the sequential equilibria are well known. The Time Homogeneous Markov Equilibrium in Duffie, et. al. (1994), Kubler and Schmedders' Markov equilibrium and Feng's Recursive equilibrium can be defined for an arbitrary set of exogenous shocks (see Duffie, et.al. page 749 and Santos and Peralta Alva page 6 respectively). The conditions for the existence of the sequential equilibria with an uncountable, atomless and iid shocks, which is essential for the results in sections 3 and 4, are presented in section 5.

The number of agents is assumed to be finite, with a typical element denoted $i \in I$. Each agent is endowed with $e^i(\sigma_t)$ units of the single (perishable) consumption good. For simplicity, the endowment process is supposed to be iid: $e^i(\sigma_t s) = e^i(s)$, where $e^i : S \to \mathbb{R}_{++}$. Further, the vector of endowments at any node will be denoted $e(\sigma_t) = \{e^i(\sigma_t)\}_{i=1}^I$.

Each agent has an additively (time) separable well behaved[5] utility function which is used to evaluate consumption streams, $c = \{c(\sigma_t)\}_{\sigma_t \in \mathfrak{T}}$:

$$U_i(c) = \sum_{t=0}^{\infty} (\beta^i)^t \sum_{\sigma_t^* s} [u_s^i(c^i(\sigma_t^* s))] \mu_t(\sigma_t^* s)$$

The asset structure is characterized by J one period assets numerarie real assets, offered in zero net supply and traded at each node of the tree, $\sigma_t \in \mathfrak{T}$. Each asset that is held by agent i is denoted $\theta_j^i(\sigma_t) \in \mathbb{R}$ and pays dividends $d_j(\sigma_t s) \in \mathbb{R}_+$, only at the S immediate successors of σ_t [6]. The portfolio of agent i at node σ_t will be denoted $\theta^i(\sigma_t) \in \mathbb{R}^J$. It is assumed that the dividend process is also iid: $d_j(\sigma_t^* s) = d_j(s)$, where $d_j : S \to \mathbb{R}_+$[7]. Further, the $J \times S$ payoff matrix, d, is supposed to have full row rank and a column of d will be denoted $d(\sigma_t)$. Consequently, market incompleteness follows directly from $J < S$. Finally, the price of security θ_j at node σ_t will be denoted $q_j(\sigma_t) \in \mathbb{R}_+$, asset prices will be collected at the vector $q(\sigma_t) \in \mathbb{R}_+^J$ and the net wealth of agent i will be written as $w^i(\sigma_t) = e^i(\sigma_t) + \theta^i(\sigma_t^*). d(\sigma_t)$.

2.2 Sequential Competitive Equilibrium[8]

An economy $\mathcal{E}$ is characterized by the endowment and payoff matrixes and the structure of preferences: $\mathcal{E} = [e, d, \{U^i\}_{i=1}^I, \{\theta_-^i\}_{i=1}^I]$. A sequential equilibrium for $\mathcal{E}$ can then be defined as follows,

[5] To the conditions stated in Duffie, et. al. (1994) page 765, Kubler and Schmedders (2002) implicitly added the assumption that u_s^i has uniformly bounded gradients. This is done to satisfy a terminal condition on the discounted expected marginal utility (see equation 1 in page 288) which in turn is used to obtain a definition of equilibria based on first order and market clearing conditions. This last definition is essential for the recursive equilibrium literature as can be seen in sections 2.3 and 2.5.

[6] Note that agents are allowed to short sale every asset θ_j. In order to define a Time Homogeneous Markov Equilibrium, Duffie, et. al. assumed that there are no short sales and a different asset structure (J Lucas trees). However, Braido (2013) recently showed that the equilibrium concepts in Duffie, et. al. still hold if short sales are permitted for a general asset structure, which includes one period real assets offered in zero net supply, provided that marginal utilities are uniformly bounded above.

[7] Except in section 2.4, where the sequential equilibrium has closed form, for economies with $\#S < \infty$, it will be assumed that the dividend structure has a riskless bond as in assumption A.6 in Magill and Quinzii (1994) (i.e. $d_j(s) = 1$ for any $s \in S$ and $j \in \{1, \ldots, J\}$.

[8] This concept is analogous to the Financial Market Equilibrium in Magill and Quinzii (1996), page 228, extended to an infinite horizon economy.

Definition 1. A sequential competitive equilibrium for ε *is a collection of consumption vectors* $[\{c^i(\sigma_t)\}_{i=1}^I]_{\sigma_t \in \mathfrak{T}}$, *portfolio holdings* $[\{\theta^i(\sigma_t)\}_{i=1}^I]_{\sigma_t \in \mathfrak{T}}$ *and asset prices* $[\,q(\sigma_t\,)]_{\sigma_t \in \mathfrak{T}}$ *that for* $s_0 \in S$ *and* $\{\theta_-^i\}_{i=1}^I$, *which is a feasible initial asset distribution, satisfy:*

a) (Feasibility) For all $\sigma_t \in \mathfrak{T}$, $\sum_{i=1}^I \theta^i(\sigma_t) = \vec{0}$, *where* $\vec{0} \in \mathbb{R}^J$.

b) (Optimality) For each agent $i \in I$ *and prices* $[\,q(\sigma_t\,)]_{\sigma_t \in \mathfrak{T}}$, $[c^i(\sigma_t), \theta^i(\sigma_t)]_{\sigma_t \in \mathfrak{T}} \in$ *argmax* $\{\, U^i(c)$ *subject to* $c(\sigma_t) = w^i(\sigma_t\,) - \theta^i(\sigma_t).\, q(\sigma_t)$ *for all* $\sigma_t \in \mathfrak{T}$ *and* $sup_{\sigma_t \in \mathfrak{T}}|\theta^i(\sigma_t).\, q(\sigma_t)| < \infty\}$.

As the payoff matrix does not depend on the price of securities, its (row) rank is constant for any period $0 \leq t \leq \infty$. Consequently, the excess demand function of all agents can be shown to be continuous (See Magill and Quinzii 1996, exercise 3, page 276 for a counterexample for the case of long-lived assets). To establish the existence of equilibria, an implicit debt constrained is added in condition b). Magill and Quinzii (1994) showed that this condition rules out Ponzi schemes, it is never binding and is sufficient for existence.

2.3 Function Based Recursive Equilibria

This section describes the branch of the recursive equilibrium literature which postulates the existence of a time invariant function that maps different state spaces (i.e. exogenous shocks, wealth, etc.) into the rest of payoff relevant variables. It will be clear in section 2.4 that the existence of this function depends on the uniqueness of equilibrium; a property that has not been proved in general equilibrium models with infinite horizons and incomplete markets.

The simplest notion of function based recursive equilibria, often called *strongly recursive,* can be found in Lucas (1978). In a representative agent economy with complete markets, Lucas is able to show that the endogenous variables in definition 1[9] can be written solely as a function of the current realization of the exogenous state variable. In this case, S constitutes a sufficient state space to describe the evolution of the economy. Unfortunately, market incompleteness and agent heterogeneity makes this equilibrium notion too restrictive. As agents will insure against each other, it is likely that asset positions will differ across agents at any given period even in the same (exogenous) state. Thus, wealth will also differ at the beginning of next period,

[9] Actually, Lucas (1978) assumed a complete set of trees. Thus, Definition 1 should be modified to account for this fact, as assets are offered in positive net supply and the payoff matrix depends on asset prices.

affecting the consumption possibility set for all $s \in S$ (see Kubler and Schmedders 2002 for a detailed discussion).

As this paper focus on non-optimal general equilibrium economies with heterogeneous agents, broader notions of recursive equilibria are required:

Definition 2: A sequential equilibrium is called Weakly Recursive if there exist continuous functions $f^i: S \times \mathbb{R}^{IJ} \to \mathbb{R}^J$ *for all* $i \in I$ *and* $g^j: S \times \mathbb{R}^{IJ} \to \mathbb{R}_+$ *for all* $j \in J$ *such that for any* $\sigma_t \in \mathfrak{T}$ *and* $s \in S$, $q_j(\sigma_t^* s) = g^j(s, \{\theta^i(\sigma_t^*)\}_{i=1}^I)$ *and* $\theta^i(\sigma_t^* s) = f^i(s, \{\theta^i(\sigma_t^*)\}_{i=1}^I)$*, where* $\{\theta^i(\sigma_t^*)\}_{i=1}^I$ *is feasible.*

Definition 3: An equilibrium is called Wealth Recursive if it is weakly recursive and if there are continuous functions $f_{WhR}^i: S \times \mathbb{R}^I \to \mathbb{R}^J$ *for all* $i \in I$ *and* $g_{WhR}^j: S \times \mathbb{R}^I \to \mathbb{R}_+$ *for all* $j \in J$ *such that* $g_{WhR}^j(s, w(\sigma_t^* s)) = g^j(s, \{\theta^i(\sigma_t^*)\}_{i=1}^I)$ *and* $f_{WhR}^i(s, w(\sigma_t^* s)) = f^i(s, \{\theta^i(\sigma_t^*)\}_{i=1}^I)$*, where* $w(\sigma_t^*) = \{w^i(\sigma_t^*)\}_{i=1}^I$.

Frequently, the macroeconomic literature (i.e. Arellano, 2008) assumes the existence of a recursive equilibrium based on several standard properties of the Bellman equation (see, for example, Stokey, Lucas and Prescott, 1989). Formally, this equilibrium notion can be thought as an extension of Mehra and Prescott's recursive competitive equilibrium (1980) to an economy with heterogeneous agents and incomplete markets. Typically, the equilibrium is defined as:

Definition 3.1[10]: A recursive equilibrium is composed by a set of I *value and price functions,* $\{V^i(s,w)\}_{i=1}^I$ *and* $\{q^i(s,w)\}_{i=1}^I$ *respectively, which satisfy the following properties:*

i) (Optimality) $V^i(s,w) = Max_{\theta^i \in \Delta}\, u_s^i(w^i - q^i(s,w)\theta^i) + E_p\left(\beta^i V^i(s',w')\right), i \in I,$ where the wealth distribution is $w' = \{w_i'\}_{i=1}^I = \{e^i(s') + d(s')\theta^i\}_{i=1}^I$, E_p is the expected value taken with respect to $p(s,.)$, and the feasible set Δ is compact[11].

ii) (Market Clearing) $\sum_i \theta^i = \vec{0}$

iii) (Expectations) $q^i(s,w) = q^l(s,w) = q(s,w)$ for all $i,j \in I$

Provided the existence of continuous price functions $\{q^i(s,w)\}_{i=1}^I$ which satisfy iii), the continuity of $\{\theta^i(s,w)\}_{i=1}^I$ follows from mild curvature conditions on u_s^i (see Stokey, Lucas and Prescott, Ch. 9 and 10). Thus, definition 3.1 is equivalent to definition 3 in the sense that both

[10] This definition does not include models of the Hugget (1993) style as this type of models does not assume the existence of aggregate uncertainty (i.e. $\#S = 1$) and the degree of heterogeneity is higher as Hugget suppose the existence of a continuum of distinct agents and idiosyncratic uncertainty.
[11] To achieve this property it is sufficient to impose a short sale constraint on assets.

imply a recursive structure based on continuous functions that depends on exogenous shocks and wealth distribution.

Unfortunately, as it will be illustrated in section in the next section, the equilibrium concepts in definitions 2, 3 and 3.1 do not always exist.

2.4 A counter example

The example is borrowed from Kubler and Schmedders (2002). It will be shown that policy functions as defined above may not exist in the presence of multiple equilibria. In particular, there will be no wealth recursive equilibria if for the same pair of states (s, w) there are at least 2 possible asset prices. This happens because wealth is not a sufficiently state variable: for 2 different portfolio distributions, wealth may the same but asset prices may differ. In this sense, wealth is insufficient to capture the heterogeneity of agents' decisions and thus constitutes an inappropriate state space for function based recursive equilibrium notions.

The authors also presented an economy with no weakly recursive equilibria. For the sake of concreteness, this paper will only discuss the first case but it should be kept in mind that multiplicity of equilibria is common in non optimal economies and affects not only endowment models and the (s, w) state space but also production economies (see Santos 2002) and more "informative" state spaces like (s, θ).

The economy is a particular parametrization of the model described in section 2.1: assume that $I = 2, J = 3, \#S = 5, \beta^i = \beta^{i'} = 5/6$. Preferences, endowments and dividends are given by:

$$u_s^i = a_s^i \frac{[C(\sigma_t^* s)]^{1-5}}{1-5}, \; a_s^1 = [1, 1024, 1], \; a_s^2 = [1, 1, 1024] \text{ for } s = 1,2,3$$

$$u_s^1 = \frac{-[C(\sigma_t^* s)]^{-2}}{2} \text{ for } s = 4,5; \; u_4^2 = \frac{-[C(\sigma_t^* s)]^{-2}}{2}, \; u_5^2 = \frac{-6.05[C(\sigma_t^* s)]^{-2}}{2}$$

$$e^1 = [e^1(1), \dots, e^1(5)] = [4,12,1,10,8.69], \;\; e^2 = [e^2(1), \dots, e^2(5)] = [4,1,12,10,11.31]$$

$$d^1 = [d^1(1), \dots, d^1(5)] = [1,0,0,0,0], \; d^2 = [0,1,0,0,0], \; d^3 = [0,0,1,0,0]$$

The transition matrix is given by:

$$[p(s,s')] = \begin{bmatrix} 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.05 & 0.05 & 0.3 \end{bmatrix}$$

As consumption of each agent is bounded above by aggregate consumption, which is in turn uniformly bound by $e^1(5) + e^2(5)$, U_i can be assumed to be bounded above without lost of generality because Bernoulli utility functions are assumed to be strictly increasing. Thus, the arguments in Duffie, et. al. (page 765) imply that consumption is uniformly bounded below by some positive constant. Consequently, marginal utilities are uniformly bounded which in turn imply that any equilibria, if it exist, can be characterized by agent's first order conditions and feasibility constraints (see Kubler and Schmedders, 2002, page 288).

The following tables contain asset prices and portfolios which satisfy the optimality (first order) and feasibility conditions in definition 1. Each table can be seen as a time independent function of the exogenous shocks and the distribution of assets. There are 2 equilibrium portfolios which are computed "by hand": $\Theta_1 = [\theta_1^1; \theta_1^2] = [0, -1.6, 1.6; 0, 1.6, -1.6]$, $\Theta_2 = [\theta_2^1; \theta_2^2] = [0, -0.98, 2.28; 0, 0.98, -2.28]$. Thus, these tables define a weakly recursive equilibrium.

Provided that the initial portfolio distribution $\{\underline{\theta}^i\}$ is either Θ_1 or Θ_2, tables 1 and 2 can be used to generate a unique sequential competitive equilibrium according to definition 1.

Asset Prices (q)					
	S=1	S=2	S=3	S=4	S=5
Θ_1	[0.25, 2.15,2.15]	[0.03, 0.25,0.25]	[0.03, 0.25,0.25]	[0.24, 2.10,2.10]	[0.10, 1.54,0.08]
Θ_2	[0.25, 3.57,1.22]	[0.01, 0.25,0.08]	[0.05, 0.73,0.25]	[0.24, 2.10,2.10]	[0.10, 1.54,0.08]

Table 1

Portfolio $[\theta^1, \theta^2]$					
	S=1	S=2	S=3	S=4	S=5
Θ_1	Θ_1	Θ_1	Θ_1	Θ_1	Θ_2
Θ_2	Θ_2	Θ_2	Θ_2	Θ_1	Θ_2

Table 2

Kubler and Schmedders (2002) showed (numerically) that the endogenous variables in tables 1 and 2 are the only ones that satisfy the optimality and feasibility conditions in definition 1 (see page 301). Then, in order to show that this economy has no wealth recursive equilibria, it suffice to show that for some pair of states (s, w), there are at least 2 possible asset prices.

Heuristically, it can be argued that the endogenous variables in the tables above define a steady state[12]: once the economy starts either at Θ_1 or Θ_2, it will never leave the state space defined by $S \times \{\Theta_1; \Theta_2\}$. Thus, in order to verify the existence of a wealth recursive equilibrium, it is useful to describe the dynamic behavior of this economy using $s \in S$ and $w^i(s,\Theta) \equiv e^i(s) + d(s)\theta^i$, where $\Theta \in \{\Theta_1; \Theta_2\}$, $i = 1,2$ and $\Theta = [\theta^1, \theta^2]$.

Suppose that $\Theta_{t=0} = \Theta_1$ and take the sequence of exogenous shocks given by $\{s_0, s_1, s_2, \dots\} = \{2,4,1,\dots\}$. Remarkably, this economy has 2 sequential competitive equilibria if definition 1 is stated in terms of an initial wealth distribution $\{w_{-}^i\}$. In particular, there are 2 different sequences of asset prices $q_t(\sigma_t)$, for $\sigma_t \in \mathfrak{T}$, that satisfy the optimality and feasibility conditions if in definition 1 $\{\theta_{-}^i\}$ is replaced by $\{w_{-}^i\}$. Note that this last change is necessary in order to allow the sequential economy to be generated out of a wealth recursive equilibrium. The figure below illustrate this result by mapping $w(s_t, \Theta_t) \equiv [w^1(s_t,\Theta_t), w^2(s_t,\Theta_t)]$[13], the wealth distribution, into $q_t^2(s,\Theta)$, the price of asset $j = 2$.

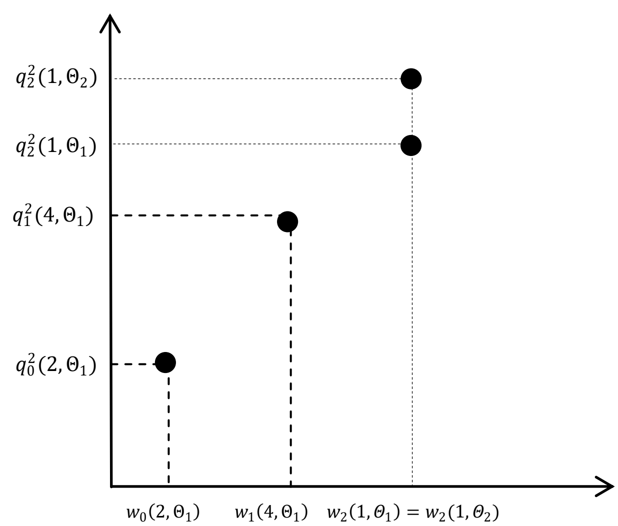

Figure 1: Wealth Equilibrium Correspondence

As $w_2(1,\theta_1) = w_2(1,\theta_2)$ but $q_2^2(1,\Theta_2) \neq q_2^2(1,\Theta_1)$, there are 2 possible images for the same element in the domain of this function. Note that in $s = 1$ the 2 admissible portfolios have $\theta_1^i = 0$ for $i = 1,2$. As dividends are 0 for the other 2 assets, *wealth does not vary with the asset*

[12] A steady state for an appropriately defined Markov representation of the sequential competitive equilibria will be formally defined in section 3.
[13] The definition of wealth in section 2.1 would imply $w_t = w(s_t, \Theta_{t-1})$. For expositional purposes, it is convenient to define $w_t = w(s_t, \Theta_t)$. The results in this section will not change using either definition.

distribution. Thus, the selected state space is insufficient to describe the evolution of endogenous variables using a *continuous function* and there is no wealth recursive equilibrium for this economy. Stated in a different way: a wealth recursive equilibrium is insufficient to fully describe the dynamic behavior of this economy. Figure 2, below, illustrate this fact.

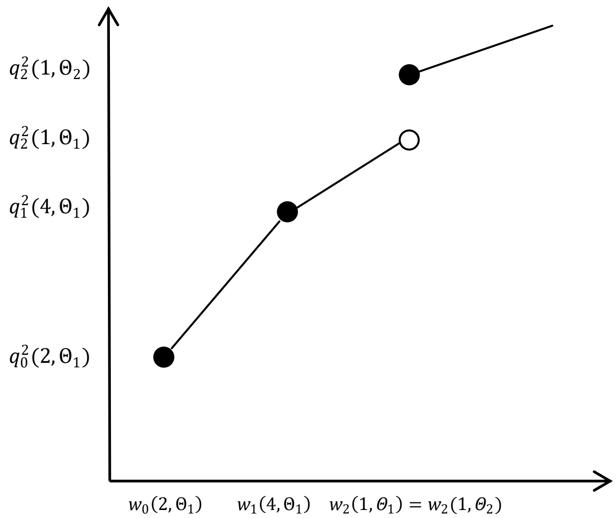

Figure 1’: Computed wealth recursive function

Figure 1’ show a function computed from the correspondence presented in figure 1. As will be discussed in section 2.5, this type of correspondences may not have a continuous selection[14]. However, as this correspondence has closed graph, it is possible to take an appropriate (i.e. measurable[15]) selection to describe the dynamic behavior of the model. Of course, because of the multiplicity of equilibrium, each selection will describe different and equally likely dynamic behavior.

As regards the genericity of this example, Hoelle (2014)[16] finds a positive measure set of economies 2 period economies, $\mathcal{E} = [e, d, \{U^i\}_{i=1}^I, \{\theta_-^i\}_{i=1}^I]$, indexed by e, which have multiple equilibria. Following definition A.2, remark A.1 and figure A.1 in the appendix, each of these

[14] The correspondence in figure 1 can be shown to be compact valued and upper hemi continuous. For a definition see for instance Stokey, Lucas and Prescott, ch. 3. There are robust examples of this type of correspondences with no continuous selections.
[15] The correspondence observed in figure 1 does not have a continuous selection in the Euclidean metric. However, because it has been established that the steady state is a set of finite cardinality, it is possible to endow the problem with the discrete metric and solve all the problems related with the lack of continuity of the transition function. Unfortunately, there are no general conditions on the cardinality of the steady state. For an example of a model with discontinuous transition functions, see Santos (2002).
[16] See Hoelle (2014), page 124. The author finds a strictly positive measure set of 2 period economies with no uncertainty (i.e. $\#S = 1$) and multiple equilibria. This type of economies can be contained in definition 1 by simply letting $p(s, s') = 0$ for $s \neq s'$ and for some $s \in S$ (i.e. s is an isolated state). As assumptions A.1 to A.6 in Magill and Quinzii (1994) does not restrict p, an economy with this “degree” of multiplicity may exist.

economies could potentially generate wealth levels with $w_0(s,\Theta) = w_0(s,\Theta')$; creating a discontinuity in the wealth map. The question is deep and, thus, a formal result on the robustness of the counter example presented in this section is left for future research.

The discussion above suggests the strength of the continuity assumption in definition 2 and 3. As this paper derives a general framework, it is necessary to derive the theoretical results without this assumption. The next section addresses this topic.

2.5 Correspondence Based Recursive Equilibria

Contrarily to the equilibrium concepts discussed in section 2.3, the "modern" recursive literature allows for multiple equilibria and, thus, requires a correspondence in order to capture the first order dynamic behavior of the economy. There are 3 seminal papers in this branch of the literature: Duffie, et. al. (1994), Kubler and Schmedders (2003) and Feng, et. al. (2013). All these papers show the existence of a time independent first order recursive structure under mild assumptions.

Section 2.5.1 introduces the results in Duffie, et. al. and discusses its usefulness and limitations for the purposes of this paper. As the recursive structure in Kubler and Schmedders (2003) uses Duffie, et. al.'s results, it share the same properties and thus will be omitted[17]. Section 2.5.2 discusses the recursive equilibrium in Feng, et. al., which is the starting point of the results in this paper.

2.5.1 Duffie's et. al. (1994) Time Homogeneous Markov Equilibria

This section illustrates how Duffie, et. al.'s results can be used to: i) show the existence of a sequential equilibrium (fact 2.5.2), a result that will be used in section 5; ii) derive a time invariant recursive structure and to generate a stationary Markov process (definition 4); iii) simulate the process (fact 2.5.1), a result that will be used in section 4.2. iv) This section also discuss the limitations of Duffie, et. al.'s results to generate numerical simulations. These facts are essential to understand how Feng, et. al.'s results fit the purposes of this paper and relate it with the recursive equilibrium literature by solving some of the problems in Duffie, et. al.

[17] One of the main contributions of Kubler and Schmedders (2003) is a correspondence based recursive structure, called Markov Equilibria, with minimal state space. As this paper is not concern with the numerical properties of the algorithms involved in the computation of the recursive structure, Kubler and Schmedders' results could be replaced with Feng, et. al.'s which are not affected by the problems in Duffie, et. al. but may have a larger state space. It would be interesting to derive Kubler and Schmedders' Markov equilibria from Feng, et. al.'s structure.

Duffie, et. al. showed that a recursive structure, called Time Homegeneous Markov Equilibria (THME), can be derived by imposing only mild assumptions on the primitives of the model if the correspondence based temporary equilibrium framework in Grandmont and Hildenbrand (1974) is applied to an enlarged state space, Z, that includes all equilibrium variables and, thus, circumvent the problems discussed in section 2.4. The virtue of this approach is its generality and its robustness to the presence of multiple equilibria.

A THME is build using 3 preliminary elements: an expectation correspondence, a self-justified set and a transition function. Let $Z = \left\{[s, \theta_{-}, c, q, \theta] \in S \times \mathbb{R}^{IJ} \times \mathbb{R}^{I} \times \mathbb{R}^{J} \times \mathbb{R}^{IJ} \,\middle|\, \sum_{i=1}^{I} \theta_{-}^{i} = \vec{0}, \sum_{i=1}^{I} \theta^{i} = \vec{0}\right\}$ be the state space.

An *expectation correspondence* is a map, $G: Z \to \mathcal{P}(Z)$, where $\mathcal{P}(Z)$ is the set of probability measures generated from Z. It will be said that $\mu \in G(z_0)$, if z_0 and any realization of the random variable z_1, which has conditional distribution given by μ, satisfy the optimality conditions implied by b) in definition 1. Typically and without loss of generality, G is supposed to have a closed graph.

The purpose of the expectation correspondence is to obtain a sequence $\{z_t\}_{t=0}^{T}$, with $T \in \mathbb{N}$, such that the conditional distribution of z_t is contained in $G(z_{t-1})$. This sequence can be constructed as follows: it will be said that $\mu \in G(z_{t-1})$ and $z_t \sim \mu$, where $z_{t-1} = [s_{t-1}, \theta_{t-2}, c_{t-1}, q_{t-1}, \theta_{t-1}]$ is feasible, if for each $i \in I$

1) $c_{t-1}^{i} = e^{i}(s_{t-1}) + \theta_{t-2}^{i} d(s_{t-1}) - \theta_{t-1}^{i} q_{t-1}$
2) $q_{t-1}\left(u_{s_{t-1}}^{i}\left(c_{t-1}^{i}\right)\right)' = \beta E_{\mu}\left[d(s_t)\left(u_{s_t}^{i}\left(c_t^{i}\right)\right)'\right]$

Where E_{μ} is the expectation with respect to μ[18], which is an arbitrary probability measure on $\mathcal{P}(Z)$, and $\left(u_{s_{t-1}}^{i}\left(c_{t-1}^{i}\right)\right)'$ is the partial derivative of $u_{s_{t-1}}^{i}$.

Let $K \subset Z$ be any measurable set such that $z_t \in K$ for any $\{z_t\}_{t=0}^{T}$ and any $T \in \mathbb{N}$. The existence of this set, typically compact, for economies with short lived assets and finite shocks is guaranteed by the results in Maguill and Quinzii (1994, see page 871). Define $C_0 \equiv K$. Then, the

[18] Duffie, et. al. add 2 technical conditions to 1) and 2). The first one restricts the marginal distribution of s_{t-1} and θ_{t-2} and the second one affects the support of μ. For a detailed discussion see for instance Duffie, et. al. pages 763 and 767.

set of all initial states of any 2 period (truncated) economy[19] is contained in the following set: $C_1 = \{z \in K | \exists\, v \in G(z)\ and\ sup_{D \subset C_0} v(D) = 1\}$, where *sup* denotes the supremum. Inductively, a sequence of nested sets $\{C_j\}$ for $j \geq 1$ can be constructed with C_j containing the initial states of any j-period economy.

It follows from Theorem 1.2 in Duffie, et. al. (page 754) that $J = \cap_{j=0}^{\infty} cl(\overline{C_j})$ is non empty and compact, where *cl* denotes the clousure of a set, if K is compact and $C_j \neq \emptyset$ for $j \geq 1$. In the present context both conditions are guaranteed to hold by corollary 5.3 in Maguill and Quinzii (page 868)[20]. J is *called self-justified set.*

Remarkably, 2 facts are worth mentioning as a conclusion of the above paragraph:

> Fact 2.5-1): J is the smallest[21] set that can be used to define an expectation correspondence as it contains all initial states of any infinite horizon sequential competitive equilibrium and is time independent. Thus, it can be used to iterate forward a *first order dynamic stochastic process with a time invariant state space* as $G(z) \cap \mathcal{P}(J) \neq \emptyset$ for all $z \in J$.
>
> Fact 2.5-2): The existence of J requires $C_j \neq \emptyset$ for $j \geq 1$ and K to be compact. While the former is typically shown in 2 steps ($C_j \neq \emptyset$ for $j \leq T < \infty$ and then extended to $j \geq 1$ by induction, see lemmas 3.4 and 3.5, page 768), the latter follows from the existence of uniform bounds on endogenous variables. These 2 facts combined with an argument on the optimality of the sequences generated from J using G (see section 3.4 in Duffie, et. al.) can be used to show the existence of a sequential competitive equilibrium. Although this result has already been applied to other incomplete market economies for the case of finite shocks (see Kubler and Schmedders, 2003, Lemma 2), it is not generally used in economies where S is assumed to be

[19] As long as Z is compact, it is clear that any $\{z_t\}_{t=0}^{T}$ contained in G is a sequential competitive equilibrium. For an arbitrary state space, a set of uniform (stationary) bounds are required. For instance, this is done by Duffie, et. al. (1994) using Radner's (1972) existence result (Lemma 3.4, page 768) and by Kubler and Schmedders (2003) using several elements of the Geanakolos and Zame (2002) existence proof (Lemma 3, page 1777).

[20] Duffie, et. al. (section 3) established the existence of a compact set K (page 767, Lemma 3.1 and 3.2) and $C_j \neq \emptyset$ for any $j \in \mathbb{N}$ (Lemma 3.4 and 3.5, page 768) in a heterogeneous agent economy with a finite number of Lucas trees and short sales constraints. Braido (2013) extended these results for a general asset structure under mild assumption on preferences.

[21] In the temporary equilibrium framework of Hildenbrand and Grandmont (1974) it is possible to set $J = Z$ as overlapping generation agents only live 2 periods. In this type of economies, agents live infinitely many periods and thus it is possible that the backward induction procedure implied by equations 1 and 2 converges to an empty set. Fact 2.5.1) show that this is not the case for economies with compact K and $C_j \neq \emptyset$ for $j \geq 1$.

uncountable and compact. For the results in this paper, the last structure of exogenous shocks turns out to be important[22]. Thus, this type of existence proof will be discussed in section 5.2. Fortunately, in the model presented in section 2.1 and 2.2, the Mas-Colell and Zame (1996) framework allows showing the existence of J and the compactness of K. Further, the optimality argument in section 3.4 of Duffie, et. al. can be straightforwardly extended in that model to the case of uncountable shocks.

A time invariant Markov process is constructed using 2 building: a state space and a Markov operator (see Stokey, Lucas and Prescott Ch. 8). In the context of Duffie, et. al., the state space is J [23]. The Markov operator is denoted π and is a selection of G (denoted $\pi \sim G$) such that $\pi: J \to \mathcal{P}(J)$. Any $\pi(.,A)$ must be measurable and $\pi(z,.)$ must be a probability measure for any measurable set A and $z \in J$ respectively. This last condition follows directly from the definition of the expectation correspondence. If J is closed, then the Kuratowski measurable selection theorem (see for instance Hildenbrand 1974, page 55) implies that the restriction of G to J has a measurable selection.

Thus, for the economy described in section 2.1, which satisfy all the relevant assumption in Magill and Quinzii (1994, see page 858), the results in Duffie, et. al. guarantee the existence of correspondence based recursive structure on an enlarged state space which the authors called Time Homogeneous Markov Equilibrium (THME):

<u>Definition 4:</u> A pair (J,π) is a <u>THME</u> for G if π is a Markov operator and J is a set that satisfies $\pi(z) \in G(z)$ for all $z \in J$.

Even though the results in Duffie, et. al. can be used to guarantee the existence of a recursive structure, a THME *is not a computable representation of the* sequential competitive equilibrium as *the time invariant transition functions of the recursive equilibrium depend on unobservable variables*. This fact is illustrated by the following lemma.

[22] Fact 2.5.2) implies that any truncated economy ($j \leq T < \infty$) which has uniformly bounded endogenous variables (contained in K) can be used to prove the existence of a sequential infinite horizon equilibria. That is, any recursive equilibrium is a sequential equilibrium. However, there may be some sequential equilibria that are not recursive or that do not have a terminal debt level equal to 0. So, fact 2.5.2) can be used to prove the existence of a subset of all possible sequential equilibria. I would like to thank A. Manelli for pointing this out to me.

[23] It is standard to assume that Z is a Borel Space. As the Cartesian product of a finite set and a finite dimensional Euclidean space is a complete, separable and metric space, the product space is a Polish space. Thus, Z is a measurable subset of a Polish space. If $\mathcal{B}_{[Z]}$ is the Borel sigma-algebra generated from Z, $(Z,\mathcal{B}_{[Z]})$ is a Borel Space. Consequently, measurable will always mean Borel measurable and any measure will be a Borel measure.

Suppose that the above defined state space, Z, can be written as a product space, $Z = S \times \hat{Z}$,where $\hat{Z} = \{[\theta_{-}, c, q, \theta] \in \mathbb{R}^{IJ} \times \mathbb{R}^{I} \times \mathbb{R}^{J} \times \mathbb{R}^{IJ} \,|\, \sum_{i=1}^{I} \theta_{-}^{i} = \vec{0}, \sum_{i=1}^{I} \theta^{i} = \vec{0}\}$.

__Lemma 1:__ If (J, π) constitute a THME and $Z = S \times \hat{Z}$, any realization of a process $\{z_t\}$ in Z satisfies as $\hat{z}_{t+1} = f(s_{t+1}, \alpha_{t+1}, z_t)$ where f is a measurable function and $\alpha_{t+1} \in [0,1]$ is uniformly distributed and i.i.d.

Proof: See Lemma 2.22 page 34 In Kallenberg (2006).

Duffie, et. al. (1994) interpret α_{t+1} as the realization of a sunspot (page 756). Note that lemma 1 implies that for each state z_t, any exogenous shock s_{t+1} could be associated with a continuum of possible continuation states in $\hat{Z}$, each one of them associated with an *unobservable variable* (α_{t+1}). Consequently, a tree structure with a finite number of branches after each node would not be an appropriate representation of $\{z_t\}_{t=0}^{\infty}$. It is clear then that the THME has limited predictive power about the evolution of the state process. Thus, a "refinement" is required to obtain a computable object. This is done in definition A.1 in section A.1.1 in the appendix which presents the notions of spotless (i.e. sunspots free) and conditionally spotless THME.

Duffie, et. al. also provided sufficient conditions for the existence of a spotless THME (see Proposition 1.3 in page 757). In particular, if S is a *finite set*, a subset of G,denoted g, is an expectation correspondence. Further, if g has a compact self-justified[24] set, then (J, π) is a spotless THME for g. As the economy described in this section has $\#S < \infty$, the conditions to guarantee the existence of J for G hold for g.

Unfortunately, *for the purposes of this* and Duffie, et. al.'s papers, the existence of a spotless recursive structure is insufficient as the Markov process associated with (J, π) may not *stationary.* Heuristically, a stochastic process is stationary if the unconditional distribution of z does not vary with time. So far, only the conditional distribution of z has been shown to be time invariant. The concept of conditionally spotless THME, also presented in section A.1.1, had to be introduced in order to address this topic and to derive a notion of steady state, called *ergodic invariant measure* (see Theorem 1.1 and Proposition 1.3 in Duffie, et. al., page 750 and 757 respectively). This measure guarantees that the Markov process associated with the THME

[24] The expectation correspondence $g \subset G$ is obtained by restricting μ in equation 2 to the set $\mathcal{P}_F(S \times \hat{Z}) \subset \mathcal{P}(Z)$ defined in setion A.1.1 in the appendix. The set of conditions on K and C_j mentioned above can still be used to guarantee the existence of a self-justified set for g.

is stationary. Further, the ergodicity of the steady state *guarantees that the process generates convergent sample paths*. These 2 issues will be briefly addressed in section 2.6 and discussed in sections 3 and 4.

The authors argued that a conditionally spotless THME implies that any sequence $\{z_t\}_{t=0}^{\infty}$ in J can be described by a function f that satisfies $\hat{z}_{t+1} = f(s_{t+1}, \alpha_t, z_t)$ for any t, where $\alpha_t \in [0,1]$ is uniformly distributed, i.i.d and represents a sunspot. The purpose of this variable is to convexify $g(z)$[25] for $z \in J$, a property that is crucial in Duffie, et. al. to establish the existence of a well behaved (i.e. ergodic) steady state. Note that f is not a computable object because of the presence of sunspots at t and thus, inappropriate for the purpose of this paper.

2.5.2 Feng, et. al.'s Recursive Equilibria

The virtue of Feng, et. al. approach is to derive a recursive structure that exists even in the presence of multiple equilibria and that generates computable time invariant transitions, that is: it can be used to derive laws of motion for the endogenous variables that do not depend neither on unobservable variables nor time. Besides, this structure has a lower dimensional state space when is compared to Duffie, et. al.'s, a property that is desirable from a numerical point of view.

In order to obtain these results, the authors restricted the number of possible recursive equilibria, as it will be clear in the next paragraph, and derived a correspondence $\Phi: \tilde{Z} \times S \rightrightarrows \tilde{Z}$ that maps $(\tilde{z}_t, s_{t+1}) \longmapsto \tilde{z}_{t+1}$ which can be used to construct the analogous of Duffie, et. al.'s expectations correspondence, G[26].

Let $Z \equiv \left\{[s, q, \theta] \in S \times \mathbb{R}^J \times \mathbb{R}^{IJ} \,\middle|\, \sum_{i=1}^{I} \theta^i = \vec{0}\right\}$, $m^{i,j} \equiv d^j(s)\left(u_s^i(c^i)\right)'$, where m is the vector of shadow values of the marginal return to investment for all assets and all agents. Assume, additionally to the hypothesis stated in section 2.1, that there exist a short sale constraint $\bar{B} > 0$ such that $\theta^{i,j} \geq -\bar{B}$. Using the budget constraint, equation 1, it is possible to define a correspondence V that maps $(z) \mapsto m$ as follows: for each $z \in Z$, $c^i \in [e^i(s) + \theta^i d(s) - I\bar{B}q, e^i(s) + \theta^i d(s) + I\bar{B}q]$ defines a selection $m \sim V(z)$ which is obtained by taking some $\theta_+^{i,j} \geq -\bar{B}$ for all $i \in I$ and $j \in J$. Provided, as discussed in section 2.5.1, that all endogenous variables in the model are (uniformly) contained in a compact set K, V is compact valued and $Gr(V)$ is compact.

[25] See Section A.1.1 in the appendix for a discussion on the convexification of the expectation correspondence using α_t.
[26] The procedure to derive the analogous of G in Duffie, et. al. from Φ will be presented at the beginning of section 3.

Then, as in the previous subsection, it is possible to derive a time invariant compact state space, which is analogous to Duffie, et. al.'s self justified set. Let $\widetilde{K} \subset K$ and $\widetilde{K} \equiv Gr(V_0)$. The first order conditions of the model can be written as:

3) $c^i = e^i(s) + \theta^i d(s) - \theta^i_+ q$

4) $\left[q\left(u^i_s(c^i)\right)' - \beta E_{p(s,.)}(m^i_+)\right][\theta^i_+ - \bar{B}] = \vec{0}$

Where $E_{p(s,.)}$ is the expectation with respect to $p(s,.)$, the conditional distribution of s_+ given s, and $m^i_+(s_+) \sim V(\theta_+, q_+, s_+)$. Thus, 4) is defined using the expected value with respect to $p(s,.)$ over $[\theta_+, q_+](s_+)$. Following equation 2), the expected value should have been taken with respect to any possible distribution of z_+, μ. Thus, *equation 4) captures a subset of all possible z for any given z_+*[27].

Then, the set of all states $\tilde{z} \in \widetilde{K}$ of any 2 period economy are contained in $Gr(V_1) = \left\{\tilde{z} \in \widetilde{K} \middle| \exists\, \tilde{z}_+ \in Gr(V_0)\ with\ \tilde{z}, \tilde{z}_+ satisfying\ eq. 3)\ and\ 4)\right\}$. That is, $[s, q, \theta, m] \in Gr(V_1)$ if $c^i(\theta^i_+)$ obtained from 3) for all $i \in I$ satisfy equation 4) for some $m^i_+(s_+) \sim V(\theta_+, q_+, s_+)$. Let $Gr(V_j) = C_j$. Iterating on this procedure, it is possible to derive a sequence of nested sets $\{C_j\}$ for $j \geq 1$ where C_j contains all $\tilde{z}_0$ of any j-period economy. Note that this procedure defines an operator $G_K: Gr(V) \to Gr(V)$. The non-emptiness and compactness of each C_j follows from the arguments discussed in section 2.5.1 as, respectively, equations 3) and 4) are identical to the optimality conditions implied by the definition of "equilibrium with explicit debt constraint" in Magill and Quinzii (page 862) and the recursive equilibria in Feng, at. al. are a subset of those in Duffie, et. al.[28]

As G_K maps compact sets to compact sets, Feng, et. al. showed (theorem 2.1 in page 6) that $V_n \to V^*$, where V^* is the analogous of Duffie, et. al.'s self justified set. Thus $Gr(V^*) = \tilde{J}$ contains all possible first period payoff relevant variables $\tilde{z}_0(\sigma_0)$ for the sequential competitive equilibrium in definition 1.

Finally, $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ is defined as follows: take any $\tilde{z} = [\tilde{s}, \tilde{\theta}, \tilde{q}, \widetilde{m}] \in \tilde{J}$, it will be said that $\tilde{z}_+ \in \Phi(\tilde{z}, \tilde{s}_+)$ if $\tilde{z}_+ \in \tilde{J}$ and $(\tilde{z}, \tilde{z}_+)$ simultaneously satisfying equations 3) and 4) with $m(s_+) \sim V^*(\theta_+, q_+, s_+)(s_+)$ and $\widetilde{m}_+ \sim V^*(\theta_+, q_+, s_+)(\tilde{s}_+)$. The following definition, which will be

[27] Duffie, et. al. also restricts $[\theta_+, q_+]$ to be a function of s_+ and that $s_+ \sim p(s,.)$. However, it is still possible to find a distribution, $z_+ \sim \mu$, which satisfy this restrictions and $E_{p(s,.)} \neq E_\mu$.
[28] Section 5.1 will provide some additional details about these facts.

addressed carefully in the appendix (see the remark before theorem 4), summarizes the preceding discussion:

Definition 5: Let $\tilde{J} = Gr(V^*)$ and $\tilde{J} \subseteq \tilde{K}$. $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ is an equilibrium correspondence if $\tilde{z}_{t+1} \in \Phi(\tilde{z}_t, s_{t+1})$ and $\{\tilde{z}_t\}_{t=0}^{\infty}$ satisfy the optimality conditions in equations 3)-4) and the feasibility restrictions in the definition of Z.

The procedure described above can be repeated an infinite number of times as $\tilde{J}$ contain all possible $\tilde{z}_0(\sigma_0)$, which are appropriate initial conditions for any $T \in \mathbb{N}$ period economy. A *time invariant transition function* is obtained by taking a selection of Φ, denoted $\varphi \sim \Phi$. This function is measurable, as Φ has closed graph and is compact valued (see Stokey, Lucas and Prescott, page 60 theorem 3.4 and 184 theorem 7.6), and does not depend on unobservable variables, thus *it constitutes the starting point of the theoretical results in section 3.*

2.6 Computability, Simulations and Empirical Validity

In some applications, it may be interesting to test the empirical performance of the predictions generated by the model. As general equilibrium economies typically lack of closed form solutions, in order to obtain these predictions it is necessary to numerically approximate the endogenous variables model.

Unfortunately, the dimension of the sequential competitive equilibrium notion hinders its computability. In particular, any sequential competitive equilibrium can be thought as an infinite sequence of measurable functions $\{\bar{z}_t(\sigma_t)\}_{t=0}^{\infty}$, $\bar{z}_t: \mathfrak{T} \to \mathbb{R}^I \times \mathbb{R}^J \times \mathbb{R}^{IJ}$, which satisfy conditions a) and b) of definition 1[29] and $\bar{z}_t(\sigma_t) = [\bar{c}_t(\sigma_t), \bar{q}_t(\sigma_t), \bar{\theta}_t(\sigma_t)]$ for some $\sigma_t \in \mathfrak{T}$. Thus, in order to compute $\{\bar{z}_t(\sigma_t)\}_{t=0}^{\infty}$, it is necessary to solve an infinite number of non linear systems of equations which came from equations 3 and 4. As discussed in Judd, et. al. (2003), this task can rarely be achieved in finite CPU time.

Moreover, the stochastic process generated from the sequential equilibrium, with typical realization $\{\bar{z}_t(\sigma_t)\}_{t=0}^{\infty}$, is generally non stationary[30]. This property follows from the associated measure μ_t defined in section 2.1, that is allowed to change over time, and turns the empirical assessment of the model a difficult task even if a closed form solution is available. For instance,

[29] Note that $\mathfrak{T} = S^{\infty}$, where $S^{\infty} = S \times S \ldots$ is the infinite Cartesian product of finite sets.
[30] If $\{s_t\}_{t=0}^{\infty}$ is an i.i.d process, a sequential competitive equilibrium $\{\tilde{z}_t\}_{t=0}^{\infty}$ is stationary (see Stokey, Lucas and Prescott (1989) page 224, exercise 8.6).

any calibration procedure may require matching the unconditional expected value of some endogenous variables against its empirical counterpart. As it was discussed in section 2.5, $\bar{z}_t(\sigma_t)$ can be proved to be uniformly bounded, which suggests that time series must be detrended. Further, it is typically assumed that the unconditional expected value of the transformed time series is time invariant (see De Jong and Dave, 2007). The non stationarity of $\{\bar{z}_t(\sigma_t)\}_{t=0}^{\infty}$ implies that $E_{\mu_t}(\bar{z}_t)$ will change over time, making the calibration of the model impossible.

An appropriate recursive structure solves one of these problems as it endows the model with a time invariant transition function that depends on a low dimensional observable state space. This function can be computed in finite time using the algorithms discussed in Feng, et. al. or Kubler and Schmedders (2003). However, the stationarity of the stochastic process generated from the recursive structure can be difficult to obtain in a correspondence based framework.

Equipped with a recursive representation of the sequential equilibrium and a time invariant state space, as it is the case in Feng, et. al., it is possible to define a Markov stochastic process, which may or may not be stationary. Formally, a necessary and sufficient condition to guarantee this last property is the existence of an *invariant measure* for the process (see Meyn and Tweedie, 2008, page 232). From an economic point of view, an invariant measure can be seen as a notion of steady state in the sense that the unconditional distribution of the payoff relevant variables $\bar{z}_t$ does not change over time. Section 4.2 finally states the relationship between the stationarity of the process and the existence of an invariant measure.

This delicate issue, which is only addressed by Duffie, et. al. among the papers mentioned in sections 2.3 and 2.5, is essential to obtain empirically meaningful dynamics as it allows to obtain time invariant unconditional moments $E_{\mu^*}(\bar{z}_t) = E_{\mu^*}(\bar{z}_{t'})$ for any $t \neq t'$, where μ^* is any invariant measure of the Markov process, and this moments can be matched with data.

Thus, a natural procedure to get empirically relevant predictions is to compute a time invariant recursive structure with an associated stationary Markov process. Then, obtain simulations that approximate the unconditional expected value of the endogenous variables in the (exact) model, which in turn can be used to match the (observed) time series behavior. This fact requires that simulations converge to unconditional moments; a property that is achieved if the invariant measures associated with the true and approximated recursive structures are *ergodic*.

Formally, it is necessary to insure that:

Fact 2.6-i) $T^{-1}\left[\sum_{t=1}^{T} f\left(\bar{z}_t^{j,\vartheta}\right)\right] \rightarrow_{a.s.} E_{\mu_{j,\vartheta}^*}(f)$ and $T^{-1}\left[\sum_{t=1}^{T} f\left(\bar{z}_t^{\vartheta}\right)\right] \rightarrow_{a.s.} E_{\mu_{\vartheta}^*}(f)$ where j denotes the j^{th} numerical approximation of the model and f is a $\bar{z}$-measurable, possibly continuous, function. Further, $\mu_{j,\vartheta}^*$, μ_{ϑ}^* is any invariant ergodic measure associated with the j^{th} approximation and associated with true model, respectively, both taking parameters values $\vartheta \in \Lambda$. Typically, ϑ contains parameters related with preferences and endowments and Λ is compact. The convergence of simulations will be assumed to be almost surely ($a.s.$) in a measure that will be defined in section 4.2.

Fact 2.6-ii) $E_{\mu_{j,\vartheta}^*}(f) \rightarrow_{Weak\,*} E_{\mu_{\vartheta}^*}(f)$ where the convergence will be assumed to be in the $weak\,*$ topology.

Fact 2.6-iii) $Min_{\vartheta \in \Lambda} \left\| E_{\mu_{\vartheta}^*}(f) - \bar{X}_f \right\|$ where $\bar{X}_f$ is the mean of the empirical analogous of $f(\bar{z})$ obtained from detrended data and $\|.\|$ is the Euclidean norm as suggested by De Jong and Dave (2007).

The purpose of this paper is to derive conditions that guarantee that facts 2.6-i) and 2.6-ii) hold in endowment general equilibrium models with incomplete markets, allowing for multiple equilibria.

Fact 2.6-i) requires approximating the recursive structure in Feng, et.al., which is the only known recursive structure that generates computable (i.e. that depend on observable variables) time independent transitions and allows for multiple equilibria. This fact also assumes that appropriately transformed exact and numerical simulations converge almost surely to unconditional moments, which are obtained by integrating against an ergodic invariant measure. The conditions to prove the existence of an invariant measure and its ergodicity will be presented in section 3.

Contrarily to what has been done in the literature, this paper derives the conditions required to prove the existence of an invariant measure *separately* from the strengthening necessary to insure its ergodicity. From a pure theoretical point of view, as the stochastic process associated with the sequential equilibria may not be stationary, establishing conditions which guarantee the first property is desirable.

Section 4 contains the implications of the existence of an ergodic invariant measure for facts 2.6-i) and 2.6-ii). Section 4.2 includes the sufficient conditions which guarantee that simulations converge almost surely, provided the existence of an ergodic invariant measure. Feng, et. al. assumed that the convergence of the approximated transitions is *uniform,* a hypothesis that will be slightly modified to prove fact 2.6-ii) in section 4.1. Thus, the ergodic nature of the invariant measure is only required for the numerical part of the paper as it insures the accuracy of simulations. Section 5 presents an economy which satisfies the conditions necessary to guarantee facts 2.6-i) and 2.6-ii).

Finally, using facts 2.6-i) to 2.6-iii) together in order to perform an empirically meaningful exercise is beyond the scope of this paper. The design of an algorithm that satisfies the conditions which guarantee fact 2.6-ii) for an economy like the one presented in section 5 are also left for future research.

3. Stationarity and Ergodicity

As was discussed in the previous section, to obtain accurate and empirically meaningful numerical simulations[31], some notion of stationarity is required. Although time homogeneity is desirably, it is clearly not enough. A reliable procedure requires an ergodic invariant measure. Heuristically, this fact was stated in section 2.6. This section formally proves the existence of an invariant measure and its ergodicity for a computable, correspondence based recursive structure.

Sections 3.1 and 3.2 together establish the existence of an invariant measure for models with at most a finite number of exogenous shocks that fit into Feng, et. al's framework. Further,

[31] The precise meaning of "numerical simulations" will be given in section 4. The results in this paper assume the existence of a uniformly convergent method to compute a recursive equilibrium in the sense of Feng, at. al.

sections 3.1 and 3.3 show the existence of an ergodic invariant measure for models with an uncountable number of shocks[32].

In the context of Duffie, et. al., section 2.5.1 discussed the 2 building blocks of any Markov process, namely: a state space and a Markov operator. Lemma 2 in section 3.1 shows that the same objects can be defined for the framework in Feng, et. al. under standard assumptions. Then, theorems 1 and 2, respectively, state properties of the underlying Markov process which guarantee the existence of an invariant measure and its ergodicity.

Section 3.2, for the case of a finite number of exogenous shocks, states conditions on the Markov operator defined in lemma 2 which guarantee the properties required in theorem 1. Section 3.3 states conditions on the Markov operator which guarantee the properties in theorem 1 and 2 for the case of uncountable exogenous shocks. Section 3.4 states sufficient conditions (in terms of the number of possible exogenous shocks, its distribution and the stationary transition $\varphi \sim \Phi$ defined in section 2.5.2) which guarantee that the related Markov operator satisfies the conditions stated in section 3.3.

The results obtained in sections 3 do not depend on the specific structure contained in a non-optimal general equilibrium economy, as the one described in section 2. As in Duffie, et. al., they could also be applied to OLG stochastic economies and repeated games. Thus, in this section and in the next one it will only be assumed the existence of a time invariant correspondence based recursive structure, $\Phi: \tilde{J} \times S \rightarrow \tilde{J}$, where Φ is closed graph and compact valued, $\tilde{J} = S \times \hat{Z}$, S is the set of exogenous shocks and $\hat{Z}$ contain the endogenous variables in some model.

The figure below illustrates the theoretical structure in the rest of section 3. This structure is composed by 3 assumptions, 3 properties of the Markov process associated with a selection of the equilibrium correspondence (definition 5), 2 theorems, 2 propositions and 3 lemmas.

[32] Economies with an infinite but countable number of shocks are intentionally left out as they represent a particularly challenging case for the purpose of this paper: the existence of equilibria requires the same strength of assumptions as in the case of uncountable shocks (see Mas Collel and Zame, 1996) and the existence of an invariant measure is as difficult to show as the case of a finite number of shocks.

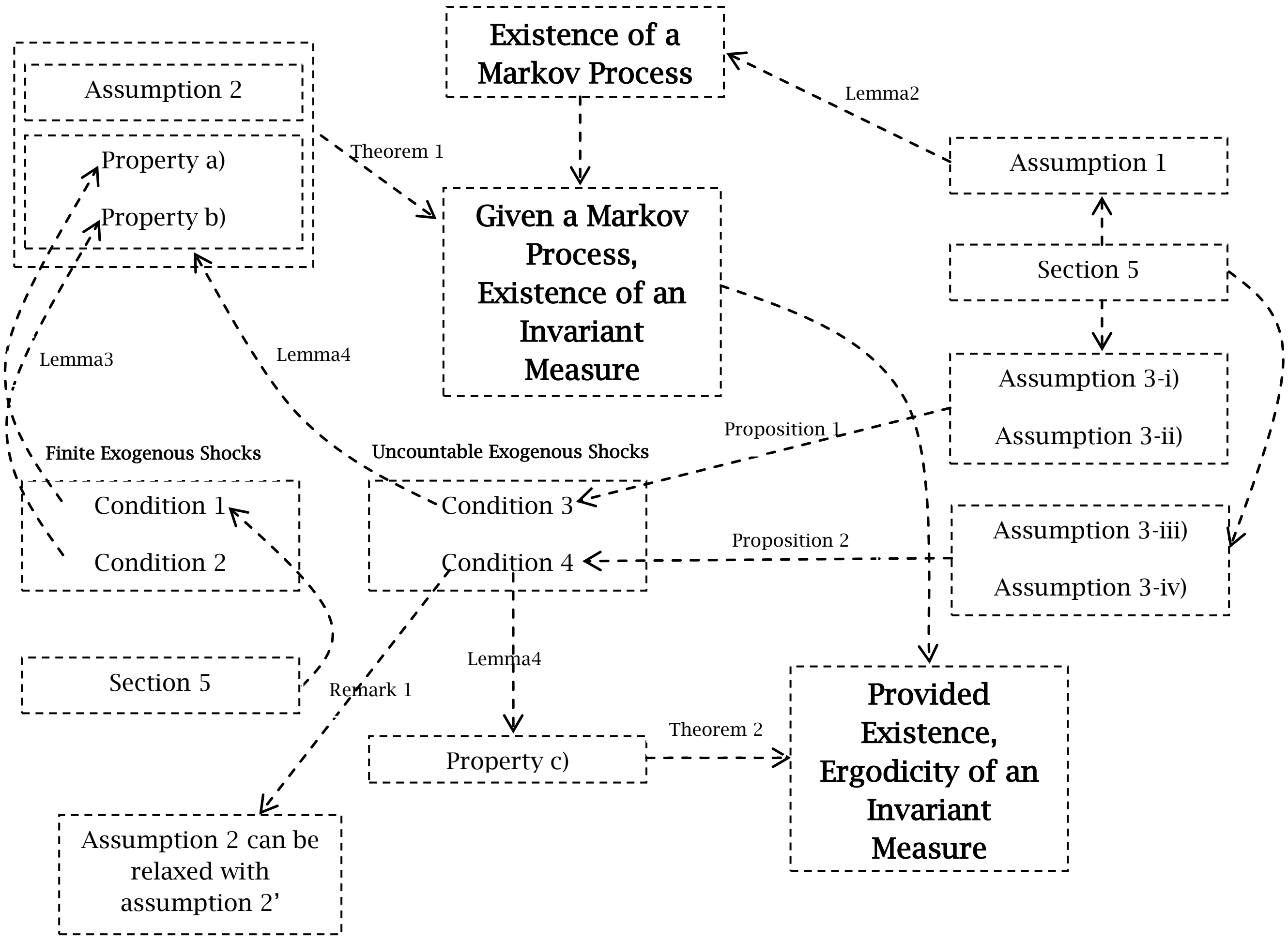

Figure 2

Just as an example of the implications of figure 2, note that assumption 3, by means of propositions 1 and 2, guarantee that conditions 3 and 4 on the Markov operator hold for the case of an uncountable number of exogenous shocks. Then, lemma 4 guarantees that condition 3 (associated with assumptions 3-i) and 3-ii)) imply properties a and b, which, together with assumption 2 prove the existence of an invariant measure using theorem 1. Also lemma 4 guarantee that condition 4 (associated with assumptions 3-iii) and 3-iv)) implies property c) that is sufficient for the ergodicity of the invariant measure. Finally, assumption 3 can be traced back to the primitive conditions of the sequential economy, which is discussed in section 5.

Finally, note that only condition 2 and assumption 2 are "disconnected" from the results in this paper. As will be discussed below, this is because the study of sufficient conditions based on the primitives of the sequential equilibrium that led to these properties are left for future research due to its difficulty.

3.1 Theorems 1 and 2: Existence of an Invariant Measure and Ergodicity

The starting point of this section is a Markov operator for exogenous shocks $p(s,A) \geq 0$ for all $s \in S$ and $A \in \mathcal{B}_S$, where S is compact and $\mathcal{B}_S$ are the Borel sets in S, together with the equilibrium correspondence in Feng, et. al., which was discussed in section 2.5.2, that is assumed to have the following properties:

Assumption 1: Let $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ be the equilibrium correspondence in definition 5. Then, $\tilde{J}$ is compact and Φ is upper hemi continuous and compact valued.

These properties can be obtained from fairly mild assumptions on the primitives of the models discuss in this paper for both the case of finite (Magill and Quinzii, 1994, page 858, assumption 1 to 5) and infinite (Araujo, et. al. 1996, page 122, assumptions 1 and 3) exogenous shocks. A detailed discussion is postponed to section 5.

Assumption 1 together with a result in Hildenbrand and Grandamont (1974) allows defining a convenient Markov operator.

Lemma 2: Let Φ satisfy assumption 1. Then, $\varphi \sim \Phi$ is a $\mathcal{B}_{\tilde{J}\times S}$ -measurable selection of Φ and $P_\varphi(\tilde{z},A) \geq 0$ is a Markov operator on $(\tilde{J},\mathcal{B}_{\tilde{J}})$, where P_φ is given by:

5) $P_\varphi(\tilde{z},A) = p(s,\{s' \in S|\varphi(\tilde{z},s') \in A\})$, *where* $\tilde{z} = [s,\hat{z}]$

Proof: See Lemma 1 in Hildenbrand and Grandamont (1974), page 260.

Note that Lemma 2 implies the existence of a $\mathcal{B}_{\tilde{J}\times S}$ - measurable function φ, which is the natural candidate to be the time invariant transition function of the process defined by $(\tilde{J},P_\varphi)$ with typical realization $\{\tilde{z}_t\}_{t=0}^{\infty}$ as it satisfies $\tilde{z}_{t+1} = \varphi(\tilde{z}_t, s_{t+1})$ for any initial condition[33].

Let $B(\tilde{J})$ and $\mathcal{P}(\tilde{J})$ be the space of bounded $\mathcal{B}_{\tilde{J}}$-measurable functions and the space of probability measures on $\tilde{J}$ respectively. Let $\hat{P}_\varphi: B(\tilde{J}) \to B(\tilde{J})$ and $P_\varphi^*: \mathcal{P}(\tilde{J}) \to \mathcal{P}(\tilde{J})$ be the semigroup and adjoint operators defined by $\hat{P}_\varphi f(\tilde{z}) = \int f(\tilde{z}')P_\varphi(\tilde{z},d\tilde{z}')$ and $P_\varphi^*\mu(A) = \int \mu(d\tilde{z})P_\varphi(\tilde{z},A)$. Standard results

[33]A careful definition of the stochastic process associated with $(\tilde{J},P_\varphi)$ will be given in section 4.

(Stokey, Lucas and Prescott, 1989, page 213 to 216) imply that $\hat{P}_\varphi f(\tilde{z}) \in B(\tilde{J})$ and $P_\varphi^* \mu(A) \in \mathcal{P}(\tilde{J})$ provided that $f \in B(\tilde{J})$ and $\mu \in \mathcal{P}(\tilde{J})$, respectively.

The purpose of this section is to establish *properties* which guarantee that the Markov process $(\tilde{J}, P_\varphi)$ has an invariant measure, $\mu \in \mathcal{P}(\tilde{J})$ with $\mu = P_\varphi^* \mu$, provided that under assumption 1, φ *may not be continuous*. Note that an invariant measure is a fixed point of the adjoint operator.

Let $C(\tilde{J})$ be the space of continuous functions on $\tilde{J}$. It will be said that P_φ has the Feller property if the associated semigroup operator maps $C(\tilde{J})$ into itself. Lemma 9.5 in Stokey, Lucas and Prescott (page 261) can be modified to show that, if $f \in C(\tilde{J})$ but $\varphi \notin C(\tilde{J} \times S)$, $\hat{P}_\varphi f(\tilde{z}) \notin C(\tilde{J})$.

The absence of the Feller property also affects the continuity of the adjoint operator, which is critical to guarantee the existence of a fixed point of it. As P_φ^* is defined over an infinite dimensional space, in order to discuss its continuity, it is necessary to select an adequate topology. The *weak** topology, the coarsest topology that makes the linear functional $\{\mu \mapsto \int f d\mu,\ f \in C(\tilde{J})\}$ continuous, is frequently chosen in this framework. This is because P_φ^* generate sequences of *weak** convergent measures under mild assumptions[34]. In particular, under assumption 1, $\tilde{J}$ is a compact subset of a finite dimensional Euclidean space. Thus, Helly's theorem (Stokey, Lucas and Prescott, page 374) implies the existence of a *weak** - convergent subsequence in $\mathcal{P}(\tilde{J})$, which is the starting point of most existence theorems.

As discussed in Aliprantis and Border (2006, page 47), the choice of a weak topology implies a trade off: there are "a lot" of weakly convergent sequences but there are "few" weakly continuous functionals. Thus, frequently, the Feller property is used to guarantee the *weak** continuity of P_φ^*. That is, $\mu_n \rightarrow_{Weak*} \mu$ implies $P_\varphi^* \mu_n \rightarrow_{Weak*} P_\varphi^* \mu$ if $\hat{P}_\varphi$ has the Feller property (see Stokey. Lucas and Prescott, page 376).

If φ can be shown to be continuous, under assumption 1, Theorem 2.9 in Futia (1982, page 383) would imply the existence of an invariant measure for P_φ^*. It only suffice to take a sequence of measures generated by applying P_φ^* iteratively on some $\mu_0 \in \mathcal{P}(\tilde{J})$ that is robust to cyclical behavior and fits into the framework of Helly's theorem. Let $\mu_{n_k} \rightarrow_{Weak*} \mu$ be the subsequence generated by Helly's theorem. The continuity of P_φ^* implies $P_\varphi^* \mu_{n_k} \rightarrow_{Weak*} P_\varphi^* \mu$. Subtracting both

[34] This is not the case of the *strong* topology, which is the topology generated by the total variation norm. Stokey, Lucas and Prescott (page 335 to 337) provides an example of a Markov process that generates sequences that converge in the *weak** topology but not in the strong (norm) topology.

subsequences, the desired result follows. *Theorem 1 in this paper shows the existence of an invariant measure for* $(\tilde{J}, P_\varphi)$ *even if* φ *is allowed to have (a certain type of) discontinuities.*

The strategy of the proof for Theorem 1 goes along the lines of Hildenbrand and Grandmont (1974). It borrows from theorem 12.10 in Stokey, Lucas and Prescott (1989) (page 376), theorem 3.5 in Molchanov and Zuyev (2011, page 15) and proposition 1 in Ito (1964, see page 155). The appendix contains a detailed description of the procedures used *up to* now to prove the existence of an invariant measure and the reasons that make them unsuitable for addressing the question at hand.

Using proposition 1 in Ito and theorem 3.5 in Molchanov and Zuyev it is possible to restore the continuity of P_φ^* in the absence of the Feller property. In particular, as P_φ^* and $\hat{P}_\varphi$ can be interchanged (see for instance Stokey, Lucas and Prescott page 216), if $\mu_{n_k} \rightarrow_{Weak\,*} \mu$, for some $f \in C(\tilde{J})$, $\int f(\tilde{z}) P_\varphi^* \mu_{n_k}(d\tilde{z}) = \int \hat{P}_\varphi f(\tilde{z}) \mu_{n_k}(d\tilde{z}) \nrightarrow \int f(\tilde{z}) P_\varphi^* \mu(d\tilde{z}) = \int \hat{P}_\varphi f(\tilde{z}) \mu(d\tilde{z})$ as $\hat{P}_\varphi f(\tilde{z})$ is may not be continuous. However, $\hat{P}_\varphi f(\tilde{z})$ is bounded and $\mathcal{B}_{\tilde{J}}$-measurable. Theorem 3.5 in Molchanov and Zuyev implies that $\int \hat{P}_\varphi f(\tilde{z}) \mu_{n_k}(d\tilde{z}) \rightarrow \int \hat{P}_\varphi f(\tilde{z}) \mu(d\tilde{z})$ if $\mu(\Delta \hat{P}_\varphi f) = 0$, where $\Delta \hat{P}_\varphi f$ is the *set of discontinuities of* $\hat{P}_\varphi f$.

Thus, it only suffices to show that the discontinuity set generated by φ *is sufficiently small under the limiting measure.* In order to achieve this property, proposition 1 in Ito is used to show that P_φ^* maps the set of *atomless measures, which will be denoted* $\mathcal{P}_0(\tilde{J}) \subset \mathcal{P}(\tilde{J})$, *into itself. The proof will be complete if it can be shown that* $\mu \in \mathcal{P}_0(\tilde{J})$ *and* $\mu(\Delta \hat{P}_\varphi f) = 0$. As a measure is atomless if and only if $\mu(\{a\}) = 0$, $\{a\} \in \tilde{J}$ (i.e. it assigns zero measure to points, see Hildenbrand and Grandmont 1974, page 45), it suffice to restrict the cardinality of $\Delta \hat{P}_\varphi f$ to be at most countable and to show that $\mathcal{P}_0(\tilde{J})$ is closed. The latter property will be insured by imposing conditions on P_φ in section 3.2 and 3.3. The former will be guaranteed by restricting the discontinuity set of φ as follows:

<u>Assumption 2:</u> Let $\varphi \sim \Phi$ *be a* $\mathcal{B}_{\tilde{J}\times S}$ – measurable selection of the correspondence defined in assumption 1 and $\Delta\varphi$ its discontinuity set. *Then,* $\Delta\varphi$ *is a collection of at most a countable number of points.*

Under assumption 1, the range of φ is uniformly bounded. Thus, provided that limits are always assumed to be well defined, assumption 2 allows φ having at most a countable number of "jump" discontinuities. Then, lemma 9.5 in Stokey, Lucas and Prescott (page 261) implies that: $\#\left(\Delta\hat{P}_{\varphi}f\right) \leq \#(\Delta f(\varphi)) \leq \#(\Delta\varphi)$, where # denotes the cardinality of a set.

The discussion in section 2.4 suggests that $\#(\Delta\varphi)$ is typically related with the cardinality of the equilibrium set of a possibly large set of economies and depends crucially on the selected state space. Often, an upper bound for the cardinality of the equilibrium set can be obtained for regular economies (see for instance, Geanakolos and Polemarchakis, 1986). Unfortunately, in the presence of short sale constraints or an uncountable number of exogenous shocks, standard regularity theorems do not hold. Moreover, the example in section 2.4 suggests that a sufficient condition which implies assumption 2 must be related with the genericity of multiple equilibria. Thus, given the state space, assumption 2 puts an upper bound on the "number of economies" that are allowed to have multiple equilibria. Section 5, which discusses applications, will take assumption 2 as given. It should be a topic of future research to relate this assumption with primitive conditions of the sequential economy.

Now, under assumptions 1 and 2, it is possible to state one of the main results in this paper:

Theorem 1: Let $\varphi \sim \Phi$ satisfies assumptions 1 and 2. Suppose additionally that a) $P_{\varphi}^{*}: \mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ and b) $\mathcal{P}_0(\tilde{J})$ is weak closed, where $\mathcal{P}_0(\tilde{J})$ is the set of atomless measures in $\mathcal{P}(\tilde{J})$. Then there is a measure $\mu \in \mathcal{P}_0(\tilde{J})$ such that $\mu = P_{\varphi}^{*}\mu$.*

Proof: See the appendix.

Note that a) and b) are "properties" of the process $(\tilde{J}, P_{\varphi})$. Sections 3.2 to 3.4 relate these properties with verifiable "conditions" on P_{φ}, φ and S. If property a) is satisfied, as the existence proof can be done for measures of the form $\mu_n = P_{\varphi}^{*}\mu_{n-1}$, it suffice to assume that the set $\{\mu_n | \mu_n = P_{\varphi}^{*}\mu_{n-1}, \mu_0 \in \mathcal{P}_0(\tilde{J})\}$ is *weak** closed. This is the strategy taken in this paper.

Let $IM(\varphi, \mathcal{P}_1) = \{\mu \in \mathcal{P}_1(\tilde{J}) | \mu = P_{\varphi}^{*}\mu\}$, where $\mathcal{P}_1(\tilde{J}) \subseteq \mathcal{P}_0(\tilde{J})$. That is, $IM(\varphi, \mathcal{P}_1)$ is a set of invariant measures of $(\tilde{J}, P_{\varphi})$, which belong to $\mathcal{P}_1(\tilde{J})$. Under assumptions 1) and 2), if properties a) and b) hold for $\mathcal{P}_1(\tilde{J})$, the non-emptiness of $IM(\varphi, \mathcal{P}_1)$ can be assured using theorem 1 as long as $\mu \in \mathcal{P}_1(\tilde{J})$, $\mu\left(\Delta\hat{P}_{\varphi}f\right) = 0$ *and* μ *is the* weak* limit of an appropriately chosen subsequence of measures generated by P_{φ}^{*}. A natural candidate for $\mathcal{P}_1(\tilde{J})$ is the set of absolutely continuous measures with respect to the Lebesgue measure on $\tilde{J}$, which will be called θ. To show that

$IM(\varphi, \mathcal{P}_1)$ is closed, which is essential for the existence of an ergodic measure, it is necessary to impose stronger conditions on P_φ, and consequently on φ, than the ones that arise from theorem 1. Once this strengthening has been made, the closedness of $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_1(\tilde{J})\}$ follows from the same result used for $\mathcal{P}_0$. Unfortunately, this result cannot be applied to show the closedness of $IM(\varphi, \mathcal{P}_1)$. A suitable proof of this result is available in the appendix. A detailed discussion of these issues is postponed to sections 3.3 and 3.4 as they can be shown only for economies with an uncountable number of exogenous shocks.

A set $A \in \mathcal{B}_J$ is called invariant under P_φ if $P_\varphi(z, A) = 1$ for any $z \in A$. An invariant measure is called ergodic if $\mu \in IM(\varphi)$ and $\mu(A) = 0$ or $\mu(A) = 1$ for any invariant set under P_φ. The next theorem presents properties of $IM(\varphi)$ which guarantee that there exist at least 1 ergodic measure.

<u>Theorem 2</u>: Let $\varphi \sim \Phi$ satisfies assumptions 1 and 2. Suppose additionally that $IM(\varphi, \mathcal{P}_1) \neq \emptyset$, where $\mathcal{P}_1$ is the set of absolutely continuous measures with respect to the Lebesgue measure. If c) $IM(\varphi, \mathcal{P}_1)$ is closed, then $IM(\varphi, \mathcal{P}_1)$ contains an ergodic measure.

<u>Proof:</u> The closedness of the set implies its compactness from proposition 2.8 in Futia (1982, page 385). As $IM(\varphi, \mathcal{P}_1)$ is convex, the Krein-Milman theorem (see Simon, 2011, theorem 8.14, page 128) implies that the set of extreme points of $IM(\varphi, \mathcal{P}_1)$, denoted $\mathcal{E}(IM(\varphi, \mathcal{P}_1))$, is non-empty. Remark 6.3 in Varadhan (2001, page 190) implies that if $\mu \in \mathcal{E}(IM(\varphi, \mathcal{P}_1))$, then μ is ergodic.

Theorems 1 and 2 are the first attempt to show the existence of an ergodic invariant measure for a computable correspondence based recursive equilibrium notion, setting aside the results in Santos and Peralta Alva (2013). These authors show that $IM(\Phi, \mathcal{P}_1) = \{\varphi \sim \Phi, \mu \in \mathcal{P}_1 | \mu = P_\varphi^* \mu\} \neq \emptyset$. Unfortunately, there are some concerns about the Santos and Peralta Alva (2013) framework. First, it is not clear if S is a finite set. If S can be characterized by a Markov process with an atomless Markov operator (i.e. $p(s,.)$ is atomless for all $s \in S$), the non-emptiness of $IM(\Phi)$ follows immediately from theorem 3.1 in Blume (1982). *This paper provides conditions which guarantee the non-emptiness of $IM(\varphi)$ for any $\varphi \sim \Phi$ that satisfies assumption 1 and 2 which is slightly stronger than $IM(\Phi) \neq \emptyset$. It is also convenient in applications as frequently it is desirably to compute only an approximation of φ.* Second, the conditions which guarantee the existence of an ergodic measure in $IM(\Phi)$ have not been established, at least explicitly. *Theorem*

2 establishes the properties of $(\tilde{J}, P_\varphi)$ *associated with the existence of an ergodic invariant measure.* Third, the critical assumptions in Santos and Peralta Alva (2013), assumption 2.3 and remark 6.2, have been stated in terms of P_φ. Thus, it may be difficult to identify this assumptions in certain applications, especially as the authors suggest the use of the implicit function theorem on φ, even though this function is allowed to be discontonuous[35]. *Sections 3.2 and 3.3 identify conditions on* P_φ *which guarantee properties a), b) and c) associated with theorems 1 and 2 that will be traced back to the primitives of certain type of economies in sections 3.4 and 5.*

3.2 The case of a finite number of shocks

In order to prove the existence of an ergodic invariant measure, theorem 1 requires 2 properties. Namely, that the adjoint operator associated with some Markov process $(\tilde{J}, P_\varphi)$ maps the set of atomless measures, $\mathcal{P}_0(\tilde{J})$, into itself (property a) and that $\mathcal{P}_0(\tilde{J})$ is closed (property b). It is interesting to explore the relationship between these properties and the Markov operator obtained from lemma 1, P_φ, because it could connect the existence of an invariant measure with primitive conditions in the model (i.e. restrictions on preferences, shocks, etc) through properties of $\varphi \sim \Phi$.

This section takes the first step towards that direction by restricting S, the set which contain the exogenous shocks, to be of finite cardinality. Let $\mu_{n,\theta}$ be a sequence of measures generated by applying P_φ^* iteratively on some $\theta \in \mathcal{P}(\tilde{J})$. Then, the following lemma states conditions on P_φ which guarantee properties a) and b).

<u>*Lemma 3*</u>*: Let* Φ *satisfy assumption 1 and* $\#S < \infty$*. Then, the measurable space* $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ *has an atomless measure* θ*. Let* $\{a\}$ *be any point in* $\tilde{J}$*. Additionally suppose that for some* $\varphi \sim \Phi$*:*

1) $\theta(\{a\}) = 0$ *implies* $P_\varphi(z, \{a\}) = 0$ θ*-almost everywhere*

2) $Sup_n \mu_{n,\theta}(\{a\}) = 0$*.*

Then, properties a) and b) in theorem 1 are satisfied.

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[35] Let $F(\tilde{J} \times S, \tilde{J})$ be the system of equations which define φ in section 2.5.2. The applicability of the implicit function theorem requires that the Jacobian of F has full rank uniformly on $\tilde{J} \times S$. This may imply the boundness of all partial derivatives of φ, which is sufficient for its continuity; a property that may not hold in the presence of multiple equilibria.

Proof: see the appendix.

Note that lemma 3 requires *conditions* 1 and 2 to hold *simultaneously* in order to guarantee properties a) and b). As can be seen in the appendix, condition 1 is associated with property a) and condition 2 with property b). While section 5 presents mild sufficient conditions on the primitives of the economy $\mathcal{E} = [e, d, \{U^i\}_{i=1}^I]$, defined in section 2.1, which guarantee condition 1, it is still an open question how to assure that condition 2 holds in a general equilibrium non-optimal economy. Thus, a strong assumption is required to assure the *weak*-closedness of* $\mathcal{P}_0(\tilde{J})$ *when the state space is of the form* $\tilde{J} = S \times \hat{Z}$, S *is finite and* $\hat{Z}$ *is uncountable.*

Section A.1.2 in the appendix provides a concrete example of θ. Ito (1964, page 177) gave an example of a discontinuous function $\varphi{\sim}\Phi$ satisfying conditions 1) and 2).

3.3 The case of an infinite number of shocks

This section presents conditions on the Markov operator P_φ for economies with an uncountable number of shocks s. In particular, lemma 4 below is analogous to lemma 3 for this type of models. However, there are 3 important differences with respect to the case presented in section 3.2. First, the existence of an invariant measure follows only from 1 condition, a strengthening of condition 1) in lemma 3. Second, it is possible to define conditions on P_φ which guarantee the ergodicity of the invariant measure. Third, it is possible to connect properties a), b) and c) in theorems 1 and 2 respectively with conditions on the set of shocks, its distribution and $\varphi{\sim}\Phi$.

Lemma 4: Let Φ *satisfy assumption 1 and 2. Further, suppose that* S *be an uncountable compact set. Then, the measurable space* $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ *has an atomless measure* θ*. Let* $\{a\}$ *and* B *be, respectively, any point and Borel measurable set in* $\tilde{J}$*. Additionally suppose that for some* $\varphi{\sim}\Phi$*:*

3) $\theta(\{a\}) = 0$ *implies* $P_\varphi(z, \{a\}) = 0$ *for any* $z \in \tilde{J}$*.*
4) $\forall \varepsilon > 0, \exists \delta > 0$ *such that* $\theta(B) < \delta$ *implies* $P_\varphi(z, B) < \varepsilon$ *for any* $z \in \tilde{J}$*.*

If condition 3) holds, then properties a) and b) in theorem 1 are satisfied.

If condition 4) holds, then property c) in theorem 2 is satisfied.

Proof: See the appendix.

Remark 1: Condition 4) implies condition 3). Further, condition 4 and proposition 2.3 in Santos and Peralta Alva (2013) implies that $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_1(\tilde{J})\}$ is weak closed. Thus, assumption 2) can be replaced with the following, milder version:*

Assumption 2'): Let $\varphi \sim \Phi$ be a $\mathcal{B}_{J \times S}$ – measurable selection of the correspondence defined in assumption 1 and $\Delta\varphi$ its discontinuity set. Then, $\Delta\varphi$ has zero Lebesgue measure.

Proof: As any point in $\tilde{J}$ has zero Lebesgue measure, the result follows from Billingslley (1995, see equation 32.4 in page 422). The possibility to replace assumption 2 by 2' once condition 4 has been imposed follows from the fact that $\mu(\Delta\varphi) = 0$ if μ is the *Weak** limit of $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_1(\tilde{J})\}$ and $\Delta\varphi$ has zero Lebesgue measure.

Condition 3) states that $P_\varphi(z,.)$ is an atomless measure for any $z \in \tilde{J}$. Note that condition 1) in lemma 3 only requires $P_\varphi(z,.)$ to be atomless almost everywhere. In this sense, condition 3) is stronger than 1). Condition 4) states that $P_\varphi(z,)$ is absolutely continuous w.r.t. θ uniformly in $z \in \tilde{J}$.

The difference between conditions 1) and 3) has 2 important consequences. First, condition 1) allows S being a finite set. This fact follows from equation 5) and is discussed extensively in the appendix (see equations A.5 and A.6 in the preliminary remark of the proof of lemma 3). As was argued in section 2, the existence of a sequential equilibrium follows from mild assumptions for this type of economies. This is the bright side. On the other hand, however, proving the existence of an invariant measure requires condition 2), which is very challenging to derive from primate conditions. Second, condition 3) allows proving the existence of an invariant measure imposing only this additional requirement to assumptions 1) and 2). Under this strengthening, condition 2 can be replaced by a recently proved result (see proposition 2.3 in Santos and Peralta Alva, 2013). Also, as will be explained in section 3.4, this condition follows from assuming that S is uncountable and from an additional mild requirement on its distribution, $p(s,.)$. However, showing the existence of a sequential equilibrium and, as stated in section 2.5.1 (see fact 2), of an appropriate recursive structure requires strong restrictions on endogenous variables. This last fact will be discussed in section 5.2.

In summary, there is a tradeoff between the strength of the conditions which guarantee the existence of a recursive structure and its stationarity or, similarly, between the mildness of the *assumptions required to prove the existence of a sequential equilibrium and to prove the existence of an invariant measure.*

From the preceding discussion it is clear that the crucial step in the existence of an invariant measure and its ergodicity is to insure that the non-atomicity / absolute continuity of a sequence of measures is preserved under *weak** limits. This can be seen by noting that properties b) and c) in theorems 1 and 2 requires, respectively, the closedness of $\mathcal{P}_0$ and $\mathcal{P}_1$ and that, as was shown in lemmas 3 and 4, these properties impose restrictions on the Markov operator. Section 3.4 will discuss how these restrictions reflect on φ and the primitives of the model and the following example illustrate the problem at hand.

<u>Example 1 (non-uniform boundness of densities)</u>: Let $P: S \times \mathcal{B}_S \to [0,1]$ be a transition function with $S = [0,1]$, $P(s, \{s/2\}) = 1$ and $\theta = U[0,1]$. Note that condition 1 is satisfied as $P(s, \{a\}) = 0$ except for $s = 2a$ with $\theta(\{2a\}) = 0$. Thus, under lemma 3, $P_\varphi^*: \mathcal{P}_0([0,1]) \to \mathcal{P}_0([0,1])$, where $\varphi(s) = s/2$. Note then that property a) in theorem 1. However, property b) will not be satisfied. Let $\mu_1 = P_\varphi^* \theta$ and $A = [0, a]$ with $0 < a < 1$. Then $\mu_1(A) = 2a$, that is, $\mu_1 = U[0,1/2]$ which has a density of 2. In general, $\mu_n = U[0,1/2^n]$ with $\mu_n = P_\varphi^* \mu_{n-1}$. Thus, $\{\mu_n\}$ has an associated sequence of densities of $\{2^n\}$, which is not a uniformly bounded sequence of functions. It is not surprising then that Kempton and Persson (2015, page 11) show that absolutely continuity is preserved under *weak** limits if the sequence of densities associated with $\{\mu_n\}$ is uniformly bounded.

This paper proved that that absolutely continuity is preserved under *weak** limits by imposing condition 4), that is slightly weaker than the uniform integrability of densities (see Diestel, 1991 for a detailed discussion), which is in turn weaker than the mentioned uniform boundness.

The example above shows that $\mathcal{P}_0([0,1])$, the subset of atomless measures in $\mathcal{P}([0,1])$ generated under the action of P_φ^*, is not closed as it has a sequence of measures in it weakly converging to a Dirac measure at 0.

Condition 2), by lemma 3, and condition 3), by lemma 4, guarantee the closedness of $\mathcal{P}_0$ for the case of finite and uncountable exogenous shocks respectively. The latter result relies on lemma 2.3 in Santos and Peralta Alva (2013) that exploits the iterative nature of μ_n, generated by applying P_φ^* successively, under a slightly stronger assumption on P_φ than condition 3).

As it is also shown in lemma 4, condition 4) assures the closedness of $\mathcal{P}_1$. This condition implies that the family of measures $\{P_\varphi(z,.)|z \in \tilde{J}\}$ is absolutely continuous w.r.t. θ and that small θ-measure sets have $P_\varphi(z,.)$-measure uniformly bounded by ε. This last condition is weaker than the uniformly integrability of densities, denoted by $\bar{p}_\varphi(z,z')$, as the latter requires $\int_B |\bar{p}_\varphi(z,z')|\theta(dz') < \varepsilon$ while the former only implies $\int_B \bar{p}_\varphi(z,z')\theta(dz') < \varepsilon$ (see Diestel, 1991). Although the distinction is subtle, it has important consequences: if $\int_B \bar{p}_\varphi(z,z')\theta(dz') < \varepsilon$ implies $\int_B |\bar{p}_\varphi(z,z')|\theta(dz') < \varepsilon$ for any $z \in \tilde{J}$, then $\bar{p}_\varphi(z,z')$ is bounded away from zero in $\tilde{J} \times \tilde{J}$. But in this case, exercise 11.4 in Stokey, Lucas and Prescott implies that P_φ satisfies the Doeblin condition (i.e. $\theta(B) < \delta$ implies $\int_B \bar{p}_\varphi(z,z')\theta(dz') < 1 - \varepsilon$ for any $z \in \tilde{J}$), which is a sufficient for the existence of an ergodic invariant measure (see page 345-8 for a detailed discussion). A similar result holds if $\bar{p}_\varphi(z,z')$ is bounded above in $\tilde{J} \times \tilde{J}$.

Consequently, by the discussion in example 1 and in the preceding paragraph, in this paper it will not be assumed that densities are neither bounded nor uniformly integrable as it suffice to restrict the Markov operator only to condition 4.

Note that assumption 2', like assumption 2, represents an upper bound on the genericity of the multiple equilibria problem discussed in section 2.4. As remark 1 suggests, condition 4 is stronger than condition 3. Thus, as any invariant measure under condition 4 is absolutely continuous with respect to the Lebesgue measure, the constraint imposed by $\mu(\Delta\varphi) = 0$ in theorem 1 is now less restrictive. Thus, $\Delta\varphi$ can be an uncountable set as long as it has zero Lebesgue measure. Section 5.2 will discuss this issue in the context of a concrete application.

Finally, the strategy in lemma 4 is different from the one used in lemma 2.3 in Santos and Peralta Alva as $IM(\Phi, \mathcal{P}_1)$ does not contain sequences of the form $\mu_n = P_\varphi^* \mu_{n-1}$. In turn, lemma 4 shows how condition 4 implies that small θ-measure sets have arbitrary small μ_n-measure, where $\{\mu_n\}$ is any sequence in $IM(\Phi, \mathcal{P}_1)$, and that this latter property guarantees that absolute continuity is preserved under *weak** limits of $\{\mu_n\}$.

3.4 Sufficient conditions for the existence of an Ergodic Invariant Measure

The main advantage of lemma 4 is that allows connecting the properties associated with the existence of an ergodic invariant measure (properties a) to c) in theorems 1 and 2) with primitive conditions in the model, described in assumption 3 below:

Assumption 3: Let S be the set that contains the exogenous shocks, $p(s,.)$ its distribution, $\Phi:\tilde{J}\times S \rightrightarrows \tilde{J}$ the equilibrium correspondence presented in definition 5 and $\Delta\varphi$ the discontinuity set of $\varphi\sim\Phi$. Assume that:

- *i)* *S is uncountable and compact*
- *ii)* *$p(s,.)$ is atomless $\forall s \in S$*
- *iii)* *Suppose that assumption 2 holds. Let $(\tilde{z},s') \in \Delta\varphi$. In addition, suppose that $lim_{(\tilde{z},s_n^{'})\to(\tilde{z},s')}\,\varphi\,(\tilde{z},s_n^{'}) = \varphi(\tilde{z},s')\ \ \forall\tilde{z} \in \tilde{J}$*
- *iv)* *$p(s,.) = U[\underline{s},\overline{s}]\ \forall s \in S$, where $U[\underline{s},\overline{s}]$ is the uniform distribution on $[\underline{s},\overline{s}]$, a closed bounded interval of $\mathbb{R}$.*

Note that assumption 3-iii) allows for some path $(\tilde{z}_n,s_n^{'})$ to be discontinuous. That is, for any $(\tilde{z},s') \in \Delta\varphi$ there may exist $(\tilde{z}_n,s_n^{'})$ with $lim_{(\tilde{z}_n,s_n^{'})\to(\tilde{z},s')}\,\varphi\,(\tilde{z}_n,s_n^{'}) \neq \varphi(\tilde{z},s')$. The purpose of this assumption is to associate rectangles in the range of $\varphi(\tilde{z},.)$ with closed sets in S. Then, the countable union of these rectangles will be associated with a small Lebesgue measure set in order to derive condition 4.

As φ is defined implicitly from a set of non-linear equations, verifying its properties requires the implicit function theorem. Unfortunately, if this theorem holds globally, it implies not only that φ is continuous but also that is has well defined partial derivatives. Assumption 3-iii) requires only the continuity of $\varphi(\tilde{z},.)$ for each coordinate of this vector valued function. The discussion on the mildness of this assumption is postponed to section 5.2 in the context of a concrete application.

The next 2 propositions connect assumption 3 with conditions 3 and 4.

Proposition 1:. Suppose that assumption 1, assumption 3-i) and 3-ii) hold. Then, condition 3) is satisfied. That is, $\theta(\{a\}) = 0$ implies $P_\varphi(z,\{a\}) = 0$ for any $z \in \tilde{J}$ for an arbitrary point $\{a\} \in \tilde{J}$

Proof: See section A.1.2 in the appendix.

Proposition 2:. Suppose that assumption 1, assumption 3-iii) and 3-iv) hold. Then, condition 4) is satisfied. That is, $\forall\varepsilon > 0, \exists\delta > 0$ such that $\theta(B) < \delta$ implies $P_\varphi(z,B) < \varepsilon$ for any $z \in \tilde{J}$.

Proof: See section A.1.2 in the appendix.

Clearly, proposition 2 calls for stronger assumptions than proposition 1. This is because it involves verifying not only the non-atomicity of $P_\varphi(z,.)$, which requires only taking care of points in $\tilde{J} \subset \mathbb{R}^K$, but also its absolute continuity, which demands proving that sets of the form $\{a_1\} \times [a_2, b_2] \times \ldots \times [a_K, b_K]$ also have zero Lebesgue measure. Each of these sets can be "matched" with a sequence of rectangles which can be traced back to $p(s,.)$ under assumption 3-iii).

<u>Remark 2</u>: proposition 2 holds under the following version of assumption 3-iv),

> *Assumption 3-iv'):* $p(s,.) = U[LB(s), UB(s)]\ \forall s \in S$, *where* $U[LB(s), UB(s)]$ *is the uniform distribution on* $[LB(s), UB(s)]$.

Proof: See section A.1.2 in the appendix.

Note that assumption 3-iv') is weaker than 3-iv) as it allows the exogenous states to follow a Markov process instead of being i.i.d. From a theoretical point of view, the remark is important as it shows the existence of a stationary (ergodic) representation of a sequential equilibrium, which, under assumption 3-iv') may not be stationary (see stokey, Lucas and Prescott, page 224).

Unfortunately, the theorem which guarantee the accuracy of simulations and the existence of an equilibrium correspondence that satisfies assumption 1 for an uncountable number of shocks, both require assumption 3-iv). The former result will be shown in section 4 and the latter in section 5. So, as this paper is concerned with numerical simulations, the relevant assumption is 3-iv).

4 Implications of Ergodicity

The first step to compute an infinite horizon process in a quite general setting is to find a time invariant transition function which was given by Feng, et. al. Theorems 1 and 2 allow taking matters a step further by proving the existence of an ergodic invariant measure in this environment. However, unconditional (in particular, invariant) measures are not easily computable. So, to complete the picture, a general result that allows approximating several characteristics of those invariant measures is required.

In this section that task is achieved by assuming the existence of a sequence of functions, $\{\varphi_j\}$ with $\varphi_j \sim \Phi_j$, *uniformly* approximating some selection of the equilibrium correspondence, $\varphi \sim \Phi$. Further, it is shown that, for *any initial condition that belongs to a positive* P_{φ_j} *-invariant measure set*[36], every average constructed using a series obtained from an approximated function converges to the mean of some P_φ-invariant measure. As discussed in section 2.6, those results imply that any model that fits this framework can be calibrated using a sufficiently large time series.

Section 4.1 proves *theorem 3*, which is an extension to non-optimal economies of a similar result in Santos and Peralta Alva (2005, see page 8). The authors suppose that the transition function has the Feller property, an assumption which does not hold in the present context. In order to compensate for this fact, *assumption 4-ii)* endows the space of approximating functions with a more restrictive norm than the one used in Santos and Peralta Alva. Further, *assumption 4-iii)* imposes a restriction on the preimage of φ_j in order to preserved the absolute continuity of the measures generated out of numerical approximations.

The results presented in section 4.2 will be stated without proof as they are standard applications of Birkhoff´s theorem and the ergodic decomposition theorem.

Using the results in the previous 2 subsections, section 4.3 proves *theorem 4* which states that, given an appropriately chosen initial condition defined in *assumption 5*, numerical simulations approximate the true steady state of the model.

The following figure illustrates the theoretical structure described above.

[36]The algorithms in Kubler and Schmedders (2003) and Feng, et. al. (2013) generate such a sequence of functions under the uniformity assumption. As those procedures aren't simulation based (like Marcet´s PEA, 1988), it is possible to first compute the equilibrium correspondence and then simulate from it.

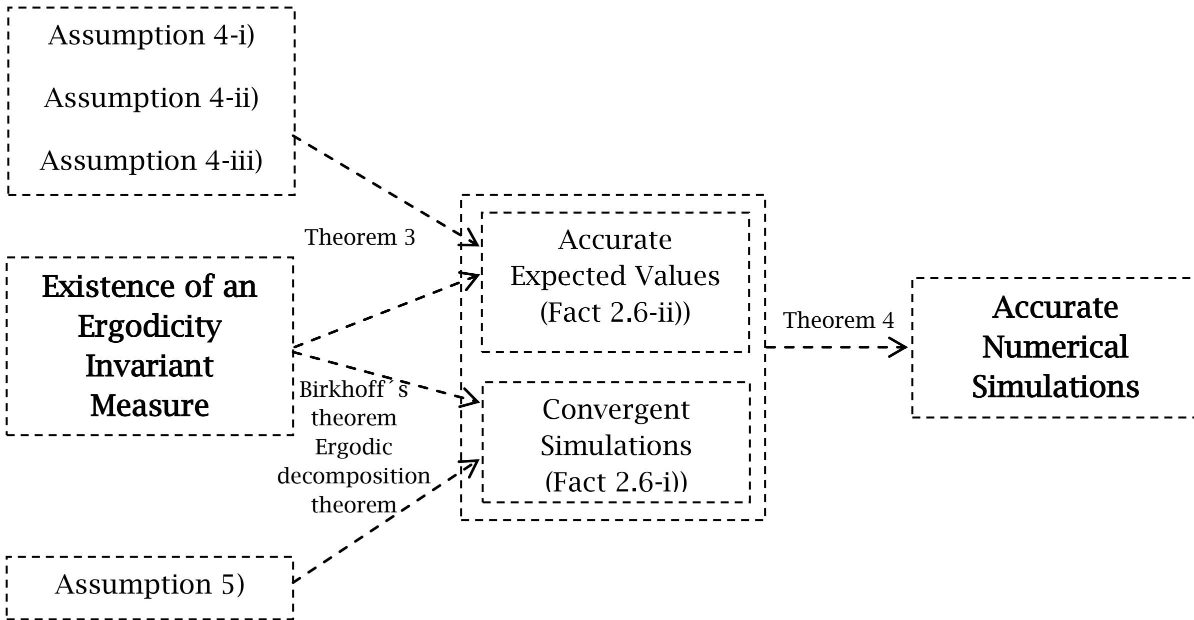

Figure 3

4.1 Accurate Expected Values.

From figure 3 and the discussion in section 2.6 it is clear that, in order to obtain accurate numerical simulations, it is necessary to insure that approximated economies have an ergodic invariant measure and that these measures converge (in some appropriate sense) to the true model. This subsection proves this fact.

Let $\varphi_j \sim \Phi_j$ be an approximation of $\varphi \sim \Phi$ with φ_j converging to φ in some metric that will be specified below; theorem 3 state conditions which insure that $E_{\mu_j}(f)$ converges weakly to $E_{\mu}(f)$, where $\mu_j \in IM(\varphi_j, \mathcal{P}_1)$, $\mu \in IM(\varphi, \mathcal{P}_1)$ and $f \in C(\tilde{J})$.

There is also a similar result in Santos and Peralta Alva (2005). In that paper the authors assumed that the Markov operators, associated with the true and all approximated economies, have the Feller property and that the computed functions converges in a metric induced by a norm slightly weaker than the sup-norm [37].

[37] The authors use the following metric: $d(\varphi_j, \varphi) = Max_{z \in \tilde{J}} \left\| \int \varphi_j(z, s') - \varphi(z, s') dU(s') \right\|$, where $\|.\|$ is the Max norm in $\mathbb{R}^K$.

Unfortunately, as discussed in section 2.4, the Feller property is not adequate for non-optimal economies. To restore the continuity of the Markov operator, section 3 imposes a series of assumptions. In particular, the absolutely continuity of the limiting measure, a key property in the proof of theorem 1, requires assumption 3-iii) (i.e. $\varphi(z,.)$ is continuous on s' for all z). Even if this assumption is imposed on all φ_j, the liming function must preserve this property in order to be an appropriate candidate for φ. As all endogenous variables are assumed to be contained in a compact set, the sup-norm serves this purpose. Assumption 4-ii) formally states this claim.

Theorem 3 will show that there is a sequence of measures $\{\mu_j\}$, with $\mu_j = P_{\varphi_j}\mu_j$, which converges weakly to μ and $\mu = P_{\varphi}\mu$. Thus, in order to be an appropriate candidate, μ must be absolutely continuous w.r.t. θ. As assumptions 3-iii) and 3-iv) are supposed to hold for φ, proposition 2 implies that P_{φ} is absolutely continuous w.r.t. θ(i.e. satisfies condition 4). Consequently, a restriction must be imposed on "the size" of $\varphi_j^{-1}(z,.)(A)$ for any open set with $\theta(A) < \delta$ in order to preserve this condition $\varphi_j \to \varphi$. This is done by imposing assumption 4-iii) which requires $\varphi_j^{-1}(z,.)(A) \subseteq \varphi^{-1}(z,.)(A)$ for any j sufficiently large.

In order to insure that $IM(\varphi_j, \mathcal{P}_1) \neq \emptyset$, assumptions 1), 2'), 3-iii), 3-iv) and 4-i) must be imposed on all approximated economies characterized by the associated equilibrium correspondence, Φ_j. The purpose of all these assumptions was extensively discussed in section 3, except assumption 4-i), which is standard in the recursive numerical literature (see for instance Feng, et. al.).

Before stating the theorem, some preliminary concepts are needed. Let $P_{\varphi_j}(z,A) = p(s,\{s' \in$ *Sφj(z,s')∈A*, which is analogous to equation 5 for *φj~Φj*.

Theorem 3: Let $\mathcal{E} = [e,d,\{U^i\}_{i=1}^I]$ *be the sequential economy in definition 1 and* K *be a compact set, with* $K \subset Z$ *and* $f \in C(K)$*. Suppose that assumption 1 hold on* $\{\Phi_j\}$ *and* Φ*. Further assume that assumptions 2'), 3-iii) and 3-iv) hold on* $\{\varphi_j\}$ *and* φ*: Finally suppose that:*

Assumption 4-i): All payoff relevant variables in $\mathcal{E} = [e,d,\{U^i\}_{i=1}^I]$ *are contained in* K*.*

Assumption 4-ii): $lim_{j\to\infty}\|\varphi_j - \varphi\| = 0$ *where* $\|\varphi\| \equiv SUP_{(K\times S)\setminus\Delta\varphi}\|\varphi\|_\infty$ *and* $\|.\|_\infty$ *is the Max-norm*

Assumption 4-iii): for any open set with $\theta(A) < \delta$, $\varphi_j^{-1}(z,.)(A) \subseteq \varphi^{-1}(z,.)(A)$ for any j sufficiently large.

Then,

1) *There is a sequence of measures $\{\mu_j\}$, with $\mu_j = P_{\varphi_j}\mu_j$, and μ_j is absolutely continuous with respect to θ*
2) *$\{\mu_j\}$ has a weakly convergent subsequence, denoted w.l.o.g. $\mu_j \to \mu$, and μ is absolutely continuous w.r.t. θ.*
3) *μ satisfies $\mu = P_\varphi \mu$.*

Proof: See section A.1.2 in the appendix.

Note that this result is in fact equivalent to the upper hemi-continuity of $\Gamma: P_j^* \to \{\mu \in \mathcal{P}(K): \mu =$ *Pj*·μ*, the correspondence of the correspondence of invariant measures with *μ∈𝒫(K)*. For a detailed discussion the reader is referred to corollary 2 in Santos and Peralta Alva (2005).

Finally, there are 2 things to be noted in theorem 3. First: the convergence is uniform on $(K \times S)\backslash\Delta\varphi$. Under assumptions 2') and 3-iii) on $\{\varphi_j\}$, this type of convergence assures that assumption 3-iii) is a valid hypothesis for φ. Further, this assumption fits the requirements of spline algorithms for discontinuous functions (see for instance Silanes, et. al. 2001) which makes the theoretical structure in figure 3 suitable for applications. Second, some algorithms (for instance Feng, et. al.) computes outer approximations of the range and domain of φ_j. Thus, assumption 4-iii) suggests that this type of procedures must be modified in order to fit into this framework. The preliminary remark of theorem 3, in the appendix, provides details on the relationship between theorem 3 and state of the art recursive algorithms.

4.2 Convergent Simulations.

The main result in this section is a direct application of Birkhoff's ergodic theorem and the ergodic decomposition theorem for Markov process. Thus, the results will be stated without proof and they will be presented just to keep the paper self-contained. This section follows closely chapter 6 of Varadhan (2001) and the reader who is familiar with the literature is invited to go directly to section 4.3.

Like in Santos and Peralta Alva (2013), Kamihigashi and Stachurski (2015) or chapter 14 of Stokey, Lucas and Prescott, it will be said that a simulation is convergent if it obey a strong law of large numbers. Contrarily to what is stated in those papers, convergence will be achieved only for a subset of all possible initial conditions.

This is because the assumptions necessary to guarantee convergence starting from an arbitrary initial condition are too strong for the purpose of this paper. In particular, Santos and Peralta Alva (2013) requires that condition 4) holds for any selection of Φ_j and Φ. Kamihigashi and Stachurski (2015) requires that φ be continuous and Breiman's theorem in Stokey, Lucas and Prescott requires a unique ergodic measure. Contrarily, theorem 3 requires only that condition 4 holds for some selection $\{\varphi_j\}$ and φ. Further, theorem 1 and 2 allow φ to be discontinuous and $(\tilde{J}, P_\varphi)$ to have multiple ergodic measures. In this kind of setting, there are no results that guarantee the "global" almost sure convergence of simulations. Thus, a "local" theorem, like Birkhoff's, has to be used.

From section 2.6 and figure 3, it should be clear that the accuracy of numerical simulations requires the existence of an ergodic invariant measure. Even though the assumptions in theorem 3 could be suitably modified in order to insure the existence of convergent expected valued for economies with only an invariant (not necessarily ergodic) measure, the convergence of simulations requires ergodicity. This is because the strong law of large numbers for stationary Markov processes (see Meyn and Tweedie, 1993, ch. 17) implies that simulations will not converge to the expected values in theorem 3. Contrarily, ergodic Markov processes will do. Thus, in this section and the next one it will be supposed that assumptions 1), 2'), 3-iii) and 3-iv') hold. Note that remark 2 allows $\{s_t\}$ to be generated by a Markov process (S, p) as long as S is an uncountable compact set of $\mathbb{R}$.

In order to present the results for this section some additional definitions are required. Let $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ be a measurable space and $\left(\tilde{J}^t, \mathcal{B}_{\tilde{J}}^t\right) = (\tilde{J} \times \ldots \times \tilde{J}, \mathcal{B}_{\tilde{J}} \times \ldots \times \mathcal{B}_{\tilde{J}})$ the associated product space. Let $A = A_1 \times \ldots \times A_t$ be a measurable rectangle (see Stokey, Lucas and Prescott page 195 for a definition) in $\mathcal{B}_{\tilde{J}}^t$. Let $\varphi \sim \Phi$ and $z_0, \ldots, z_t \in \tilde{J}$. As long as t is finite, by virtue of the Caratheodony and Hahn theorems and theorem 7.13 in Stokey, Lucas and Prescott, $\mu^t(z_0, A)$, defined by

$\mu^t(z_0, A) = \int_{\cdot A_1} \dots \int_{\cdot A_t} P_\varphi(z_{t-1}, dz_t) \dots P_\varphi(z_0, dz_1)$, can be uniquely extended to a probability measure in any set of $\mathcal{B}_{\tilde{J}}^t$. Note that $\int_{\cdot A_i}$ denotes integration with respect to $P_\varphi(z_{i-1}, dz_i)$.

Analogously, let $B = A_1 \times \dots \times A_T \times \tilde{J} \times \dots$be a finite measurable rectangle (see page 221 of Stokey, Lucas and Prescott for a definition) and $\mathcal{L}$ its power set. Let $\mathcal{M}$ be the algebra generated by finite unions in $\mathcal{L}$ and $\mathcal{F} = \mathcal{B}_{\mathcal{M}}$, that is $\mathcal{F}$ is the sigma field generated by $\mathcal{M}$. Further, $\mu^\infty(z_0, B) = \int_{\cdot A_1} \dots \int_{\cdot A_T} P_\varphi(z_{T-1}, dz_T) \dots P_\varphi(z_0, dz_1)$ can be shown to be extended to $\mathcal{F}$ in 2 steps. First, using the Caratheodony and Hahn theorems it is possible to extend $\mu^\infty(z_0, B)$ to $\mathcal{M}$ and then to $\mathcal{F}$. Later, using standard arguments for processes with a finite dimension distribution (see Shiryaev 1996, Ch. 9), $\mu^\infty(z_0, B)$ can be shown to be countably additive.

Standard results (see for instance exercise 8.6 in Stokey, Lucas and Prescott) imply that $(\Omega, \mathcal{F}, \mu^\infty(z_0, .))$ is a Markov process with stationary transitions P_φ. Let $\Omega = \tilde{J} \times \tilde{J} \times \dots$ with typical realization $\omega \in \Omega$. As Ω is the space of sequences, it is natural to define a $\mathcal{F}_t$-measurable random variable $z_t: \Omega \to \tilde{J}$, where $\omega(t) = z_t = z_t(\omega)$ denotes a typical realization and $\{\mathcal{F}_t\}$ is a sequence of nested sigma algebras on $\{\times_{i=1}^t \tilde{J}(i)\}$, where $\tilde{J}(i) = \tilde{J}$ for $i \geq 1$. The shift operator is denoted by $T: \Omega \to \Omega$. A set $A \in \mathcal{F}$ is called T-*invariant* if $TA = A$[38].

Let $\mu^\infty(z_0, B) \equiv \boldsymbol{P}_{\varphi, z_0}(B)$ and note that under the same assumptions $\boldsymbol{P}_{\varphi_j, z_0}(B)$ can be analogously defined if $\tilde{J}$ is replaced by K, which was supposed to be compact in assumption 4-i). Further, $\boldsymbol{P}_{\varphi,\mu} \equiv \int_{A_0} \boldsymbol{P}_{\varphi, z_0} \mu(dz_0)$ can be used to define a stochastic process $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$ which allows to randomize z_0 as μ is a measure on $(\tilde{J}, \mathcal{B}_{\tilde{J}})$. $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$ is said to be *stationary* if $\boldsymbol{P}_{\varphi,\mu}[C(t,n)] = \boldsymbol{P}_{\varphi,\mu}[C(t',n)]$ for all $n \geq 0$ and $t \neq t'$ with $C(t,n) = \{\omega \in \Omega: [z_{t+1}(\omega), \dots, z_{t+n}(\omega)] \in C\}$. $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$ is said to be ergodic if $\boldsymbol{P}_{\varphi,\mu}(A) \in \{0,1\}$, where A is a T-invariant set.

The following facts follow from the discussion in Varadhan (2001, pages 179 and 187-192):

Fact 4.2-i): $\mu \in IM(\varphi)$ then $\boldsymbol{P}_{\varphi,\mu}$ is stationary

Fact 4.2-ii): μ is ergodic if and only if $\boldsymbol{P}_{\varphi,\mu}$ is ergodic

Fact 4.2-iii): $\boldsymbol{P}_{\varphi,\mu} = \int \boldsymbol{P}_{\varphi,v} Q(dv)$, where v is an ergodic measure in $IM(\varphi)$ and $Q: \mathcal{P}(\tilde{J}) \to [0,1]$ a measure on $\mathcal{E}(IM(\varphi))$.

[38] Exercise 6.2 in Varadhan shows that this definition can be used w.l.o.g.

Fact 4.2-iv): $lim_{n\to\infty}[\sum_{t=1}^{n} f(z_t)]n^{-1} = \int f(z)\mu(dz)$ for almost every $\{z_t\} = \omega$ with respect to $\boldsymbol{P}_{\varphi,z_0}$ if z_0 belong to a set of positive μ-measure and $\mu \in IM(\varphi)$.

Fact 4.2-iv) can be seen as a consequence of the previous 2 facts: as the ergodicity of μ is equivalent to the ergodicity of $\boldsymbol{P}_{\varphi,\mu}$ (fact 4.2-ii), theorem 2 suffices to show the existence of a Markov ergodic process $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$. Then fact 4.2-iii), the ergodic decomposition theorem for Markov processes, implies that Birkhoff's ergodic theorem can be applied to any initial condition in a positive μ-measure with $\mu \in IM(\varphi)$. That is, μ can be assumed to be ergodic w.l.o.g. Santos and Peralta Alva (2013, page 30) uses fact 4.2-iii) to obtain a similar result.

As was discussed in section 2.6, the stochastic process derived directly from a sequential equilibrium may not be stationary. Fact 4.2-i) illustrates the importance of the results in section 3.2: even if an invariant measure cannot be shown to be ergodic, it suffices to prove the existence of a stationary process associated with the sequential equilibrium.

4.3 Accurate numerical Simulations.

This section puts connects all the pieces and proves one of the main theorems of the paper under an additional assumption.

For any selection $\varphi \sim \Phi$, given $z_0 \in \tilde{J}$ and $\{s_t\}$ generated from (S, p), which is a draw from a stochastic process defined analogously to $(\Omega, \mathcal{F}, \mu^\infty(z_0, .))$ as $z_0 = [s_0, \hat{z}_0]$, it is possible to define a sample path $\{z_t(\omega)\}$ inductively as follows: $z_1(z_0, \omega, \varphi) = \varphi(z_0, s_1)$ and for any t, z_t satisfies $z_t(z_0, \omega, \varphi) = \varphi(z_{t-1}(z_0, \omega, \varphi), s_t)$. $\{z_t^j(\omega)\}$ can be defined in a similar way by replacing φ with φ_j and $\tilde{J}$ with K. The preliminary remark of theorem 4 in the appendix provides further details on this procedure. Finally, for any $f \in C(\tilde{J})$, it is possible to define a "time average" as $(N)^{-1}\sum_{i=1}^{N} f(z_i(z_0, \omega, \varphi))$.

The following theorem follows directly from the results in sections 4.1 and 4.2.

<u>Theorem 4</u>: Suppose that all the assumptions made in theorem 3 hold. Additionally suppose:

<u>Assumption 5</u>: z_0^j belong to a set of positive μ_j -measure and $\mu_j \in IM(\varphi_j, \mathcal{P}_1)$, where j is sufficiently large, $\mu_j \to_{Weak*} \mu$ and $\mu \in IM(\varphi, \mathcal{P}_1)$.

Then

$$\left|(N)^{-1}\sum_{i=1}^{N} f\left(z_i^j\left(z_0^j,\omega,\varphi_j\right)\right)-\int f(z)\mu(dz)\right|\to 0 \text{ as } N\to\infty$$

Where the convergence is achieved for almost every $\{z_t^j(z_0^j,\omega,\varphi_j)\}$ with respect to $\boldsymbol{P}_{\varphi_j,z_0^j}$ and $\mu\in IM(\varphi,\mathcal{P}_1)$.

Proof: See section A.1.2 in the appendix.

Note that the results above imply, by the convexity of $IM(\varphi,\mathcal{P}_1)$, that every limit point of $(N)^{-1}\sum_{i=1}^{N} f\left(z_i^j\left(z_0^j,\omega,\varphi_j\right)\right)$ converges to some "state average" generated by a measure in $IM(\varphi,\mathcal{P}_1)$ provided that z_0^j *have been appropriately chosen*. Intuitively, if μ_j defines a "steady state" of the economy denoted by φ_j, assumption 5 implies that z_0^j fluctuates around its steady state values. This result can be used to justify the usual practice in the applied numerical literature where the first "1000" simulations are "thrown away" to assure that the simulated paths are already fluctuating around some ergodic set (see Guerron-Quintana, Fernandez-Villaverde, Rubio-Ramirez and Uribe, 2011)[39].

It should be clear that assumption 5 is the strongest in this paper. It is easy to construct an example of an economy that has a transient and an ergodic set (see for example 2 in Stokey, Lucas and Prescott, page 322). In this setting, simulations will fluctuate around the steady state once the transient set has been abandoned. Unfortunately, the first "1000" may be insufficient to leave the transient set with probability 1 and, although infrequently, it is possibly that z_0^j may not reflect the long run behavior of the model as desired. Thus, the "practitioners approach" must be repeated several times in order to avoid this type of problems.

5 Applications

This section applies the results in section 3 to a concrete parametrization of the economy described in section 2. Following figure 2 the requirements to achieve the existence of an invariant measure can be categorized in 3: properties (a-c), conditions (1-4) and assumptions (1-

[39] I would like to thank H. Seoane for pointing this out to me.

3). Section 3.2 and 3.3 connected conditions (mostly on the Markov operator P_φ) with properties of the associated Markov process $(\tilde{J}, P_\varphi)$.

Section 3.4 shows that conditions 1-4 can be generated by assumptions (mostly on the structure of exogenous variables) for the case of uncountable shocks. While the majority of these assumptions -1, 3-i), 3-ii) and 3-iv) - are stated in terms of the primitives of the model, there are 2 which are still stated in terms of endogenous variables of the model. This section connects one of these assumptions, 3-iii), with primitives of a version of the model presented in section 2 that is borrowed from Mas – Colell and Zame (1996). Unfortunately, assumption 2', which restricts the cardinality of the discontinuity set, cannot be associated with primitive conditions. It is a matter of future research to investigate the relationship between the cardinality of the discontinuity set and the equilibrium set as suggested in section 3.

For the case of finite shocks the existence of an invariant measure is guaranteed by 2 conditions: the first one connects the Markov process with the set of atomless measures, the second one guarantees the closedness of this last set. Section 5.1 shows that it is possible to derive the first condition from the curvature of the utility function using the implicit function theorem, which are assumed to hold almost surely. The second condition however, cannot be derived from primitive conditions of the model and thus deserves to be study in detail. Recently, Martinez and Pierri (2017) provide an example of an economy which illustrates the difficulty of the question at hand. If the economy has finite shocks and discontinuous Markov equilibria in the natural state space (as for instance in Santos 2002), it is possible to prove the existence of an invariant measure by enlarging the state space as in Duffie, et. al. (1994).

The requirements that insure the accuracy of numerical simulations, described in assumptions 4 and 5, are outside the scope of this paper as they require developing an algorithm which is capable of computing the equilibrium correspondence in definition 5 while keeping track of the requirements that preserves the absolute continuity of the measures involved in the successive computations. This type of algorithm has not been developed yet and thus requires a careful separate treatment.

5.1 Finite Shocks

The model is the same as the one described in section 2.1. Following figure 2, the first step to prove the stationarity of the model is to derive a recursive representation for the sequential equilibria. As discussed in section 2.5.2, *the existence of a recursive structure is guaranteed by the existence of the sequential competitive equilibria and the compactness of the equilibrium* *set*. In the present framework, these properties will be shown to be implied by assumptions 6.1-i) to 6.1-v) listed below.

Moreover, all the assumptions required for the existence of an invariant measure are presented below. As mentioned, 6.1-i) to 6.1-v) insure the existence of a non-empty compact equilibrium set which will be shown to be sufficient to derive a Markov representation of equilibria. Provided this representation, in order to show the existence of an invariant measure it suffices to impose assumption 2 and properties a) and b) (presented in section 3.1, theorem 1). The first and the last are stated as a hypothesis below (assumptions 6.1-vi and 6.1-vii respectively) and the second one will be derived from primitive conditions of the model which are implicit in assumptions 6.1-i) to 6.1-v).

Assumption 6.1). *Suppose an incomplete market economy as the one described in section 2.1. To that structure add the following assumptions:*

i) *The utility function in the optimality condition of definition 1 (sequential competitive equilibrium) is:*

$$U_i(c) = \sum_{t=0}^{\infty} (\beta^i)^t \sum_{\sigma_t^* s} [u_s^i(c^i(\sigma_t^* s))] \mu_t(\sigma_t^* s)$$

Where the instantaneous return function is given by $u_s^i[c^i(\sigma_t^* s)] = 1 - \lambda exp[-\lambda c^i(\sigma_t^* s)]$ *with* $\lambda > 0$ *and* $exp[x] \equiv e^x$.

ii) *The realizations of the exogenous shock* s_t *lie in set* S *of finite cardinality for any time period* $t = 0,1, \dots$.

iii) *Endowments satisfy:* $e^i(\sigma_t) > 0$ *and* $\sum_{i=1}^{I} e^i(\sigma_t) < K$ *with* $K > 0$ *for any agent* $i \in \{1, \dots, I\}$ *and node* σ_t. *That is, idiosyncratic endowments are strictly positive and aggregate endowments are uniformly bounded.*

iv) *There is a finite number,* J, *of numerarie short lived assets with (uniformly) bounded dividends and short sale constraints. That is, for each agent* i *and any node* σ_t *the portfolio is given by* $\theta^i(\sigma_t) \geq -B$, $B \in \mathbb{R}_+^J$, *the associated dividends by* $d(\sigma_t s) \in M \subset \mathbb{R}_+^J$, *where* M *is uniformly bounded, and the budget equation by*

$$c^i(\sigma_t) = e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t) - \theta^i(\sigma_t).q(\sigma_t)$$

Where $q(\sigma_t)$ *is the price of the portfolio in terms of the numerarie for every node* σ_t *and* σ_t^* *is the predecessor of* σ_t.

v) *There is a riskless bond. That is, there is an asset l which has associated dividends given by $d_l(\sigma_t s) = 1$ for any $s \in S$ and any node σ_t.*

vi) *Assumption 2 holds (i.e. the discontinuity set of any measurable selection of the equilibrium correspondence has at most finite cardinality).*

vii) *Condition 2 holds (i.e. provided that the adjoint operator maps the set of atomless measures into itself, this set is weakly closed).*

Except the assumption on u_s^i and the short sale constraints, 6.1-vi) and 6.1-vii), the rest are standard in the literature. The results in Magill and Quinzii (1994) imply that under assumptions 6.1-i) to 6.1-v), excluding the restriction on u_s^i, the economy describe in section 2.1 has a non-empty compact equilibrium set (see assumption A.1 to A.6 and the discussion that follows them in pages 858-60).

The chosen return function on assumption 6.1-i) guarantees that marginal utility is bounded on the entire feasible consumption set which, because of assumption 6-iii), is given by $[0, K]$. Kubler and Schmedders (2002) shows that assumptions 6.1-i) to 6.1-v), including the restriction on the return function but excluding the short sale constraints, imply that any sequence of consumption bundles $\{\{c^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$, portfolios $\{\{\theta^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$ and prices $\{q(\sigma_t)\}_{\sigma_t\in\mathfrak{T}}$ which satisfy the feasibility requirement $\sum_{i=1}^{I}\theta^i(\sigma_t) = \vec{0}$, where $\vec{0} \in \mathbb{R}^J$ for any $\sigma_t \in \mathfrak{T}$, and the Kuhn Tucker conditions listed in equation 3 and 4 meet the optimality and feasibility conditions in definition 1 and thus constitutes a sequential competitive equilibrium. The compactness of the equilibrium set follows from Magill and Quinzii (1994).

Short sale constraints are standard in the recursive literature since Duffie, et. al. (1994). Braido (2013) showed that a recursive equilibrium in the sense of Duffie, et. al. exists even if explicit short sale constraints are removed. This is possible as Magill and Quinzii (1994) showed that there is a uniform bound on assets even in the absence of short sale constraints. However, the theoretical results in this paper depend on Feng, et. al. (2013) recursive equilibria which, as discussed in section 2.5.2, are a subset of all possible recursive equilibria in Duffie, et. al. It is not clear that Braido's results hold in Feng, et. al.'s framework. Thus, short sale constraints are imposed in order to guarantee the existence of an appropriate (sunspots free) recursive equilibria.

As discussed in section 2.5.2 (see also Feng, et. al. 2013 section 2.2 for a detailed discussion), if the equilibrium set is compact and can be generated by the set of equations implied by the Kuhn Tucker and feasibility conditions, the equilibrium correspondence Φ in definition 5 satisfy the assumptions in lemma 2 and thus P_φ, as defined in equation 5, is a well defined Markov operator and $(\tilde{J}, P_\varphi)$ defines a (compact) Markov process with typical state $\tilde{z} = [s, \theta, q, m] \in \tilde{J}$ and $m_j^i = d^j(s)(u_s^i(c^i))'$.

Now, given the existence of a Markovian representation of the sequential equilibrium $(\tilde{J}, P_\varphi)$, theorem 1 implies that to prove the existence of an invariant measure, it suffices to impose assumption 2 and conditions a) and b). The first and the last are listed in assumptions 6.1-vi) and 6.1-vii) as they cannot be derived from primitive conditions of the model.

The discussion in the preliminary remark of lemma 3 in the appendix implies that property a), namely that the adjoint operator associated with P_φ maps the space of atomless measures into itself, is guaranteed to hold if the implicit function theorem can be applied to the system of equations defined by equations 3, 4 and $\sum_{i=1}^{I} \theta^i = \vec{0}$ in a full lebesgue measure set. More precisely, let $z = [s, \theta, q]$ and $F(z, z_+) = \vec{0}$ be the system of $J + J \times I$ equations that can be obtained by replacing equation 3 into 4 and considering only interior solutions[40]. Section A.2.1) in the appendix will show that, under assumptions 6.1-i) to 6.1-v), $D_{z_+}F(z, z_+)$ has full rank a.e. in z, where $D_{z_+}F$ is the Jacobian matrix of F with respect to z_+.

Once this property has been established, it suffices to apply lemma 3. That is, lemma 3 connects condition 1 (i.e. $\mu(\{a\}) = 0$ *implies* $P_\varphi(\tilde{z}, \{a\}) = 0$ z*-a.e. with respect to an atomless measure* μ) with property a) (i.e. $P_\varphi^*: \mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ where P_φ^* is the adjoint operator associated and $\tilde{J}$ the state space of the process with P_φ and $\mathcal{P}_0(\tilde{J})$ the (sub)space of atomless measures in $\mathcal{P}(\tilde{J})$). The arguments in the preliminary remark of lemma 3 and section A.2.1 in the appendix show that the full rank of $D_{z_+}F(z, z_+)$ is sufficient to guarantee condition 1.

Notice that the implicit function theorem is required to hold a.e. in $\tilde{z}$. Thus there is no contradiction between this property and the possible discontinuity of φ as, taking into account assumption 2, the discontinuity set of φ has finite cardinality and thus zero measure on μ.

[40] The discussion in section A.2.1 in the appendix connects Φ with F and $\tilde{z}$ with z. Once Φ is defined, it suffice to note that $\tilde{z} = [z, m]$ and m is defined by the additional equation given above.

While assumptions 6.1-i) to 6.1-vi) are relatively mild, assumption 6.1-vii) is quite strong as it directly implies the weak-closedness of $\mathcal{P}_0(\tilde{J})$ (i.e. property b). Further, this assumption cannot be connected with primitive conditions of the model. Fortunately, it is possible to obtain properties a) and b) jointly by strengthening condition 1. This is done by lemma 4, that requires only condition 3, which strengthens condition 1 by requiring it to hold *uniformly in* $\tilde{z}$. However, proposition 1 shows that condition 3 holds if the model is allowed to have an uncountable number of exogenous shocks s. Taking into account the distinctive nature of this type of economies, they must be treated separately. Section 6.2 below addresses this point.

5.2 Uncountable Shocks

The discussion in the preceding section sets a trade off: in order to get rid of unverifiable assumptions like property b), the structure of exogenous shocks must be modified. Unfortunately, proving the existence of the sequential equilibria (and thus the existence of an appropriate recursive structure in the sense of Feng, et. al.) with uncountable shocks requires imposing an additional assumption on 6.1-i) to 6.1-v). This assumption, labeled 6.2-ii) below, was extensively discussed in the literature (see for instance Mas-Colell and Zame, 1996, or Araujo, et. al. 1996). While 6.2-ii) was considered unsatisfactory by the sequential equilibrium literature, it have been implicitly assumed in recursive models as can be seen in the preliminary remark of section A.1.2) in the appendix. Thus, in the present context, assumption 6.2-ii) is rather mild.

Once this additional hypothesis has been imposed there is an important gain in terms of the predictive power of the model developed in section 2.1 and 2.2 as the theory developed in section 3 allows showing not only that the model has a well behaved steady state (i.e. an invariant measure, see theorem 1) but also that it is ergodic (see theorem 2). Further, with the notable exception of assumption 2') and 6.2-ii), the remaining hypothesis can be directly traced back to primitive conditions of the model. This last fact can be obtained by proving an additional lemma which allows getting rid of assumption 3-iii) in section 3.4. Once this lemma has been shown, propositions 1 and 2 con be used to derive conditions 3 and 4 and thus theorems 1 and 2 by means of conditions 3 and 4.

As in section 5.1, assumption 6.2 below contained all the sufficient conditions to show the existence of an ergodic invariant measure in the model discussed in sections 2.1 and 2.2 except assumption 3-iii) which will be treated separately in a lemma below.

<u>Assumption 6.2).</u> Suppose an incomplete market economy as the one described in section 2.1. To that structure add the following assumptions:

i) *Assumptions 6.1-i), 6.1-iii) and 6.1-iv) hold.*

ii) $e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t) > 0$, $\sigma_t \in \mathfrak{T}$

iii) *Assumptions 3-i) and 3-iv) hold (i.e. the set of exogenous shocks is* $S = [\underline{S}, \overline{S}] \subset \mathbb{R}$ *and* $p(s,.) = U[\underline{S}, \overline{S}]$*, where* U *is the uniform distribution).*

iv) *Assumption 2' holds (i.e. the discontinuity set is at most of zero lebesgue measure).*

Assumptions 6.2-i) to 6.2-iii) guarantees the existence of the sequential equilibria. The proof follows immediately by extending the induction argument in Mas-Colell and Zame (1996) for $T = \infty$ as in Duffie, et. al. (1994, see fact 2.5-2 in section 2.5.1). In particular, theorem 4.1 in Mas-Colell and Zame allows proving the non-emptiness C_j for $1 \leq j \leq T < \infty$, where C_j is the set of initial states of a $j+1$ period economy defined in section 2.5.2. The compactness of K, the set that includes all payoff relevant states, follows from theorem 4.2 also in Mas-Collel and Zame. The induction argument in section 5 of that paper can be used to set $T = \infty$. The optimality argument in Duffie, et. al. (section 3.4) can be immediately extended to the Mas-Colell and Zame framework as theorem 4.1 and 4.2 hold $\mu_s^\infty(s_0,.)$-a.e. for $s_0 \in S$ and θ_-^i satisfying assumption 6.2-ii), where $(\Omega, \mathcal{F}, \mu_s^\infty(s_0,.))$ is the stochastic process defined in section 4.2 above but restricting the state space Ω to contain only an infinite sequences of exogenous shocks $\{s_t\}$.

The compactness of K and the continuity (in $\tilde{z}_+$) of the system of equations defined by 3), 4) and the feasibility of assets guarantees that the equilibrium correspondence, Φ in definition 5, satisfies the assumptions required by lemma 2. Thus, there is at least 1 measurable selection $\varphi \sim \Phi$ and $(\tilde{J}, P_\varphi)$ defines a Markov process.

Once an appropriate Markov process have been shown to exist, proposition 2 implies that assumptions 6.2-iii), 6.2-iv) and 3.iii) are sufficient to show the ergodicity of the process $(\tilde{J}, P_\varphi)$. The following lemma shows that if there is only 1 asset, assumption 3-iii) can be omitted.

Lemma 5: Suppose that assumption 3-iii) holds or $J = 1$ *(i.e. there is just 1 asset). Then, under assumptions 6.2-i) to 6.2-iv),* $(\tilde{J}, P_\varphi)$ *has an ergodic invariant measure.*

Proof: see section A.2.2 of the appendix.

6 Conclusions and directions for future research

This paper develops the theoretical foundations for an accurate calibration method for incomplete markets general equilibrium models with aggregate uncertainty. Taking into account the lack of robustness of frequently used procedures, as illustrated for instance by De Groot, et. al. (2013) or Hatchondo, et. al. (2010), the results in this paper are relevant as they provide a set of assumptions which insure that empirically relevant models can be taken to data accurately. The parameters obtain are then reliable to perform policy experiments which could be, taking into account the incomplete markets nature of the model, welfare enhancing.

From a theoretical point of view, the paper provides a set of results which allow characterizing incomplete markets general equilibrium models beyond existence. Further, it distinguishes between the predictive performances of models with different degree of uncertainty as measured by the cardinality of the set which contains exogenous shocks.

From a practical point of view, the paper presents a set of sufficient conditions that guarantee that the parameters obtained by appropriately designed algorithms reflects accurately the long and short run behavior of the general equilibrium model originally proposed.

Although the results are quite general and assumptions rather mild, there is scope for future research both in models with a finite number or with uncountable shocks. For the former, condition 2, which insures the existence of an invariant measure, must be connected with primitive conditions. Further, these conditions must also guarantee the ergodicity of the measure as theorem 2 requires even stronger assumptions than theorem 1 as illustrated by properties b) and c). For the case of uncountable shocks, an extensive numerical test must be performed on the algorithm design in section 4. Although the results in Feng, et. al. (2013) and in Silanes, et. al. (2001) have been tested separately, they have not been used jointly in models with uncountable shocks. Moreover, the conditions for the accuracy of simulations, especially assumption 4-iii), must be implemented in an algorithm which combines the Feng, et. al. and the Silanes, et. al. procedures.

Bibliographical References

Akyol, A. (2004): “Optimal monetary policy in an economy with incomplete markets and idiosyncratic risk”, Journal of Monetary Economics, 51(6), 1245-1269, September.

Aliprantis, C. and Border, K. (2006): “Infinite Dimensional Analysis”, Springer, Third Edition.

Arellano, C. (2008): “Default Risk and Income Fluctuations in Emerging Economies”, American Economic Review, 98(3), 690-712, June.

Aruoba, S., Fernandez-Villaverde, J. and Rubio-Ramirez, J. (2006): “Comparing solution methods for dynamic equilibrium economies", Journal of Economic Dynamics and Control, 30(12), 2477-2508, December.

Attanasio, O. and Davis, S. (1996): “Relative Wage Movements and the Distribution of Consumption,” Journal of Political Economy, 104 (6), 1227–62, December.

Azevedo, E., Weyl, G. and White, A. (2013): “Walrasian equilibrium in large, quasilinearmarkets”, Theoretocal Economics, 8 (2), 281-290.

Bewley, t. (1972): “Existence of Equilibria with Infinitely Many Commodities”, Journal of Economic Theory, 4, 514-540.

Braido, L. (2013): “Ergodic Markov Equilibrium with Incomplete Markets and Short Sales”, Theoretical Economics, vol. 8 (1), 41-57

Bogachev, V. (2007): “Measure Theory, Volume 1”, Springer.

Chari, V. V., Kehoe, P. J. and McGrattan, E. R. (2002): “Can Sticky Price Models Generate Volatile and Persistent Exchange Rates?”, Review of Economic Studies, 69, 533-563.

Cochrane, J. (1991): “A Simple Test of Consumption Insurance,” Journal of Political Economy, 99 (5), 957–76, October.

Conesa, J. C. and Krueger, D. (1999): “Social Security Reform with Heterogeneous Agents”, Review of Economic Dynamics, 2(4), 757-795, October.

Constantinides, G. (1982): “Intertemporal Asset Pricing with Heterogeneous Consumers and Without Demand Aggregation,” Journal of Business, 55, 253–267

Cooley, T. and E. Prescott (1995) “Economic Growth and Business Cycles” en Cooley, T. (ed.). Frontiers in Business Cycle Research, Princeton University Press, Ch. 1.

Cooley, T. F. and Quadrini, V. (2001): “Financial Markets and Firm Dynamics”, The American Economic Review, 91(5) , 1286-1310, December.

Covarrubias, E. (2013): “The number of equilibria of smooth infinite economies”, Journal of Mathematical Economics, 49 (4), p.p. 263-265.

Dana, R. (1993) “Existence and Uniqueness of Equilibria when Preferences are Additively Separable”, Econometrica 61, pp. 953-957.

De Jong, D. and Dave, C. (2007) “Structural Macroeconometrics”, Princeton University Press.

De Nardi, M. (2004): “Wealth Inequality and Intergenerational Links”, Review of Economic Studies, 71, 743-768.

Deaton, A. and Paxson, C. (1994): “Intertemporal Choice and Inequality,” Journal of Political Economy, 102 (3), 437–67, June.

Den Haan, W. and Marcet, A. (1990): “Solving the Stochastic Growth Model by Parameterizing Expectations”, Journal of Business and Economic Statistics, 8(1), 31-34.

De Groot, O., Durdu, C. and Mendoza, E. (2013): “Solving DSGE Models with Incomplete Asset Markets: Global vs. Local Methods'', mimeo.

Duffie, D., Geanakopolos, J., Mas-Colell, A., McLennan, A. (1994): “Stationary Markov Equilibria”, Econometrica, 62 (4), July, 745-81.

Fernandez-Villaverde, J., Guerron-Quintana, P., Rubio-Ramirez, J. and Uribe, M. (2011): "Risk Matters: The Real Effects of Volatility Shocks," American Economic Review, 101(6), October, 2530-61.

Fernández-Villaverde, J., Rubio-Ramírez, J. and Santos, M. (2006) “Convergence Properties of the Likelihood of Computed Dynamic Models”, Econometrica, vol. 74(1), p.p. 93-119, January.

Gaspar, J. and Judd, K. (1997): "Solving Large Scale Rational Expectations Models", Macroeconomic Dynamics, 1 (1), January, 45-75.

Geanakolopos, J. and Zame, W. (2002): "Collateral and the Enforcements of Intertemporal Contracts," Discussion Paper, Yale University.

Grandmont, J-M (1983): "Money and Value", Cambridge University Press.

Grandmont, J-M.andHildenbrand, W. (1974): "Stochastic Process of Temporary Equilibria", Journal of Mathematical Economics, 1, 247-277.

Gottardi, P. and Kubler, F. (2013): "Dynamic Competitive Economies with Complete Markets and Collateral Constraints", mimeo.

Guvenen, F. (2009): "An Empirical Investigation of Labor Income Processes," Review of Economic Dynamics, 12 (1), 58–79, January.

Guvenen, F. (2011): "Macroeconomics with Heterogeneity: A Practical Guide", NBER Working Paper Nº 17622.

Hellwig, M. (1980): "Stochastic Processes of Temporary Equilibria – A note". Journal of Mathematical Economics, 7, 287-299.

Hoelle, M. (2014). The relation between sunspot effects and multiplicity in incomplete markets models with numeraire assets. Journal of Mathematical Economics, 50, 119-127.

Huggett, M. (1993): "The risk-free rate in heterogeneous-agent incomplete-insurance economies," Journal of Economic Dynamics and Control, Elsevier, vol. 17(5-6), pages 953-969.

Ito, Y. (1964): "Invariant Measures for Markov Processes". Transactions of the American Mathematical Society, (110), 152-184

Judd, K., (1996): "Approximation, perturbation, and projection methods in economic analysis" Handbook of Computational Economics in Handbook of Computational Economics, vol. 1, ch. 12, 509-585. Elsevier.

Judd, K., Kubler, F. and Schmedders, K. (2002): "A solution method for incomplete asset markets with heterogeneous agents", mimeo.

Jordan, J. (1977): “The Continuity of Optimal Dynamic Decision Rules”, Econometrica, (45), 6, 1365-1376.

Kallenberg, O. (2006). Foundations of modern probability. Springer Science & Business Media.

Kehoe, T. (1991): “Computation and Multiplicity of Equilibria”, Handbook of Mathematical Economics, 4, 2049-2143, North Holland.

Kempton, T., and Persson, T. (2015). Bernoulli convolutions and 1D dynamics. Nonlinearity, 28(11), 3921.

Kubler, F. and Schmedders K. (2002): “Recursive Equilibria in Economies with Incomplete Markets”, Macroeconomic Dynamics, 6, 284-306.

Kubler, F. and Schmedders K. (2003): “Stationary Equilibria in Asset Pricing Models with Incomplete Markets and Collateral”, Econometrica, 71, 6, November, 1767-93

Kubler, F. and Polemarchakis, H. (2004): “Stationary Markov equilibria for overlapping generations”, Economic Theory, 24(3), 623-643.

Krusell, P. and Smith, A. (1998): “Income and Wealth Heterogeneity in the Macroeconomy”, Journal of Political Economy, 106, 867-896.

Kydland, F. E. and Prescott, E. (1982): “Time to Build Aggregate Fluctuations”, Econometrica, 50, 1345-1370.

Hatchondo, J. C., Martinez, L. and Sapriza, H. (2010): “Quantitative properties of sovereign default models: solution methods matter”, Review of Economic Dynamics, 13(4), 919–933, October.

Heathcote, J. (2005): “Fiscal Policy with Heterogeneous Agents and Incomplete Markets” Review of Economic Studies, 72(1), 161-188.

Hildenbrand, W. (1974): “Core and Equilibria of a Large Economy”, Princeton University Press.

Lucas, R. (1976): "Econometric policy evaluation: A critique," Carnegie-Rochester Conference Series on Public Policy, 1(1), January, 19-46.

Lucas, R. (1978): “Asset Prices in an Exchange Economy”, Econometrica, 46, 1429-1445.

Lucas, R. and Sargent, T. (1979): "After Keynesian macroeconomics," Quarterly Review, Federal Reserve Bank of Minneapolis, issue Spr.

Magill, M. and Shafer, W. (1991): “Incomplete Markets”, Handbook of Mathematical Economics, Vol. IV, North Holland.

Magill, M. and Quinzii, M (1994): “Infinite Horizon Incomplete Markets”, Econometrica, 62 (4), July, 853-80.

Magill, M. and Quinzii, M (1996): “Theory of Incomplete Markets, Volume 1”, MIT Press.

Magill, M. and Quinzii, M (2008): “Theory of Incomplete Markets, Volume 2”, Edward Elgar Publishing.

McGrattan, E. and Prescott, E. (2009) “Unmeasured investment and the puzzling U.S. boom in the 1990s" Staff Report 369, Federal Reserve Bank of Minneapolis.

Mehra., R. and Prescott, E. (1980) “Recursive Competitive Equilibria: The Case of Homogeneous Households.” Econometrica 48:1365-79

Mehra, R. and Prescott, E. (1985): “The equity premium: A puzzle,” Journal of Monetary Economics, March 1985, 15 (2), 145–161.

Miao, J. (2006): “Competitive Equilibria of economies with a continuum of consumers and aggregate shocks”, Journal of Economic Theory, 128, 274-298.

Molchanov, I and Zuyev, S. (2011): Advanced Course in Probability: Weak Convergence and Asymptotics, Mimeo

Ostroy, J. and Zame, W. (1994): “Nonatomic Economies and the Boundaries of Perfect Competition”, Econometrica, 62 (3), May, 593-633.

Parthasarathy, K. (1967): “Probability Measures on Metric Spaces”, New York, Academic Press.

Pijoan-Mas, J. (2007): “Pricing Risk in Economies with Heterogeneous Agents and Incomplete Markets”, Journal of the European Economic Association, 5(5), 987-1015.

Santos, M. (2002): “On Non-existence of Markov Equilibria in Competitive-Market Economies”, Journal of Economic Theory 105, 73–98

Santos and Peralta Alva (2005): "Accuracy of Simulations for Stochastic Dynamic Models", Econometrica, 73 (6), Nov, 1939-1976.

Santos and Peralta Alva (2013): "Ergodic Invariant Measures for Non-optimal Dynamic Economies", mimeo.

Shiryaev, A. (1996): "Probability", Springer.

Singleton, K. (1990): "Specification and Estimation of Intertemporal Asset Pricing Models", Handbook of Monetary Economics, North Holland.

Sturgeon, M. (1974): "The ergodic decomposition of conservative Baire measures", Proceedings of the American Mathematical Society, 44, 141-146.

Raad, R. (2013): "Approximate Recursive Equilibria with Minimal State Space", working paper Nº 737, Getulio Vargas Foundation.

Radner, R. (1972): "Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets", Econometrica, 40, 289-303.

Stokey, N., Lucas, R. and Prescott, E. (1989): "Recursive Methods in Economic Dynamics", Harvard University Press.

Appendix

A.1) Sections 3 and 4.

A.1.1) Related results on the existence of Invariant measures.

Because the expectation correspondence in Duffie, et. al. G is defined as map $z \mapsto \mu$, with $\mu \in \mathcal{P}(Z)$, using the arguments presented in section 2.5.1 and 3.1 it is easy to show that a THME (J,π) can be used to define a sequence of measures $\{\lambda_t\}_{t=0}^{\infty}$ in $\mathcal{P}(J)$ such that $\lambda_{t+1} = \pi^*\lambda_t$, where π^* is the adjoint operator associated with (J,π).

Grandmont and Hildenbrand showed that the continuity of π^* is sufficient to show the existence of an invariant measure λ, provided that J is a compact set and G is constructed from an equilibrium correspondence similar to the one presented in definition 6: every $\pi \sim G$ satisfies $\pi = \pi_\varphi$ with $\varphi \sim \Phi$ and $\Phi: J \times S \rightarrow J$. Provided that assumption 1 holds, π_φ follows from Lemma 2. As discussed in section 3.1, π^* is continuous iff $\hat{\pi}$ has the Feller property, where $\hat{\pi}$ is the semigroup operator associated with (J,π). Unfortunately, the authors could not show that $\hat{\pi}$ has this property and had to assume it (see Lemma 2 in Grandmont and Hildenbrand, page 263).

The arguments discussed in section 2.4 imply that φmay not be continuous, thus the result in Hildenbrand and Grandmont was considered unsatisfactory. Blume (1982) dispense with this assumption and took a rather different approach. Given a Markovian structure with time homogeneous transitions, the author used Fan's fixed point theorem to show the existence of an invariant measure for $G_B: \mathcal{P}(Z) \rightarrow \mathcal{P}(Z)$, where $G_B = \{\pi_\varphi^* : \pi = \pi_\varphi, \varphi \sim \Phi\}$. As G_B was assumed to be nonempty, to have closed graph and Z to be compact, the required upper-hemicontinuity followed immediately. However, to apply Fan's theorem, G_B has to be convex valued. That is, if $\lambda_1', \lambda_2' \in G_B(\lambda)$, with $\lambda_1' = \pi_{\varphi_1}^*\lambda$, $\lambda_2' = \pi_{\varphi_2}^*\lambda$ and $\varphi_1, \varphi_2 \sim \Phi$, then $\lambda' \in G_B(\lambda)$ with $\lambda' = (\alpha)\pi_{\varphi_1}^*\lambda + (1 - \alpha)\pi_{\varphi_2}^*\lambda$, $(\alpha)\pi_{\varphi_1}^* + (1-\alpha)\pi_{\varphi_2}^* \in G_B$ and $\alpha \in [0,1]$. To guarantee this latter property, Blume assumed that S is characterized by an atomless measure. Clearly, if S is a finite set, this last assumption is not realistic. The arguments in section 3.2 try to fill this gap. Even if S is a compact, uncountable set and $p(s,.)$ is atomless $\forall s \in S$, as discussed at the end of section 3.1, the results in Blume only shows that G_B has a fixed point, which is equivalent to $IM(\Phi) \neq \emptyset$ but weaker than $IM(\varphi) \neq \emptyset$ for any $\varphi \sim \Phi$ satisfying assumptions 1, 2 and the additional hypothesis presented in section 3.3. As discussed in section 4, this last fact has important numerical implications as it allows approximating φ instead of Φ.

The results in Blume highlighted the necessity of a "convexified" correspondence, G_B, in order to prove the existence of an invariant measure. This was the approach taken by Duffie, et. al. (1994), theorem 1.1, to show the existence of an *ergodic* invariant measure. As discussed in section 2.5.1, provided the existence of self-justified set and that G is convex valued, Duffie, et. al. (1994) showed that a refinement of a THME, called conditionally spotless, has an ergodic invariant measure. The following definition states this notion of equilibrium formally:

Definition A1: Let $\mathcal{P}_F(S \times \hat{Z}) = \{\mu \in \mathcal{P}(S \times \hat{Z}) | \exists \text{ a function } h, h: S \to \hat{Z}, \text{with } Supp(\mu) = Gr(h)\}$. *A THME* (J, π) *is spotless if* $\pi(z) \in \mathcal{P}_F(S \times \hat{Z})$ *for all* $z \in J$. *A THME* (J, π) *is called conditionally spotless if for all* $z \in J$, $\exists\, M \subset \mathcal{P}_F(S \times \hat{Z}) \cap G(z)$, $\eta \in \mathcal{P}(M)$, $\pi(z) = \int v \, d\eta(v)$ *and* G *is convex valued.*

Note that the existence of a spotless THME removes the possibility of sunspots discussed in Lemma 1: given $z_t \in J$, there is a measure $\mu_{z_t} \in G(z_t) \cap \mathcal{P}_F(S \times \hat{Z})$, which gives the conditional distribution of z_{t+1}, $\hat{z}_{t+1} = h(s_{t+1})$ and $\mu_{z_t}(Gr(h)) = 1$. Intuitively, each pair (z_t, s_{t+1}) is associated with a unique $\hat{z}_{t+1}$ or equivalently $\hat{z}_{t+1} = h_{\mu_{z_t}}(s_{t+1})$. Note that it is possible to refine even more a spotless THME by letting $\hat{z}_{t+1} = h_{z_t}(s_{t+1})$, where the measurability of f has to be shown and $z_{t+1} \sim \mu \in \mathcal{P}(S \times \hat{Z})$ has to be defined accordingly. *The results in section 3 and 4 hold for this last type of equilibria.*

To show the existence of an ergodic invariant measure for a spotless THME the authors proceeded in 2 steps. First, they applied Fan's fixed point theorem to $T \equiv E \circ m_2 \circ m_1^{-1}: \mathcal{P}(J) \to \mathcal{P}(J)$, where $m_1: \mathcal{P}(Gr(G_J)) \to \mathcal{P}(J)$, $m_2: \mathcal{P}(Gr(G_J)) \to \mathcal{P}(\mathcal{P}(J))$ give the marginals of $\mathcal{P}(Gr(G_J))$ and $E\eta \equiv \int \mu d\eta(\mu), \eta \in \mathcal{P}(\mathcal{P}(J))$ is the mean of η, which is uniquely defined by the Riesz representation theorem for continuous function[41]. As T is a continuous linear functional and G_J is upper-hemicontinuous. This was assumed in Duffie, et. al. In the context of this paper, a similar property follows from theorem 3.1 in Blume under assumption 1 provided that G_J is constructed from Φ using Lemma 2. However, as discussed in section 2.5.2, this procedure only captures a subset of all possible recursive equilibria. Under these 2 properties, T is also upper

[41] See Theorem 14.12 in Aliprantis and Border (2006, page 496).

hemi-continuous[42]. As J is a self-justified set, G_J is nonempty. T is convex valued as G assumed to be so. Finally, as $\mathcal{P}(J)$ is nonempty, (weakly) compact and convex, T has a fixed point. Second, the authors showed that any λ with $\lambda = T(\lambda)$ also satisfies $\lambda = \pi \cdot \lambda$. In order to derive this result, Duffie, et. al. (1994) defined a transition function $P: J \to \mathcal{P}(\mathcal{P}(J))$ and showed that $E \circ P(z) \in G_J(z)$ λ-a.e. This last fact implies $\pi(z) = \int v \, d\eta(v)$ almost everywhere for $\eta \in \mathcal{P}(\mathcal{P}(J))$.

To obtain an ergodic invariant measure for a conditionally spotless THME, which is defined for economies with a finite number of exogenous shocks, $\mathcal{P}(J)$ should be replaced with $G_J(z) \cap \mathcal{P}_F(S \times \hat{Z})$. This implies that G_J is convex valued: definition A1 assures that for any $z \in J$, there exist an expectation correspondence $\hat{g}$ which is convex valued as it contains any possible randomization $\mathcal{P}(M)$ over spotless transitions $M \subseteq \mathcal{P}_F(S \times \hat{Z}) \cap G(z)$ for any $z \in J$. A selection $\pi(z) \sim \hat{g}(z)$ is constructed by changing M, $\eta \in \mathcal{P}(M)$ and computing $\pi(z) = \int v \, d\eta(v)$. The assumption that G is convex valued can be done w.l.o.g. provided that it can be replaced by $\hat{g}$ once transitions f are allowed to depend on "contemporaneous" sunspots (α_t) which select among randomized spotless transitions.

Unfortunately, the discussion above implies that the transition functions generated by a conditionally spotless THME are affected by sunspots; a fact that affects the computability of the recursive structure. The authors did not prove the existence of an ergodic invariant measure for some spotless THME (definition 4), which generate sunspots free stationary transition function. The purpose of this paper is to show this result for a restriction of all possible spotless THME (i.e. those generated from Feng, et. al.'s recursive structure).

A.1.2) Proofs

Preliminary Remark on $\tilde{J}$

As theorem 1 will show that there exist $\mu \in \mathcal{P}_0$ with $\mu = P_\varphi^* \mu$ (i.e. an invariant measure exists and it is atomless), it is necessary for the state space of the process defined by $(\tilde{J}, P_\varphi)$ to be uncountable. This is because the candidate measure μ_N, with $\mu_{N_k} \to_{Weak*} \mu$, satisfies $Supp(\mu_K) \subseteq \tilde{J}$ as it is constructed applying iteratively P_φ^*.

[42] See Grandmont (1983, page 158).

Fortunately, the results used to guarantee the non emptiness of C_j for $j \geq 1$ (i.e. the set which contains all initial states, $\tilde{z}_0$, of any j-period economy) which were discussed in sections 2.5.1 (fact 2), 2.5.2 and 5.1 can be used to guarantee the desired result. In particular, Theorems 25.1 in Magill and Quinzii (1996) and theorem 4.1 together with section 5 in Mas-Colell and Zame (1996) for economies with finite and infinite number of shocks respectively can be used to show the existence of a sequential competitive equilibrium (see Definition 1) for a truncated economy $\mathcal{E} = [e, d, \{U^i\}_{i=1}^I, T]$, with $T < \infty$. The optimality conditions in Definition 1 for this economy are:

A1) $c^i = e^i(s) + \theta^i d(s) - \theta_+^i q$

A2) $\left[q\left(u_s^i(c^i)\right)' - \beta E_{p(s,.)}(m_+^i)\right][\theta_+^i - \bar{B}] = \vec{0}$

Where short sale constraints $\bar{B}$ are assumed to hold (see sections 2.5.2 and 5.1) and $\theta_+^i = 0$ if $\theta^i = \theta^i(\sigma_{T-1})$. In the sequential economic literature, if $\theta_+^i = \theta^i(\sigma_0)$, it is customary to assume that $\theta_-^i \equiv \theta^i = 0$ and $\sigma_0 \equiv s_0$ is supposed to be fixed. However, in the recursive literature, both θ_-^i and σ_0 are allowed varying as $\tilde{z}_0 = [s_0, \theta_-^i, \hat{z}_0]$, where $\hat{z}_0$ contains the rest of the state space.

Moreover, the existence of equilibria for $\mathcal{E} = [e, d, \{U^i\}_{i=1}^I, T]$ requires that $e^i(s_0) > 0$ (see assumption A.2 in Magill and Quinzii, page 858). Thus, provided the rest of the assumptions mentioned in sections 2.1, 2.5.1 and 5.1 hold, as noted by Duffie, et. al. (Lemma 3.4), θ_-^i and s_0 can be chosen arbitrarily as long as $e^i(s_0) + \theta_-^i d(s_0) > 0$, which can be considered the initial endowment of goods if the exogenous state is s_0. Formally, it suffices to assume that:

Definition A2: *The initial distribution of assets θ_- will be called admissible and denoted $\theta_- \in \Lambda$ if is feasible and satisfies $Min_{i \in I, s \in S} e^i(s) + \theta_-^i d(s) > 0$.*

Remark A1: $\tilde{J} = S \times \Lambda \times \hat{Z}$, where $\Lambda \times \hat{Z}$ is uncountable because and has no isolated points: i) Λ is uncountable and has no isolated points according to definition A2, ii) under the assumptions made in sections 2.1, 2.5.2 and 5.1, $C_j \neq \emptyset$ independently of the cardinality of S (i.e. an equilibrium for $\mathcal{E} = [e, d, \{U^i\}_{i=1}^I, \theta_-]$ exists independently of the cardinality of S) for any $\theta_- \in \Lambda$ (i.e. for any admissible θ_-).

Remark 1 is frequent in applications: see for instance Duffie, et. al. (1994) section 3 and Kubler and Schmedders (2003) page 1777. Typically θ_- describes individual wealth, any predetermined level of asset holdings or the capital stock. Consequently, in numerical approximations θ_- is supposed to be contained in an uncountable subset of $\mathbb{R}$ and its properties (i.e. compactness) can be defined independently of those characterizing $\hat{Z}$ as (s, θ_-) are initial conditions of some sequential competitive equilibrium. Thus, Λ is compact if and only if it is closed. This last property is easily verifiable as can be seen in Kubler and Schmedders (2003) (see lemma 1, page 1776). As will be seen in the proof of lemma 3, the crucial property of Λ, besides its cardinality, is the lack of isolated points. This property follows w.l.o.g. from definition A2.

In all the proof in the appendix, except that it is mentioned explicitly, it will be assumed that the state space can be written as $\tilde{J} = S \times \Lambda \times \hat{Z}$ and that Λ is admissible.

Theorem 1

Preliminary Remark

As discussed in section 3.1, theorem 1 will fail if any selection $\varphi \sim \Phi$ has an uncountable discontinuity set. Fortunately, there are no examples in economics where such a function characterizes the (recursive) equilibrium set. In fact, the literature (see for instance Santos, 2002) has only found examples with jump discontinuities. As will be claimed in the preliminary remark of theorem 2, there are no available methods to numerically approximate a function with an uncountable discontinuity set. Consequently, if a model does not satisfy assumption 2 it can be said to be *non-computable.*

Theorem 3.5 in Molchanov and Zuyev (2011) only requires the discontinuity set to have zero measure under the limiting measure (i.e. $\mu_n \rightarrow_{Weak*} \mu$ and $\mu(\Delta\varphi) = 0$). Thus, it is only necessary, under assumption 2, for μ to be atomless. The arguments in sections 3.2 and 3.3 illustrate the usefulness of properties a) and b) to achieve this purpose. In particular, proposition 1 in Ito (1964) holds under quite mild assumptions on the primitives and assures property a). The critical property is then b), which hold under rather different assumptions depending on the cardinality of S.

As discussed in section 3.1, theorem 3.5 in Molchanov and Zuyev restores the continuity of the adjoint operator by extending the set of adequate functions in the *weak** topology from continuous to Borel measurable if the limiting function is atomless and assumption 2 holds.

The following example illustrates the importance of the atomless assumption when dealing with a Borel measurable function in the $weak^*$ topology.

<u>*Example A.1 (atomic measures and tight spaces)*[43]</u>. Let $P: S \times \mathcal{B}_S \to [0,1]$ be a transition function with $S = [0,1]$ and $P(s, \{s/2\}) = 1$. Let $\{\lambda_n\}$ be a sequence of Dirac measures with $\lambda_n = \delta_{(1/2)^n}$. Thus, $\lambda_n \to \delta_0$, where the convergence is in distribution. Define the bounded Borel measurable function $f(s) = \{1 \; if \; s = 0 \; ; 0 \; otherwise\}$and $\delta_0 \equiv \lambda$. Then $\int f(s)\, \lambda_n(ds)$ =0 and $\int f(s)\, \lambda(ds) = 1$ which in turn implies that $\lambda_n \nrightarrow_{weak^*} \lambda$. The reason behind the lack of $weak^*$ convergence is the impossibility to reduce the measure of the discontinuous part of f.

Proof of theorem 1

Let Φ be an equilibrium correspondence according to definition 6 which satisfies assumption 1. By Lemma 2 $P_\Phi = \{P_\varphi : \varphi \sim \Phi\} \neq \emptyset$ and upper hemi continuous (see for instance proposition 2.2. in Blume, 1982). If P_Φ is convex valued, an ergodic invariant measure can be shown to exist using proposition 1.3 in Duffie, et. al. (1994) (see page 757).

If P_Φ is not convex valued, suppose that assumption 2 together with properties a) and b) in theorem 1 hold. Choose any $\lambda_0 \in \mathcal{P}(\tilde{J})$ and construct a non-oscillating sequence of measures $\{\mu_N\}$ with $\mu_N = h(\{\lambda_n\})$, where h averages the first N-1 elements of $\{\lambda_n\}$ and λ_n satisfies $\lambda_n = P_\varphi^* \lambda_{n-1}$. The dependence of $\{\mu_N\}$ on λ_0 can be omitted w.l.o.g. as the initial condition is arbitrary.

As $\mu_N \in \mathcal{P}(\tilde{J})$ for any N, Helly's theorem (see Stokey, Lucas and Prescott (1989) page 372 and 374) implies that$\{\mu_N\}$ has a weakly convergent subsequence. That is, $\{\mu_{N_k}\} \to_{weak\,*} \mu$.

For notational simplicity $P_\varphi^* \lambda$ and $\hat{P}_\varphi f$ will be replaced by $\pi \cdot \lambda$ and $\pi \cdot f$ as P_φ with $\varphi \sim \Phi$ will be held constant throughout the proof.

For any $f \in C(\tilde{J})$ note that:

[43] This example borrows from Stokey, Lucas and Prescott (1989), page 336. Note that $\{\lambda_n\}$ satisfies $\lambda_n = P \cdot \lambda_{n-1}$. That is, it is possible to generate a sequence of non-atomic measures out of the action induced by P. I would like to thank Prof. R. Fraiman for pointing this out to me.

$$\left|\int f(z)\mu(dz) - \int (\pi \cdot f)(z)\mu(dz)\right|$$
$$\leq \left|\int f(z)\mu(dz) - \int f(z)\mu_{N_k}(dz)\right| + \left|\int f(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu_{N_k}(dz)\right| + \left|\int (\pi \cdot f)(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu(dz)\right| \quad (A.3)$$

From the corollary of theorem 8.1 in Stokey, Lucas and Prescott (1989) (page 215), $(\pi \cdot f): Z \to \mathbb{R}$ is a bounded $\mathcal{B}_{[\tilde{J}]}$-measurable function. Further, from property a) and b), μ is atomless. Under assumption 2, $\mu(\Delta\varphi) = 0$. Then, from theorem 3.5 in Molchanov and Zuyev (2011, fact f), the third term in A3 can be made arbitrarily small. Further, noting that $\{\mu_{N_k}\} \to_{weak*} \mu$ and $f \in C(\tilde{J})$, the first and the third term in A.3 can be made arbitrarily small.

Following the same reasoning as in Stokey, Lucas and Prescott (1989) page 377, the second term satisfies:

$$\left|\int f(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu_{N_k}(dz)\right| \leq 2\|f\|/N \quad (A.4)$$

Where $\|.\|$ is the sup-norm. Thus, for an N arbitrarily large, $\int f(z)\mu(dz) = \int (\pi \cdot f)(z)\mu(dz) = \int f(z)(\pi \cdot \mu)(dz)$, where the last equality follows from theorem 8.3 in Stokey, Lucas and Prescott (1989) (see page 216). Thus, $\int f(z)\mu(dz) = \int f(z)(\pi \cdot \mu)(dz)$. As f was arbitrary, by virtue of corollary 2 of theorem 12.6 in Stokey, Lucas and Prescott (1989) (page 364) $\mu = \pi \cdot \mu$, QED.

■

Lemma 3

Preliminary remark

The proof of this lemma requires π to be θ-nonsingular. A transition function is said to be θ-nonsingular if for any measurable set B, $\theta(B) = 0$ implies $\pi(z, B) = 0$ θ-a.e. As θ is atomless this is equivalent to say that the set D, defined below, is a finite set.

$$D = \{z \in \tilde{J}: \pi(z, B) > 0 \; if \; \theta(B) = 0\} \quad (A.5)$$

Additionally B was restricted to be a point. For those transition functions defined by lemma 2, Ito (1964) show that any non-constant possibly discontinuous many-to-one function $\varphi \sim \Phi$ will generate a θ-nonsingular transition function. This can be seen by written π_φ in lemma 2 as

$$\pi_{\vartheta}(z,B) = p\{s|s' \in S: \varphi(z,s') = a\} = p\{s|s' \in S: \{s_i^{'}\} \cap \tilde{\varphi}^{-1}(z,.)(a_{\hat{z}})\} \quad (A.6)$$

Where $z = [s, \hat{z}]$, $p(s|.)$ is the s^{th} row of the transition matrix which defines the evolution of the exogenous process $\{s_t\}$, $B = \{s_i^{'}\} \times B_{\hat{z}}$ was restricted to a point $a = \{s_i^{'}\} \times a_{\hat{z}}$, $\varphi(z,s') = [s^{'}, \tilde{\varphi}(z,s^{'})]$ is a vector valued function and $\tilde{\varphi}^{-1}(z,.)(B_{\hat{z}})$ is the z-section of the pre-image of $\tilde{\varphi}$ on $B_{\hat{z}}$.

From A.5 and A.6, it is clear that under assumption 2, $\#D < \infty$ provided that $\tilde{\varphi}(.,s^{'})$ is non-constant in z for all $s' \in S$. In section 5, the implicit function theorem is used to show that the model defined in section 2.1 generates θ-nonsingular transition functions.

Proof of lemma 3

Let $(\tilde{J}, \mathcal{B}_{\tilde{J}}, m)$ be a measure space. By assumption 1, $\tilde{J}$ is compact and by remark A1 this set could be written as $\tilde{J} = S \times \Lambda \times \hat{\mathrm{Z}}$, where Λ contain all admissible states and $\Lambda \times \hat{\mathrm{Z}}$ is uncountable and has no isolated points. Further, note that any measure in $(\Lambda, \mathcal{B}_{\Lambda})$, denoted m_{Λ}, is a Radon measure as Λ is a Hausdorff metric space and m_{Λ} is: i) defined over a Borel sigma-algebra ($\mathcal{B}_{\Lambda}$), ii) regular as it is a measure on a Hausdorff (compact) metric space (Λ), iii) $\mathcal{B}_{\Lambda}$-finite as it is a probability measure. Thus, as Λ has no isolated points, $(\Lambda, \mathcal{B}_{\Lambda})$ has an atomless measure m_{Λ}^{A} (see Bogachev 2007, page 136) which in turn implies by remark A1 that there is a measure m^{A} in $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ that is also atomless. The first part of the lemma is completed by setting $m^{A} \equiv \theta$.

Let $\mathcal{P}_0(\tilde{J}) \subset \mathcal{P}(\tilde{J})$ be the set of atomless measures in $\mathcal{P}(\tilde{J})$ generated by π, starting from θ. It follows from proposition 1 in Ito (1964) that π maps $\mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ as π is θ-nonsingular by condition 1. Finally, condition 2 is just the definition of a *weak*-closed* set applied to $\mathcal{P}_0(\tilde{J})$.

■

Example A.2 ($\theta \in \mathcal{P}(\tilde{J})$ and θ is atomless). The reference measure θ could be a mixed joint density: $\theta(s \times A) = P(s = \{s\}, \hat{z} \in A) = \int_A p_{S,\hat{Z}}(s,\hat{z})d\hat{z}$ where $p_{S,\hat{Z}}(s,\hat{z}) = \theta(s \times \{\hat{z}\}) = 0$ is a density function on $\hat{Z}$ which may vary with any $s \in S$. From fact 14 page 45 in Hildenbrand and Grandmont (1974), θ is atomless.

Lemma 4

Preliminary Remark

The implication of condition 4) requires to show the *weak** closedness of $IM(\varphi,\mathcal{P}_1)$. The proof below shows that $IM(\varphi,\mathcal{P}_1)$ is *weak** sequentially compact (i.e. that every bounded sequence in $IM(\varphi,\mathcal{P}_1)$ has a *weak** convergent subsequence). As $\mathcal{P}_1$ can be endowed with the Prohorov metric (see Hildenbrand and Grandmont 1974, page 49), sequential compactness implies that $IM(\varphi,\mathcal{P}_1)$ is not only closed but also compact.

Proof of lemma 4

For the existence of an atomless measure on $\tilde{J} = S \times \Lambda \times \hat{Z}$ with S uncountable and compact, let θ be the uniform measure on $\tilde{J}$.

For property a), note that condition 3) implies that P_φ is θ-nonsingular. Thus, proposition 1 in Ito (1964) applies just as in the proof of lemma 3.

In order to prove property b), note that any point $\{a\} \in \tilde{J}$ has zero Lebesgue measure. Thus, under condition 4), proposition 2.3 in Santos and Peralta Alva (2013, page 8) can be used to guarantee the desired result.

Property c) will be proved in 3 parts: i) $IM(\varphi,\mathcal{P}_1) \neq \emptyset$. As $\tilde{J}$ is compact, Helly's theorem implies the existence of a weak* converging subsequence in $IM(\varphi,\mathcal{P}_1)$ denoted w.l.o.g. $\mu_n \rightarrow_{weak\,*} \mu$. It will be shown that: ii) μ is absolutely continuous w.r.t θ, iii) $\mu \in IM(\varphi,\mathcal{P}_1)$.

In what follows it will be assumed w.l.o.g. that $\theta(dz) = dz$. This is done for expositional purposes only.

i) Standard results (See Billingsley 1968, page 422) imply that condition 4) is equivalent to the following statement: for any measurable set B, $\theta(B) = 0$ implies $SUP_{z\in J}[\pi_\vartheta(z,B)] = 0$. Thus, π_ϑ is θ-nonsingular. By proposition 1 in Ito (1964), $\pi_\vartheta: \mathcal{P}_1 \rightarrow \mathcal{P}_1$. Under the same condition, lemma 2.3 in Santos and Peralta Alva (2013) also holds, which implies that $\mathcal{P}_1$ is *weak** closed. Under assumption 2, theorem 1 implies that $IM(\varphi,\mathcal{P}_1) \neq \emptyset$.

ii) By the characterization of absolutely continuity in Billingsley (1968, page 422), it suffice to show that for any $\varepsilon > 0$, $\exists \delta > 0$ such that $\theta(B) < \delta$ implies $\mu(B) < \varepsilon$. Condition 4) implies that $\pi_\vartheta(z,.)$ is absolutely continuous w.r.t. θ for any $z \in \tilde{J}$. That is, $\pi_\vartheta(z,dz') = \bar{\pi}_\vartheta(z,z')dz'$ where $\bar{\pi}_\vartheta(z,.)$ is the density associated with $\pi_\vartheta(z,dz')$. Take any sequence $\{\hat{\mu}_n\} \in IM(\varphi,\mathcal{P}_1)$. Note that $\{\pi_\vartheta\hat{\mu}_n\}$ is a family of measures that satisfies the hypothesis of Helly's theorem and $\{\pi_\vartheta\hat{\mu}_n\} \in IM(\varphi,\mathcal{P}_1)$.

Let $\pi_\vartheta \hat{\mu}_n \equiv \mu_n$ and note that $\{\mu_n\} \in IM(\varphi, \mathcal{P}_1)$ and has a *weak** limit denoted (passing to a subsequence if necessary) μ.

Note that $\mu_n(B) = \int_B h_n(z')\theta(dz')$ where $h_n(z') = \int \bar{\pi}_\vartheta(z, z')\,\mu_n(dz)$. But now note that $\mu_n(B)$ could be written as:

$$\mu_n(B) = \int_B h_n(z')\theta(dz') = \int \left[\int_B \bar{\pi}_\vartheta(z, z')dz'\right] \mu_n(dz)$$

Condition 4) implies that $\left[\int_B \bar{\pi}_\vartheta(z, z')dz'\right] < \varepsilon$ uniformly in z. Thus $\mu_n(B) < \varepsilon$. The arguments in the first part of lemma 3 imply that $\{\mu_n\}$ and μ are regular measures. Thus, B can be assumed to be open w.l.o.g. Now, the definition of *weak** convergence implies (see theorem 12.3-c in Stokey, Lucas and Prescott, page 358) $\mu(B) \leq liminf_n \mu_n(B)$. In order to complete the proof, by the preliminary remark of this lemma, it suffice to note that $liminf_n \mu_n(B) < \varepsilon$.

iii) It remains to show that $\mu \in IM(\varphi, \mathcal{P}_1)$. Take $\mu_n \rightarrow_{weak *} \mu$. Note that for any $f \in C(\tilde{J})$:

$$\lim_n \int f(z)\mu_n(dz) = \int f(z)\mu(dz) = \lim_n \int f(z)[\pi\mu_n](dz) = \lim_n \int [\pi f](z)\mu_n(dz) = \int [\pi f](z)\mu(dz)$$
$$= \int f(z)[\pi\mu](dz) \quad (A.7)$$

Where the first equality in A.7 follows from the definition of *weak** convergence of $\mu_n \rightarrow_{weak *} \mu$, the second from $\{\mu_n\} \in IM(\varphi, \mathcal{P}_1)$, the third from theorem 8.3 in Stokey, Lucas and Prescott, the forth from theorem 3.5 in Molchanov and Zayev as μ is absolutely continuous w.r.t. θ (and thus atomless) and the last equality from theorem 8.3 in Stokey, Lucas and Prescott again. Note that A.7 implies $\int f(z)\mu(dz) = \int f(z)[\pi\mu](dz)$. As $f \in C(\tilde{J})$ is arbitrary, the proof is complete.

■

Proof of Proposition 1

Under assumption 1, lemma 2 implies that π is well defined (i.e. is a Markov operator). Under assumptions 3-i) and 3-ii) the result follows from equation A.6) by noting that $\{s_i'\} \cap \tilde{\varphi}^{-1}(z,.)(a_{\hat{z}})$ is either a point in S or $\emptyset$ for any $z \in \tilde{J}$.

■

Proposition 2

Preliminary remark

Arbitrarily selecting $z \in \tilde{J}$, it will be shown that $\forall \varepsilon(z) > 0, \exists \delta(z) > 0$ such that $\theta(B) < \delta(z)$ implies $\pi(z,B) < \varepsilon(z)$. As $\tilde{J}$ is compact and $\varepsilon(z), \delta(z)$ are finite (real) numbers, it suffices to take $max_{z \in \tilde{J}}\, \varepsilon(z) = \varepsilon$ and $max_{z \in \tilde{J}}\, \delta(z) = \delta$.

For the first part of the proof the following fact will be useful: let θ be the Lebesgue measure and $R \subseteq \tilde{J} \subset \mathbb{R}^K$ a rectangle and μ^V its volume. That is, $R = [a_1, b_1] \times \dots \times [a_K, b_K]$ and $\mu^V(R) = [b_1 - a_1] \dots [b_K - a_K]$. Then, $\theta(B) = 0$ if $\forall \gamma > 0$, $\exists \{R_i\}_{i=1}^{\infty}$ with $B \subseteq \bigcup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$. The proof of the first part the proposition will be completed if it can be shown that for each $\varepsilon(z) > 0$, there exist an $\gamma > 0$ such that $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$ implies $\sum_{i=1}^{\infty} \pi(z, R_i) \leq \varepsilon(z)$ because $\theta(B) = 0$ as long as $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$.

Proof

Note that any positive $\pi_\varphi(z,.)$-measure rectangle, R_i, could be written as

$$R_i = \left[\varphi_1\left(z, s'_{1,i} - 2^{-1}h_{1,i}\right), \varphi_1\left(z, s'_{1,i} + 2^{-1}h_{1,i}\right)\right] \times \dots \times \left[\varphi_K\left(z, s'_{K,i} - 2^{-1}h_{K,i}\right), \varphi_K\left(z, s'_{K,i} + 2^{-1}h_{K,i}\right)\right]$$

where the first coordinate is just $\left[s'_{1,i} - 2^{-1}h_{1,i}, s'_{1,i} + 2^{-1}h_{1,i}\right]$, φ_k and $s'_{k,i}$ denote any coordinate of φ for $1 \leq k \leq K$ and the elements of S that generates coordinate k of rectangle i.

Note assumption 3-iii) implies that $\varphi_k(z,.)$ is allowed to oscillate continuously, not necessarily forming a straight line, between $\varphi_k(z,x)$ and $\varphi_k(z,y)$ where $x = s'_{k,i} - 2^{-1}h_{k,i}$ and $y = s'_{k,i} + 2^{-1}h_{k,i}$. Thus, by theorem 2.27 in Aliprantis and Border (2006), $h_{k,i}$ is the length of the interval in the pre-image of $\varphi_k(z,.)$, where $\varphi_k(z,x)$ and $\varphi_k(z,y)$ are exactly the endpoints of the k^{th} coordinate of rectangle R_i.

Now equation A.6) implies that $\pi(z, R_i) \leq p\left(s, \bigcap_{k=1}^{K}\left[s'_{k,i} - 2^{-1}h_{k,i}, s'_{k,i} + 2^{-1}h_{k,i}\right]\right) = p\left(s, \bigcap_{k=1}^{K}\left[0, h_{k,i}\right]\right)$, where the inequality follows from the preceding discussion and the equality from assumption 3-iv) after normalizing $p(s,.)$ to be in the unit interval.

Now note that assumption 1 implies that $\mu^V(R_i)$ is finite as the range of any $\varphi \sim \Phi$ is bounded, and $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$ implies $lim_{i \to \infty} \mu^V(R_i) = 0$. Thus,

$$\pi_\varphi(z, R_i) \leq \varepsilon(z)_i 2^{-i} \quad \text{A.8)}$$

where $\varepsilon(z)_i = min_k h_{k,i}$.

Also from $lim_{i\to\infty}\mu^V(R_i) = 0$, equation A.8) implies that $lim_{i\to\infty}(\varepsilon(z)_i)$ is finite. Thus, $SUP_i\varepsilon(z)_i = max_i\varepsilon(z)_i = \varepsilon(z)$ and $\sum_{i=1}^{\infty}\pi(z,R_i) \leq \varepsilon(z)$, as $\sum_{i=1}^{\infty} 2^i = 1$.

Now to prove the dependence of γ on $\varepsilon(z)$, let $R_{i,k}$ be the k^{th} coordinate of rectangle R_i. Note that assumption 3-iii) implies, by theorem 2.34 in Aliprantis and Border, that for all $i, \exists k$ with $R_{i,k} = [\varphi_1(z,x), \varphi_1(z,y)]$ and $[x,y]$ has length smaller or equal to $\varepsilon(z)$. Consequently, $\varepsilon(z)$ could be made arbitrarily small as desired and there will always be an associated γ such that A.8) holds. As z is arbitrary, the proof is complete.

■

Proof of remark 2: the result follows from replacing $p(s, \cap_{k=1}^{K}[0, h_{k,i}2^{-i}])$ by $p(s, \cap_{k=1}^{K}[LB(s), h_{k,i}2^{-i}])$ in equation A.8) and noting that $\varepsilon(z)_i = min_k \frac{h_{k,i-LB(s)}}{UB(s)-LB(s)}$, where z is a vector of the form $z = [s, \hat{z}]$, is a finite number for all $z \in \tilde{J}$.

Theorem 3

Preliminary Remark

As in the case of theorem 1, theorem 3 has to be applied to economies that has at least 1 selection φ~Φ with at most a zero Lebesgue measure discontinuity set. This restriction must be extended to all approximated economies which are characterized by $\varphi_j \sim \Phi_j$. This is because, as discussed in section 3.1, $IM\{\varphi_j, \mathcal{P}_1\}$ may be empty if the discontinuity set is allowed to have positive Lebesgue measure.

The discussion in section 2.4 suggests that the cardinality of the discontinuity set is associated with the number of possible equilibria. Thus, even though it is theoretically possible to have a discontinuity set with positive Lebesgue measure, *the endogenous laws of motions in this economy may not be computable even using state of the art procedures*.

In particular, the algorithm in Feng, et. al. (2013) computes an outer approximation of the equilibrium correspondence (i.e. $Gr(\Phi_j) \supseteq Gr(\Phi)$). Thus, assumption 4-iii) may not hold for this procedure. Further, in this procedure it is not clear how to impose assumption 3-iii) because the interpolation method used to convexify the computed equilibrium correspondence is not specified in the paper (see page 11 for an outline of the algorithm and pages 39 to 41 for details). The procedure in Kubler and Schmedders (2003) circumvent some of these problems as it provides a convenient spline-based interpolation method. Unfortunately, the sequence of approximating functions is assumed to be continuous and to converge in the sup-norm on K. Both facts taken together imply that the limiting function is continuous on K (see page 1782), which may be inadequate in the context of this paper.

There are spline based procedures which allows computing functions with an uncountable discontinuity set (see for instance Silanes, et. al. 2001). These procedures converge uniformly on $(K \times S)\backslash\Delta\varphi$. Unfortunately, the arguments in the proof of theorem 3 will show that this type of convergence is inadequate under assumption 4-ii) if $\Delta\varphi$ has positive Lebesgue measure.

It is worth noticing that in an algorithm that approximates $\tilde{J}$ using a sequence of correspondences or sets, theorem 3.5 in Santos and Peralta Alva (2013) can be used to prove the desired upper hemi-continuity and compact valuedness of Φ_j (assumption 1 applied to theorem 3). This is the case of the recursive equilibrium algorithm in Feng, et. al. However, as mentioned before, this procedure generates a sequence of correspondence Φ_j with $Gr(\Phi_j) \supseteq Gr(\Phi)$, which may be inadequate under assumption 4-iii). Finally, it is possible to construct φ_j using a policy function $\varrho_j : S \times Z_1 \to \tilde{Z}$ with $Z = S \times Z_1 \times \tilde{Z}$ as in Kubler and Schmedders (2003). This procedure lowers the dimension of the state space and thus the computational burden, measure in CPU time, of the algorithm. The authors provided a detailed spline procedure, but they did not take care of $\Delta\varphi$. It is a matter of future research to establish if the spline procedure in Silanes, et. al., which addresses $\Delta\varphi$ appropriately, fits into the framework of theorem 3. In particular, assumption 4-iii) should be carefully enforced as it involves all zero Lebesgue measure sets which, because of assumption 2'), do not belong to $(K \times S)\backslash\Delta\varphi$.

Finally, the other known recursive algorithms (see for instance Raad, 2013) may not suitable for simulations as it is not clear how to fit those procedures into the theoretical framework outlined in this paper or in Santos and Peralta Alva (2013).

Consequently, if all stationary laws of motion (i.e. all $\varphi \sim \Phi$) have a positive Lebesgue measure set of discontinuities, this economy may not be accurately computable and thus is beyond the scope of this paper.

The proof of this theorem will proceed in 3 steps: first it will be shown that there is a sequence of absolutely continuous measures $\{\mu_j\}$, with $\mu_j = \pi_{\varphi_j}\mu_j$, which has a *weak** limit μ that is also absolutely continuous. Second, using the first result, it will be shown that the evaluation map, $Ev(\varphi,\mu) \equiv \pi_\varphi \mu$, is jointly continuous when φ is endowed with the sup-norm topology and μ with the weak topology. Finally, using the second result, it will be shown that $\mu = \pi_\varphi \mu$.

Proof of theorem 2

i) Assumptions 1, 2'), 3-iii) and 3-iv) applied to $\{\varphi_j\}$ implies, by theorem 2, that $IM\{\varphi_j, \mathcal{P}_1\} \neq \emptyset$ for all j. Assumption 4-i) implies that the sequence $\{\mu_j\}$, with $\mu_j = \pi_{\varphi_j}\mu_j$, satisfies the hypothesis of Helly's theorem as $\mathcal{P}_1 \subset \mathcal{P}(K)$. Thus, $\{\mu_j\}$ has a subsequence weakly converging to μ. As assumption 3-iii) and 3-iv) hold for φ, proposition 2 implies $\pi_\varphi(z,A) < \varepsilon$ for any open set A with $\theta(A) < \delta$. Assumption 4-iii) implies that $lim_{j\to\infty}\pi_{\varphi_j}(z,A) \le \pi_\varphi(z,A)$, which in turn implies that $lim_{j\to\infty}\pi_{\varphi_j}\mu_j(A) < \varepsilon$. The same arguments used in lemma 4-ii) implies that μ is absolutely continuous w.r.t. θ as desired.

ii) Let $\mu_j \to_{Weak*} \mu$. It has to be shown that $Ev(\varphi_j,\mu_j) \to_{Weak*} Ev(\varphi,\mu)$. The arguments in Blume (1982, page 63) implies that it suffice to take an arbitrary test function in the unit ball generated by the sup-norm on $C(\tilde{J})$. Thus, the proof will be completed if it can be shown that:

$$\left|\int f(z)(\pi_{\varphi_j}\mu_j)(dz) - \int f(z)(\pi_\varphi\mu)(dz)\right| \to 0 \text{ as } j \to \infty \qquad \text{A.9)}$$

Using theorem 8.3 in Stokey, Lucas and Prescott, A.9) could be written as:

$$\left|\int (\pi_{\varphi_j} f)(z)\mu_j(dz) - \int (\pi_\varphi f)(z)\mu(dz)\right|$$
$$= \left|\int\left[\int f\big(\varphi_j(z,s')U(ds')\right]\mu_j(dz) - \int\left[\int f(\varphi(z,s')U(ds')\right]\mu(dz)\right|$$

Adding and subtracting $\int[\int f(\varphi(z,s')U(ds')]\mu_j(dz)$ and using the triangle inequality the above expression could be written as

$$\left|\int\left[\int f\big(\varphi_j(z,s')U(ds')\right]\mu_j(dz) - \int\left[\int f(\varphi(z,s')U(ds')\right]\mu(dz)\right|$$
$$\le \left|\int\left[\int f\big(\varphi_j(z,s')\big)U(ds') - \int f(\varphi(z,s')U(ds')\right]\mu_j(dz)\right|$$
$$+ \left|\int\left[\int f(\varphi(z,s')U(ds')\right]\mu_j(dz) - \int\left[\int f(\varphi(z,s')U(ds')\right]\mu(dz)\right|$$

Because of assumption 2') is supposed to hold for $\{\varphi_j\}$ and φ, the first term is bounded above by $SUP_{(K\times S)\setminus\Delta\varphi}\|\varphi_j(z,s^{'})-\varphi(z,s^{'})\|_{\infty}$, which converges to zero by assumption 4-ii). The arguments in the proof of theorem 1 implies that the second term also converges to zero as μ is absolutely continuous w.r.t. θ and assumption 2') holds on φ. These 2 facts taking together implies $Ev(\varphi_j,\mu_j)\rightarrow_{Weak*} Ev(\varphi,\mu)$ as f is arbitrary.

iii) Equation A.9) and $\mu_j = \pi_{\varphi_j}\mu_j$ for any j implies

$$\left|\int f(z)\mu_j(dz)-\int f(z)(\pi_{\varphi}\mu)(dz)\right|\rightarrow 0 \text{ A.10)}$$

Also $\mu_j \rightarrow \mu$ implies

$$\left|\int f(z)\mu_j(dz)-\int f(z)\mu(dz)\right|\rightarrow 0 \text{ A.11)}$$

Now, taking $\left|\int f(z)\mu(dz)-\int f(z)(\pi_{\varphi}\mu)(dz)\right|$ and adding and subtracting $\int f(z)\mu_j(dz)$, the triangle inequality implies

$$\left|\int f(z)\mu(dz)-\int f(z)(\pi_{\varphi}\mu)(dz)\right| \leq \left|\int f(z)\mu_j(dz)-\int f(z)\mu(dz)\right|+\left|\int f(z)\mu_j(dz)-\int f(z)(\pi_{\varphi}\mu)(dz)\right|$$

Finally, equation A.10) and A.11) implies $\left|\int f(z)\mu(dz)-\int f(z)(\pi_{\varphi}\mu)(dz)\right|\rightarrow 0$ which proves the last part of the theorem.

■

Proof of Theorem 4 (LLN)

Preliminary Remark on the equilibrium correspondence Φ (definition 5)

In section 2.5.2 $\Phi:\tilde{J}\times S\rightrightarrows\tilde{J}$ was defined as containing any $\tilde{z}_+=[\tilde{s}_+,\tilde{\theta}_+,\tilde{q}_+,\tilde{m}_+],\tilde{z}=[\tilde{s},\tilde{\theta},\tilde{q},\tilde{m}]\in\tilde{J}$ simultaneously satisfying equations A.1) and A.2); implying $\tilde{z}_+\in\Phi(\tilde{z},\tilde{s}_+)$ with $m(s_+)\sim V^*(\theta_+,q_+,s_+)(s_+)$ and $\tilde{m}_+\sim V^*(\theta_+,q_+,s_+)(\tilde{s}_+)$. This remark explores this claim in detail as it is essential to understand the meaning of $\{\tilde{z}_t\}$ as a realization $\omega\in\Omega$ of the process $(\Omega,\mathcal{B}_{\Omega},P_{\mu})$ defined in section 4.2.

For simplicity take a 5 period economy with only 2 exogenous shocks $S = \{s_1, s_2\}$ as will suffice to illustrate the iterative procedure that generates $\{\tilde{z}_t\}$. The figure below illustrates a sequence of $\{c_j\}$, as defined in section 2.5.2, obtained from equations A.1) and A.2):

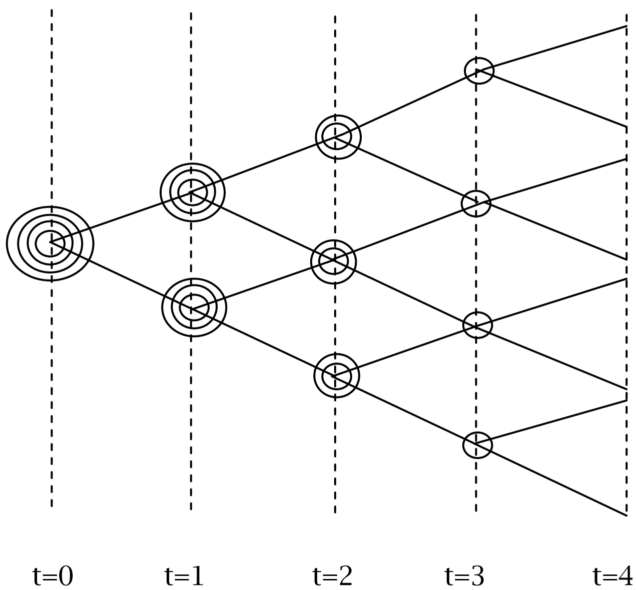

Figure A.1

The nodes with at least 1 circle belong to C_1 (i.e. the set that contains the initial conditions of all 2 period economies in figure A.1), the nodes with at least 2 circles belong to C_2 and so on. Remarkably, note that the only node at t=0 has 4 circles. Thus, *any element* of C_4 has the initial conditions *not only of a 5 period economy but also some of all possible initial conditions of any other economy depicted by the figure.* Further, $\tilde{J} = \bigcap_{j=1}^{4} C_j$.

Take any pair of elements in $\tilde{J}$, $[\tilde{z}^i, \tilde{z}^l]$ and let $c_j^i \in C_j$. Note that $[\tilde{z}^i, \tilde{z}^l] = [c_3^i, c_4^l] = [\tilde{z}_+, \tilde{z}]$ where the first equality follows from the definition of $\tilde{J}$ and the second from the definition of C_3 and C_4. Now, w.l.o.g., let $\tilde{z}_+ = [s_1, \tilde{\theta}_+, \tilde{q}_+, \widetilde{m}_+]$. From definition 5, $\tilde{z}_+ \in \Phi(\tilde{z}, s_1)$ if $[\tilde{z}_+, \tilde{z}]$ satisfies equations A.1) and A.2). But this fact follows from theorem 1.2 in Duffie, et. al. as, following the discussion in section 2.5.2, the recursive equilibrium in Feng, et. al. are a subset of all possible THME (see definition 4) implying $G(\tilde{z}) \cap \mathcal{P}(\tilde{J}) \neq \emptyset$, $G(\tilde{z}) = \{P_\varphi(\tilde{z}, .) : \varphi \sim \Phi\}$ and $\tilde{z} \in \tilde{J}$ as it can be seen from fact 1) in section 2.5.1. Note that theorem 1.2 in Duffie, et. al. requires G to be closed graph. This property follows from standard results in Blume (1982) under assumption 1.

In order to iterate the process forward using Φ in definition 5, take $\{\tilde{s},\tilde{q},\tilde{\theta},\tilde{m}\} \in \tilde{J}$ and $\tilde{s}_+$. Given $\{\tilde{s},\tilde{q},\tilde{\theta},\tilde{m}\}$ use $m^{i,j} \equiv d^j(s)\left(u_s^i(c^i)\right)'$ and equation A.1 to compute c and $\tilde{\theta}_+$. Take a sequence $\{m_+(s_+)\}_{s_+}$ with $m_+(s_+) \sim V^*(\theta_+, q_+, s_+)(s_+)$ and $\tilde{\theta}_+ = \theta_+(s_+)$. If c and $\{m_+(s_+)\}_{s_+}$ satisfy equation A.2, then $\{\tilde{s}_+, q_+(\tilde{s}_+), \tilde{\theta}_+, m_+(\tilde{\theta}_+, \tilde{s}_+)\}$ is the next state.

The existence of $\{m_+(s_+)\}_{s_+}$ satisfying these properties is guaranteed by proposition 1.3 in Duffie, et. al. The fact that m_+ is a function of s_+ follows from the definition of Spotless THME (see definition A.1) applied to this type of economies as the vector $[\theta_+, q_+]$ is allowed to depend measurably on s_+ (see Duffie, et. al. page 767). Note that in this case, $\theta_+(s_+) = \tilde{\theta}_+$ once $\tilde{s},\tilde{q},\tilde{\theta},\tilde{m}$ has been fixed. Thus θ_+ is continuous on s_+, as required by assumption 3-iii).

Proof

The assumptions of theorem 3 implies that $\mathcal{E}\left(IM\left(\varphi_j, \mathcal{P}_1\right)\right) \neq \emptyset$ for any j sufficiently large. Let z_0^j be an initial condition which satisfies assumption 5. Then, fact 4.2-iv) implies that $\boldsymbol{P}_{\varphi_j, z_0^j}$ -almost surely, $lim_{N\to\infty}\left[\sum_{t=1}^{N} f(z_i^j(z_0^j, \omega, \varphi_j))\right]N^{-1} = \int f(z)\mu_j(dz)$ with $\mu_j \in IM\left(\varphi_j, \mathcal{P}_1\right)$. Finally, By the assumptions in theorem 3, $\mu_j \to_{weak*} \mu$ and $\mu \in IM(\varphi, \mathcal{P}_1)$ or equivalently $\left|\int f(z)\mu_j(dz) - \right.$ *f(z)μ(dz)=0* for *j* sufficiently large. Then,*t=1Nf(zijz0j,ω,φj)N−1−f(z)μ(dz)=0* for *N,j* sufficiently large as desired.

■

A.2) Section5.

A.2.1) Finite Shocks

This section proves that under assumptions 6.1-i) and 6.1-vi) the implicit function theorem can be applied to the system of equations that is equivalent to the sequential competitive equilibrium in definition 1.

The results in Magill and Quinzii (1994) and Kubler and Schmedders (2002) imply that under assumptions 6.1-i) to 6.1-v) the following system of equations defines a sequence of consumption bundles $\{\{c^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$, portfolios $\{\{\theta^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$ and prices $\{q(\sigma_t)\}_{\sigma_t\in\mathfrak{T}}$ which satisfy the feasibility and optimality requirements in definition 1:

A.12) $\sum_{i=1}^{I}\theta_{+}^{i}=\vec{0}$ with $\vec{0}\in\mathbb{R}^{J}$

A.13) $q_j u_s^i(e^i(s)+\theta^i d(s)-\theta_{+}^{i}q)'-\beta\sum_{s_+\in S}d_j(s_+)p(s,s_+)u_s^i(e^i(s_+)+\theta_{+}^{i}d(s_+)-\theta_{++}^{i}q)'=0,\ j\in J, i\in I$

Let $z=[s,\theta,q]$ with $\sum_{i=1}^{I}\theta^{i}=\vec{0}$ and $m^i=d(s)u_s^i(e^i(s)+\theta^i d(s)-\theta_{+}^{i}q)'$. Also let $F(z,z_+)=\vec{0}$ be the system of equations defined by A.12) and A.13), where $\vec{0}\in\mathbb{R}^{J+J\times I}$.

The discussion in section 2.5.2 and 6.1 imply that under assumptions 6.1-i) to 6.1-v) $[z_+,m_+]\in\tilde{J}$ if $[z,m]\in\tilde{J}$, where $\tilde{J}$ is the expanded equilibrium state space in definition 5. Moreover, the same results imply that each θ_{++} implicit in m_+ define a different selection $m_+\sim V^*(z_+)$, where $\tilde{J}=Gr(V^*)$. Thus, as A.12) and A.13) can be used to define a particular selection $\varphi\sim\Phi$, θ_{++} can be assumed to be constant throughout the analysis.

Further, because $s,s_+\in S$ and $\#S<\infty$ and condition 1 is required to hold a.e. in an atomless measure μ, the discussion in the preliminary remark of lemma 3 implies that it suffice to show that $D_{z_+}F(z,z_+)$ has full rank μ -a.e. in z as this implies that $\mu(D)=0$, where $D=\{[z,m]\in$ *J:Pφ(z, m,a)>0 if μ(a)=0* was defined in equation A.6). Moreover, assumption 6.1-vi) guarantees that $D_{z_+}F(z,z_+)$ is well defined μ -a.e. in z as the discontinuity set of φ is allowed to have up to finite cardinality and F is defined for interior solutions only.

To complete the proof it suffice to write $D_{z_+}F(z,z_+)$ explicitly in order to note that this matrix has full rank under assumptions 6.1-i) and 6.1-v) provided that there is more than 1 asset[44].

A.2.1) Finite Shocks

Preliminary remark of Lemma 5

As Discussed in section 3.4), the existence of an ergodic invariant measure can be shown under a slightly weaker assumption than 3-iv). The results holds under assumption 3.iv') which allows $p(s,.)$, the distribution of exogenous shocks, to depend on s. Assume further that,

Assumption A.1) : *Let* $p(s,.)$ *satisfy assumption 3-iv'). Then,* $p(s,.)$ *has the Feller property.*

[44] $D_{z_+}F(z,z_+)$ is available under request.

The proof below assumes that $p(s,.)$ satisfies assumption A.1) provided the existence of a recursive structure Φ. The discussion in section 5.2 and the results in Mas-Colell and Zame (1996) imply that assumption 3.4) is required to insure the existence Φ in definition 5. Of course, 3.4) implies A.1). However, the proof will be done imposing the less restrictive assumptions in case Φ can be derived under milder restrictions for a different type of economy.

Under assumptions 6.2-i) to 6.2-iv) and 3-iii) the result in lemma 5 follows from proposition 1 and 2 and theorems 1 and 2. Thus the proof of the lemma will only take care of the case of only 1 asset which allows to dispense with assumption 3-iii). It will be shown that there exist a selection $\varphi \sim \Phi$, with $\varphi(\tilde{z}, s_+) = [s_+, \theta_+(\tilde{z}, s_+), q_+(\tilde{z}, s_+), m_+(\tilde{z}, s_+)]$, that is continuous in each coordinate in s_+. Moreover, taking into account the incomplete markets nature of the model, $\theta_+(\tilde{z}, s_+)$ will be assumed to be constant. That is, $\theta_+(\tilde{z}, s_+) = \theta_+(\tilde{z})$ for each $s_+ \in [LB(s), UB(s)]$. Once the continuity of $q_+(\tilde{z}, s_+)$ has been shown below, the continuity of $m_+(\tilde{z},.)$ follows from definition.

Proof

Assume that $\theta_+(\tilde{z}, s_+)$ is constant in in s_+ for any given $\tilde{z} \in \tilde{J}$. In order to complete the proof it suffice to show that $q(\tilde{z}, s_+)$ is continuous in s_+ for any given $\tilde{z} \in \tilde{J}$.

Under assumptions 6.2-i) to 6.2-iii) any equilibria in this economy exists satisfies equation A.12 together with

A.14) $q_j u_s^i(e^i(s) + \theta^i d(s) - \theta_+^i q)' - \beta K(s) \int d_j(s_+) u_s^i(e^i(s_+) + \theta_+^i d(s_+) - \theta_{++}^i q)' ds_+ = 0, j \in J, i \in I$

Where $K(s)$ is the constant associated with the uniform distribution in assumption 3-iv').

Now suppose that assumption A.1 holds. Then, as mentioned in the preliminary remark, $p(s,.)$ has the Feller property. Then:

A.15) $\lim_{s^n \to s^1} \beta K(s^n) \int m_{++}^{i,j}(x) dx = \beta K(s^1) \int m_{++}^{i,j}(x) dx = q_+^j(s^1) u(e^i(s^1) + \theta_+^i d(s^1) - \theta_{++}^i q_+(s^1))'$

The last equality in A.15) follows because, under assumption 6.2-i) to 6.2-iii), there is a sequential competitive equilibrium for each s^1 which satisfies equation A.14).

After the discussion in section 5.2 above, the last equality follows from theorem 4.1, 4.2 and section 5 in Mas-Colell and Zame (1996). Further, under the special form $u_s^i = u$ in assumption 6.2-i), equation A.14 and A.15 implies:

$$\text{A.16)}\ \lim_{s^n \to s^1} \frac{\beta K(s^n) \int m_{++}^{i,j}(x)dx}{u\left(e^i(s^n)+\theta^i d(s)\right)'} = \lim_{s^n \to s^1} q_+^j(s^n) u(-\theta_{++}^i q_+(s^n))' = q_+^j(s^1) u(-\theta_{++}^i q_+(s^1))'$$

Note that equation A.14 implies the first equality in A16) under u in assumption 6.2-i). Then, as $u\left(e^i(s^n) + \theta^i d(s^n)\right)'$ is bounded above and bounded away from zero for any admissible value of $e^i(s^n) + \theta^i d(s^n)$ under assumptions 6.2-i), equation A.15 implies the last equality.

Now, setting $\lambda = 1$ in u w.l.o.g., the continuity of ln implies

$$\text{A.17)}\ \underbrace{lim_{s^n \to s^1}[-\theta_{++}^i q_+(s^n)] + \theta_{++}^i q_+(s^1)}_{A} + \underbrace{ln\,[\lim_{s^n \to s^1} q_+^j(s^n)] - ln\,[q_+^j(s^1)]}_{B} = 0$$

Suppose that $B = 0$, then as $\theta_{++}^i \neq 0$ w.l.o.g., A implies $lim q_+(s^n) = q_+(s^1)$ as desired.

Suppose that $B \neq 0$. The compactness of the equilibrium set implied by theorem 4.2 in Mas-Colell and Zame (1996) under assumptions 6.2-i) to 6.2-iii) implies that $B \in \mathbb{R}$. Then A.17 under $J = 1$ (i.e. there is only 1 asset) implies:

$$q_+^j(s^1) = \frac{B}{\theta_{++}^i(1 - exp(B))}$$

As B depends on s^1 for each $s^1 \in [LB(s), UB(s)]$, the equation above implies that $q_+^j(.)$ is continuous in s^1; implying a contradiction with $B \neq 0$ as $\theta_{++}^i \neq 0$ w.l.o.g.

■