Overborrowing and the Slow Moving Debt Crisis*

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Abstract

Governments can force a widespread default on private external debt. We show that policy-induced default may destabilize the economy when atomistic borrowers fail to internalize their contribution to the economy-wide risk. This pecuniary externality produces overborrowing and an explosive self-reinforcing loop of higher debt, higher interest payments, and further borrowing: a slow-moving debt crisis. We establish the mechanism in a two-period model with closed-form solutions and extend it to an infinite horizon. Calibrated to Argentina (1982–2016), the model generates a slow-moving debt crises with one-period debt and without shifts in expectations. A centralized benchmark instead borrows little, pays higher spreads in normal times, and defaults mainly after adverse income shocks. Cheaper private credit in tranquil times thus coexists with greater fragility.

Keywords: Default; Private external debt; Overborrowing; Slow-Moving Debt Crisis.

JEL Codes: F34, F32, C62.

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1 Introduction

A landmark regulatory intervention in Argentina's financial landscape came in December 2020, when the Central Bank issued Communication A7106. The measure placed binding constraints on firms carrying external debt obligations above USD 1 million, limiting their access to official foreign exchange for principal repayments to 40 percent of the amount due, and forcing them to refinance the outstanding 60 percent under terms stipulating a minimum maturity of two years. This provision, in practice, constituted a centralized and compulsory restructuring of private external liabilities. Such government influence over private external debt is not unique to Argentina — it varies systematically across countries and is explicitly priced in by rating agencies such as Moody's and Fitch (See Arellano et al. (2016)). [1] Centralized default of private external debt is, as such, a source of macroeconomic instability in its own right.

In this paper, we investigate the destabilizing effects of private debt with incomplete markets and centralized default. Because default is coordinated by the government, private agents fail to internalize that their borrowing affects the equilibrium interest rate, resulting in overborrowing relative to the debt levels a constrained social planner would choose. Ex ante, the possibility of government intervention destabilizes the economy, generating a slow-moving debt crisis in which leverage accumulates gradually in a low-interest-rate environment along an unsustainable trajectory; ex post, authorities are forced into drastic intervention to reset the debt cycle. Building on Zame (1993), where default acts as a market-completing mechanism, we demonstrate that it can also function as a macroeconomic stabilizer when private external borrowing would otherwise lead to unsustainable debt trajectories. A central implication is that low borrowing costs need not signal financial stability. When borrowing is decentralized but default is centralized, low interest rates can sustain long episodes of debt accumulation and consumption smoothing, while default provides the nonlinear adjustment that ultimately restores global stability.

We illustrate the core contribution of the paper through a stylized, two-period general equilibrium model with a closed-form solution, featuring private borrowing and centralized default. This

[1] Additionally, policymakers can indirectly influence the private sector's ability to repay its debt through market mechanisms. There are several reasons that account for their significance for the aggregate behavior of the economy. For instance, if the private sector issues debt in foreign currency, a nominal depreciation may imply a de facto massive private default; a case along these lines happened for Argentina in 2001 with the abandonment of the currency peg. These type of episodes represent an event where the government induces a default of privately issued external debt.

framework allows us to highlight three key mechanisms: (i) the overborrowing that arises when private agents fail to internalize the impact of debt issuance on interest rates, (ii) the resulting destabilizing effects of private debt accumulation under policy-induced default risk, and (iii) the stabilizing role of sovereign default as a corrective device.

Building on the intuitions provided by the two-periods model, we develop a quantitative infinite-horizon model of a small open economy populated by atomistic households, a government, and competitive, risk-neutral international investors. In our setting, households issue foreign debt, while the government retains the ability to coordinate default decisions. Thus, borrowing is decentralized—similar to the framework in Bianchi (2011)—but instead of relying on a collateral constraint to introduce market frictions, we assume that private external debt is defaultable and that default is centralized, in line with Kim and Zhang (2012). The infinite-horizon extension allows us to characterize the global dynamics of the economy and to conduct a robust quantitative exercise aimed at providing both a qualitative and quantitative description of the type of crisis induced by the interaction between centralized default and decentralized borrowing. The resulting instability is therefore not a rare transition triggered by an exceptional shock, but an endogenous feature of equilibrium debt dynamics: normal times are characterized by recurrent, slow-moving debt buildups, whereas default is the relatively infrequent event that terminates them. This characterization is underpinned by a theorem establishing the ergodicity of the recursive equilibrium, which guarantees that the long-run dynamics and the associated quantitative moments are well defined. In particular, the framework allows us to characterize in detail the emergence, persistence, and resolution of slow-moving debt crises.

In indebted countries, we demonstrate that pecuniary externalities generated by default risk induce an overborrowing loop that destabilizes the economy in the absence of default. We illustrate this result using a stochastic phase diagram, a novel graphical device which extends the results in Azariadis and Lambertini (2003). The critical difference between this paper and ours is the steady state. As we work in a stochastic environment, the steady state is different with respect to Azariadis and Lambertini (2003), and thus the notion of required stability is always global. Moreover, we formally confirm the intuitions behind the two-period model and the phase diagram by showing the ergodicity of the stochastic steady state, a property that induces a suitable form of global stability for this class of models. Ergodicity also extends the quantitative analysis in a particularly useful

direction for characterizing slow-moving debt crises: it allows us to study the distribution of crisis arrival times, or equivalently the duration of normal times before the economy reaches a crisis state. In particular, we can show that this duration is finite almost surely, so that the timing of crisis onset can be characterized as an endogenous random variable with a well-defined distribution.

We organize our main results around three contributions:

Overborrowing and Destabilization.—Our first contribution is to characterize the conditions under which decentralized private borrowing leads to socially excessive debt accumulation when default is centralized. When atomistic households fail to internalize the impact of their individual borrowing decisions on the equilibrium interest rate, the resulting pecuniary externality generates an overborrowing loop, which arises because individual asset holdings are globally increasing functions of aggregate wealth. This property is generated by default risk in a recursive stationary equilibrium under rational expectations: when decentralized decisions result in higher indebtedness, a symmetric rational expectations equilibrium with default risk entails an increase in the market interest rate, which generates negative income effects for indebted countries and induces further borrowing, creating the mentioned loop. Unlike models in which the relevant transmission channel operates only when a collateral constraint binds, as in Bianchi (2011), the debt instability in our framework is conditioned by sovereign default incentives and is operative at all debt levels. We show that this mechanism is sufficient to destabilize the economy, especially in recessions. As the economy approaches the default region, however, the increase in the equilibrium interest rate also generates an intertemporal substitution effect that works in the opposite direction: higher borrowing costs induce households to reduce current borrowing, partially offsetting the negative income effect underlying the pecuniary externality. Our numerical phase diagram shows that this mechanism can locally slow down debt accumulation near the default threshold, providing a degree of endogenous stabilization before default occurs. Yet this force is not sufficient to stabilize the economy globally. For some realizations of the income shock, equilibrium debt dynamics still cross the endogenous default threshold, so that the government optimally chooses to default despite the increase in borrowing costs. Thus, the model features a tension between a locally stabilizing substitution effect and a globally destabilizing income effect, with centralized default ultimately providing the reset mechanism required to prevent unbounded debt dynamics. We illustrate the instability generated

by the pecuniary externality in the two-periods model and further characterize it through the phase diagram analysis of the infinite-horizon economy.

Default as a Stabilizer.—Our second contribution is to show that centralized default can function as a macroeconomic stabilizer in this environment. When the overborrowing loop would otherwise push the economy along an explosive debt trajectory, the government's ability to coordinate a default resets the debt cycle. Building on Zame (1993), where default completes markets, we show that in our setting the default corrects the inefficiency created by the overborrowing externality. Importantly, the default decision is also normative in our framework: the government compares the social value of continuing repayment with the value of default, and chooses to suspend repayment only when the latter yields higher social welfare. This welfare criterion generates a distinctive form of intervention. Rather than continuously discouraging borrowing through higher financing costs, the government allows households to retain access to relatively cheap credit during normal times. Consistent with our quantitative comparison with the centralized economy of Arellano (2008), interest rates are substantially lower away from default in the decentralized economy, allowing households to sustain longer borrowing cycles and to smooth consumption over time. The counterpart of this policy is a much deeper accumulation of private debt. Instead of preventing this buildup ex ante by making borrowing persistently more expensive, the government tolerates low borrowing costs and intervenes only when the economy reaches sufficiently adverse states. At that point, centralized default temporarily suspends access to international capital markets and resets the stock of private external debt. The resulting equilibrium therefore combines long debt cycles and relatively high private consumption during normal times with an infrequent but drastic welfare-improving intervention that restores global stability. We provide a formal characterization of the ergodic paths along which default occurs, establishing a suitable notion of global stability for this class of models.

Slow-Moving Debt Crisis.—Our third contribution is quantitative. We calibrate the infinite-horizon model to Argentina over 1982–2016, a sample with a single private default episode: the 2001 collapse. In this calibration, the pecuniary externality generates overborrowing and a slow-moving debt crises in the sense of Lorenzoni and Werning (2019). Debt accumulates gradually, rising interest payments, inducing further borrowing until the economy crosses the default threshold. We formally show that, in the recursive equilibrium, future debt is increasing in the economy's existing

indebtedness. This monotonicity is a central property of the equilibrium for two reasons. First, it reflects the self-enforcing nature of rational expectations: when agents expect higher aggregate indebtedness, their optimal borrowing decisions validate those expectations, reinforcing the overborrowing loop and generating a stable feedback mechanism in which high debt leads endogenously to further debt accumulation. Second, monotonicity is crucial for the global stability of the recursive equilibrium in the presence of default, because it generates recurrent debt-accumulation cycles between successive default episodes. Slow-moving debt crises are therefore an inherent feature of the equilibrium dynamics rather than isolated low-probability events. What remains rare is the default event itself, which constitutes the endogenous termination of some of these recurrent debt buildups. Three findings characterize these crises. First, the buildup is systematic: across simulated episodes, the distribution of debt narrows toward a common threshold in the quarters before default, while the median interest rate rises only gradually. Second, default is slow even from high debt: less than 20% of the paths starting from the most indebted states default within ten quarters, against two-thirds in a centralized benchmark. Third, private borrowers pay lower spreads than the centralized benchmark in normal times while the gap closes only as default approaches. Lower borrowing costs in tranquil times therefore do not indicate greater stability.

The comparison with the centralized economy of Arellano (2008) clarifies the mechanism. A government that internalizes the effect of borrowing on bond prices has a low level of debt and defaults mainly after adverse income realizations. In our economy, by contrast, default is the end point of a debt-driven buildup. Unlike in Lorenzoni and Werning (2019), these dynamics arise with one-period debt and within a stationary recursive equilibrium, without long-term debt or shifts in investors' expectations.

Relation to the literature

Debt instruments play a central role in macroeconomic theory and policy. Traditionally, both private and sovereign debt have been viewed as mechanisms for smoothing consumption and taxation. Foundational contributions such as the Permanent Income Hypothesis underscore the role of private debt in intertemporal consumption smoothing, while Barro (1979) examines sovereign debt as a tool for optimal tax smoothing. In settings with incomplete markets, the international macroeconomics literature has emphasized the destabilizing potential of sovereign debt accumula-

tion. Seminal works such as Eaton and Gersovitz (1981) and Cole and Kehoe (2000) demonstrate how sovereign borrowing can lead to explosive implicit returns and self-fulfilling crises. Nevertheless, despite the empirical relevance of these facts and the importance of private debt, the literature has not studied its destabilizing effects or the role of policy-induced default in depth so far.

Our paper connects the literature on sovereign defaults, such as Eaton and Gersovitz (1981), with the results on overborrowing by Bianchi (2011). The papers closest to ours are Jeske (2006) and Kim and Zhang (2012). The former one studies international private borrowing with repudiation risk, but in contrast to us, the author considers a complete markets environment. In this context, capital controls are analogous to the case of centralized borrowing and default. Instead, the latter develops a quantitative model with private debt and sovereign defaults with incomplete markets. Here, the authors explore the pecuniary externality and show that decentralized borrowing generates higher spreads than centralized borrowing, i.e. Arellano (2008). Our paper, even though to some extent complementary to theirs, differs in many aspects regarding the modeling assumption, the objective and scope. In fact, our contribution is that we move one step forward. First, we develop a theoretical 2-period model with closed form solution that allows us to isolate the channels behind the debt instability and the role of centralized default. This global instability analysis is not evident from the Euler equation analysis in Kim and Zhang (2012), but is clear both in the two period example and in the phase diagram analysis of the infinite horizon version of the economy that we develop later. Second, we show that, unlike Bianchi (2011), the debt instability is conditioned by sovereign default incentives, while in the case of a collateral constraint, the relevant transmission channels operate only when the constraint binds. Finally, we show how this environment that combines private unstable debt dynamics due to centralized default risk amplifies the response to different international conditions, indicating the potential destabilizing role of the combination of default incentives and overborrowing. On a more technical aspect, we also differ from Kim and Zhang (2012) in our assumption about the debt markets. In our case, debt is not sold at a discount price, in a sense that our debt markets are closer to those in Ayres et al. (2018). We avoid the problems associated with the well-known multiplicity of equilibria highlighted in that paper by taking a stationary selection. Since our stability results are robust to discontinuities, it suffices to take a measurable selection that is self-sustaining under rational expectations. This allows us to derive theoretical results for the two period and infinite horizon model and provide a

theoretical analysis in which we formally demonstrate the existence of ergodic paths with default. Our environment also provides a microfounded theory of the Debt Elastic Interest Rate device in Schmitt-Grohé and Uribe (2003), Aguiar and Gopinath (2007), and Garcia-Cicco et al. (2010), that implies a positive stationary relationship between private external debt and interest rates.

Our paper is also related to Lorenzoni and Werning (2019). The authors study slow moving debt crises in which self-fulfilling expectations of future default generate, through the feedback between high interest rates and debt accumulation, a gradual increase in public debt. A key assumption is that the borrowing government cannot commit to a fixed issuance quantity, so that debt dynamics are driven by the shape of the debt Laffer curve and, in the case of long-term bonds, by the intertemporal dimension of the coordination problem. Our paper shares with theirs the focus on slow moving debt crises—in the sense that debt accumulates gradually before a crisis materializes—and on the role of the interest rate as a destabilizing feedback mechanism. However, our framework differs in three fundamental respects. First, whereas Lorenzoni and Werning (2019) study *sovereign* debt crises arising from investor coordination failures, we study *private* external debt crises arising from a pecuniary externality among decentralized optimizing borrowers. Second, the mechanism driving debt instability in Lorenzoni and Werning (2019) requires the presence of long-term debt and the associated dilution channel, and operates through the shape of the debt Laffer curve; in our model, instability arises even with short-term debt and is driven instead by the failure of atomistic households to internalize the effect of their borrowing on the equilibrium spread. Third, whereas in Lorenzoni and Werning (2019) the government defaults when cannot raise enough revenue; we show that centralized default itself plays a stabilizing role, resetting the debt cycle when private accumulation becomes unsustainable or unstable. In this sense, the optimal nature of centralized default is essential because it reveals a policy trade-off between consumption smoothing in incomplete markets and the global stability of the current account, with implications for the stability of the overall economy. Together, these differences imply that our paper provides a complementary and distinct account of slow moving debt crises, one grounded in the interaction of decentralized private borrowing, pecuniary externalities, and policy-induced default in small open economies with incomplete markets.

The remainder of this paper goes as follows. In section 2, we briefly present evidence on the relevance of private defaultable debt and policy interventions. In Section 3, we present the model.

Section 4 describes the infinite horizon model. Section 5 presents the main theoretical results. Section 6 contains the quantitative implementation and the main numerical results. Section 7 concludes. All proofs are available in the appendix.

2 Motivating evidence: private debt and policy interventions

Empirically, private debt constitutes a significant share of defaultable external debt in most emerging economies. Table 1 presents evidence along this dimension for Argentina and Chile. The table shows that total public debt with private lenders is 87 and 46 billion U$S while private debt is 73 and 192 billion U$S, both respectively. The sum of these items accounts for the defaultable debt (excluding debt with senior lenders, which is virtually non-defaultable). It turns out that 46% for Argentina and 81% for Chile of the total defaultable debt is private. For this reason, we focus on private debt, and we abstract from sovereign debt, which in turn allows us to derive theoretical results that the literature could not address in other frameworks.

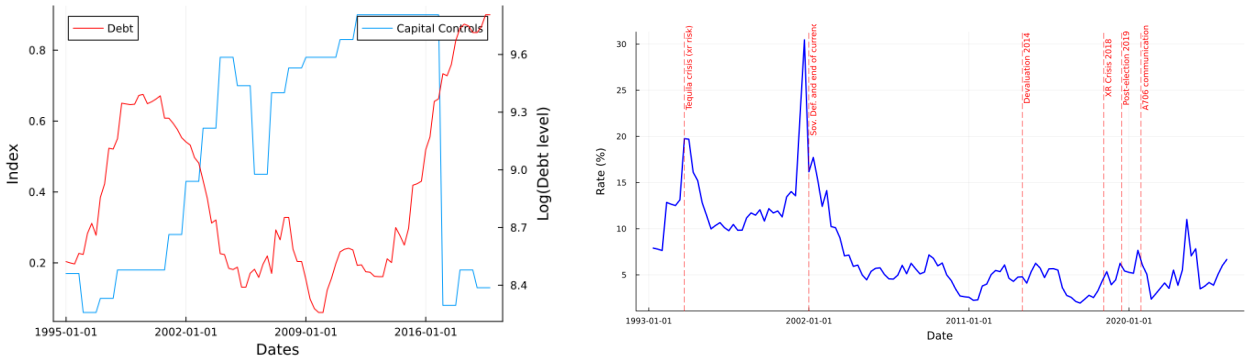
Table 1: External debt in Argentina and Chile (2021)

bln of Dollars	$Public_{PV}$ (I)	$Public_{IO}$ (II)	I+II	*Private* (III)	I+III	III/(I+III)
Argentina	87	74	161	73	160	46%
Chile	46	0	46	192	238	81%

Note: "$Public_{PV}$" refers to public external debt with private lenders, "*IO*" stands for international organisations, "I+II" is total public debt. *Private* is private external debt. "I+III" is the total debt subject to default, which excludes debt with IO. The last column computes the share of private debt to total defaultable debt. Source: Central Bank of Chile and National Institute of Public Statistics for Argentina.

The dynamics of private debt accumulation have received comparatively less attention than the dynamics of sovereign debt, partly due to the influence of the Washington Consensus, which posited that rational agents operating in frictionless capital markets would make optimal borrowing decisions. However, empirical evidence suggests that episodes of capital account liberalization are often accompanied by surges in private sector debt. This pattern is suggestive in the Argentine experience since the early 1990s, where periods of liberalization (before 2002 and the period 2015–2019) coincide with an increase in private debt, while the period 2003–2015 is characterized by

the reintroduction of capital controls, exchange rate management, and exclusion from international financial markets. The intervention of the Argentinean Central Bank through Communication A7106 confirms that the centralized default of private external debt is empirically relevant and plausible. Also, Arellano et al. (2016) show that it does not only affect Argentina, it is a common fact also in advanced economies. If the private sector issues debt in foreign currency, a nominal depreciation may imply a de facto massive private default—as happened in Argentina in 2001 with the abandonment of the currency peg.



(a) Private sector debt and capital controls

(b) Private sector lending rates

Figure 1: Empirical motivation

Panel (a): Private borrowing is the stock of debt of non-financial corporations in USD from the BIS (IDS: Debt securities / International debt securities (BIS-compiled)) and capital control index from Fernández et al. (2016). *Panel (b):* Private lending rates in USD and large devaluations; lending rates are USD rates for private lending from the IMF.

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Figure 1 documents both patterns. Panel (a) plots private sector debt alongside the capital control index, illustrating the tight co-movement between liberalization episodes and debt surges. Panel (b) shows the evolution of corporate lending rates in USD together with episodes of large devaluations or currency tensions; the spikes do not reflect UIP dynamics but rather the impact of currency market tensions on private sector repayment capacity.

3 The two-period model

This section develops the main ideas of the paper using a two-period model for a small open economy. We present two versions of the model, with and without internalization of the effect that borrowing has on the default choice and the interest rate.

3.1 The case without internalization

Consider a 2-period economy with decentralized private debt and centralized public default. That is, in the presence of private debt, the Government may force the private sector to default. Assume that the private sector's objective function is

$$u(c_1) + \beta \mathbb{E}_0 u(c_2),$$

where u is a standard, strictly increasing, and concave utility function, over consumption of a unique good at each point in time with discount factor β. The constraints the agent faces are,

$$c_1 = \omega + b_1,$$

and

$$c_2 + R(B_1)b_1 = y,$$

where $b_1 > 0$ denotes the individual debt issued in period 1, that matures in period 2, and B_1 denotes the aggregate debt. ω denotes the first period initial wealth, and y is the endowment in period 2, which is a random variable with a process to be defined later.

The government may choose to default in period 2. This choice is not internalized by the household, who is atomistic. That is, aggregate borrowing B_1, and not private borrowing b_1, determines default risk $R - R^*$, where R^* is the gross risk-free rate. In equilibrium we have that $b_1 = B_1$.

Default happens when the utility in autarchy $u(y^d)$, where y^d is the endowment after default, is greater than or equal to the utility when the government chooses to repay $u(y - b_1 R(B_1))$. As u is strictly increasing, this is equivalent to default when output in period 2 is low enough, that is, lower

than the autarky level y^d plus debt services $b_1 R(B)$. As y is a random variable with a c.d.f. given by F, these facts imply in turn that default probability is the probability that the output level is lower than the outside option given by $y^d + b_1 R(B_1)$:

$$P(D) = F(y < y^d + b_1 R(B_1)).$$

The objective function for the household becomes:

$$V^c(\omega) = u(\omega + b_1) + \beta \left[F(y^d + b_1 R(B_1)) u(y^d) + \int_y u(y - b_1 R(B_1)) f(y) dy \right],$$

where the integral is taken over the repayment set, that is, for $y > y^d + b_1 R(B_1)$. The household determines the optimal borrowing taking into account marginal costs and returns, which follows from standard F.O.Cs. with respect to b_1 using the objective function described above.

Lenders are risk-neutral and deep-pocketed investors characterized by a standard profit maximization condition. Then, the equilibrium in this economy is determined by the following three equations

$$u'(\omega + b_1) = \beta \, R(B_1) \int_y u'\big(y - b_1 R(B_1)\big) \, f(y) \, dy \tag{1}$$

$$R^* = R(B_1) \left[1 - F\big(y^d + b_1 R(B_1)\big) \right] \tag{2}$$

$$b_1 = B_1 \tag{3}$$

Functional forms

Assume that the utility follows an exponential function given by $u(c) = -e^{(-\alpha c)}$, where $\alpha > 0$, and assume that income follows an exponential distribution $f(y) = \lambda e^{(-\lambda y)}$, with $\lambda > 0$.

The equilibrium conditions boil down to:

$$r(B_1) \approx r^s = \frac{r^* + \lambda y^d + \lambda B_1}{1 - \lambda B_1}, \tag{4}$$

$$r(B_1) \approx r^d = \frac{(\lambda+\alpha)y^d - \omega\alpha - \ln\left(\frac{\lambda}{\alpha+\lambda}\right) + (\lambda-\alpha)B_1}{1-\lambda B_1}. \tag{5}$$

In Equation (5) we set $ln(\beta) = 0$ as β is approximately 1. Following Ayres et al. (2018), we call Equation (4) the supply curve, denoted r^s, and Equation (5) the demand curve, labeled r^d. As $log(\beta)$ is not exactly equal to zero, we say that these expressions approximate ($\approx$) the actual values for the interest rate consistent with the investor's and households' problem.[2] The figure below illustrates the equilibrium at point A.

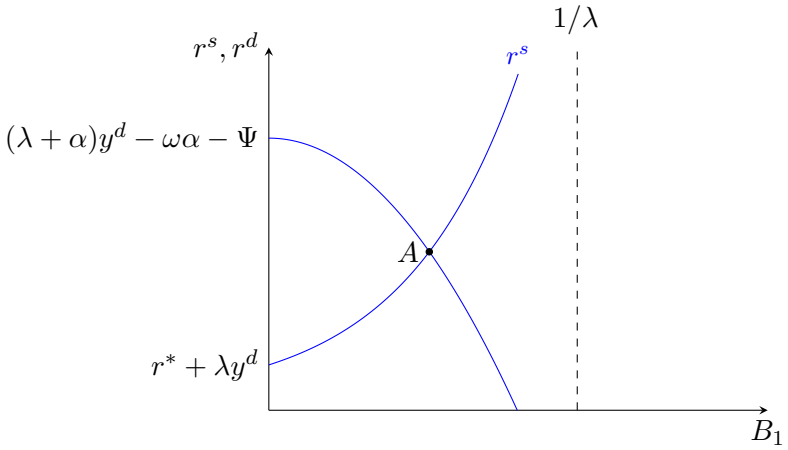


Figure 2: Equilibrium in the two-period model

Note: In the vertical axis, we depict r^s and r^d, the interest rate for the investor and borrower respectively. In the horizontal axis, we depict the level of equilibrium debt $b_1 = B_1$. The vertical dotted line depicts the asymptotic limit of both curves. We define $\Psi \equiv ln(\lambda/(\alpha+\lambda))$ for simplicity.

Figure 2 depicts Equations (4) and (5) as a function of equilibrium debt $B_1 = b_1$. There are several things to note in Figure 2. First, the intersections of r^s and r^d with the horizontal axis both depend on the default punishment y^d. Second, the borrower's interest rate tends to infinity for some boundary value of debt, $B_1 = 1$. Because of this last fact, after a negative wealth shock (i.e., a drop in initial wealth ω), equilibrium borrowing increases, pushing the equilibrium interest rate close to infinity. Moreover, the first fact implies that it is possible for some values of y^d that the curves may

[2]We derived all the expressions in the appendix. On top of the mentioned restriction on $log(\beta)$, the closed forms for r^s and r^d are approximated only because we assume $ln(R) = r$. That is, the *log* of the gross rate equals the net rate. The closed-form expressions for the equilibrium interest rate and debt are available upon request.

not intersect. In an infinite-dimensional setting, these facts are relevant as we must restrict the values of the default punishment to guarantee the existence of an equilibrium. Additionally, from a welfare perspective, the Government has incentives to rule out the explosive interest rate paths that follow a negative wealth shock. This implies an additional policy instrument, which we refer to as *early default*, and we define as follows.

Early default

Consider the following slight extension of the model. Assume that initial wealth ω, is given by:

$$\omega = y_0 - R(B_0)b_0.$$

The initial wealth is the current income realization net of an initial debt repayment. By *early default*, we refer to the possibility of defaulting on the initial debt $b_0 = B_0$ in period 1. We assume that the government is able to generate a haircut of 100% on B_0. Early default is, consequently, a new instrument for the government that can prevent the private sector from issuing debt in period 1 as, even with default in period two, Figure 2 shows that the decentralized economy could generate an interest rate $R(B_1)$ arbitrarily close to infinity, affecting consumption and thus overall welfare.

As the government is benevolent, the default decision in period 1 is based on a comparison between the value of moving to autarky in period 1, which we denote as V^d and define below, and the continuation value, previously introduced as $V^c(\omega)$.

$$V(\omega) = \max\left\{V^d(y^d), V^c(\omega)\right\}$$

with

$$V^d(y^d) = u(y^d) + \beta u(y^d),$$

where *early default* implies defaulting on b_0 and moving to autarky, $b_1 = 0$, as we are not allowing for reentry.

There is a value for $b_0 = B_0$ such that early default is preferred. We call this value $\bar{B}$. Intuitively, suppose that b_0 is such that the economy starts with a low ω. As figure 2 indicates, the lower the

ω, the higher the $r(B_1)$ leading to an "explosive" interest rate (debt) path. Hence, there is a b_0 large enough such that the government may prefer autarky from $t = 1$. In this case, the government chooses to default early. Formally, the continuation value in V^c decreases as higher debt implies higher debt services $B_1 R(B_1)$ because, as shown below, R is increasing in B.[3] In the infinite horizon version of the model, we formally prove this result.

3.2 The case with internalization

The equilibrium under internalization is different. The Euler equation in this case is

$$u'(\omega + B_1) = \beta \underbrace{[R(B_1) + B_1 R'(B_1)]}_{\text{MCC}} \underbrace{\left[\int_R u'(y - B_1 R(B_1)) f(y) dy\right]}_{\text{MCD}} \tag{6}$$

and

$$R^* = R(B_1)\left[1 - F(y^d + B_1 R(B_1))\right] \tag{7}$$

Equation 6 takes as given the default set (as in Kim and Zhang (2012)). The term MCC comes from the internalization of borrowing decisions. The government recognized that borrowing affects the market rate. The relationship between internalization and debt issuance is linked to the overborrowing literature, which has been widely studied in international macroeconomics to provide a rationale for macro-prudential policies. In our context, with default, the key for an overborrowing result is the value of $B_1 R'(B_1) > 0$. In our case,

$$r(B_1) \approx \frac{r^* + \lambda y^d + \lambda B_1}{1 - \lambda B_1}$$

the derivative with respect to B_1 is

$$\frac{dr(B_1)}{dB_1} = \frac{\lambda(1 - B_1) + \lambda(r^* + y^d + B_1)}{(1 - \lambda B_1)^2} = \lambda \frac{1 + r^* + y^d}{(1 - \lambda B_1)^2} > 0$$

[3] As R is increasing in the two-period model, a sufficient condition for this is to assume that the utility function u is unbounded below and consumption is bounded below by zero. Examples of this type of preferences are: $u(c) = ln(c)$ and $u(c) = c^{(1-\sigma)}/(1 - \sigma)$ with $\sigma > 1$.

As $r(B_1)$ is an affine transformation of $R(B_1)$, the properties of the derivative are common. Hence, in this case, we always have overborrowing, as $B_1 R'(B_1) > 0$, which implies a higher cost of borrowing with respect to the solution without internalization. In the model with infinite horizons, we prove this property using a two-step operator robust to the presence of discontinuous equilibria.

We present this information in Figure 3, where we plot the relationship between the decentralized and centralized equilibria, which arise from the right-hand side of the Euler equation, Equation (6), which in turn determines the demand for debt. In the centralized equilibrium, the effects of indebtedness on the risk premium are internalized by the Government, leading authorities to issue less debt when compared with the decentralized equilibrium.

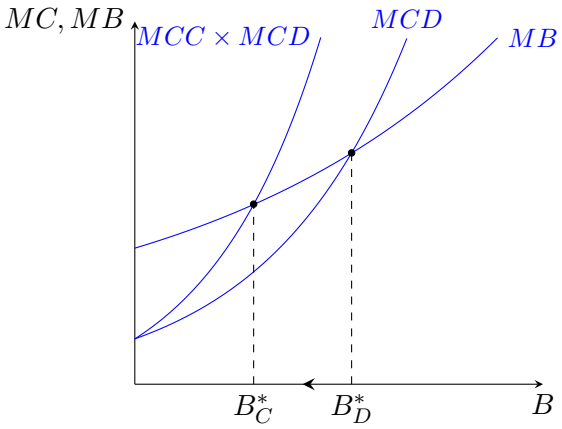


Figure 3: Indebtedness in the centralized and decentralized equilibrium

Note: In the vertical axis, we depict the marginal benefit (MB) - the left-hand side of equation (6)- and the marginal cost (MC) - the right-hand side of the same equation. The MC has two components, the centralized (MCC) and the decentralized (MCD). By internalizing the effects of indebtedness on the risk premium, the Government chooses a smaller level of debt with respect to the decentralized equilibrium, moving the equilibrium from B_D^* to B_C^*.

3.3 The centralized default

How does early centralized default operate in these economies? This section argues that centralized default stabilizes unstable debt dynamics.

We can study the stabilizing role of centralized default in the two-period model with early default as follows. The next figure shows a supply schedule r^s, and a collection of demand functions of debt,

which depend on the wealth level ω. The lower ω, as seen in the previous equations, the larger the equilibrium debt levels b. Suppose that at point A the wealth level is ω_A. Then at point B wealth is lower, $\omega_B < \omega_A$, and debt is higher. From figure 3, at point B_1, debt is lower than at point A for the same level of wealth due to pecuniary externalities.

In a dynamic setting, ω would be the initial income net of maturing debt. Hence, the higher the debt that matures today, the lower ω that would imply a higher debt issuance today.

As seen in Figure 2, for large levels of debt, approaching 1, the interest rate explodes, and at some point, the government finds it optimal to default. In Figure 4, without changing the timing of the game between the borrower and the lender, we allow for default *also* in period one. Previously, we referred to this policy as *early default*. Note that in Figure 2, the default was only possible in period two. The default threshold is shown in the red vertical line, which implies a full haircut. Thus, the economy with decentralized debt jumps from either points A or B to B_2, suffering a complete deleverage. Thus, after a negative wealth shock, the indebtedness cycle starts from zero, and Government policies create recurrent debt cycles, stabilizing the debt dynamics. Note, remarkably, that in an infinite-horizon economy without an early default, a sufficiently large streak of bad shocks would drive the debt to infinity. Thus, points A, B, and even B_1 are not necessarily stable in the absence of early default.

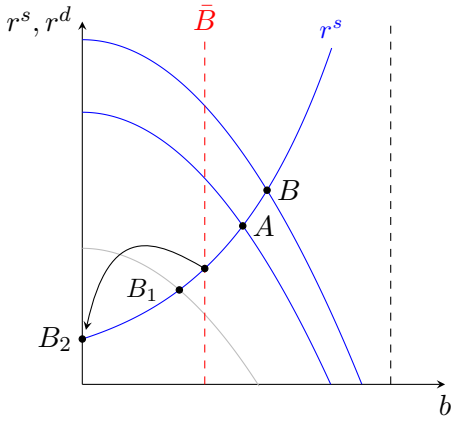


Figure 4: The stabilizing role of early default

Note: we depict equations (4) and (5) (for two different values of ω, with $\omega_A > \omega_B$), generating points A and B, both with overborrowing as R is increasing in B. Point B_1 is the equilibrium without overborrowing, with public (centralized) debt. Allowing for early default generates point B_2 as the government enforces a full haircut when private indebtedness is above $\bar{B}$.

3.4 Pecuniary externality and instability

The previous discussion suggests a connection between overborrowing and instability. Different initial conditions $\omega_A = y_0 - R(b_0^A)b_0^A > \omega_B = y_0 - R(b_0^B)b_0^B$, imply different future debt services $R(b_1^A)b_1^A < R(b_1^B)b_0^B$. Thus, depending on the initial net wealth of the private sector, decentralized decisions will drive the economy along an explosive path, approaching the asymptotic boundary in Figure 2. If we can imagine that points A and B do not characterize two different economies, but the same economy in two different periods, the notion of instability arises immediately. Moreover, in this case, the crisis has the properties of a slow moving debt crisis where the economy's debt increases over time inducing sequentially larger interest rates that eventually exploit.

In the arguments above, overborrowing acts to accelerate the transition to instability. As R is increasing, the private economy loops faster to the edge. The economy with public debt partially internalizes this fact, as it slows down the transition to an explosive path. Typically, this solution is implemented through contingent taxes on the private economy, even with capital controls. However, for a sufficiently large sequence of bath shocks, the private economy with taxes or capital controls will ultimately reach the explosive path.

Early default, as it induces a full haircut, is more effective in controlling instability as it regenerates the indebtedness cycle entirely. In the infinite-horizon economy, we formally prove that this policy instrument is sufficient to globally stabilize the economy, as it regenerates the dynamical system by forcing the private sector to undergo a sudden and complete deleveraging. This fact contrasts with the deleveraging induced by price-dependent collateral constraints, which the Balance of Payments crises literature refers to as Fisherian Deflation. We claim that in the presence of default risk, overborrowing could generate an explosive accumulation of debt services in the future and the government, by foreseeing the welfare effects of this fact, prefers to act preventively, forcing an instantaneous and full deleverage that creates a recurrent and thus stable behavior of private debt in the presence of pecuniary externalities, which are natural in this environment due to the cumulative interaction between default risk and private debt.

3.5 Summary

In summary, the two-period economy in this section is in line with Ayres et al. (2018), but depicts a unique equilibrium. This model of centralized default and decentralized borrowing generates overborrowing, which differs from Arellano (2008), but is in line with Bianchi (2011). Extending the model to allow for early default enables us to capture the existence of explosive debt dynamics that induce the government to prevent borrowing from period 1 onward. Early default is a macroprudential tool that stabilizes debt levels and is welfare-improving.

Based on this two-period model, we next consider the extension to an infinite-horizon economy. The findings of the two-period economy can be extrapolated to the infinite horizon. Although it is not possible to derive closed-form solutions, and although we can provide a sharp characterization of endogenous variables, we will need to rely on computational methods to obtain accurate results.

4 The infinite horizon economy

In this section, we generalize the previous two-period model to an infinite horizon economy populated by households, the government, and international investors. Households are atomistic and risk-averse agents that issue non-contingent foreign debt in real terms, consume, and receive an exogenous endowment. The benevolent government decides every period whether the private sector repays or

defaults their debt. The international investors are deep-pocketed, risk-neutral agents that purchase external debt and understand the default risk. The remainder of this section presents the details of each agent.

4.1 The households

The domestic economy is populated by a large number of identical households that can borrow or save using a one-period real net asset, $b < 0$ denotes debt holdings, with a non-contingent gross interest rate $R(B)$ subject to default risk. Each of these households receives a positive stochastic endowment y that follows an identically distributed (i.i.d.) process.[4] Preferences are standard and represented by a strictly increasing, strictly concave, and differentiable instantaneous return function u. In line with the two-period model, the recursive problem of the agent is:[5]

$$V^c(b, B, y; h) = \max_{b_+ \geq -\bar{b}} u(F(b, B, y) - b_+) + \beta \mathbb{E}[V(b_+, h(B, y), y'; h)]. \tag{8}$$

Here, c denotes consumption, $F(b, B, y) = y + bR(B)$, and h is the aggregate law of motion for assets B_+ that we define below, V^c is the value function under repayment, and V is the option value between default and repayment, which depends on the early default decision of the government formalized through h. The policy function for this problem is $b_+^*(b, B, y; h)$. Our objective is to characterize the dynamic stochastic equilibrium using the interaction between b_+ and b provided that h exists.

To characterize households' decisions, we use the analogous of Equation (1) for the infinite-horizon economy. Taking R as given, as the households are atomistic, the characterization of this problem is straightforward, with and Euler equation is given by:[6]

[4] We only need the i.i.d. assumption to prove ergodicity of the equilibrium. The existence of a stationary equilibrium can be proved under Markov shocks.

[5] Ayres et al. (2018) shows that it is possible to have multiple equilibria in similar environments. Our definition of stability does not require the existence of a continuous equilibrium. Thus, if there are multiple equilibria, we assume the existence of a measurable, possibly discontinuous selection, which follows from standard assumptions provided compactness. This property follows from the presence of early default, as shown in Figure 4. Yet, the existence of multiplicity studied in Ayres et al. (2018) may not arise as it depends on additional assumptions.

[6] We provide further details about the first order condition in the appendix.

$$u'(c(b,B,y;h)) \geq R(B_+)\beta\mathbb{E}[u'(c_+(b_+,B_+,y';h))], \tag{9}$$

where $B_+ = h(B;y)$ and b_+ arguments are $(b,B,y;h)$. Equation (9) may hold with strict inequality if the upper bound on debt $-\bar{b}$ is binding and $c(b,B,y;h) = y - b_+(b,B,y;h) + bR(B)$. [7] As R is decreasing in B, we obtain the same pecuniary externality of the 2-period model, which in turn implies further debt issuance as the law of motion for b is monotonic in equilibrium. These interactions generate a destabilizing spiral. As Figure 4 suggests, early default may also stabilize the infinity horizon economy. In the next section, we prove that this intuition extends naturally to more general environments.

If B is sufficiently low, which implies that $R(B)$ is sufficiently large after a given sequence of bad shocks, we may have:[8]

$$u'(c(b,B,y;h)) \geq \mathbb{E}[u'(c_+(b_+(b,B,y;h),h(B,y),y';h))]. \tag{10}$$

Equation (10) is at the heart of the instability of the decentralized equilibrium in the infinite horizon. Formally, it demonstrates that, for some sequence of bad shocks, we can characterize the private economy using a supermartingale, which, since Ljungqvist and Sargent (2012), is well known to generate explosive paths. Lemma 5 in the Supplementary appendix for this section presents conditions that lead to explosive paths. Taking the behavior of households as given, the government decides whether to allow repayment or force default in the private sector. We specify the government's problem in the following section.

[7] From a technical point of view, this equation is useful for us in order to prove compactness and monotonicity of the equilibrium policy function. The reader can find the details in the appendix.

[8] To obtain equation (10) it suffices to impose $R(h(B,y))\beta \geq 1$. Note that $R(B)$ could be arbitrarily large if repayment occurs with probability zero. However, as we suggest in Figure 2 and will demonstrate below, this may occur in off-equilibrium paths that are not relevant to the solution of the model. Even in the most general Markov equilibrium notions, such as Duffie et al. (1994) and Phelan and Stacchetti (2001), the consistency of general equilibrium implies restricting the stationary equilibrium to those paths that satisfy individual optimization with positive probability. If we assume that u is bounded above and unbounded below, something that is natural in CRRA preferences, the existence of a uniform lower bound for consumption follows from standard results. Thus, the fact that $R(B')$ can be factored out implies that any unbounded endogenous interest rate would violate individual optimization. Thus, these paths cannot be considered a part of the recursive stationary equilibria with minimal state space. This type of equilibria, which is the notion used in this paper, is a subset of the recursive equilibrium notions in Duffie et al. (1994) and Phelan and Stacchetti (2001).

4.2 Centralized default

As in the two-period model, we assume that the default decision is centralized. The government may induce default in any state and we abstract from the specific instruments that could lead to a massive default of the private sector. Nevertheless, we provided some examples in the introduction. To focus on stationary equilibrium, we set $b = B$. As the government does not choose the level of debt but only the decision to induce a private default, the problem of the government is:

$$\text{Default if:} \quad V^c(B, y) \leq V^{def}(y). \tag{11}$$

Here, $V^c(B, y)$ represents the continuation value, i.e., the reward for repayment the outstanding debt, that satisfies

$$V^c(B, y) = u(y - b_+(y, B; h) + BR(B)) + \beta \mathbb{E} \max \left\{ V^c(b_+(y, B; h), y'), V^{def}(y')) \right\}, \tag{12}$$

while $V^{def}(y)$ stands for the value of default and satisfies

$$V^{def}(y) = u(y^{def}(y)) + \beta \mathbb{E} \left\{ \theta V^c(0, y') + (1 - \theta) V^{def}(y') \right\}, \tag{13}$$

where θ is the probability of regaining access to the market after default occurs.

Note that, *if consumption and assets in the next period are both increasing in B for each y*, we have the following characterization of default sets:

$$\left\{ \begin{array}{llll} \text{Repay} & if\ B > \bar{B}(y) & \text{as this implies} & V^c(B, y) > V^{def}(y) \\ \text{Default} & if\ B \leq \bar{B}(y) & \text{as this implies} & V^c(B, y) \leq V^{def}(y) \end{array} \right\}. \tag{14}$$

Equation (14) presents a formal definition of $\bar{B}$, which was first depicted in Figure 4. Given the existence of a stationary equilibrium, equation (14) shows that, if consumption and assets are both increasing in aggregate states, *private debt induces state-dependent default sets* as in Arellano (2008). This fact will enable us to characterize stochastic dynamics using a traditional approach in the stochastic processes literature (see, for instance, Meyn and Tweedie (1993)). Moreover,

state-dependent default sets were used only in models with public debt, which are not suitable for macro-prudential analysis. These state-dependent thresholds, in the presence of overborrowing, are essential for connecting the literature on Balance of Payments crises, as in Bianchi (2011), with the literature on default risk; both of which are salient features in emerging economies. We claim that the presence of pecuniary externalities may not only lead to a Sudden Stop, but also to a default, as they both generate excessive private sector borrowing, ultimately taking the economy to unsustainable debt levels. In the Balance of Payments literature, macro-prudential taxes and/or capital controls are used to replicate a constrained planner solution. As Figure 4 shows, these instruments are not enough to globally stabilize the economy, as it is possible to generate a sequence of shocks that takes the economy to excessively high debt services from a welfare perspective. We demonstrate that early default, as formally defined by Equation (14), is sufficient to stabilize the economy, as it forces a sudden and instantaneous deleveraging, thereby resetting the indebtedness cycle and creating a stable orbit for debt.

4.3 The international investors

International investors in this model are risk-neutral, deep-pocket agents whose objective is to break even in expectations. In line with the standard assumptions in the literature, these agents price debt internalizing the default probability. Denoting the (gross) risk-free rate by R^*, and π for the transition probabilities for shocks, the (gross) interest rate subject to default risk is:

$$R(B_+) = R^* \left[\sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y') \mathbb{I} \left\{ V^c(B_+(B,y), y') > V^{def}(y') \right\} \right]^{-1}. \tag{15}$$

Here $B_+(B,y)$ is the aggregate debt in equilibrium and $\mathbb{I}\left\{V^c(B_+(B,y),y') > V^{def}(y')\right\}$ is an indicator function that takes the value of 1 in states in which the government does not induce a default. As is clear from the equation, the assumption of incomplete markets implies that investors cannot condition the return on future debt on future states. Hence, the equilibrium interest rate does not depend on future income, but rather on current income, only if the income process is persistent, $\pi(y', y)$, which we eliminate by assumption. Thus, we denote the interest rate relevant for future

debt as $R(B_+)$. Further, as future equilibrium debt is a function of current states, $B_+(B, y)$, we can factor out the interest rate from the Euler equation, (9).

4.4 Definition of Stationary Equilibrium

Below, we connect all the pieces of the model together in a definition of Stationary Equilibrium.

Definition 1. ***Stationary Equilibrium.*** *A stationary equilibrium is a set of policy functions h, c, b_+, a set of value functions V^{def}, V^c and V, and a price function R, such that:*

1. *Households. Given y, $R(B_+)$, and the government's default choice, $B_+ = h(B, y)$, c and b_+ solve (8), maximizing utility V^c.*

2. *Government. Given y, the household's choices, and the value function under default V^{def}, a benevolent government optimally decides to $\bar{B}(B, y)$, solving (14), generating the equilibrium value of early default V.*

3. *Investors. International investors maximize profits, solving (15) and pricing debt subject to default risk according to R.*

4. *Market clearing-Rational Expectations. For each state, (B, y), with $b = B$, individual decisions are consistent with general equilibrium, $b_+ = B_+ = h(B, y)$. Hence, the aggregate debt function in equilibrium is*

$$\left\{ \begin{array}{c} if\ B \geq \bar{B}(B,y),\ h(B,y) = b_+(B,y;h) \\ if\ B < \bar{B}(B,y),\ with\ prob.\ \theta,\ h(0,y) = b_+(0,y;h) \\ if\ B < \bar{B}(B,y),\ with\ prob.\ 1-\theta,\ h(B,y) = 0 \end{array} \right\} \tag{16}$$

Regarding Definition 1, the model assumes that the government has an enforcement technology to keep the private economy away from individual optimization (as described by equation (9) and formally captured by h when $b_+(B, y; h) < \bar{B}(B, y)$) as long as re-entry is not possible.

Equation (16) is the representation in the space of policy functions of Equation (14), which we define using exclusively value functions. Both equations reflect the state-contingent nature of the default threshold $\bar{B}$, but the latter emphasizes the discontinuity of the equilibrium laws of motion,

which is essential to understand the global stochastic stability discussed in the next section. This characterization depends on the existence of a stationary equilibrium. Provided the existence of this equilibrium, Definition 1 describes an aggregate law of motion, h. In this paper, for expository purposes, we do not include an existence proof. We present those results in a separate, more technical paper.

Note that, if the economy hits the default threshold for a positive measure subset of the state space (B, y), h is discontinuous even if it is unique. So the tools used to prove the existence of equilibrium must be robust to the presence of discontinuities. Even though we present this proof in a separate paper, it is worth noticing that showing existence in this environment allows us to accommodate all the concerns associated with the presence of multiple equilibria as discussed in Ayres et al. (2018). We will always assume the existence of a possible discontinuous measurable aggregate law of motion h. This assumption addresses the issues associated with multiplicity, which can lead to a discontinuous selection rule. The existence proof of the stationary equilibrium relies on the fact that equation (9) induces an order structure, which will allow us to use suitable theorems. It turns out that Coleman (1991), Mirman et al. (2008), and Aguiar and Amador (2019) serve this purpose. Moreover, the default restrictions associated with (16) are not internalized by the household. Thus, as $\bar{b}$ can be assumed to be arbitrarily large, we can prove the results using a standard Euler operator without taking into account inequality constraints.

5 Global stochastic stability

In this section, we extend the dynamic analysis implicit in Figure 4 to the infinite-horizon version of the model. First, we will depict several stochastic paths to illustrate the instability inherent in private debt in the presence of default risk, which arises primarily due to pecuniary externalities. Then, we will formally generalize those results, showing the connection between the stabilizing effects of sovereign default and the existence of a stochastic steady state.

We say that the model is *globally stable* if, regardless of the initial condition, it remains contained in a compact state space almost everywhere and the joint distribution of endogenous variables, such as net external assets and the real interest rate, is time-invariant. Moreover, we will say that the

model is *ergodic* if it generates finite recurrent sets (i.e., positive probability paths generate orbits), the state space is irreducible (i.e., it does not break into different isolated islands), and both these elements can be characterized using a unique probability measure for all possible initial conditions. Then, *ergodicity requires some form of global stochastic stability as it endows the model with an invariant measure and a compact state space that characterizes all positive probability paths.* In this section, we graphically characterize these paths and demonstrate that the model is ergodic, thereby establishing its global stability in a stochastic sense. Moreover, the ergodic invariant measure of the model is its stochastic steady state.

Following Azariadis and Lambertini (2003) and Brock and Hommes (1997), we will use a *phase diagram* in discrete time to characterize global dynamics. This diagram depicts iterative dynamics (i.e., from t to $t+1$ for all $t \geq 0$), mapping $B = b$ to $B_+ = b_+$ using the equilibrium law of motion for debt, h. However, there is a critical difference between those papers and our approach: our analysis is *stochastic*. Thus, the notion of *steady state* is different. In our case, a steady state is a stationary (i.e., time-invariant) measure characterizing endogenous variables in any period.

To construct the phase diagram, we will use different demarcation curves, one for each possible shock: $h(., y)$ for all y lying in a finite set. As shocks are finite, every time the state of nature changes, we move from one demarcation curve to the other. If the economy remains in the same stochastic state, it slides along the same demarcation curve continuously from $B = b$ to $B_+ = b_+$. Moreover, because there is a possible default, debt may change even for the same shock if the economy crosses the threshold $\bar{B}$. This fact is critical for stability as it allows us to adapt the standard definition from the non-stochastic analysis to our notion of steady state. For this purpose, we use the stochastic version of an orbit in dynamical systems, known as a recurrent set. We will show that h is monotonic. Then, to obtain recurrence, we must prevent the model from diverging to infinity. As Figure 2 suggests, this is possible even if debt is finite, as debt services may explode. Unstable dynamics may arise from non-monotonic laws of motion. In our case, the monotonicity of h follows from the fact that R is decreasing. This fact, in turn, generates pecuniary externalities and overborrowing. Thus, these properties endow the model with a tractable form of instability, which is essential for our formal results. To stabilize the economy, we must *only* address the rapid growth of debt. Default, by forcing a full haircut on the face value of debt, resets the leverage cycle, creating

a new orbit. We will demonstrate that this behavior generates recurrent sets for all possible initial conditions, thereby stabilizing the economy provided that it reaches default thresholds in finite time.

5.1 Phase Diagrams and global characterization

Figure 5 represents the way in which default induces global stochastic stability in the infinite-horizon model. The figure presents the transition for the case of 2 shocks and one of the 2 possible initial conditions $B_0 : b_+(B_0, Y_{UB}) > B_0$. It contains a candidate for a non-stochastic steady state, i.e. a point B^N satisfying $B^N = b_+(B^N, y)$ for any $y \in Y$,[9] which allows us to establish that the default set is non-empty and, as we will show below, stable.[10]

[9]Note that as we are considering an equilibrium, we have $b = B$. On top of that $B_+ = b_+(B, y)$ and along the 45° line $B = b_+(B, y)$

[10]This means that it does not contain transient sets. Let P^n_φ be the n-th iterate of a Markov kernel. A transient set A satisfies $P^n_\varphi(z, A) \to 0$ for all $z \in A$, and in practice, they are eliminated by throwing away the first 1,000/10,000 simulations before computing any long run average. This property implies that the process does not create islands, and it is connected. Moreover, we need to generate well-behaved orbits or recurrent sets. We will demonstrate this property by proving that the process reaches default thresholds in finite time, a fact that, in turn, depends on the monotonicity of b_+.

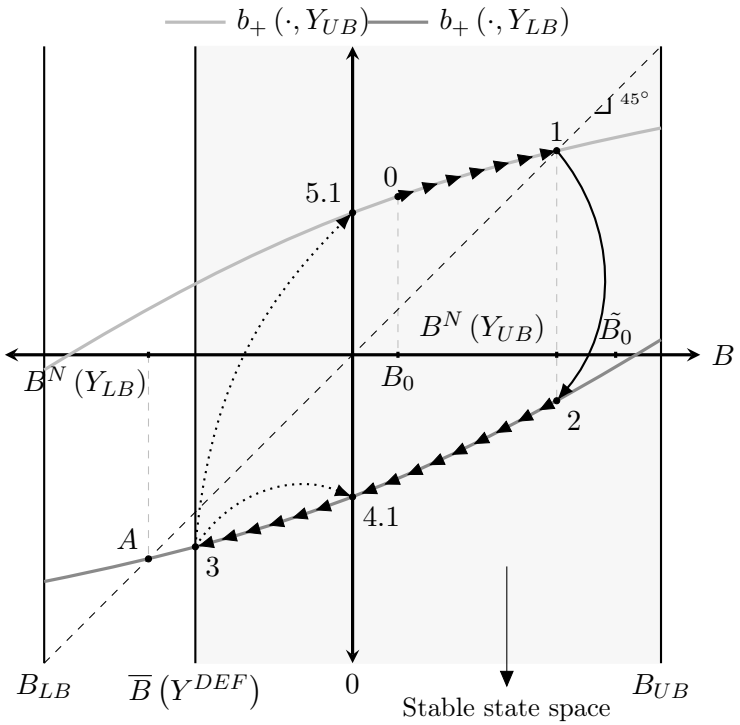


Figure 5: Non-empty and (saddle path) stable default set

Note: Start with the initial state B_0, Y_{UB} in point 0, an initial condition which implies $b_{+,*}(B_0, Y_{UB})$. A good endowment shock increases asset accumulation to point 1, where a bad endowment realization occurs, inducing the economy to issue debt to smooth consumption (point 2). If a sequence of Y_{LB} occurs, the economy goes straight to default in point 3. The figure illustrates that when returning to asset markets, the economy can transition to point 5.1 if Y_{UB} or 4.1 if Y_{LB}.

This Figure 5 is closely related to Figure 4. To some extent, it represents the dynamic (infinite horizon) version of the two-period model. The solid lines indicate the policy functions for two different endowment levels (the upper endowment produces the light gray line and the lower endowment produces the dark gray line). The arrows indicate the accumulation or deleveraging of debt. In the figure, the optimal debt issuance moves from point 1 to 2 in the presence of a negative endowment shock. When the economy reaches point 3, the government defaults and regenerates the cycle, to point 4.1 with a negative endowment state, or to point 5.1 in the case of a good realization of the endowment. In terms of Figure 4, moving from A to B, due to a decreasing sequence of negative wealth shocks ω, is analogous to sliding through $b(., Y_{LB})$ from points 2 to 3. Moreover, defaulting

corresponds to reaching $\bar{B}$, and then jumping to points 5.1 (or Section 4.1). These points correspond to the point B_2 in Figure 4.

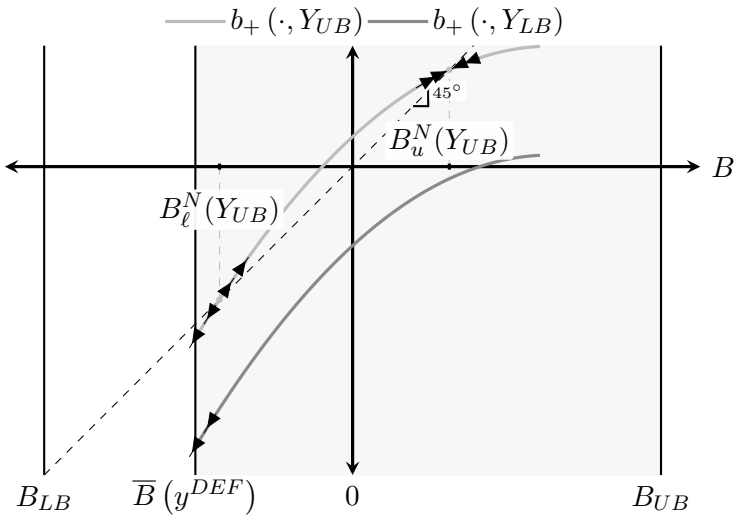


Figure 6: Unstable Transitions

Note: Instability is possible without default. If demarcation curves are concave, then for indebted countries, the economy is unstable even for high levels of GDP.

Figure 6 depicts another situation. If demarcation curves are concave, in the absence of default, indebted countries may be unstable regardless of their GDP level. This situation could arise either because there is no intersection between the 45° line and demarcation curves or because these curves cross when the county has a negative net external assets position. In Figure 6, by resetting the indebtedness cycle, default takes the economy back to accumulating assets in the good state. If the economy enters a recession, then it prevents debt services from exploding. As noted by Brock and Hommes (1997), global instability is typical for non-monotonic models. In our case, even with increasing demarcation curves, instability arises due to penuniary externalities. Entering into a spiral of high debt, high interest rates, that induce higher debt tomorrow happens if the economy is to the left of the unstable equilibrium on the left of the $B_\ell^N(Y_{UB})$ point. The dynamics to the left of that point requires a default to regenerate the cycle.

5.2 Existence of a stable stochastic steady state

The economics for the previous section are determined by the dynamics around the stochastic steady state. In this section, we discuss its existence and properties. To formally characterize the steady state, we need two main standard assumptions:

Assumption 1 (Finite i.i.d. endowments)**.** *All* $y \in \{Y_{LB}, ..., Y_{UB}\} \equiv Y$ *with* $Y_{LB} > 0$, $, Y_{UB} < \infty$ *and* $\pi(y) > 0$*; where* π *is a probability measure.*

Assumption 2 (Preferences)**.** $u : \mathbb{X} \to \mathbb{R}$*, where* $\mathbb{X} = \mathbb{R}_+$ *is the consumption space,* u *is once differentiable with derivative given by* $u'(c)$*, strictly increasing, strictly concave, unbounded below and bounded above. Moreover,* u' *satisfies Inada:* $lim_{c\to\infty}\ u'(c) = 0$ *and* $lim_{c\to 0}\ u'(c) = \infty$*. Finally,* $\beta R^* < 1$

Assumption 1 allows us to define a process $(\Omega, \Sigma, \mu_{y_0})$ with a typical element in the sequence space $\{y_0, y_1, ...\}$ and an associated process in the space of random variables for $[c, b_+, R](\omega)$, which maps Ω to $\mathbb{R}^3$ (see Lucas et al. (1989), chapters 7 to 9). Assumption 2 is a mild restriction of preferences. For instance, a CRRA utility function with a risk aversion parameter above 1 satisfies all the mentioned requirements. As shown below, under these assumptions, c, b_+, and R are bounded almost everywhere in μ_{y_0}, for all $y_0 \in Y$. The following two Lemmas formally characterize endogenous variables.

Lemma 1 (Bounds)**.** *Under assumptions 1 and 2,* $[c, b_+, R](\omega) \in \mathbb{K}$ *almost everywhere in* Ω*, where* $\mathbb{K} \in \mathbb{R}^3$ *and is compact. Moreover,* $c(\omega)$ *is bounded below almost everywhere in* Ω *by* $\underline{c} > 0$*.*

Proof. See the Appendix □

Lemma 1 implies that debt services are bounded. However, this fact does not preclude default from being welfare-enhancing. Provided some qualitative properties of policy functions, Equation (14) implies that the government may prefer to default. Thus, characterizing policy functions, induced by equation (9), is essential for our purposes as they guarantee that the value of repayment behaves accordingly.

Lemma 2 (Policy Functions)**.** *Under assumptions 1 and 2, if $R(B)$ is decreasing in B, then $c(b, B, y; h)$ and $b_+(b, B, y; h)$ are both weakly increasing in b when $b = B$ for any $y \in Y$ and h.*[11] *Moreover, either c or b_+ is strictly increasing.*

Proof. See the Appendix □

Lemma 2 not only characterizes policy functions qualitatively, but it also implies that a persistent recession may lead to explosive debt paths that end in a default episode. The supplementary appendix to this section formally states this result (see lemma 5). Of course, there is a "limit" to the degree of instability that allows us to keep the equilibrium tractable. To curve the instability inherent in the model, we assume $R\beta < 1$.

Definition 2. ***[Equilibrium state space Z]*** *Let $\overline{B}(y)$ be the upper bounds for debt implied by h. The minimal state space for the equilibrium process generated by the Markov kernel P_φ, to be defined below, is given by: $Z_1 \equiv \left[\overline{B}(y_{LB}), B_{UB}\right] \times Y$. Then, there exist a function φ mapping (B, y, c, R) to (B_+, y_+, c_+, R_+) and these elements belong to a Stationary Equilibrium. The state space Z is composed of all possible (B, y, c, R) spanned by Z_1 using φ.*

Equipped with Definition 2, we can project Z_1 onto Z using φ, which allows us to derive the following Markov kernel:

$$P_\varphi(z, A) = \left\{\pi\left(y' \in Y : \left[c(h(B, y), y'), R(h(B, y)), h(B, y), y'\right] \in A\right)\right\} \tag{17}$$

We show that P_φ has an invariant measure, which is also the stable stochastic steady state for the model summarized by (Z, P_φ).

To prove that a possibly discontinuous equilibrium has an invariant measure, we cannot use standard tools (see, for instance, Lucas et al. (1989) chapter 12). Following Meyn and Tweedie (1993), we must define a regeneration point. Fortunately, this point arises almost naturally in default models. If we can prove that the model hits the default set $\bar{B}(y)$ with positive probability for all $y \in Y$, then the default states generate a regeneration set. However, this set must have at most one point (i.e., the set must be a singleton), which implies that we must refine the default

[11] Both c and h are contained in s set of function C defined in the appendix.

punishment. The assumption below imposes these restrictions (for details, see the supplementary appendix of this section).

Assumption 3. ***[Ergodic punishment]*** *Assume that* $y^{def}(y) = y^{def}$ *for all* $y \in Y$. *Let* $B^N(y_{LB}) \equiv B = b_{+,*}(B, y_{LB})$. *Assume that* $B^N(y_{LB}) < 0$.

Note that the second part of assumption 3 imposes restrictions on endogenous variables. We need to assume the existence of a non-stochastic steady state B^N for the lowest shock y_{LB}.

The restriction on $B^N(y_{LB})$ in Assumption 3 has a minimal consistency requirement: we suppose that it is negative. As we are modeling default, we require that in the non-stochastic steady state, households hold negative net assets in the worst possible scenario (i.e., $y = y_{LB}$). Then, as lemma 2 shows that b_+ is increasing in B, the definition of y^{def} will allow us to construct a point satisfying: $B^N(y_{LB}) < \bar{B}(y_{LB}) < 0$, where $\bar{B}(y_{LB})$ is defined in equation (14). This inequality is necessary to prove the connectedness of the state space in definition 2. Then, we use the connected state space to prove the equilibrium's global stability, and the existence of a well-defined stochastic steady state.

Theorem 1. *Under assumptions 1, 2, 3, there exist* y^{def} *such that* $B^N(y_{LB}) < \bar{B}(y^{def}) < 0$ *and* (Z, P_φ) *has an unique invariant probability measure* μ_*^{def}.

Proof. See the appendix. □

Theorem 1 proves the existence of a stable stochastic steady state, as it endows the process (Z, P_φ) with an orbit of finite size for any possible initial condition in the state space defined before. The appendix contains the proof of this theorem and additional technical details to ensure the paper's self-containment.

6 A quantitative exercise

So far, we highlighted the implications of decentralized debt in an economy with centralized default using a stylized two-period model, with a closed-form solution, and its extension to an infinite horizon stochastic economy, showing the generality of our results. We use stochastic phase diagrams,

a useful tool to deal with the qualitative implications of our general model. In this section, we study the quantitative implications of centralized default on a decentralized economy with pecuniary externalities generated by default risk.

We solve the stochastic infinite horizon model and take it to the data. The solution algorithm is constructive, based on the theoretical results of Section 4. Thus, our numerical solution is accurate as we use an algorithm which converges to the recursive equilibrium presented in Definition 1.[12] Our purpose is to measure the destabilizing impact of private debt in response to shocks using a quantitatively plausible model calibrated to match Argentinean data between 1982 and 2016, as this sample includes only 1 event, the default of 2001. The re-entry probability θ is fixed at a standard value for quarterly sovereign-default models (Table 2). We calibrate the discount factor β, the international risk-free rate r^* and a numerical cap on the equilibrium interest rate, r_{max}, to match three moments: the annual default frequency, the debt-to-GDP ratio B/Y, and the current-account-to-GDP ratio CA/Y, shown in Table 3 together with their model counterparts, the annual default frequency to 3% and the average B/Y and CA/Y. We also compare the model's performance with respect to mean debt services RB/Y, which is not targeted. As seen in the Table 2, parameters are in a standard range. As discussed before, the shock is assumed to be iid.

Table 2: Parameters

Parameter	Value	Description	Source
σ	2.0	Risk aversion param.	Arellano (2008)
θ	0.10	Re-entry prob.	Standard quarterly value
β*	0.9835	Discount factor	Calibrated
ρ_e	0.001	Persis. (endowment)	Assumpt. ($\approx$ i.i.d.)
STD_e	0.02	St. dev. (endowment)	Arellano (2008)
r**	0.75%	Net risk-free interest rate	Calibrated
r_{max}*	6.6%	Interest rate cap	Calibrated

Note: the second column contains the values of the parameters used in this paper as a benchmark calibration. * denotes parameters calibrated as described above to the three starred moments in Table 3: β, r^*, and r_{max}). ρ_e and STD_e are the coefficients of the AR(1) process that was discretized using a grid of 21 points. r_{max} is also calibrated to bound the equilibrium interest rate.

[12] We discretize the bond position on a grid of 300 equally spaced points that contains $B = 0$ exactly, the position of the economy after a default, and we evaluate continuation values at next-period debt by linear interpolation between grid points.

Table 4 reports a set of additional, non-targeted moments—business-cycle statistics that played no role in the calibration—alongside their empirical counterparts.[13] We report the data over two samples: the full calibration window 1982–2016, which—like the model samples—contains one default, and the pre-default window 1982–2001. Note that, due to our ergodic results, we can compare the model's performance against the pre and post-default sample. We show that the ergodic distribution contains at most one default and we can also characterize the model's behavior after the crises.

Three results are worth emphasizing. First, the model's external account and consumption are less volatile in the model than in the data: $\sigma(C/Y)$ is 1.61% /1.51% in the model's full sample / 40 years window against 2.98%, and $\sigma(CA/Y)$ is 1.38% / -1.29% against 3.03%. Both are closer to the pre-default window (1.19% and 1.82%), whose volatilities are lower because the 2001 collapse is a large outlier. We also obtain less volatile GDP and consumption growth rates ($\sigma(g_y)$ and $\sigma(g_c)$ respectively. However, consumption growth is more volatile than output growth, as in the data: $\sigma(g_c)/\sigma(g_y) = 1.26$ and 1.18, in the full sample and 40 years windows respectively, against 1.04. These gaps follow from the endowment process, which is calibrated to the near-i.i.d. specification of Arellano (2008) and is considerably smoother than Argentine GDP; with income nearly unforecastable and only mildly variable, consumption in the model is driven primarily by the endogenous dynamics of debt rather than by the income shock itself. Second, the model does a good job in matching the correlation between consumption and growth. It also replicates the pro-cyclical borrowing pattern observed in the correlation between the current account and GDP growth. Finally, as we do not model investment, the consumption-to-GDP ratio is higher in our simulations than in the data. However, we are close to the $82-16$ current account to GDP average.

[13] We compute external private debt using a novel database which classifies external indebtedness according to: international organizations (i.e., non-subject-to default), subject-to default public, and subject-to default private.

Table 3: Results

Variable	$(R(B)B)/Y$	Def. freq.*	B/Y*	CA/Y*
Data	−0.6%	3.0%	−8.2%	−0.8%
Model, 40-year windows	−0.7%	3.4%	−12.5%	−0.4%
Model, full simulated sample	−0.7%	3.4%	−12.2%	−0.5%

Note: $*$ denotes targeted moments; interest payments, $(R(B)B)/Y$, are not targeted. Def. freq. is the annual frequency of default, while B/Y and CA/Y denote net external debt and the current account relative to GDP, respectively. Model moments are computed from a 50,000-quarter ergodic simulation. The 40-year windows are centered on individual defaults and contain exactly one default, replicating the data sample (120 windows). The full-sample results use all simulated quarters. Both specifications include default and exclusion periods, during which debt and interest payments are zero, output and consumption equal y^{def}, and the current account is zero.

Table 4: Additional non-targeted moments

Moment	Data 82–16	Data 82–01	Model, 40 years windows	Model, full sample
C/Y (mean)	78.44%	77.65%	99.72%	99.82%
CA/Y (mean)	−0.92%	−2.44%	−0.41%	−0.49%
$\sigma(g_y)$	6.04%	5.93%	1.90%	1.93%
$\sigma(g_c)$	6.31%	6.19%	2.25%	2.43%
$\sigma(g_c)/\sigma(g_y)$	1.04	1.04	1.18	1.26
$\sigma(C/Y)$	2.98%	1.19%	1.51%	1.61%
$\sigma(CA/Y)$	3.03%	1.82%	1.29%	1.38%
$Corr(g_c, g_y)$	0.98	0.98	0.78	0.81
$Corr(CA/Y, g_y)$	−0.10	−0.28	−0.15	−0.23

Note: C/Y denotes consumption relative to GDP; $g_x = \ln x_t - \ln x_{t-1}$ denotes the growth rate of variable x. Argentine data are annual and cover 1982–2016 and the pre-default period 1982–2001. Model moments are computed from the same simulation and 40-year default-centered windows as in Table 3. Quarterly simulated series are aggregated to annual frequency before computing moments. Default and exclusion periods are included. Since the model abstracts from investment consumption-to-GDP levels are not directly comparable with the data.

Figure 7 presents the numerical counterpart of the phase diagram in Figure 5, computed using the benchmark calibration reported in Table 2. Unlike the theoretical phase diagram, which provides a qualitative representation of the equilibrium dynamics, this figure is constructed directly from the policy functions obtained from the numerical solution of the model. It focuses on the region surrounding the endogenous default thresholds, where the interaction between debt accumulation, default risk, and borrowing decisions becomes particularly relevant. The figure plots the demarcation curves for different income realizations, together with the 45-degree line, the default thresholds, and the range of debt positions visited in the ergodic simulation. This representation allows us to

examine more precisely the local behavior of equilibrium debt dynamics as the economy approaches the default region.

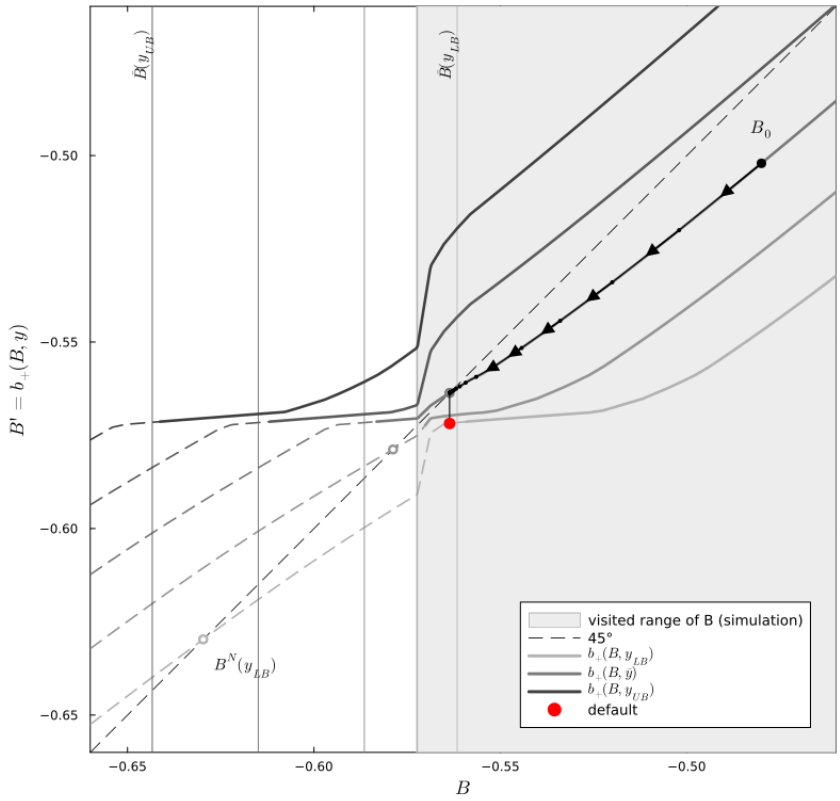


Figure 7: Numerical phase diagram near the default thresholds

Note: Demarcation curves $b_+(B, y) = y + R(B, y)B - c(B, y)$, the net foreign assets chosen for next quarter as a function of current assets B, for five points of the income grid, $y \in \{0.942, 0.970, 1.000, 1.030, 1.062\}$, from light gray ($y_{LB} = 0.942$) to dark gray ($y_{UB} = 1.062$); $\bar{y} = 1.000$ is the median. The horizontal axis is restricted to $B \in [-0.66, -0.46]$. Solid segments are repayment states; dashed segments are states in the default set, $V^c(B, y) < V^{def}(y)$. Vertical lines mark the default thresholds $\bar{B}(y)$, the largest bond position in the default set for each income level, from $\bar{B}(y_{LB}) = -0.562$ to $\bar{B}(y_{UB}) = -0.643$. Open circles mark $B^N(y)$, the crossing of $b_+(\cdot, y)$ with the 45° line, when it falls in the window: $B^N(y_{LB}) = -0.630$ and $B^N(0.970) = -0.579$ lie inside the default set, while $B^N(\bar{y}) = -0.564$ lies in the repayment region, just above $\bar{B}(\bar{y}) = -0.587$. The shaded band is the range of B visited in the 50,000-quarter simulation of Table 3, $[-0.573, 0]$. The arrows follow the equilibrium law of motion from $B_0 = -0.48$ under $\bar{y}$ for 25 quarters, which takes the economy to $B^N(\bar{y})$; a single realization of y_{LB} then places it in the default set (red dot).

An interesting feature of Figure 7 is that the demarcation curves suggest the presence of a local attractor in the neighborhood of the default region. In particular, as debt accumulates, the policy functions approach the 45-degree line, and their intersections identify non-stochastic fixed points around which debt dynamics may exhibit local convergence. This behavior reflects the interaction

between two opposing forces operating through the Euler equation. On the one hand, higher indebtedness increases the equilibrium interest rate and debt-service obligations, reducing disposable resources and inducing further borrowing through a negative income effect. This is the destabilizing mechanism associated with the pecuniary externality. On the other hand, higher borrowing costs also generate an intertemporal substitution effect, encouraging households to reduce current borrowing and limiting further debt accumulation. When the latter effect becomes sufficiently strong relative to the former, the policy function approaches the 45-degree line from above, generating a locally stabilizing drift in debt dynamics. Thus, default risk may induce a degree of local stabilization even before default actually occurs.

However, local convergence does not guarantee global stability in a stochastic environment. The fixed points identified by the intersections with the 45-degree line are conditional on particular income realizations and do not represent a stochastic steady state. Even when debt dynamics converge locally under a given income realization, an adverse sequence of shocks may shift the economy toward a different demarcation curve, pushing indebtedness beyond the region of local attraction and eventually into the default set. Consequently, the stabilizing substitution effect is not sufficient to prevent the accumulation of debt or to ensure that equilibrium trajectories remain bounded for all possible shock realizations. This is where centralized default plays its essential role. By imposing a full haircut on outstanding private external debt, default interrupts the destabilizing feedback between borrowing, interest rates, and debt-service obligations, resetting the debt cycle and generating recurrent paths. Hence, while the Euler equation may generate local convergence near the default region, it is the endogenous default decision that provides the regeneration mechanism required for global stochastic stability and the existence of the invariant distribution characterized in Theorem 1.

6.1 Comparison with the centralized benchmark

The quantitative results that follow contrast our decentralized economy with the centralized benchmark of Arellano (2008), in which the government chooses debt issuance and default jointly and therefore internalizes the effect of its borrowing on the equilibrium bond price. We solve the centralized economy using the canonical implementation of Arellano (2008).

Each economy is evaluated under its own calibration. We find this approach appropriate because both models are calibrated to Argentina over broadly comparable periods (1982–2001 in Arellano (2008) and 1982–2016 in our case). The decentralized economy follows the calibration reported in Table 2, while the centralized economy retains that of Arellano (2008): $\beta = 0.953$, $\sigma = 2$, $r^* = 1.7\%$, a persistent endowment process with $\rho = 0.945$ and $\eta = 0.025$ (discretized using a 21-point Tauchen grid), a re-entry probability $\theta = 0.282$, and income-contingent output during exclusion given by $\min\{0.969\,\mathbb{E}[y], y\}$. We retain the original calibration of the centralized model rather than imposing our parameter values, as the two frameworks differ in both their calibration strategies and equilibrium mechanisms. In particular, the near-i.i.d. endowment process and endogenous, income-independent default output y^{def} are features of the equilibrium with private debt, not the centralized framework. Finally, the interest-rate ceiling r_{max} has no counterpart in the centralized model.

We solve both economies using the same grid for bonds, corresponding to our baseline calibration (Table 2). This grid is wider than that used by Arellano (2008), as decentralized borrowing leads to debt levels that are not reached in the centralized economy.

We conduct the trajectory comparison in Figure 8 as the first controlled experiment. Both economies start from the same initial debt position and face the same sequence of 38 realized income levels, drawn from the decentralized economy. Then, fully simulate the models to obtain the ergodic distributions in Figures 9 and 10 by simulating each economy under its own endowment process. As we show below, these figures illustrate the anatomy of default in both models, highlighting their contrasting debt dynamics. In particular, the decentralized economy exhibits a slow-moving debt crisis, characterized by a gradual buildup of debt and financial fragility that eventually culminates in default.

6.2 A slow moving debt crisis

Following Lorenzoni and Werning (2019), we characterize a slow-moving debt crisis as a process in which the interaction between default risk, borrowing costs, and debt accumulation generates a gradual deterioration in debt sustainability before default occurs. In their model of public debt, pessimistic expectations about future repayment raise current borrowing costs. Since the government cannot commit to a fixed quantity of bond issuance, higher interest rates induce additional

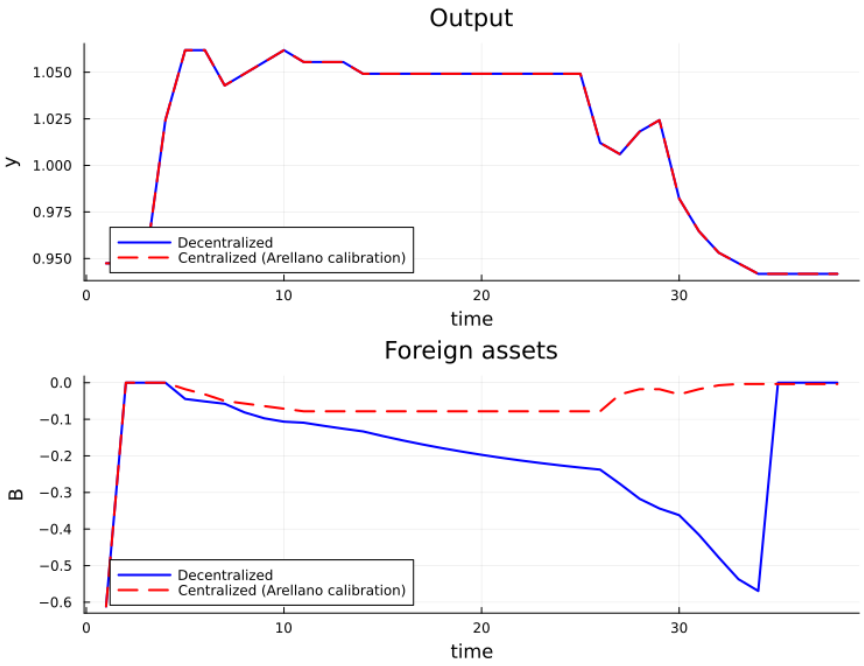
borrowing to meet its financing needs. The resulting increase in debt raises future default probabilities, validating investors' initial expectations. This feedback can sustain a gradual accumulation of debt and rising borrowing costs, rather than an immediate loss of market access. Lorenzoni and Werning (2019) show that such dynamics may arise in self-fulfilling equilibrium and, even when the equilibrium is unique, identify a tipping point separating stable debt trajectories from paths characterized by increasing debt and default risk.

Our model generates a similar pattern through a different mechanism. As in Lorenzoni and Werning (2019), default risk affects equilibrium borrowing costs, and the resulting increase in debt-service obligations induces further borrowing, potentially leading to default. However, rather than relying on self-fulfilling changes in investors' expectations or coordination failures, the source of instability in our framework is a pecuniary externality arising from decentralized private borrowing and centralized default. Individual borrowers take the equilibrium interest rate as given and do not internalize how their collective indebtedness increases the probability of default and the associated risk premium. For an indebted economy, higher interest payments reduce disposable resources and encourage additional borrowing to smooth consumption, generating a feedback loop between debt accumulation and borrowing costs. This mechanism operates within a stationary recursive equilibrium and can produce a slow-moving debt crisis even with one-period debt, without requiring equilibrium switches or long-term debt dilution. Centralized default, which generates the pecuniary externality, eventually interrupts this process by resetting the private debt cycle.

This mechanism also distinguishes our model from the standard sovereign default framework of Arellano (2008). In that model, the government jointly determines debt issuance and default and therefore internalizes the effect of additional borrowing on equilibrium bond prices. This internalization limits debt accumulation and eliminates the pecuniary externality associated with decentralized borrowing. Consequently, although default remains possible, the model does not generate the same systematic, debt-driven buildup observed in our decentralized economy. Our quantitative results illustrate this distinction.

Figure 8 compares the debt dynamics of the baseline economy with decentralized debt to those of the model in Arellano (2008). The figure traces the evolution of income and asset positions over 38 periods, including the endogenous default decision.

Figure 8: Debt dynamics in the centralized and decentralized economies



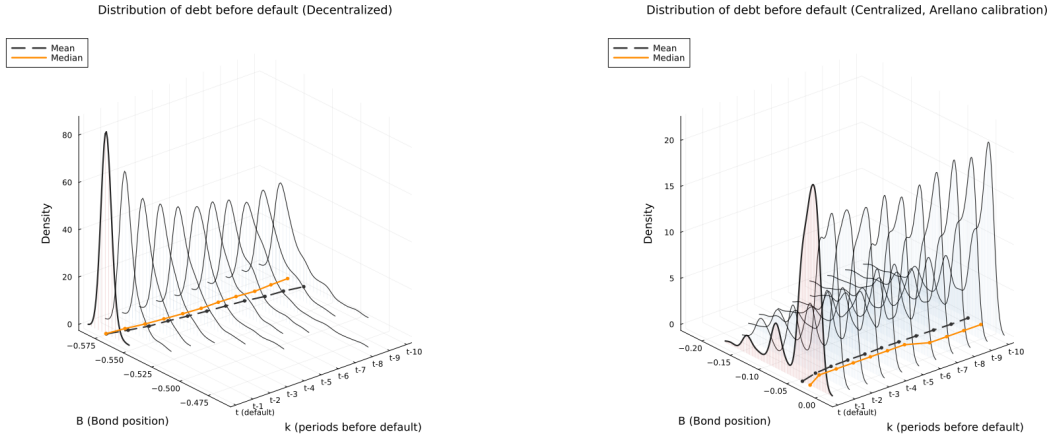
Note: Equilibrium dynamics for a path of income shocks simulation of 38 periods. The decentralized economy is calibrated according to Table 2; the centralized economy is solved at the calibration of Arellano (2008), on the same bond grid. Both economies receive the same initial debt, the same sequence of income levels (the endowments realized in our economy) and the same re-entry rule.

As the figure shows, by accounting for the impact of borrowing on debt prices, the centralized economy limits its debt accumulation to approximately -0.08, maintains this position while income remains high, and gradually reduces its debt when income declines after $t = 26$, avoiding a second default. In contrast, the decentralized economy starts borrowing immediately after regaining market access and continues accumulating debt throughout the episode. Rather than adjusting its debt position when income falls after $t = 26$, it accelerates borrowing, reaching approximately -0.57 before defaulting again at $t = 34$.

A striking difference to highlight is how shallow the borrowing cycle is in the case of centralized debt (peak debt of about -0.08, against -0.57). The government in this case internalizes the borrowing effect on the price of debt, reducing the incentives to borrow. The general equilibrium effect is absent in the non-internalization case, inducing a much larger and profound debt cycle that generates a slow moving debt crisis in private debt markets.

Figure 9 complements this single-trajectory illustration with a distribution of trajectories. For each economy (centralized and decentralized debt issuance), we simulate a long ergodic path, isolate every default episode preceded by at least ten consecutive quarters of uninterrupted market access, and plot the distribution of the bond position at each of the ten quarters leading up to default. The build up of the debt crisis is remarkably different between the two economies. In the decentralized economy (panel (a)) the distribution of debt visibly channelizes, with a small dispersion, toward a common threshold as default approaches: the mean and median bond position fall from -0.546 and -0.560, respectively, ten quarters before default to -0.569 and -0.570 at the default quarter itself—the same gradual, debt-driven buildup documented in Figure 8, now shown to be systematic across several episodes. In the centralized economy (panel (b)), by contrast, the mean and median bond position stay within a few hundredths of zero at every horizon before default (between -0.042 and -0.018) (panel (b) has a much narrower horizontal scale), confirming that internalizing the borrowing externality all but eliminates the slow, debt-driven buildup: default in the centralized economy is closer to an income-driven event than a debt-driven one.

Figure 10 performs the same exercise for the interest rate paid on maturing debt, using $k = 1, \ldots, 10$ (the default-declaration quarter itself, $k = 0$, carries no priced rate by construction and is omitted). Consistent with Figure 9, the decentralized economy's rate rises only gradually as default approaches—its median moves from 0.88% ten quarters before default to 1.87% one quarter



(a) Decentralized

(b) Centralized

Figure 9: Distribution of the bond position in the quarters leading up to default

Note: each panel pools every default episode, drawn from a 50,000-quarter ergodic simulation (decentralized economy: calibration in Table 2; centralized economy: calibration of Arellano (2008)), that is preceded by at least ten consecutive quarters of uninterrupted market access, and plots the kernel-smoothed distribution of the bond position B at $k = 0, 1, \ldots, 10$ quarters before the default declaration ($k = 0$, front row of each panel). The dashed gray line traces the mean and the solid orange line the median of each distribution, both evaluated at their true (B, k) coordinates. Panel (a): decentralized economy, $n = 427$ qualifying episodes. Panel (b): centralized economy, $n = 331$ qualifying episodes.

before—whereas the centralized economy shows a late spike concentrated almost entirely in the last quarter before default, with its mean rising from 2.47–2.88% at $k = 3, \ldots, 10$ to 4.55% at $k = 1$, while its median moves only from 2.55% to 2.98%. In the decentralized economy, the higher median interest rate is accompanied by a fat upper tail in the distribution of sovereign spreads, reflecting episodes of severe financial distress. This pattern is consistent with the empirical evidence on sovereign debt crises. Pescatori and Sy (2007) document the importance of extreme spread realizations in identifying debt crises, while Paluszynski and Stefanidis (2023) highlight the difficulty of standard quantitative sovereign default models in reproducing the high spreads observed during crisis episodes.

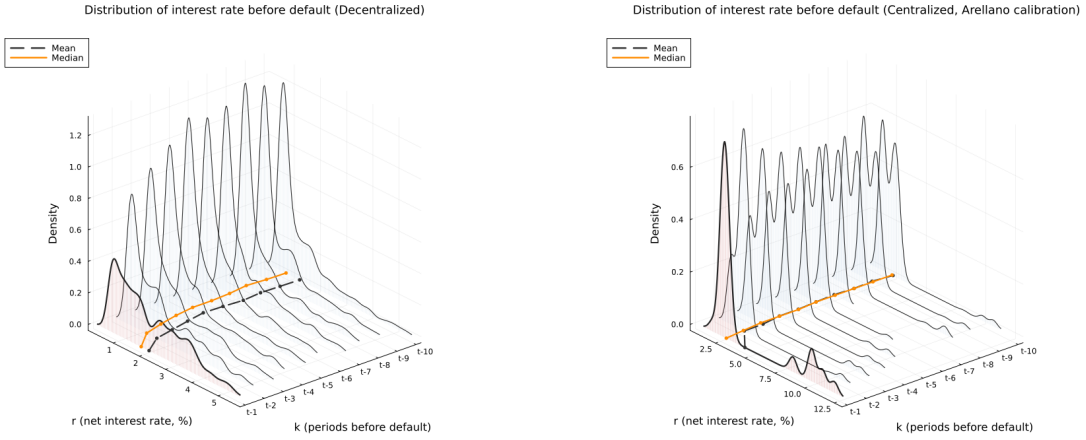
An additional interesting feature is that the centralized economy pays a higher risk premium than the decentralized economy far from default. The gap closes as default approaches. The decentralized economy is therefore cheaper to finance in normal times but exposed to a much larger debt buildup once a crisis approaches, while the centralized economy pays a persistently higher ambient risk premium but avoids the slow-moving debt spiral documented above. Numerically, this result is documented also in Table 5.

Table 5: Interest rate build-up before default: median rate, far vs. last-priced quarter

	$t-10$ (far)	$t-1$ (last priced)
Decentralized median rate	0.88%	1.87%
Centralized median rate	2.55%	2.98%

Note: median net interest rate r at $k = 10$ and $k = 1$ quarters before default, from the same sample and qualification criterion as Figure 10.

This last result is particularly interesting in light of the empirical evidence on sovereign and private borrowing costs. In general, corporate bonds carry higher risk premia than comparable sovereign bonds, reflecting the conventional view that governments are more creditworthy than domestic private issuers. However, this relationship is not universal. Durbin and Ng (2005) document several cases in which emerging-market firms borrow at lower spreads than their sovereigns, particularly when they have substantial foreign revenues. More strikingly, Grandes et al. (2017) find that, while corporate spreads generally exceed sovereign spreads in Latin America, Argentina stands out as an exception, with corporate spreads averaging approximately 400 basis points below sovereign spreads during their sample period.



(a) Decentralized

(b) Centralized

Figure 10: Distribution of the interest rate in the quarters leading up to default

Note: same sample and qualification criterion as Figure 9 (at least ten consecutive quarters of uninterrupted market access before default). Each panel plots the kernel-smoothed distribution of the net interest rate r paid on maturing debt at $k = 1, \ldots, 10$ quarters before the default declaration; $k = 0$ is omitted since the rate is zero by construction in the default-declaration quarter in both simulators. The dashed gray line traces the mean and the solid orange line the median of each distribution, both evaluated at their true (r, k) coordinates. Panel (a): decentralized economy, $n = 427$ qualifying episodes. Panel (b): centralized economy, $n = 331$ qualifying episodes.

This evidence suggests that sovereign borrowing costs need not be lower than those faced by private borrowers, particularly in economies with a history of sovereign default. Our model provides a related but distinct perspective. By internalizing the effect of borrowing on bond prices, the centralized economy limits debt accumulation and reduces the likelihood of severe debt crises, yet it faces a higher risk premium in normal times. The decentralized economy, in contrast, enjoys lower borrowing costs far from default but is more vulnerable to debt accumulation and the subsequent deterioration of financing conditions. Thus, lower borrowing costs in normal times do not necessarily imply greater financial stability.

6.3 Time to default and initial debt

The distributions in Figures 9 and 10 condition on a default occurring. We now ask the converse question: starting from a given level of debt, how long does each economy take to default? We measure the hitting time τ, the number of quarters until the first default declaration, grouped into ten sets of equal size by the level of debt. Table 6 and Figure 11 report the results; their notes describe the construction.

Table 6 examines how initial debt affects the timing and probability of default in both economies. In the decentralized economy, moving from the least indebted states (D1) to the most indebted (D10) reduces the median time to default from 73 to 50 quarters and the mean from 96.7 to 76.0 quarters. The probability of default within ten quarters rises from 0.9% to 19.8%, while the probability within twenty quarters increases from 8.7% to 29.0%. The effects are considerably stronger in the centralized economy. The median time to default falls from 95 quarters in D1 to just one quarter in D10, while the probability of default within ten quarters increases from 5.2% to 65.8% and that within twenty quarters from 11.6% to 68.5%. Thus, although initial debt matters in both economies, its relationship with default timing differs substantially.

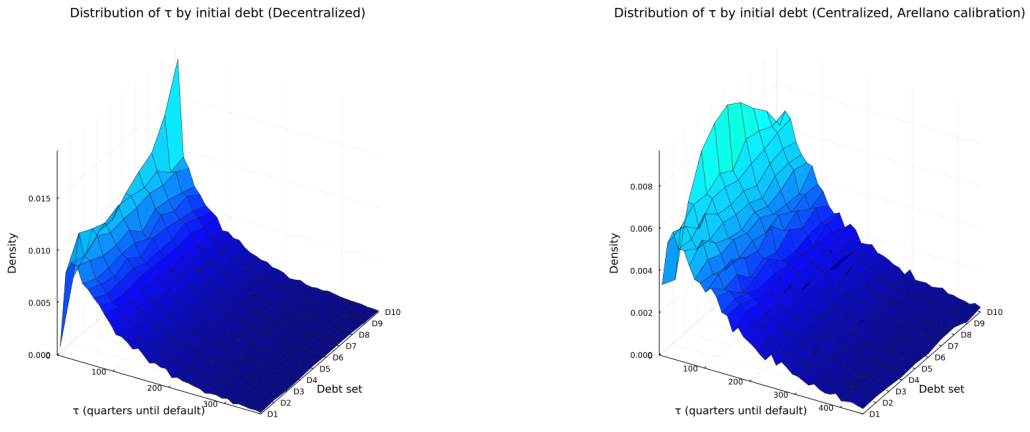
Table 6: Time to default by initial debt: least vs. most indebted states

	Median τ (quarters)			Mean τ (quarters)			$P(\tau \leq 10)$			$P(\tau \leq 20)$		
	D1	D10	D10/D1	D1	D10	D10/D1	D1	D10	D10−D1	D1	D10	D10−D1
Decentralized	73	50	0.68	96.7	76.0	0.79	0.9%	19.8%	+19.0 pp	8.7%	29.0%	+20.3 pp
Centralized	95	1	0.01	126.7	43.1	0.34	5.2%	65.8%	+60.6 pp	11.6%	68.5%	+56.9 pp

Note: τ denotes the number of quarters until the first default, with $\tau = 1$ indicating immediate default. For each economy, market-access observations from a 50,000-quarter ergodic simulation are sorted by initial debt into ten equally sized groups. $D1$ and $D10$ represent the 10% least and most indebted states, respectively. From each group, 5,000 independent paths are simulated, drawing initial debt from the corresponding group and initial income from its stationary distribution. Paths without default are excluded from mean and median calculations but included in the denominators of default probabilities. $P(\tau \leq 10)$ and $P(\tau \leq 20)$ measure default probabilities within 2.5 and 5 years, respectively. $D10/D1$ reports the ratio of default times, while $D10 - D1$ reports differences in default probabilities (percentage points). Each economy uses its own calibration and debt distribution.

The results for the decentralized economy are consistent with a slow-moving, debt-driven crisis mechanism. Higher initial debt shifts the distribution of default times toward earlier defaults, as illustrated in Figure 11a, but this process remains gradual. Even among the most indebted states (D10), only 19.8% of paths default within ten quarters and 29.0% within twenty quarters, compared with 65.8% and 68.5%, respectively, in the centralized economy. Thus, rapid defaults are substantially more frequent in the centralized economy, despite the stronger role of debt accumulation in the decentralized one. Moreover, the distributions in Figure 11a retain substantial mass at longer horizons across debt groups, suggesting that default in the decentralized economy is generally preceded by a prolonged deterioration rather than an immediate collapse. Together with the gradual debt buildup documented above, these findings support the interpretation of decentralized crises as slow-moving processes in which rising indebtedness progressively increases financial vulnerability.

The centralized economy presents a different pattern. At first glance, the sharp decline in default times across debt groups appears difficult to reconcile with our earlier finding that crises in this economy are primarily income-driven rather than debt-driven. However, these results capture different mechanisms. As shown in Figure 9b, the planner internalizes the effect of borrowing on bond prices and strongly limits debt accumulation, making high debt levels relatively uncommon in the ergodic distribution. Conditional on reaching these highly indebted states, default risk can nevertheless be very high, particularly when initial income is drawn independently of debt, as in Table 6. Figure 11 provides a complementary perspective by preserving the observed joint distribution of debt and income, which substantially reduces the apparent sensitivity of default times to initial debt. In the



(a) Decentralized

(b) Centralized

Figure 11: Distribution of the time to default by initial debt

Note: same debt sets as in Table 6 (D1: the 10% least indebted market-access states of each economy's ergodic simulation; D10: the 10% most indebted), but each of the 5,000 paths per set starts from an observed pair (B_0, y_0) drawn at random from the set, so that initial debt and income are jointly distributed as in the ergodic simulation. τ is the number of quarters until the first default declaration ($\tau = 1$: immediate default). For each set, the vertical axis shows the share of its 5,000 paths whose τ falls in each bin, divided by the bin width (a density per quarter). Bins are 10 quarters wide in panel (a) and 12 quarters wide in panel (b), up to the 99th percentile of each economy's pooled distribution.

figure, each curve parallel to the τ axis is the distribution of the time to default for one debt set. Thus, a strong conditional relationship between debt and default risk does not imply that debt accumulation is the main mechanism generating crises in equilibrium. In the centralized economy, adverse income realizations remain the primary trigger of default, while the planner's borrowing decisions prevent the persistent debt buildup that characterizes the decentralized economy.

7 Concluding remarks

Our theoretical and quantitative analysis reveals that private external debt accumulation generates pecuniary externalities, leading to systematic overborrowing and potentially explosive debt dynamics in the presence of policy-induced default with incomplete markets. Here, ex-post, the default option can serve as a stabilization mechanism in economies.

In this context, centralized default acts by resetting the debt cycle and removing unsustainable trajectories. The two-period model illustrates the core mechanisms behind this result: the inability

of private agents to internalize the impact of their borrowing on interest rates, the resulting instability, and the welfare-improving role of early default. Extending these insights to an infinite-horizon setting, we show that centralized default ensures global stochastic stability and ergodicity, creating recurrent debt cycles that maintain the economy within a compact state space.

Our quantitative exercise, calibrated to Argentina, underscores the empirical relevance of these findings. The analysis shows the role of the pecuniary externality in the dynamics of debt and the generation of a slow moving debt crisis that is absent in centralized debt models with default. Particularly, we document the slower debt build-up, accompanied by interest rate increase and its subsequent vulnerability. In contrast to the centralized debt economy, a slow moving debt crisis is a crisis triggered by a slow accumulation of debt and increasing interest rates rather than an income driven crisis.

Our findings suggest that, in the absence of effective macroprudential regulation that internalizes the pecuniary externality of sovereign risk, private debt cycles in emerging markets will remain inherently fragile and subject to forced corrections through policy-induced defaults.

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Appendix

We will show each result separately. First, we will present some preliminary comments regarding Section 4. Then, we will prove the results, incorporating additional concepts to ensure the paper remains self-contained.

Proofs for section 5.2

Before proving the results, we add some details regarding the Euler equation (9). Typically, in default models with public debt, the differentiability of the value function requires additional assumptions (see Clausen and Strub (2020)). Fortunately, in the presence of private debt, the differentiability of the value function follows from standard results. Critically, the price-taking assumption, which is essential for the presence of pecuniary externalities, implies that individual decisions cannot affect the default set, as otherwise they will influence prices. Then, differentiability is immediate. After stating those remarks carefully, we prove lemmas 1 and 2. The reader familiar with the discussion on the differentiability of the value function in default models can skip the Preliminary Remarks and proceed directly to the proofs.

Preliminary remarks

The purpose of this subsection is to carefully derive the Euler equation (9), which is the same used by Kim and Zhang (2012). As regards this equation, there are at least to major difference with respect to the centralized default literature (see, for instance, Arellano (2008)). First, the agent that issues debt / purchases assets does not internalize her portfolio decisions on market prices. This was already noted by Kim and Zhang (2012). Second, the Government chooses to default or repay but does not issue debt. Then, to compute the Euler equation, as we don't assume partial default, we must take the derivative with respect to the individual state b_+, not the aggregate state B_+, which only affects the value of repaying $V^c(b, B, y; h)$, not the option value $V(b, B, y; h) = max\{V^c(b, B, y; h), V^{def}(y)\}$. Thus, we can use the standard envelope theorem as we take the derivative with respect to b on $V^c(b, B, y; h)$ and then forward this expression 1 period. To take care of the interaction between default the the marginal value repaying, we use the fact

that the interest rate depends only on the aggregate state. Thus, the discontinuities happened at equilibrium (or aggregate) level.

For exposition purposes, we first present the relevant value functions. The value for the household of the repaying option for the Government is given by:

$$V^c(b, B, y; h) = \max_{b_+ \geq -\bar{b}} u(F(b, B, y) - b_+) + \beta \mathbb{E}[V(b_+, h(B, y), y'; h)]. \tag{18}$$

Where $F(b, B, y) = y + bR(B)$ and h is the aggregate law of motion for assets B_+ in equation (16). Note that, due to the i.i.d. structure in assumption 2, the interest rate is not a function of the exogenous shocks. However, even if we allow for Markov shocks, they will imply that the interest rate relevant for b_+ is a function of y, not y_+. This is clear from equation (15). Intuitively, because there are incomplete markets, the return on net assets, b_+, must be independent of y_+. Moreover, the interaction between default and private welfare is captured by the derivative of $V^c(b_+, B_+, y_+; h)$ with respect to b_+. The interest rate only enters into the Euler equation after replacing this derivative using the envelope theorem.

We need to address the differentiability of the value function, which involves equilibrium and off-equilibrium paths. We will say that h^n is the nth iteration of the operator used to prove the existence of an equilibrium. We derive this operator in a separate paper. Formally, the general value (i.e., with Markov shocks and outside the recursive equilibrium) of the interest rate affected by default risk is given by:

$$\begin{aligned} &R(B', y) \equiv R^*(B, y; h^n) = \\ &R^* \left[\sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y', y) \mathbb{I} \left\{ V^c(b_+(b, B, y; h^n), h^n(B, y), y') > V^{def}(y') \right\} \right]^{-1}. \end{aligned} \tag{19}$$

Note that outside the recursive equilibrium, during iterations, we have $b_+(b, B, y; h^n) \neq h^n(B, y)$ but in a recursive equilibrium we get by definition $b_+(b, B, y; h^*) = h^*(B, y)$. As we are modeling a competitive equilibrium, individual states do affect the interest rate. Thus, marginal changes in b_+ must not affect equation (19). To enforce this fact, we must set a strict inequality inside the

indicator function (i.e., when the Government is indifferent, defaults). Now, using this notation, we rewrite the right-hand side of equation (9):

$$\begin{aligned}
&\mathbb{E}_y[V(b_+(b,B,y;h^n),h^n(B,y),y';h^n)] = \sum_{y'\in[Y_{LB},Y_{UB}]} \pi(y',y) \\
&[\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),h^n(B,y),y';h^n) > V^{def}(y')\right\} V^c(b_+(b,B,y;h^n),h^n(B,y),y';h^n) \\
&\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),h^n(B,y),y';h^n) \leq V^{def}(y')\right\} V^{def}(y')].
\end{aligned} \tag{20}$$

Note that V^{def} and the indicator functions $\mathbb{I}$ in Equation (20) do not change with b_+. The latter follows from the absence of partial default, and the latter from the price-taking assumption, because individual decisions cannot affect the Government's default choice, as otherwise they will affect the equilibrium interest rate. This is not the case in models with public debt and centralized default as in Arellano (2008). Then, we can rewrite Equation (9) as:

$$\begin{aligned}
&u'\left(F(b,B,y) - b_+(b,B,y;h^n)\right) \geq \\
&R(B,y;h^n) \sum_{y'\in[Y_{LB},Y_{UB}]} \pi(y',y)\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),h^n(B,y),y') > V^{def}(y')\right\} \\
&u'\left(F(b_+(b,B,y;h^n),h^n(B,y),y') - b_+(b_+(b,B,y;h^n),h^n(B,y),y';h^n)\right),
\end{aligned} \tag{21}$$

where the second and the third lines capture the interaction between the default and the marginal utility of consumption. That is, even though there are incomplete markets and the interest rate can be factored out of the expectation in the Euler equation (i.e., it does not depend on y' even outside the recursive equilibrium), the number of future exogenous states in which the Government repays affects consumption decisions today through the Euler equation. It can be shown that $R(B,y;h^n)$ is decreasing in B for all h^n, provided that h^0 is appropriately chosen such that an equilibrium exists. We defer all technical details related to the existence proof of a stationary equilibrium to a separate paper. The monotonicity of R for admissible equilibrium laws of motion h^n is essential to show the monotonicity of individual policy functions, b_+ and c, which in turn will help us to show the existence of a stable steady state.

Proofs

To prove Lemma 1, we need an additional assumption that is standard in the literature: the Government will never default on its assets. Following Aguiar and Amador (2019) (see Assumption 4 in that paper), we impose this restriction.

Proof of Lemma 1. Any sequence of consumption is valued by $U = \sum \beta^t u(c_t(\omega))\mu_{y_0}(\omega)$ and u bounded above and unbounded below. Under these assumptions, it is standard to show (see Duffie et al. (1994) page 765) that any utility maximizing sequence $\hat{c}_t(\omega) > \underline{c}$ with $\underline{c} > 0$ almost everywhere in Ω. This fact implies that marginal utility is uniformly bounded above. Under assumption 2 of this paper, Assumption 4 in Aguiar and Amador (2019) and Lemma 1 in Braido (2013) implies that there exists $\rho \in (0,1)$ with: $b_+ \leq R^*/(1-\rho)$ almost everywhere in Ω, where R^* is the risk-free gross rate. The lower bound on b follows from the restrictions on problem 8. Thus, $b_+ \in [B_{LB}, B_{UB}]$. Because there is a uniform lower bound in consumption, $c \geq C_{LB} \equiv \underline{c}$.

Given these results, it is easy to show that $R(B)$ is bounded above. Suppose not. We will deal only with the case $b = B$ (i.e., only equilibrium candidates) and $R(B) = +\infty$ when $B = b < 0$. Under Assumption 4 in Aguiar and Amador (2019), all default thresholds are negative. Then, $R(B) = +\infty$ when $B = b > 0$ is not possible, as the Government would never choose to default with assets. Then, equations (19) and (21) implies that $u'(c(y,b,B;h)) \geq 0$, which satisfies $c, c_+ \geq \underline{c}$. If $R(B)$ is unbounded and if the expected value of the marginal utility of future consumption is finite, then consumption today must be zero, which is a contradiction as consumption is uniformly bounded away from zero. Then, as $c_+ \geq \underline{c}$, the right-hand side of the Euler equation (21) is equal to 0 as the marginal utility of future consumption is zero. As we must have $c = F(b,B,y) - b_+ \geq \underline{c} > 0$ and $F(b,B,y) = bR(B) + y$, b_+ must be unbounded below, which contradicts the uniform lower bound on b_+ in equation (18) (i.e., $b_+ \geq -\bar{b}$). The lower bound on $R(B)$ is given by R^*, which is standard under risk-neutral pricing. Thus, $R(B) \in [R_{LB}, R_{UB}]$. Finally, the upper bound on c is given by: $Y_{UB} + R_{UB}B_{UB} - B_{LB}$. Thus, $c \in [C_{LB}, C_{UB}]$

□

Proof of Lemma 2. We begin by defining an appropriate space of functions for c and h. Let C be the space of candidate functions for h. As in in Coleman (1991), we require:

$$C(\mathbb{B}\times Y)=\left\{\begin{array}{c} 0\leq C(B,y)\leq F(B,y) \\ 0\leq C(B',y)-C(B,y)\leq F(B',y)-F(B,y)\ if\ B'\geq B \end{array}\right\} \tag{22}$$

Where $F(B,y)=y+R(B)B$ with $B\in[B_{LB},B_{UB}]\equiv\mathbb{B}$ from lemma 1. Further, equation (22) implies that both c and b_+ are (weakly) increasing in B for each $y\in Y$. We will now define an operator on $C(\mathbb{B}\times Y)$, A, and we will show that $Ac\in C(\mathbb{B}\times Y)$. This is sufficient to demonstrate that h^n is increasing for all n. This is the first step to derive an existence proof, which we present in a separate paper. The equation below defines operator A, implicitly from the Euler Equation:

$$u'(Ac(B,y)))=\beta E\left[u'\left(c\left(F(B,y)-Ac(B,y),y'\right)\right)R\left(F(B,y)-Ac(B,y),y'\right)\right], \tag{23}$$

where Ac defined the *Coleman-Reffett* operator and it may not be equal to c. Equation (23) simply defines the candidates for the fixed point $Ac=c$.

To show the first part of Lemma 2, we must prove that A maps $C(\mathbb{B}\times Y)$ into itself.

Take $c\in C(\mathbb{B}\times Y)$. Let $B'(B,y)=y+R(B)B-Ac(B,y)$. Thus, for any $\hat{c},\tilde{c}\in C(\mathbb{B}\times Y)$, with $\hat{c}\leq\tilde{c}$, we must show that $A\hat{c}\leq A\tilde{c}$ and $\hat{B}'\leq\tilde{B}'$. In order to do so, notice that:

$$\begin{array}{c} u'(A\hat{c}(B,y)))=\beta E\left[u'\left(\hat{c}\left(\hat{B}',y'\right)\right)R\left(\hat{B}',y'\right)\right]\geq\beta E\left[u'\left(\tilde{c}\left(\hat{B}',y'\right)\right)R\left(\hat{B}',y'\right)\right]\geq \\ \beta E\left[u'\left(\tilde{c}\left(\tilde{B}',y'\right)\right)R\left(\hat{B}',y'\right)\right]\geq\beta E\left[u'\left(\tilde{c}\left(\tilde{B}',y'\right)\right)R\left(\tilde{B}',y'\right)\right]=u'(A\tilde{c}(B,y))), \end{array}$$

where the first inequality follows from $\hat{c}\leq\tilde{c}$, the second from the optimality of consumption, and the third because R is decreasing in B as any candidate function for consumption is increasing in B. Note that the last inequality implies $\hat{B}'\leq\tilde{B}'$ and the first and the last terms together imply $A\hat{c}\leq A\tilde{c}$ as desired. Thus, $AC(\mathbb{B}\times Y)\subseteq C(\mathbb{B}\times Y)$, which in turn implies that h^n, $b_+(b,B,y;h^n)$ and $c(b,B,y;h^n)$ are increasing for any $(B,y)\in\mathbb{B}\times Y$ with $b=B$ and any $n=0,1,....$

It remains to show that either c or b_+ is strictly increasing. Suppose not. Then, for some $y\in Y$ and $\tilde{B},B\in\mathbb{B}$, with $\tilde{B}>B$, we have $b_+(B,y)=b_+(\tilde{B},y)$ and $c(B,y)=c(\tilde{B},y)$. From equation (8) we know that:

$$V(B,y)=u(c(B,y))+\beta E[V(b_+(B,y),y')]\text{ with }c(B,y)+b_+(B,y)=y+R(B)B$$

Suppose that $B > 0$. This is without loss of generality as, from lemma 1, $B_{UB} > 0$. Thus, $c(\tilde{B}, y) + b_+(\tilde{B}, y) < y + R(\tilde{B})\tilde{B}$. This inequality implies that there is a basket $\tilde{c}(\tilde{B}, y) > c(\tilde{B}, y)$ which is also feasible and:

$$u(\tilde{c}(\tilde{B}, y)) + \beta E[V(b_+(\tilde{B}, y), y')] > V(\tilde{B}, y)$$

The strict inequality implies a contradiction and it follows that $b_+(B, y) = b_+(\tilde{B}, y)$ *or* $c(B, y) = c(\tilde{B}, y)$ but not both. As y and $\tilde{B}, B$ are arbitrary, we can extend the result for any $y \in Y$ and any strictly ordered pair $\tilde{B}, B \in \mathbb{B}$. □

To show the stability of the equilibrium, we prove that there is an ergodic invariant measure characterizing the steady state of the model. Ergodicity requires recurrence and connectedness (i.e., irreducibility). From Meyn and Tweedie (1993), a discontinuous Markov process has an invariant measure if it has a regeneration point called *atom*. The atom is a point that the process hits with positive probability, starting from any initial condition in an appropriate state space. In our case, it is the default state $(0, y^{def})$, which resets the leveraging cycle and creates an orbit. As hits to the default threshold happen in a finite time, the orbit has a finite radius, generating recurrent sets. As the process is monotonic and does not diverge to infinity due to the finiteness of the orbits, it is stable as it never leaves the state space, which at the same time is the support of the invariant measure. Under standard results, after demonstrating recurrence, we can prove the connectedness of the process, thereby obtaining joint ergodicity and global stability.

To prove the existence of an invariant measure, thus, we first need to define the equilibrium state space of the Markov process. We use Definition 2 to describe Z_1 and Z.

Equation (17) implies that we can construct Z using φ. We will assume the existence of a stationary equilibrium, which implies that there is at least one measure function φ. We present the conditions for the uniqueness of this equilibrium in a separate, more technical paper. Thus, given an element in Z_1 and $y_+ \in Y$ we can find at most 1 vector $(c, R, B_+, y_+, c_+, R_+)$ associated with it. That is, iterating this procedure, it is possible to construct a *finite time path from the Stationary Equilibrium* using φ. Using these paths, we will demonstrate that this equilibrium is also ergodic, despite being discontinuous.

Let us start by formally defining an "accessible atom", which can be thought as a point that is non-negligible from a probabilistic perspective and gets "hit" frequently. Let $P_{\varphi}^{n}(z, A)$ be the probability that the Markov chain goes from z to any point in A in n steps with A being measurable, let ψ be some measure, and $B(Z)$ be the Borel sigma algebra generated by Z. Then the set $A \in B(Z)$ is *non-negligible* if $\psi(A) > 0$. A chain is called *irreducible* if, starting from any initial condition, the chain hits all non-negligible sets with positive probability in finite time (i.e. $\psi(A) > 0 \rightarrow P_{\varphi}^{n}(z, A) > 0$.) Intuitively, irreducibility is a notion of connectedness for the Markov process as it implies non-negligible sets are visited with positive probability in finite time.

We are now in position to define an atom and state an important intermediate result.

Definition 3 (Accessible Atom)**.** *A set $\alpha \in B(Z)$ is an atom for (Z, P_{φ}) if there exists a probability measure μ such that $P_{\varphi}(z, A) = \mu(A)$ with $z \in \alpha$ for all $A \in B(Z)$. The atom is accessible if $\psi(\alpha) > 0$.*

Intuitively an atom is a set containing points in which the chain behave like an i.i.d. process. Any singleton $\{\alpha\}$ is an atom. Note that there is a trade off: if the atom is a singleton, the i.i.d. requirement is trivial but, taking into account that the state space is uncountable, the accessibility clause becomes an issue as it is not clear how to choose ψ. The same happens with irreducibility: when the state space is finite, it suffices to ask for a transition matrix with positive values in all its positions. In the general case, we need to define carefully what is a meaningful set. Fortunately, when the state space Z is a product space between a finite set (Y) and an uncountable subset of [3], containing (B, c, R), there is a well know results that help us find an accessible atom in an irreducible chain (for proof, see Proposition 5.1.1 in Meyn and Tweedie (1993).)

Lemma 3 (Irreducibility and accesible atoms)**.** *Suppose that $P_{\varphi}^{n}(z, \alpha) > 0$ for all $z \in Z$. Then α is an accessible atom and (Z, P_{φ}) is a $P_{\varphi}(\alpha, .)-$irreducible.*

Proposition 3 follows directly from standard results in Meyn and Tweedie (1993) [14]. Note the relevance of the atom, $\alpha = z_* = (0, y_{LB}, y^{def}, R^*)$ for the stochastic stability of the process: we

[14] If in In proposition 5.1.1 we assume that the atom is a singleton, we still have to deal with the reference measure ψ. Typically, ψ is set to be the "maximal" measure. Fortunately, if the chain is irreducible with respect to some measure, say $P_{\varphi}(\alpha, .)$, then it can be "expanded" to ψ (e.g, see Meyn and Tweedie (1993), Proposition 4.2.2)

define a meaningful set to be the one that can be hit by the chain starting from it. In this sense, it is similar to a saddle path point in a phase diagram in non-stochastic models where endogenous variables can only take 1 initial condition that leads to convergence to the steady state state.

To apply Proposition 3, the finiteness of Y in assumption 1 and the definition of the Markov kernel P_φ in equation (17) are essential. As we are considering a point, in order to show that $P_\varphi^\tau(z, \{z_*\}) > 0$, it suffices to find a finite sequence $\{y_0, ..., y_\tau\}$ such that the economy defaults when $y_\tau = y_{LB}$.

The effect of an atom in the recurrence structure of the chain is essential to define an invariant measure (i.e. a measure μ which satisfy $\mu = \int P_\varphi(z, A)\mu(dz)$). Suppose that the atom is hit for the first time with positive probability in period $\tau_{z_*} < \infty$ starting from z_0. Then, it is possible to define a (not necessarily probability) measure μ which gives the expected number of visits to a particular set in $B(Z)$, called it A, before τ_{z_*}. Then $\mu(A)$ gives the sum of the probabilities of hitting A *avoiding* the atom. In period $\tau_{z_*} - 1$ when "forward" μ 1 period (i.e. by applying the Markov operator to it, $\int P_\varphi(z, A)\mu(dz)$) the expected number of visits to A avoiding the atom is the same as the chain will hit z_* in period $t = \tau_\alpha$. Thus, μ must not change or equivalently $\mu = \int P_\varphi(z, A)\mu(dz)$. That is, μ is an invariant measure. Provided that $\tau_{z_*} < \infty$, it is possible to normalize μ to be a probability measure. Further, as the accessibility of the atom comes together with the irreducibility of the chain (see lemma 3), the invariant measure is unique as the chain does not break into different "unconnected islands". Finally, the Krein-Milman theorem guarantees the ergodicity of the chain provided its uniqueness (see Futia (1982)).

We first show that (Z, P_φ) satisfy the conditions of proposition 3.

Lemma 4 (Accessible atom in the default model)**.** *Let the atom be $z_* = (0, y_{LB}, y^{def}, R^*)$. Then, under assumptions 1, 2 and 3, for any $(B_0, y_0) \in Z_1$, $P_\varphi^{\tau(B_0,y_0)}(z, \{z_*\}) > 0$ and $\tau(B_0, y_0) < \infty$.*

Proof. To show that $B^N(y_{LB}) < \bar{B}(y^{def}) < 0$ note that under assumption 3, $B^N(y_{LB}) < 0$. Then, equation 14 and lemma 2, implies that there is at least 1 y^{def} with such a property.

We now show that starting from any initial condition, the chain hits the atom. We will first show that for any $y \neq y_{LB}$ and any B_0, there is a positive probability path $\{y_0, y_1, ..., y_\tau\} = \{y_0, y_{LB}..., y_{LB}\}$, and an associated sequence $z(y^t)$ for which the economy defaults when $y_\tau = y_{LB}$.

If $b_{+,*}(B_0, y_0) < B_0$ using lemma 2 we know that: $b_{+,*}(B_0, y_{LB}) < b_{+,*}(B_0, y_0) = B_1$. Then, $B_2 = b_{+,*}(B_1, y_{LB}) < b_{+,*}(B_0, y_{LB}) < B_1$. Then, $B_3 = b_{+,*}(B_2, y_{LB}) < b_{+,*}(B_1, y_{LB})$. Continuing with this logic, as $\bar{B}(y^{def})$ is finite, $B_{\tau+1} < \bar{B}(y^{def})$ as the chain would have converged to $B^N(y_{LB})$ in the absence of default. By the definition of h, then the planner chooses to default at period τ, which in turn implies that the economy hits the atom in this time period.

If $b_{+,*}(B_0, y_0) \geq B_0$, choose $y_t = y_0$ until $B^{y_0} = b_{+,*}(B^{y_0}, y_0)$. Because of remark **??**, we know that this point exist for every $y \in Y$ and is finite. Thus, the chain hits B^{y_0} in finite time. Call this period $s = t$ and thus $B^{y_0} = B_{s+1}$. Choose $y_{s+1} = y_{LB}$. Then, $B_{s+2} = b_{+,*}(B_{s+1}, y_{LB}) < B_{s+1}$ and $B_{s+3} = b_{+,*}(B_{s+2}, y_{LB}) < b_{+,*}(B_{s+1}, y_{LB}) = B_{s+2}$. Continuing with this logic, the chain will hit $\bar{B}(y^{def})$ and thus the atom in finite time.

If $y_0 = y_{LB}$, choose $y_1 = y$ with $y \neq y_{LB}$ and repeat the previous reasoning. □

Now using lemma 4, we show that the chain has a unique invariant measure.

Proof of theorem 1. Note that lemma 4 imply that $P^{\tau}_{\varphi}(z_*, \{z_*\}) > 0$ with $\tau < \infty$. The results in Remark 4.2.1, proposition 4.2.2, theorem 8.2.1 and theorem 10.2.1 in Meyn and Tweedie (1993) imply that (Z, P_{φ}) has an unique invariant measure. As $\tau < \infty$ for any initial condition in Z, theorem 10.2.2 in Meyn and Tweedie (1993) implies that the invariant measure is a probability measure. As it is unique, the Krein-Milman theorem (See Futia (1982)) implies that this measure is ergodic.

□

Supplementary Appendix

Supplementary Appendix for Section 3

The equilibrium, as characterized in the two-period model without internalization, is obtained as follows.

Algebraic details

Operating on the pricing function

$$\ln(R^*) - \ln(R(B_1)) = -\lambda y^d - \lambda b_1 R(B_1)$$

$$r^* - r(B_1) = -\lambda y^d - \lambda B_1 R(B_1)$$

$$r(B_1) = \frac{r^* + \lambda y^d + \lambda B_1}{1 - \lambda B_1}$$

Operating the FOC

$$\alpha e^{-\alpha(\omega+b_1)} = \beta R(B_1)\alpha \int_R e^{-(\lambda+\alpha)y+\alpha b_1 R(B_1)} dy$$

we can factorize the exponent and solve the integral (for instance, using Barrow's method):

$$\alpha e^{-\alpha(\omega+b_1)} = \beta R(B_1)\alpha e^{(\alpha b_1 R(B_1))}\lambda \int_R e^{(-(\lambda+\alpha)y)} dy$$

Note that the Barrow method is such that

$$\int_A^K e^{-(\lambda+\alpha)y} dy = \frac{-\lambda}{\lambda+\alpha} e^{-(\lambda+\alpha)K} - \frac{-\lambda}{\lambda+\alpha} e^{-(\lambda+\alpha)A}$$

but as here the upper limit is ∞,

$$\lambda \int_A^\infty e^{-(\lambda+\alpha)y} dy = \frac{\lambda}{\lambda+\alpha} e^{-(\lambda+\alpha)(y^d+b_1 R(B_1))}$$

Then

$$\alpha e^{-\alpha(\omega+b_1)} = \beta R(B_1)\alpha e^{(\alpha b_1 R(B_1))} \frac{\lambda}{\lambda+\alpha} e^{-(\lambda+\alpha)(y^d+b_1 R(B_1))}$$

Simplify

$$e^{-\alpha(\omega+b_1)} = \frac{\lambda\beta R(B_1)}{\lambda+\alpha} e^{-(\lambda+\alpha)y^d - \lambda b_1 R(B_1)}$$

$$e^{(\lambda+\alpha)y^d - \alpha\omega} = \frac{\lambda\beta R(B_1)}{\lambda+\alpha} e^{\alpha B_1 - \lambda B_1 R(B_1)}$$

Taking logs

$$(\lambda+\alpha)y^d - \alpha\omega = \ln\left(\frac{\lambda\beta}{\lambda+\alpha}\right) + \ln(R(B_1)) + B_1(\alpha - \lambda R(B_1))$$

$$(\lambda+\alpha)y^d - \alpha\omega = \ln\left(\frac{\lambda\beta}{\lambda+\alpha}\right) + r(B_1) + B_1(\alpha - \lambda - \lambda r(B_1))$$

Supplementary Appendix for Section 4.1

Lemma 5 (Unstable paths)**.** *Under assumptions 1 and 2, if $R(B)$ is decreasing in B, then there exist some $\hat{B} < 0$ such that for any $y \in Y$ with $-\hat{B} > y$ and $\beta R(\hat{B}) > 1$, $B \leq \hat{B}$ implies that $b_+(B, y)$ converges to $\bar{B}$ for any weakly decreasing path $[y\downarrow, ..., y_T]$.*

Proof of Lemma 5. Under the assumptions of these lemma, (9) implies $u(c) > E[u'(c_+)]$, where the dependence on $b, B, y; h$ was omitted for simplicity. Then, as $-\hat{B} > y$ and consumption is uniformly bounded away from zero because of lemma 1, $b_+ - \hat{B} < y + (R(\hat{B}) - 1)\hat{B} < 0$. Equivalently, $b_+(\hat{B}, y) < \hat{B}$. Because of 2, b_+ is increasing in B and by assumption R is decreasing in B. Thus, $\beta R(b_+(b_+(\hat{B}, y), y)) > \beta R(b_+(\hat{B}, y)) > 1$. As problem (8) is a standard savings problem and y follows a weakly decreasing path, we know that $u(c_+) > E[u'(c_{++})]$. Taking expectations on the second inequality and using the first we get $u(c) > E[u'(c_{++})]$. Continue with these logic and noting that u' is bounded below by zero, we get $lim_{T\to\infty} E\left[u'(c_T)\right] = 0$ (A1). Because of lemma 1, we know that $c_T + b_{T+1} \leq Y_{UB} + R_{UB}B_{UB}$, which then implies that $b_{T+1} \to \bar{B}$. As y and $\hat{B}$ were arbitrary and A1 was obtained after taking expectations for every period, the convergence is in finite time, which in turn implies that the weakly decreasing path has positive probability.

□

The intuition for lemma 5 follows from the conditions $-\hat{B} > y$, a debt to GDP ratio bigger than 100%, and $\beta R(\hat{B}) > 1$, a sufficiently high interest rate. By noting that a weakly decreasing path represents a persistent recession, we can say that sufficiently high debt levels coupled with a poor growth prospect lead to a default. Depending on the level of GDP during default, $y^{def}(y)$, its welfare effects can vary significantly. We will see the values of $y^{def}(y)$ are critical to show the existence of equilibrium and to generate an ergodic representation. Notice that there is a clear connection between $\hat{B}$, $\bar{B}$ and the type of recession required to induce default. Clearly, $\bar{B}(y)/y$ for any $y \in Y$ defines an upper bound for the debt to GDP ratio. If in any period t, $\hat{B}/y_t$ is close to $\bar{B}(y_t)/y_t$, then it takes a short and mild recession to cause a default.