Overborrowing and instability under default risk*

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Abstract

We examine policy-induced defaults—government actions that trigger widespread private external debt default—and argue they generate overborrowing. We develop a model of decentralized private external borrowing under sovereign default risk, showing that pecuniary externalities lead to overborrowing and unstable dynamics. Centralized default resets the debt cycle, globally stabilizing the economy. A quantitative exercise calibrated to Argentina reveals that a 100-basis-point increase in the international risk-free rate more than triples debt service and nearly doubles the default probability. These findings highlight the ex-ante destabilizing impact of default risk, as well as the stabilizing role of policy-induced default in emerging markets.

Keywords: Default; Private external debt; Overborrowing.

JEL Codes: F41, E61, E10, C02

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1 Introduction

This paper investigates the destabilizing effects of private external debt when it is subject to default risk in small open economies with incomplete markets. We consider a setting in which optimizing private borrowers fail to internalize the impact of their borrowing decisions on the equilibrium interest rate, resulting in overborrowing relative to the debt levels a constrained social planner would choose under default risk. In this framework, centralized default by a benevolent government serves as a stabilizing device, effectively resetting the debt cycle when the economy becomes unsustainable due to cumulative pecuniary externalities. Ex-ante, the possibility of Government intervention destabilizes the decentralized economy, forcing authorities to take drastic measures ex-post. Our findings offer a strong economic interpretation: building on the insights of Zame (1993), where default acts as a market-completing mechanism, we demonstrate that default can also function as a macroeconomic stabilizer in environments where private external borrowing would otherwise lead to unstable and unsustainable debt trajectories.

We illustrate the core contribution of the paper through a stylized, two-period general equilibrium model with a closed-form solution, featuring private borrowing and centralized default. This framework allows us to highlight three key mechanisms: (i) the overborrowing that arises when private agents fail to internalize the impact of debt issuance on interest rates, (ii) the resulting destabilizing effects of private debt accumulation under policy-induced default risk, and (iii) the stabilizing role of sovereign default as a corrective device. Building on this intuition, we develop a quantitative infinite-horizon model of a small, open-endowment economy populated by atomistic households, a government, and competitive, risk-neutral international investors. In our setting, households issue foreign debt, while the government retains the ability to coordinate default decisions. Thus, borrowing is decentralized—similar to the framework in Bianchi (2011)—but instead of relying on a collateral constraint to introduce market frictions, we assume that private external debt is defaultable and that default is centralized. In Bianchi's model, crises are triggered by shocks that bind a price-dependent inequality constraint, leading to abrupt deleveraging. In contrast, our model features default as the mechanism that initiates deleveraging, thereby stabilizing the economy when debt becomes unsustainable.

In indebted countries, we demonstrate that pecuniary externalities generated by default risk induce an overborrowing loop that, regardless of the level of GDP, destabilizes the economy in the absence of default. We illustrate this result using a stochastic phase diagram, a novel graphical device which extends the results in Azariadis and Lambertini (2003). The critical difference between this paper and ours is the steady state. As we work in a stochastic environment, the steady state is different with respect to Azariadis and Lambertini (2003), and thus the notion of required stability is always global. Moreover, we formally confirm the intuitions behind the two-period model and the phase diagram by showing the ergodicity of the stochastic steady state, a property that induces a suitable form of global stability for this class of models.

We calibrate the model to match the private defaults in Argentina in 2001. Our findings imply a quantitatively strong result. A 100 basis point increase in the international risk-free rate exacerbates private overborrowing, more than triples yearly capital debt services and almost doubles the probability of default. Hence, private defaultable debt works as an important transmission channel of systemic shocks in Emerging Markets in the short run.

Table 1: External debt in Argentina and Chile (2021)

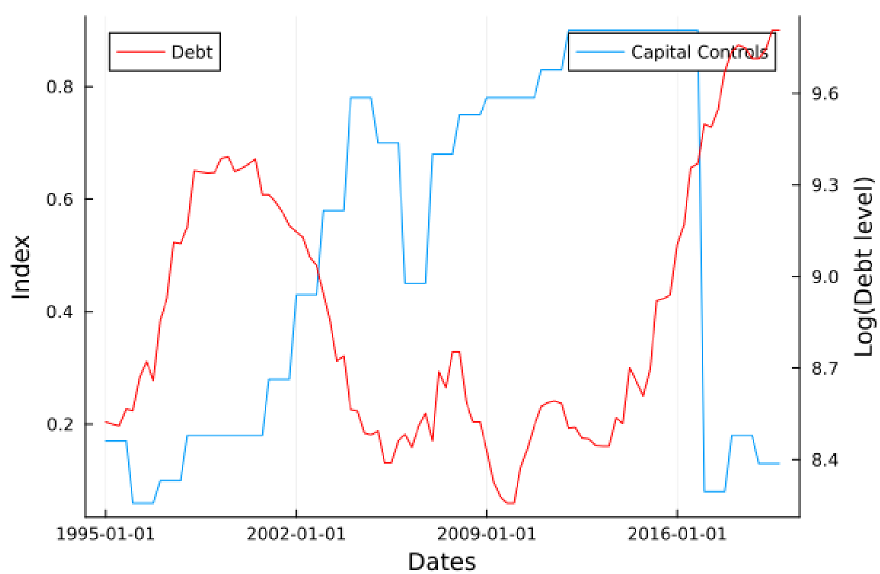
bln of Dollars	$Public_{PV}$ (I)	$Public_{IO}$ (II)	I+II	*Private* (III)	I+III	III/(I+III)
Argentina	87	74	161	73	160	46%
Chile	46	0	46	192	238	81%

Note: "$Public_{PV}$" refers to public external debt with private lenders, "IO" stands for international organisations, "I+ II" is total public debt. *Private* is private external debt. "I+III" is the total debt subject to default, which excludes debt with IO. The last column computes the share of private debt to total defaultable debt. Source: Central Bank of Chile and National Institute of Public Statistics for Argentina.

Empirically, private debt constitutes a significant share of defaultable external debt in most emerging economies. Table 1 presents evidence along this dimension for Argentina and Chile. The table shows that total public debt with private lenders is 87 and 46 billion U$S while private debt is 73 and 192 billion U$S, both respectively. The sum of these items accounts for the defaultable debt (excluding debt with senior lenders, which is virtually non-defaultable). It turns out that 46% for Argentina and 81% for Chile of the total defaultable debt is private. For this reason, we focus on private debt, and we abstract from sovereign debt, which in turn allows us to derive theoretical results that the literature could not address in other frameworks.

The dynamics of private debt accumulation have received comparatively less attention than the dynamics of sovereign debt, partly due to the influence of the Washington Consensus, which posited that rational agents operating in frictionless capital markets would make optimal borrowing decisions. However, empirical evidence suggests that episodes of capital account liberalization are often accompanied by surges in private sector debt. This pattern is suggestive in the Argentine experience since the early 1990s, where periods of liberalization (before 2002 and the period 2015-2019) coincide with an increase in private debt, while the period 2003–2015 is characterized by the reintroduction of capital controls, exchange rate management, and exclusion from international financial markets. This is shown in Figure 1.

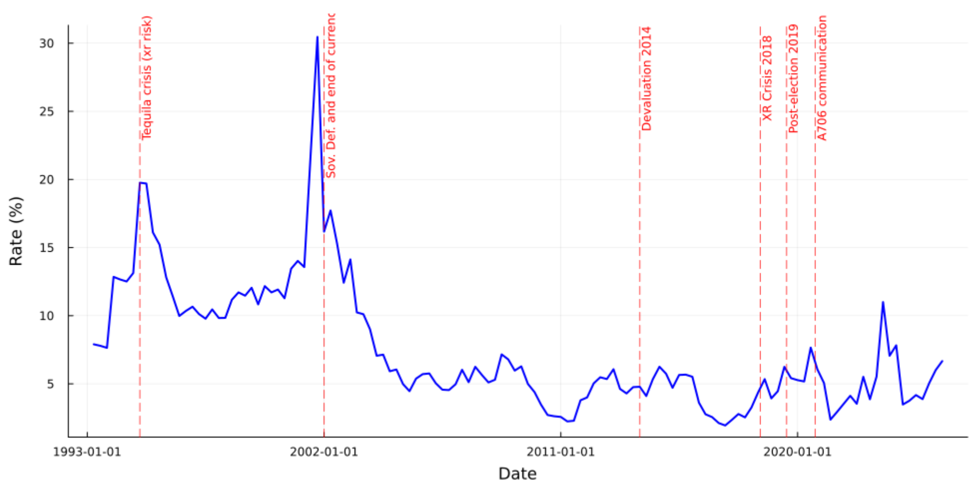
Figure 1: Private sector Debt and capital controls



Note: Private Borrowing is the stock of debt of non-financial corporations in USD from the BIS (IDS:Debt securities / International debt securities (BIS-compiled)) and capital control index from Fernández et al. (2016).

Additionally, considering the centralized default of private external debt is empirically relevant and plausible. Arellano et al. (2016) demonstrates that the government's ability to directly influence private external debt varies across countries. Additionally, policymakers can indirectly influence the private sector's ability to repay its debt through market mechanisms. There are several reasons that account for their significance for the aggregate behavior of the economy. For instance, if the private sector issues debt in foreign currency, a nominal depreciation may imply a de facto massive private default; a case along these lines happened for Argentina in 2001 with the abandonment of the currency peg. On the other hand, we can consider the imposition of capital controls: in 2001, the Argentinean government unilaterally changed the conditions of the private credit market, while in 2021, the Central Bank of Argentina forced firms to renegotiate their external debt obligations, even without passing a bill through Congress. Each of these episodes represents an event where the government induces a default of privately issued external debt. Arellano et al. (2016) highlights that these risks are priced in by Credit Rating institutions such as Moody's or Fitch, as they explicitly mention in their briefings. Figure 2 presents the evolution of corporate interest rates measured in USD together with episodes of large devaluations/depreciations or currency tensions for Argentina since 1993. As the interest rate is in USD, the spikes associated with large devaluations do not respond to the UIP logic; instead, they are related to the fact that tensions in currency markets may affect the repayment opportunities of the private sector.

Figure 2: Private sector lending rates



Note: Private lending rates in USD and large devaluations. Lending rates are USD rates for private lending from the IMF.

Debt instruments play a central role in macroeconomic theory and policy. Traditionally, both private and sovereign debt have been viewed as mechanisms for smoothing consumption and taxation. Foundational contributions such as the Permanent Income Hypothesis underscore the role of private debt in intertemporal consumption smoothing, while Barro (1979) examines sovereign debt as a tool for optimal tax smoothing. In settings with incomplete markets, the international macroeconomics literature has emphasized the destabilizing potential of sovereign debt accumulation. Seminal works such as Eaton and Gersovitz (1981) and Cole and Kehoe (2000) demonstrate how sovereign borrowing can lead to explosive implicit returns and self-fulfilling crises. Nevertheless, despite the empirical relevance of these facts and the importance of private debt, the literature has not studied its destabilizing effects or the role of policy-induced default in depth so far. Hence, our paper fills an important gap.

Our paper connects the literature on sovereign defaults, such as Eaton and Gersovitz (1981), with the results on overborrowing by Bianchi (2011). The closest paper to ours is Kim and Zhang (2012), which develops a quantitative model incorporating private debt and sovereign defaults. However, their analysis is purely quantitative and focuses on the impact of decentralized borrowing. Our approach is also similar to Jeske (2006), which studies international private borrowing with repudiation risk, but in contrast to us, the author considers private borrowing and default in a complete markets environment. In this context, capital controls are analogous to the case of centralized borrowing and default. Our environment also provides a microfounded theory of the Debt Elastic Interest Rate device in Schmitt-Grohé and Uribe (2003), Aguiar and Gopinath (2007), and Garcia-Cicco et al. (2010), that implies a positive stationary relationship between private external debt and interest rates.

We also closely relate to Arellano et al. (2016). The authors compare the US and Euro area experiences and legislation to examine the relationship between sovereign debt crises and external (private) debt crises. They study how the government, under different legislation, can destabilize the economy by inducing a sovereign debt crisis or an external debt crisis. We contribute to this literature by explicitly connecting sovereign default of private debt with the overborrowing result in Bianchi (2011) and the destabilizing role of centralized default risk. In contrast to considering the stabilizing role of capital controls, we analyze the role of policy-induced private default. In this way, we relate to Zame (1993), who highlights the welfare-improving benefit of default by completing markets. In our case, default improves welfare in the presence of overborrowing, as the private sector may end up inducing an external debt crisis that is resolved by the centralized default choice.

The remainder of this paper goes as follows. In Section 2, we present the model. Section 3 describes the main theoretical results. Section 4 contains the quantitative implementation and the main numerical results. Section 5 concludes. All proofs are available in the appendix.

2 The two-period model

This section develops the main ideas of the paper using a two-period model for a small open economy. We present two versions of the model, with and without internalization of the effect that borrowing has on the default choice and the interest rate.

2.1 The case without internalization

Consider a 2-period economy with decentralized private debt and centralized public default. That is, in the presence of private debt, the Government may force the private sector to default. Now, we assume the first case. We assume that the private sector's objective function is

$$u(c_1) + \beta \mathbb{E}_0 u(c_2),$$

where u is a standard, strictly increasing, and concave utility function, over consumption of a unique good at each point in time with discount factor β. The constraints the agent faces are,

$$c_1 = \omega + b_1,$$

and

$$c_2 + R(B_1)b_1 = y,$$

where $b_1 > 0$ denotes the individual debt issued in period 1, that matures in period 2, and B_1 denotes the aggregate debt. ω denotes the first period initial wealth, and y is the endowment in period 2, which is a random variable with a process to be defined later.

The government may choose to default in period 2. This choice is not internalized by the household, who is atomistic. That is, aggregate borrowing B_1, and not private borrowing b_1, determines default risk $R - R^*$, where R^* is the gross risk-free rate. In equilibrium we have that $b_1 = B_1$.

Default happens when the utility in autarchy $u(y^d)$, where y^d is the endowment after default, is greater than or equal to the utility when the government chooses to repay $u(y - b_1 R(B_1))$. As u is strictly increasing, this is equivalent to default when output in period 2 is low enough, that is, lower than the autarky level y^d plus debt services $b_1 R(B)$. As y is a random variable with a c.d.f. given by F, these facts imply in turn that default probability is the probability that the output level is lower than the outside option given by $y^d + b_1 R(B_1)$:

$$P(D) = F(y < y^d + b_1 R(B_1)).$$

The objective function for the household becomes:

$$V^c(\omega) = u(\omega + b_1) + \beta \left[F(y^d + b_1 R(B_1)) u(y^d) + \int_y u(y - b_1 R(B_1)) f(y) dy \right],$$

where the integral is taken over the repayment set, that is, for $y > y^d + b_1 R(B_1)$. The household determines the optimal borrowing taking into account marginal costs and returns, which follows from standard F.O.Cs. with respect to b_1 using the objective function described above.

Lenders are risk-neutral and deep-pocketed investors characterized by a standard profit maximization condition. Then, the equilibrium in this economy is determined by the folloiwng three equations

$$u'(\omega + b_1) = \beta\, R(B_1) \int_y u'\big(y - b_1 R(B_1)\big)\, f(y)\, dy \tag{1}$$

$$R^* = R(B_1) \left[1 - F\big(y^d + b_1 R(B_1)\big) \right] \tag{2}$$

$$b_1 = B_1 \tag{3}$$

Functional forms

Assume that the utility follows an exponential function given by $u(c) = -e^{(-\alpha c)}$, where $\alpha > 0$, and assume that income follows an exponential distribution $f(y) = \lambda e^{(-\lambda y)}$, with $\lambda > 0$.

The equilibrium conditions boil down to:

$$r(B_1) \approx r^s = \frac{r^* + \lambda y^d + \lambda B_1}{1 - \lambda B_1}, \tag{4}$$

$$r(B_1) \approx r^d = \frac{(\lambda + \alpha) y^d - \omega\alpha - \ln\left(\frac{\lambda}{\alpha+\lambda}\right) + (\lambda - \alpha) B_1}{1 - \lambda B_1}. \tag{5}$$

In Equation (5) we set $ln(\beta) = 0$ as β is approximately 1. Following Ayres et al. (2018), we call Equation (4) the supply curve, denoted r^s, and Equation (5) the demand curve, labeled r^d. As $log(\beta)$ is not exactly equal to zero, we say that these expressions approximate ($\approx$) the actual values for the interest rate consistent with the investor's and households' problem.[1] The figure below illustrates the equilibrium at point A.

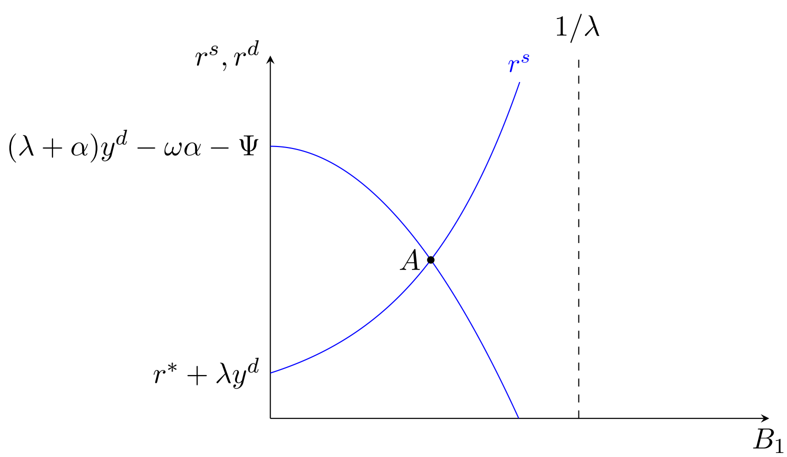


Figure 3: Equilibrium in the two-period model

Note: In the vertical axis, we depict r^s and r^d, the interest rate for the investor and borrower respectively. In the horizontal axis, we depict the level of equilibrium debt $b_1 = B_1$. The vertical dotted line depicts the asymptotic limit of both curves. We define $\Psi \equiv ln(\lambda/(\alpha + \lambda))$ for simplicity.

[1]We derived all the expressions in the appendix. On top of the mentioned restriction on $log(\beta)$, the closed forms for r^s and r^d are approximated only because we assume $ln(R) = r$. That is, the log of the gross rate equals the net rate. The closed-form expressions for the equilibrium interest rate and debt are available upon request.

Figure 3 depicts Equations (4) and (5) as a function of equilibrium debt $B_1 = b_1$. There are several things to note in Figure 3. First, the intersections of r^s and r^d with the horizontal axis both depend on the default punishment y^d. Second, the borrower's interest rate tends to infinity for some boundary value of debt, $B_1 = 1$. Because of this last fact, after a negative wealth shock (i.e., a drop in initial wealth ω), equilibrium borrowing increases, pushing the equilibrium interest rate close to infinity. Moreover, the first fact implies that it is possible for some values of y^d that the curves may not intersect. In an infinite-dimensional setting, these facts are relevant as we must restrict the values of the default punishment to guarantee the existence of an equilibrium. Additionally, from a welfare perspective, the Government has incentives to rule out the explosive interest rate paths that follow a negative wealth shock. This implies an additional policy instrument, which we refer to as *early default*, and we define as follows.

Early default

Consider the following slight extension of the model. Assume that initial wealth ω, is given by:

$$\omega = y_0 - R(B_0)b_0.$$

The initial wealth is the current income realization net of an initial debt repayment. By *early default*, we refer to the possibility of defaulting on the initial debt $b_0 = B_0$ in period 1. We assume that the government is able to generate a haircut of 100% on B_0. Early default is, consequently, a new instrument for the government that can prevent the private sector from issuing debt in period 1 as, even with default in period two, Figure 3 shows that the decentralized economy could generate an interest rate $R(B_1)$ arbitrarily close to infinity, affecting consumption and thus overall welfare.

As the government is benevolent, the default decision in period 1 is based on a comparison between the value of moving to autarky in period 1, which we denote as V^d and define below, and the continuation value, previously introduced as $V^c(\omega)$.

$$V(\omega) = \max \left\{ V^d(y^d), V^c(\omega) \right\}$$

with

$$V^d(y^d) = u(y^d) + \beta u(y^d),$$

where *early default* implies defaulting on b_0 and moving to autarky, $b_1 = 0$, as we are not allowing for reentry.

There is a value for $b_0 = B_0$ such that early default is preferred. We call this value $\bar{B}$. Intuitively, suppose that b_0 is such that the economy starts with a low ω. As figure 3 indicates, the lower the ω, the higher the $r(B_1)$ leading to an "explosive" interest rate (debt) path. Hence, there is a b_0 large enough such that the government may prefer autarky from $t = 1$. In this case, the government chooses to default early. Formally, the continuation value in V^c decreases as higher debt implies higher debt services $B_1 R(B_1)$ because, as shown below, R is increasing in B. [2] In the infinite horizon version of the model, we formally prove this result.

2.2 The case with internalization

The equilibrium under internalization is different. The Euler equation in this case is

$$u'(\omega + B_1) = \beta \underbrace{[R(B_1) + B_1 R'(B_1)]}_{\text{MCC}} \underbrace{\left[\int_R u'(y - B_1 R(B_1)) f(y) dy\right]}_{\text{MCD}} \tag{6}$$

and

$$R^* = R(B_1)\left[1 - F(y^d + B_1 R(B_1))\right] \tag{7}$$

Equation 6 takes as given the default set (as in Kim and Zhang (2012)). The term MCC comes from the internalization of borrowing decisions. The government recognized that borrowing affects the market rate. The relationship between internalization and debt issuance is linked to the overborrowing literature, which has been widely studied in international macroeconomics to provide a rationale for macro-prudential policies. In our context, with default, the key for an overborrowing result is the value of $B_1 R'(B_1) > 0$. In our case,

$$r(B_1) \approx \frac{r^* + \lambda y^d + \lambda B_1}{1 - \lambda B_1}$$

[2] As R is increasing in the two-period model, a sufficient condition for this is to assume that the utility function u is unbounded below and consumption is bounded below by zero. Examples of this type of preferences are: $u(c) = ln(c)$ and $u(c) = c^{(1-\sigma)}/(1-\sigma)$ with $\sigma > 1$.

the derivative with respect to B_1 is

$$\frac{dr(B_1)}{dB_1} = \frac{\lambda(1 - B_1) + \lambda(r^* + y^d + B_1)}{(1 - \lambda B_1)^2} = \lambda \frac{1 + r^* + y^d}{(1 - \lambda B_1)^2} > 0$$

As $r(B_1)$ is an affine transformation of $R(B_1)$, the properties of the derivative are common. Hence, in this case, we always have overborrowing, as $B_1 R'(B_1) > 0$, which implies a higher cost of borrowing with respect to the solution without internalization. In the model with infinite horizons, we prove this property using a two-step operator robust to the presence of discontinuous equilibria.

We present this information in Figure 4, where we plot the relationship between the decentralized and centralized equilibria, which arise from the right-hand side of the Euler equation, Equation (6), which in turn determines the demand for debt. In the centralized equilibrium, the effects of indebtedness on the risk premium are internalized by the Government, leading authorities to issue less debt when compared with the decentralized equilibrium.

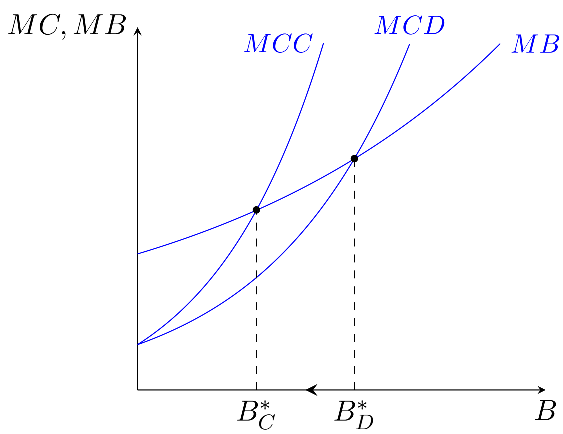


Figure 4: Indebtedness in the centralized and decentralized equilibrium

Note: In the vertical axis, we depict the marginal benefit (MB) - the right-hand side of equation (6)- and the marginal cost (MC) - the left-hand side of the same equation. The MC has two components, the centralized (MCC) and the decentralized (MCD). By internalizing the effects of indebtedness on the risk premium, the Government chooses a smaller level of debt with respect to the decentralized equilibrium, moving the equilibrium from B_D^* to B_C^*.

2.3 The centralized default

How does early centralized default operate in these economies? This section argues that centralized default stabilizes unstable debt dynamics.

We can study the stabilizing role of centralized default in the two-period model with early default as follows. The next figure shows a supply schedule r^s, and a collection of demand functions of debt, which depend on the wealth level ω. The lower ω, as seen in the previous equations, the larger the equilibrium debt levels b. Suppose that at point A the wealth level is ω_A. Then at point B wealth is lower, $\omega_B < \omega_A$, and debt is higher. From figure 4, at point B_1, debt is lower than at point A for the same level of wealth due to pecuniary externalities.

In a dynamic setting, ω would be the initial income net of maturing debt. Hence, the higher the debt that matures today, the lower ω that would imply a higher debt issuance today.

As seen in Figure 3, for large levels of debt, approaching 1, the interest rate explodes, and at some point, the government finds it optimal to default. In Figure 5, without changing the timing of the game between the borrower and the lender, we allow for default *also* in period one. Previously, we referred to this policy as *early default*. Note that in Figure 3, the default was only possible in period two. The default threshold is shown in the red vertical line, which implies a full haircut. Thus, the economy with decentralized debt jumps from either points A or B to B_2, suffering a complete deleverage. Thus, after a negative wealth shock, the indebtedness cycle starts from zero, and Government policies create recurrent debt cycles, stabilizing the debt dynamics. Note, remarkably, that in an infinite-horizon economy without an early default, a sufficiently large streak of bad shocks would drive the debt to infinity. Thus, points A, B, and even B_1 are not necessarily stable in the absence of early default.

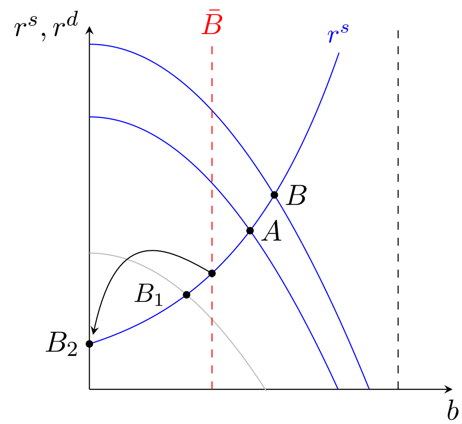


Figure 5: The stabilizing role of early default

Note: we depict equations (4) and (5) (for two different values of ω, with $\omega_A > \omega_B$), generating points A and B, both with overborrowing as R is increasing in B. Point B_1 is the equilibrium without overborrowing, with public (centralized) debt. Allowing for early default generates point B_2 as the government enforces a full haircut when private indebtedness is above $\bar{B}$.

2.4 Pecuniary externality and instability

The previous discussion suggests a connection between overborrowing and instability. Different initial conditions $\omega_A = y_0 - R(b_0^A)b_0^A > \omega_B = y_0 - R(b_0^B)b_0^B$, imply different future debt services $R(b_1^A)b_1^A < R(b_1^B)b_0^B$. Thus, depending on the initial net wealth of the private sector, decentralized decisions will drive the economy along an explosive path, approaching the asymptotic boundary in Figure 3. If we can imagine that points A and B do not characterize two different economies, but the same economy in two different periods, the notion of instability arises immediately.

In the arguments above, overborrowing acts to accelerate the transition to instability. As R is increasing, the private economy loops faster to the edge. The economy with public debt partially internalizes this fact, as it slows down the transition to an explosive path. Typically, this solution is implemented through contingent taxes on the private economy, even with capital controls. However, for a sufficiently large sequence of bath shocks, the private economy with taxes or capital controls will ultimately reach the explosive path.

Early default, as it induces a full haircut, is more effective in controlling instability as it re-generates the indebtedness cycle entirely. In the infinite-horizon economy, we formally prove that this policy instrument is sufficient to globally stabilize the economy, as it regenerates the dynamical system by forcing the private sector to undergo a sudden and complete deleveraging. This fact contrasts with the deleveraging induced by price-dependent collateral constraints, which the Balance of Payments crises literature refers to as Fisherian Deflation. We claim that in the presence of default risk, overborrowing could generate an explosive accumulation of debt services in the future and the government, by foreseeing the welfare effects of this fact, prefers to act preventively, forcing an instantaneous and full deleverage that creates a recurrent and thus stable behavior of private debt in the presence of pecuniary externalities, which are natural in this environment due to the cumulative interaction between default risk and private debt.

2.5 Summary

In summary, the two-period economy in this section is in line with Ayres et al. (2018), but depicts a unique equilibrium. This model of centralized default and decentralized borrowing generates overborrowing, which differs from Arellano (2008), but is in line with Bianchi (2011). Extending the model to allow for early default enables us to capture the existence of explosive debt dynamics that induce the government to prevent borrowing from period 1 onward. Early default is a macro-prudential tool that stabilizes debt levels and is welfare-improving.

Based on this two-period model, we next consider the extension to an infinite-horizon economy. The findings of the two-period economy can be extrapolated to the infinite horizon. Although it is not possible to derive closed-form solutions, and although we can provide a sharp characterization of endogenous variables, we will need to rely on computational methods to obtain accurate results.

3 The infinite horizon economy

The previous economy can be generalized to an infinite horizon as follows. Consider a small open economy populated by households and a government. Households are atomistic and risk-averse agents that issue non-contingent foreign debt in real terms, consume, and receive an exogenous endowment. The benevolent government decides every period whether the private sector repays or defaults on its debt.[3] The international investors are deep-pocketed, risk-neutral agents that purchase external debt and understand the default risk. The remainder of this section presents the details of each agent.

3.1 The households

The domestic economy is populated by a large number of identical households that can borrow or save using a one-period real net asset, $b < 0$ denotes debt holdings, with a non-contingent gross interest rate $R(B)$ subject to default risk. Each of these households receives a positive stochastic endowment y that follows an identically distributed (i.i.d.) process.[4] Preferences are standard and represented by a strictly increasing, strictly concave, and differentiable instantaneous return function u. In line with the two-period model, the recursive problem of the agent is:[5]

$$V^c(b, B, y; h) = \max_{b_+ \geq -\bar{b}} u(F(b, B, y) - b_+) + \beta \mathbb{E}[V(b_+, h(B, y), y'; h)]. \tag{8}$$

Here, c denotes consumption, $F(b, B, y) = y + bR(B)$, and h is the aggregate law of motion for assets B_+ that we define below, V^c is the value function under repayment, and V is the option value between default and repayment, which depends on the early default decision of the government formalized through h. The policy function for this problem is $b_+^*(b, B, y; h)$. Our objective is to characterize the dynamic stochastic equilibrium using the interaction between b_+ and b provided that h exists.

To characterize households' decisions, we use the analogous of Equation (1) for the infinite-horizon economy. Taking R as given, as the households are atomistic, the characterization of this

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[3]The Government can prevent excessive borrowing in any state by forcing the private sector to default on its debt. The intuition for this is that in emerging economies, this is typically achieved by a drastic change in the economic environment surrounding private indebtedness, such as a domestic currency depreciation, suspending access to the exchange rate markets, direct capital controls, and other means.

[4]We only need the i.i.d. assumption to prove ergodicity of the equilibrium. The existence of a stationary equilibrium can be proved under Markov shocks.

[5]Ayres et al. (2018) shows that it is possible to have multiple equilibria in similar environments. Our definition of stability does not require the existence of a continuous equilibrium. Thus, if there are multiple equilibria, we assume the existence of a measurable, possibly discontinuous selection, which follows from standard assumptions provided compactness. This property follows from the presence of early default, as shown in Figure 5. Yet, the existence of multiplicity studied in Ayres et al. (2018) may not arise as it depends on additional assumptions.

problem is straightforward. We give details about the first order condition in the appendix (see the subsection containing preliminary remarks). The Euler equation is given by:

$$u'(c(b, B, y; h)) \geq R(B_+)\beta\mathbb{E}[u'(c_+(b_+, B_+, y'; h))], \tag{9}$$

where $B_+ = h(B; y)$ and b_+ arguments are $(b, B, y; h)$. Equation (9) may hold with strict inequality if the upper bound on debt $-\bar{b}$ is binding and $c(b, B, y; h) = y - b_+(b, B, y; h) + bR(B)$.[6] As R is decreasing in B, there is a pecuniary externality of private indebtedness on future debt services, which in turn implies further debt issuance as the law of motion for b is monotonic in equilibrium. These interactions generate a destabilizing spiral. As Figure 5 suggests, early default may also stabilize the infinity horizon economy. In the next section, we prove that this intuition extends naturally to more general environments.

Note that, as in Kim and Zhang (2012), we are modeling a perceived law of motion h that maps $(B \times Y) \mapsto B_+$. Thus, as the integral in the expectation operator is taken with respect to y', $B_+ = h(B, y)$ is constant with respect to that variable and thus can be factored out in the Euler equation. This is one important difference with respect to Arellano (2008): as there is decentralized credit, the effects of borrowing on the interest rate are not internalized by the private agent. Thus, given (B, y), the agent expects a constant interest rate in the future, which is a reflection of the price-taking assumption.

If B is sufficiently low, which implies that $R(B)$ is sufficiently large after a given sequence of bad shocks, we may have:[7]

[6] From a technical point of view, this equation is useful for us in order to prove compactness and monotonicity of the equilibrium policy function. The reader can find the details in the appendix.

[7] To obtain equation (10) it suffices to impose $R(h(B, y))\beta \geq 1$. Note that $R(B)$ could be arbitrarily large if repayment occurs with probability zero. However, as we suggest in Figure 3 and will demonstrate below, this may occur in off-equilibrium paths that are not relevant to the solution of the model. Even in the most general Markov equilibrium notions, such as Duffie et al. (1994) and Phelan and Stacchetti (2001), the consistency of general equilibrium implies restricting the stationary equilibrium to those paths that satisfy individual optimization with positive probability. If we assume that u is bounded above and unbounded below, something that is natural in CRRA preferences, the existence of a uniform lower bound for consumption follows from standard results. Thus, the fact that $R(B')$ can be factored out implies that any unbounded endogenous interest rate would violate individual optimization. Thus, these paths cannot be considered a part of the recursive stationary equilibria with minimal state space. This type of equilibria, which is the notion used in this paper, is a subset of the recursive equilibrium notions in Duffie et al. (1994) and Phelan and Stacchetti (2001).

$$u'(c(b,B,y;h)) \geq \mathbb{E}[u'(c_+(b_+(b,B,y;h),h(B,y),y';h))]. \tag{10}$$

Equation (10) is at the heart of the instability of the decentralized equilibrium in the infinite horizon. Formally, it demonstrates that, for some sequence of bad shocks, we can characterize the private economy using a supermartingale, which, since Ljungqvist and Sargent (2012), is well known to generate explosive paths. Lemma 5 in the Supplementary appendix for this section presents conditions that lead to explosive paths. Taking the behavior of households as given, the government decides whether to allow repayment or force default in the private sector. We specify the government's problem in the following section.

3.2 Centralized default

As in the two-period model, we assume that default decision is centralized. The government may induce default in any state and we abstract from the specific instruments that could lead to a massive default of the private sector. Nevertheless, we provided some examples in the introduction. To focus on stationary equilibrium, we set $b = B$. As the government does not choose the level of debt but only the decision to induce a private default, the problem of the government is:

$$\text{Default if:} \quad V^c(B,y) \leq V^{def}(y). \tag{11}$$

Here, $V^c(B,y)$ represents the continuation value, i.e., the reward for repayment the outstanding debt, that satisfies

$$V^c(B,y) = u(y - b_+(y,B;h) + BR(B)) + \beta\mathbb{E}\max\left\{V^c(b_+(y,B;h),y'),V^{def}(y')\right\}, \tag{12}$$

while $V^{def}(y)$ stands for the value of default and satisfies

$$V^{def}(y) = u(y^{def}(y)) + \beta\mathbb{E}\left\{\theta V^c(0,y') + (1-\theta)V^{def}(y')\right\}, \tag{13}$$

where θ is the probability of regaining access to the market after default occurs.

Note that, *if consumption and assets in the next period are both increasing in B for each y*, we have the following characterization of default sets:

$$\begin{Bmatrix} \text{Repay} & if\ B > \bar{B}(y) & \text{as this implies} & V^c(B,y) > V^{def}(y) \\ \text{Default} & if\ B \leq \bar{B}(y) & \text{as this implies} & V^c(B,y) \leq V^{def}(y) \end{Bmatrix}. \tag{14}$$

Equation (14) presents a formal definition of $\bar{B}$, which was first depicted in Figure 5. Given the existence of a stationary equilibrium, equation (14) shows that, if consumption and assets are both increasing in aggregate states, *private debt induces state-dependent default sets* as in Arellano (2008). This fact will enable us to characterize stochastic dynamics using a traditional approach in the stochastic processes literature (see, for instance, Meyn and Tweedie (1993)). Moreover, state-dependent default sets were used only in models with public debt, which are not suitable for macro-prudential analysis. These state-dependent thresholds, in the presence of overborrowing, are essential for connecting the literature on Balance of Payments crises, as in Bianchi (2011), with the literature on default risk; both of which are salient features in emerging economies. We claim that the presence of pecuniary externalities may not only lead to a Sudden Stop, but also to a default, as they both generate excessive private sector borrowing, ultimately taking the economy to unsustainable debt levels. In the Balance of Payments literature, macro-prudential taxes and/or capital controls are used to replicate a constrained planner solution. As Figure 5 shows, these instruments are not enough to globally stabilize the economy, as it is possible to generate a sequence of shocks that takes the economy to excessively high debt services from a welfare perspective. We demonstrate that early default, as formally defined by Equation (14), is sufficient to stabilize the economy, as it forces a sudden and instantaneous deleveraging, thereby resetting the indebtedness cycle and creating a stable orbit for debt.

3.3 The international investors

International investors in this model are risk-neutral, deep-pocket agents whose objective is to break even in expectations. In line with the standard assumptions in the literature, these agents price debt internalizing the default probability. Denoting the (gross) risk-free rate by R^*, and π for the transition probabilities for shocks, the (gross) interest rate subject to default risk is:

$$R(B_+) = R^* \left[\sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y') \mathbb{I} \left\{ V^c(B_+(B, y), y') > V^{def}(y') \right\} \right]^{-1} . \tag{15}$$

Here $B_+(B, y)$ is the aggregate debt in equilibrium and $\mathbb{I} \left\{ V^c(B_+(B, y), y') > V^{def}(y') \right\}$ is an indicator function that takes the value of 1 in states in which the government does not induce a default. As is clear from the equation, the assumption of incomplete markets implies that investors cannot condition the return on future debt on future states. Hence, the equilibrium interest rate does not depend on future income, but rather on current income, only if the income process is persistent, $\pi(y', y)$, which we eliminate by assumption. Thus, we denote the interest rate relevant for future debt as $R(B_+)$. Further, as future equilibrium debt is a function of current states, $B_+(B, y)$, we can factor out the interest rate from the Euler equation, (9).

3.4 Definition of Stationary Equilibrium

Below, we connect all the pieces of the model together in a definition of Stationary Equilibrium.

Definition 1. ***Stationary Equilibrium.*** *A stationary equilibrium is a set of policy functions h, c, b_+, a set of value functions V^{def}, V^c and V, and a price function R, such that:*

1. *Households. Given y, $R(B_+)$, and the government's default choice, $B_+ = h(B, y)$, c and b_+ solve (8), maximizing utility V^c.*

2. *Government. Given y, the household's choices, and the value function under default V^{def}, a benevolent government optimally decides to $\bar{B}(B, y)$, solving (14), generating the equilibrium value of early default V.*

3. *Investors. International investors maximize profits, solving (15) and pricing debt subject to default risk according to R.*

4. *Market clearing-Rational Expectations. For each state, (B, y), with $b = B$, individual decisions are consistent with general equilibrium, $b_+ = B_+ = h(B, y)$. Hence, the aggregate debt function in equilibrium is*

$$\begin{Bmatrix} if\ B \geq \bar{B}(B,y),\ h(B,y) = b_{+}(B,y;h) \\ if\ B < \bar{B}(B,y),\ with\ prob.\ \theta,\ h(0,y) = b_{+}(0,y;h) \\ if\ B < \bar{B}(B,y),\ with\ prob.\ 1-\theta,\ h(B,y) = 0 \end{Bmatrix} \tag{16}$$

Regarding Definition 1, the model assumes that the government has an enforcement technology to keep the private economy away from individual optimization (as described by equation (9) and formally captured by h when $b_{+}(B,y;h) < \bar{B}(B,y)$) as long as re-entry is not possible.

Equation (16) is the representation in the space of policy functions of Equation (14), which we define using exclusively value functions. Both equations reflect the state-contingent nature of the default threshold $\bar{B}$, but the latter emphasizes the discontinuity of the equilibrium laws of motion, which is essential to understand the global stochastic stability discussed in the next section. This characterization depends on the existence of a stationary equilibrium. Provided the existence of this equilibrium, Definition 1 describes an aggregate law of motion, h. In this paper, for expository purposes, we do not include an existence proof. We present those results in a separate, more technical paper.

Note that, if the economy hits the default threshold for a positive measure subset of the state space (B,y), h is discontinuous even if it is unique. So the tools used to prove the existence of equilibrium must be robust to the presence of discontinuities. Even though we present this proof in a separate paper, it is worth noticing that showing existence in this environment allows us to accommodate all the concerns associated with the presence of multiple equilibria as discussed in Ayres et al. (2018). We will always assume the existence of a possible discontinuous measurable aggregate law of motion h. This assumption addresses the issues associated with multiplicity, which can lead to a discontinuous selection rule. The existence proof of the stationary equilibrium relies on the fact that equation (9) induces an order structure, which will allow us to use suitable theorems. It turns out that Coleman (1991), Mirman et al. (2008), and Aguiar and Amador (2019) serve this purpose. Moreover, the default restrictions associated with (16) are not internalized by the household. Thus, as $\overline{b}$ can be assumed to be arbitrarily large, we can prove the results using a standard Euler operator without taking into account inequality constraints.

4 Global stochastic stability

In this section, we extend the dynamic analysis implicit in Figure 5 to the infinite-horizon version of the model. First, we will depict several stochastic paths to illustrate the instability inherent in private debt in the presence of default risk, which arises primarily due to pecuniary externalities. Then, we will formally generalize those results, showing the connection between the stabilizing effects of sovereign default and the existence of a stochastic steady state.

We will say that the model is *globally stable* if, regardless of the initial condition, it remains contained in a compact state space almost everywhere and the joint distribution of endogenous variables, such as net external assets and the real interest rate, is time-invariant. Moreover, we will say that the model is *ergodic* if it generates finite recurrent sets (i.e., positive probability paths generate orbits), the state space is irreducible (i.e., it does not break into different isolated islands), and both these elements can be characterized using a unique probability measure for all possible initial conditions. Then, *ergodicity requires some form of global stochastic stability as it endows the model with an invariant measure and a compact state space that characterizes all positive probability paths.* In this section, we graphically characterize these paths and demonstrate that the model is ergodic, thereby establishing its global stability in a stochastic sense. Moreover, the ergodic invariant measure of the model is its stochastic steady state.

Following Azariadis and Lambertini (2003) and Brock and Hommes (1997), we will use a *phase diagram* in discrete time to characterize global dynamics. This diagram depicts iterative dynamics (i.e., from t to $t+1$ for all $t \geq 0$), mapping $B = b$ to $B_+ = b_+$ using the equilibrium law of motion for debt, h. However, there is a critical difference between those papers and our approach: our analysis is *stochastic*. Thus, the notion of *steady state* is different. In our case, a steady state is a stationary (i.e., time-invariant) measure characterizing endogenous variables in any period.

To construct the phase diagram, we will use different demarcation curves, one for each possible shock: $h(., y)$ for all y lying in a finite set. As shocks are finite, every time the state of nature changes, we move from one demarcation curve to the other. If the economy remains in the same stochastic state, it slides along the same demarcation curve continuously from $B = b$ to $B_+ = b_+$. Moreover, because there is a possible default, debt may change even for the same shock if the economy crosses the threshold $\bar{B}$. This fact is critical for stability as it allows us to adapt the standard definition from the non-stochastic analysis to our notion of steady state. For this purpose, we use the stochastic version of an orbit in dynamical systems, known as a recurrent set. We will show that h is monotonic. Then, to obtain recurrence, we must prevent the model from diverging to infinity. As Figure 3 suggests, this is possible even if debt is finite, as debt services may explode. Unstable dynamics may arise from non-monotonic laws of motion. In our case, the monotonicity of h follows from the fact that R is decreasing. This fact, in turn, generates pecuniary externalities and overborrowing. Thus, these properties endow the model with a tractable form of instability, which is essential for our formal results. To stabilize the economy, we must *only* address the rapid growth of debt. Default, by forcing a full haircut on the face value of debt, resets the leverage cycle, creating a new orbit. We will demonstrate that this behavior generates recurrent sets for all possible initial conditions, thereby stabilizing the economy provided that it reaches default thresholds in finite time.

4.1 Phase Diagrams and global characterization

Figure 6 represents the way in which default induces global stochastic stability in the infinite-horizon model.

In the figure described above, we present the transition for the case of 2 shocks and one of the 2 possible initial conditions $B_0 : b_+(B_0, Y_{UB}) > B_0$. The figure contains a candidate for a non-stochastic steady state, i.e. a point B^N satisfying $B^N = b_+(B^N, y)$ for any $y \in Y$,[8] which allows us to establish that the default set is non-empty and, as we will show below, stable.[9]

This Figure 6 is closely related to Figure 5. To some extent, it represents the dynamic (infinite horizon) version of the two-period model. The solid lines indicate the policy functions for two different endowment levels (the upper endowment produces the light gray line and the lower endowment produces the dark gray line). The arrows indicate the accumulation or deleveraging of debt. In the figure, the optimal debt issuance moves from point 1 to 2 in the presence of a negative endowment shock. When the economy reaches point 3, the government defaults and regenerates the cycle, to

[8]Note that as we are considering an equilibrium, we have $b = B$. On top of that $B_+ = b_+(B, y)$ and along the 45º line $B = b_+(B, y)$

[9]This means that it does not contain transient sets. Let P^n_φ be the n-th iterate of a Markov kernel. A transient set A satisfies $P^n_\varphi(z, A) \to 0$ for all $z \in A$, and in practice, they are eliminated by throwing away the first 1,000/10,000 simulations before computing any long run average. This property implies that the process does not create islands, and it is connected. Moreover, we need to generate well-behaved orbits or recurrent sets. We will demonstrate this property by proving that the process reaches default thresholds in finite time, a fact that, in turn, depends on the monotonicity of b_+.

point 4.1 with a negative endowment state, or to point 5.1 in the case of a good realization of the endowment. In terms of Figure 5, moving from A to B, due to a decreasing sequence of negative wealth shocks ω, is analogous to sliding through $b(.,Y_{LB})$ from points 2 to 3. Moreover, defaulting corresponds to reaching $\bar{B}$, and then jumping to points 5.1 (or Section 4.1). These points correspond to the point B_2 in Figure 5.

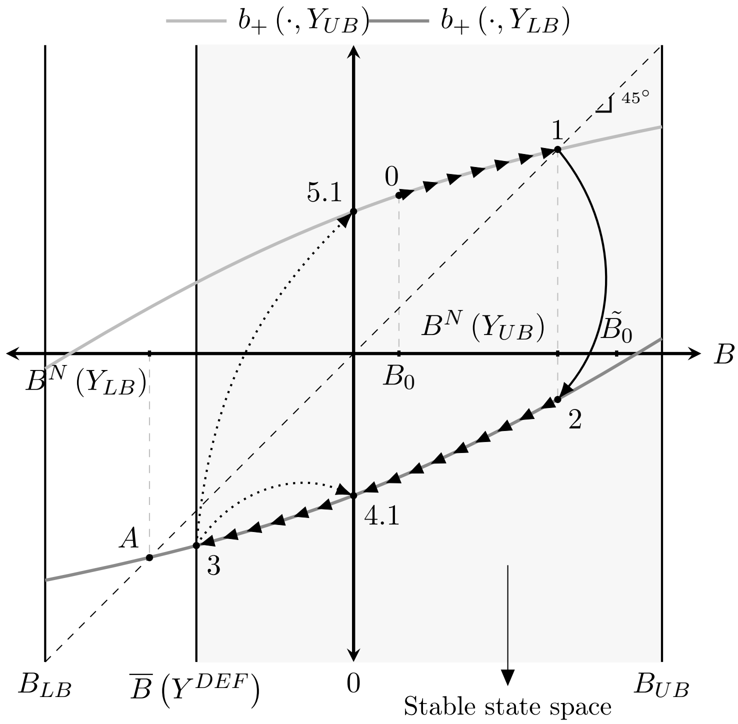

Figure 6: Non-empty and (saddle path) stable default set

Note: Start with the initial state B_0, Y_{UB} in point 0, an initial condition which implies $b_{+,*}(B_0, Y_{UB})$. A good endowment shock increases asset accumulation to point 1, where a bad endowment realization occurs, inducing the economy to issue debt to smooth consumption (point 2). If a sequence of Y_{LB} occurs, the economy goes straight to default in point 3. The figure illustrates that when returning to asset markets, the economy can transition to point 5.1 if Y_{UB} or 4.1 if Y_{LB}.

Figure 7 depicts another situation. If demarcation curves are concave, in the absence of default, indebted countries may be unstable regardless of their GDP level. This situation could arise either because there is no intersection between the 45° line and demarcation curves or because these curves cross when the county has a negative net external assets position. In Figure 7, by resetting the indebtedness cycle, default takes the economy back to accumulating assets in the good state. If the economy enters a recession, then it prevents debt services from exploding. As noted by Brock and Hommes (1997), global instability is typical for non-monotonic models. In our case, even with increasing demarcation curves, instability arises due to penuniary externalities. Entering into a spiral of high debt, high interest rates, that induce higher debt tomorrow happens if the economy is to the left of the unstable equilibrium on the left of the $B_\ell^N(Y_{UB})$ point. The dynamics to the left of that point requires a default to regenerate the cycle.

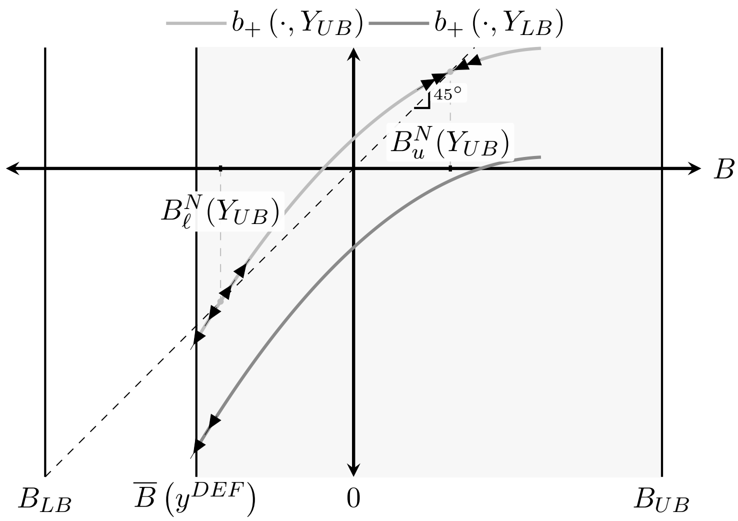

Figure 7: Unstable Transitions

Note: Instability is possible without default. If demarcation curves are concave, then for indebted countries, the economy is unstable even for high levels of GDP.

4.2 Existence of a stable stochastic steady state

The economics for the previous section are determined by the dynamics around the stochastic steady state. In this section, we discuss its existence and properties. To formally characterize the steady state, we need two main standard assumptions:

Assumption 1 (Finite i.i.d. endowments)**.** *All* $y \in \{Y_{LB}, ..., Y_{UB}\} \equiv Y$ *with* $Y_{LB} > 0,\ , Y_{UB} < \infty$ *and* $\pi(y) > 0$*; where* π *is a probability measure.*

Assumption 2 (Preferences)**.** $u : \mathbb{X} \to \mathbb{R}$*, where* $\mathbb{X} = \mathbb{R}_+$ *is the consumption space,* u *is once differentiable with derivative given by* $u'(c)$*, strictly increasing, strictly concave, unbounded below* *and bounded above. Moreover, u' satisfies Inada: $lim_{c\to\infty}\ u'(c) = 0$ and $lim_{c\to 0}\ u'(c) = \infty$. Finally, $\beta R^* < 1$*

Assumption 1 allows us to define a process $(\Omega, \Sigma, \mu_{y_0})$ with a typical element in the sequence space $\{y_0, y_1, ...\}$ and an associated process in the space of random variables for $[c, b_+, R](\omega)$, which maps Ω to $\mathbb{R}^3$ (see Lucas et al. (1989), chapters 7 to 9). Assumption 2 is a mild restriction of preferences. For instance, a CRRA utility function with a risk aversion parameter above 1 satisfies all the mentioned requirements. As shown below, under these assumptions, c, b_+, and R are bounded almost everywhere in μ_{y_0}, for all $y_0 \in Y$. The following two Lemmas formally characterize endogenous variables.

Lemma 1 (Bounds)**.** *Under assumptions 1 and 2, $[c, b_+, R](\omega) \in \mathbb{K}$ almost everywhere in Ω, where $\mathbb{K} \in \mathbb{R}^3$ and is compact. Moreover, $c(\omega)$ is bounded below almost everywhere in Ω by $\underline{c} > 0$.*

Proof. See the Appendix □

Lemma 1 implies that debt services are bounded. However, this fact does not preclude default from being welfare-enhancing. Provided some qualitative properties of policy functions, Equation (14) implies that the government may prefer to default. Thus, characterizing policy functions, induced by equation (9), is essential for our purposes as they guarantee that the value of repayment behaves accordingly.

Lemma 2 (Policy Functions)**.** *Under assumptions 1 and 2, if $R(B)$ is decreasing in B, then $c(b, B, y; h)$ and $b_+(b, B, y; h)$ are both weakly increasing in b when $b = B$ for any $y \in Y$ and h.*[10] *Moreover, either c or b_+ is strictly increasing.*

Proof. See the Appendix □

Lemma 2 not only characterizes policy functions qualitatively, but it also implies that a persistent recession may lead to explosive debt paths that end in a default episode. The supplementary appendix to this section formally states this result (see lemma 5). Of course, there is a "limit" to

[10] Both c and h are contained in s set of function C defined in the appendix.

the degree of instability that allows us to keep the equilibrium tractable. To curve the instability inherent in the model, we assume $R\beta < 1$.

Definition 2. ***[Equilibrium state space*** Z***]*** *Let* $\overline{B}(y)$ *be the upper bounds for debt implied by* h*. The minimal state space for the equilibrium process generated by the Markov kernel* P_φ*, to be defined below, is given by:* $Z_1 \equiv \left[\overline{B}(y_{LB}), B_{UB}\right] \times Y$*. Then, there exist a function* φ *mapping* (B, y, c, R) *to* (B_+, y_+, c_+, R_+) *and these elements belong to a Stationary Equilibrium. The state space* Z *is composed of all possible* (B, y, c, R) *spanned by* Z_1 *using* φ*.*

Equipped with Definition 2, we can project Z_1 onto Z using φ, which allows us to derive the following Markov kernel:

$$P_\varphi(z, A) = \left\{\pi\left(y' \in Y : \left[c(h(B, y), y'), R(h(B, y)), h(B, y), y'\right] \in A\right)\right\} \tag{17}$$

We show that P_φ has an invariant measure, which is also the stable stochastic steady state for the model summarized by (Z, P_φ).

To prove that a possibly discontinuous equilibrium has an invariant measure, we cannot use standard tools (see, for instance, Lucas et al. (1989) chapter 12). Following Meyn and Tweedie (1993), we must define a regeneration point. Fortunately, this point arises almost naturally in default models. If we can prove that the model hits the default set $\bar{B}(y)$ with positive probability for all $y \in Y$, then the default states generate a regeneration set. However, this set must have at most one point (i.e., the set must be a singleton), which implies that we must refine the default punishment. The assumption below imposes these restrictions (for details, see the supplementary appendix of this section).

Assumption 3. ***[Ergodic punishment]*** *Assume that* $y^{def}(y) = y^{def}$ *for all* $y \in Y$*. Let* $B^N(y_{LB}) \equiv B = b_{+,*}(B, y_{LB})$*. Assume that* $B^N(y_{LB}) < 0$*.*

Note that the second part of assumption 3 imposes restrictions on endogenous variables. We need to assume the existence of a non-stochastic steady state B^N for the lowest shock y_{LB}.

The restriction on $B^N(y_{LB})$ in Assumption 3 has a minimal consistency requirement: we suppose that it is negative. As we are modeling default, we require that in the non-stochastic steady state, households hold negative net assets in the worst possible scenario (i.e., $y = y_{LB}$). Then, as lemma 2 shows that b_+ is increasing in B, the definition of y^{def} will allow us to construct a point satisfying: $B^N(y_{LB}) < \bar{B}(y_{LB}) < 0$, where $\bar{B}(y_{LB})$ is defined in equation (14). This inequality is necessary to prove the connectedness of the state space in definition 2. Then, we use the connected state space to prove the equilibrium's global stability, and the existence of a well-defined stochastic steady state.

Theorem 1. *Under assumptions 1, 2, 3, there exist* y^{def} *such that* $B^N(y_{LB}) < \bar{B}(y^{def}) < 0$ *and* (Z, P_φ) *has an unique invariant probability measure* μ_*^{def}.

Proof. See the appendix. □

Theorem 1 proves the existence of a stable stochastic steady state, as it endows the process (Z, P_φ) with an orbit of finite size for any possible initial condition in the state space defined before. The appendix contains the proof of this theorem and additional technical details to ensure the paper's self-containment.

5 A quantitative exercise

This section applies the model to the data to measure the destabilizing impact of private debt in response to foreign shocks. We calibrate the model to Argentina between 1982 and 2016, as this sample includes only 1 event, the default of 2001.

Table 2: Results

Variable	$(R(B)B)/Y$*	Def. freq.*	B/Y	CA/Y	$C.V.(CA/Y)$
Data	-0.6%	3.0%	-1.4%	-0.8%	3.6
Model	-0.6%	2.4%	-2.0%	-1.3%	4.2

Note: * denotes moments that are matched using the simulated method of moments. The rest of the statistics are non-targeted moments. $(R(B)B)/Y$ are (yearly) interest payments of private external debt with respect to GDP. "Def. freq." is the frequency of default for events that were preceded by 19 years (between 1983 and 2001) of open access to the international credit markets. B/Y are yearly capital payments (i.e., amortizations) of foreign private debt over GDP. CA/Y is the current account to GDP and C. V. is the coefficient of variation of CA/Y, its standard deviation divided by its mean.

Because the model is highly stylized, we target two unconditional statistics using the simulated method of moments with two parameters. We borrow the remaining parameters from the literature, and we then test the empirical fit of the model by comparing non-targeted moments with their empirical counterparts. Table 2 presents the targeted (with $*$) and non-targeted moments.[11] The model can match the main data moments.

The parametrization is in Table 3. Calibrated parameters are indicated with $*$ while the parameters that are not calibrated within the model are borrowed from the Kim and Zhang (2012) and Arellano (2008).

Table 3: Parameters

Parameter	Value	Description	Source
σ	2.0	Risk aversion param.	Arellano (2008)
θ^*	0.0725	Re-entry prob.	Calibrated
β^*	0.935	Discount factor	Calibrated
ρ_e	0.001	Persis. (endowment)	Assumpt. ($\approx$ i.i.d.)
STD_e	0.02	St. dev. (endowment)	Arellano (2008)
r^*	1.7%	Net risk-free interest rate	Arellano (2008)

Note: the second column contains the values of the parameters used in this paper as a benchmark calibration. * denotes parameters that are used in the simulated method of moments. ρ_e and STD_e are the coefficients of the AR(1) process that was discretized using a grid of 15 points. $\beta R^* = \beta(1 + r^*) < 1$ as required by assumption 2.

5.1 The risk free rate: quantitative impact of a long run change

We use the model to measure the long-run implications of a change in the international risk free interest rate. This exercise is empirically relevant as there is a consensus that low interest rate environment is over, partly as a consequence of an inflation stabilization policy in the developed world. We want to understand how this may affect the defaultable private debt in emerging markets. We assume, for this purpose, that the change in the risk free is an unexpected once and for all shock, and we study the long run implications of such a change.

The increase in the risk-free rate pushes up the risk premium and the probability of default, thus increasing average debt, $\mu(B/Y)$, and the standard deviation of debt. Accordingly, the threshold value of debt, $\bar{B}(Y^{def})$ must decrease.

Second, as the interest rate increases demarcation curves rotate up, which implies more assets tomorrow for the same level of assets today. This is the typical Euler equation effect, as we observe more savings. As during autarky, the country is not allowed to save, higher interest rates and more

[11] We compute external private debt using a novel database which classifies external indebtedness according to: international organizations (i.e., non-subject-to default), subject-to default public, and subject-to default private.

Table 4: Comparative statics of moments

Sim.	$\bar{B}(Y^{def})$	Y^{def}	$\mu(B/Y)$	$STD(B/Y)$	Def. freq	$E((B/Y)^2)$	$E(B/Y)^2$	$R-1$
Baseline (BE)	0.24	1.00	-2.0	7.7	2.4	0.006	0.001	1.7
High R (P1)	-0.05	0.98	-7.0	14.7	4.1	0.027	0.005	2.7

Note: The first row contains the benchmark calibration. $\mu(B/Y)$ is the long-run mean of the ratio of net external assets to GDP and is expressed in percentage points. Def. freq and $STD(B/Y)$ are also expressed in percentage points. $\bar{B}(Y^{def})$ and Y^{def} are the threshold for debt and the value of GDP during default, respectively.

assets increase the value of continuation in the stationary equilibrium, V^c, pushing down $\bar{B}(Y^{def})$ even further.

Thus, the model suggests that a worsening of international capital markets represented by *a 100 basis points increase in the risk-free rate more than triples net external debt and almost doubles the probability of default and the volatility of the economy as measured by the standard deviation of net external assets.*

6 Concluding remarks

Our theoretical and quantitative analysis reveals that private external debt accumulation generates pecuniary externalities, leading to systematic overborrowing and potentially explosive debt dynamics in the presence of policy-induced default with incomplete markets. Here, ex-post, the default option can serve as a stabilization mechanism in economies.

In this context, centralized default acts by resetting the debt cycle and removing unsustainable trajectories. The two-period model illustrates the core mechanisms behind this result: the inability of private agents to internalize the impact of their borrowing on interest rates, the resulting instability, and the welfare-improving role of early default. Extending these insights to an infinite-horizon setting, we show that centralized default ensures global stochastic stability and ergodicity, creating recurrent debt cycles that maintain the economy within a compact state space.

Our quantitative exercise, calibrated to Argentina, underscores the empirical relevance of these findings. A modest increase in the international risk-free rate significantly amplifies debt volatility and default probabilities, underscoring the vulnerability of emerging markets to external shocks through the private debt channel, particularly in the context of policy-induced default risk.

Our findings suggest that, in the absence of effective macroprudential regulation that internalizes the pecuniary externality of sovereign risk, private debt cycles in emerging markets will remain inherently fragile and subject to forced corrections through policy-induced defaults.

7 Bibliography

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Appendix

We will show each result separately. First, we will present some preliminary comments regarding Section 3. Then, we will prove the results, incorporating additional concepts to ensure the paper remains self-contained.

Proofs for section 4.2

Before proving the results, we add some details regarding the Euler equation (9). Typically, in default models with public debt, the differentiability of the value function requires additional assumptions (see Clausen and Strub (2020)). Fortunately, in the presence of private debt, the differentiability of the value function follows from standard results. Critically, the price-taking assumption, which is essential for the presence of pecuniary externalities, implies that individual decisions cannot affect the default set, as otherwise they will influence prices. Then, differentiability is immediate. After stating those remarks carefully, we prove lemmas 1 and 2. The reader familiar with the discussion on the differentiability of the value function in default models can skip the Preliminary Remarks and proceed directly to the proofs.

Preliminary remarks

The purpose of this subsection is to carefully derive the Euler equation (9), which is the same used by Kim and Zhang (2012). As regards this equation, there are at least to major difference with respect to the centralized default literature (see, for instance, Arellano (2008)). First, the agent that issues debt / purchases assets does not internalize her portfolio decisions on market prices. This was already noted by Kim and Zhang (2012). Second, the Government chooses to default or repay but does not issue debt. Then, to compute the Euler equation, as we don't assume partial default, we must take the derivative with respect to the individual state b_+, not the aggregate state B_+, which only affects the value of repaying $V^c(b, B, y; h)$, not the option value $V(b, B, y; h) = max\{V^c(b, B, y; h), V^{def}(y)\}$. Thus, we can use the standard envelope theorem as we take the derivative with respect to b on $V^c(b, B, y; h)$ and then forward this expression 1 period. To take care of the interaction between default the the marginal value repaying, we use the fact that the interest rate depends only on the aggregate state. Thus, the discontinuities happened at equilibrium (or aggregate) level.

For exposition purposes, we first present the relevant value functions. The value for the household of the repaying option for the Government is given by:

$$V^c(b,B,y;h) = \max_{b_+ \geq -\bar{b}} u(F(b,B,y) - b_+) + \beta \mathbb{E}[V(b_+, h(B,y), y'; h)]. \tag{18}$$

Where $F(b,B,y) = y + bR(B)$ and h is the aggregate law of motion for assets B_+ in equation (16). Note that, due to the i.i.d. structure in assumption 2, the interest rate is not a function of the exogenous shocks. However, even if we allow for Markov shocks, they will imply that the interest rate relevant for b_+ is a function of y, not y_+. This is clear from equation (15). Intuitively, because there are incomplete markets, the return on net assets, b_+, must be independent of y_+. Moreover, the interaction between default and private welfare is captured by the derivative of $V^c(b_+, B_+, y_+; h)$ with respect to b_+. The interest rate only enters into the Euler equation after replacing this derivative using the envelope theorem.

We need to address the differentiability of the value function, which involves equilibrium and off-equilibrium paths. We will say that h^n is the nth iteration of the operator used to prove the existence of an equilibrium. We derive this operator in a separate paper. Formally, the general value (i.e., with Markov shocks and outside the recursive equilibrium) of the interest rate affected by default risk is given by:

$$\begin{aligned} &R(B',y) \equiv R^*(B,y;h^n) = \\ &R^* \left[\sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y',y) \mathbb{I} \left\{ V^c(b_+(b,B,y;h^n), h^n(B,y), y') > V^{def}(y') \right\} \right]^{-1}. \end{aligned} \tag{19}$$

Note that outside the recursive equilibrium, during iterations, we have $b_+(b,B,y;h^n) \neq h^n(B,y)$ but in a recursive equilibrium we get by definition $b_+(b,B,y;h^*) = h^*(B,y)$. As we are modeling a competitive equilibrium, individual states do affect the interest rate. Thus, marginal changes in b_+ must not affect equation (19). To enforce this fact, we must set a strict inequality inside the indicator function (i.e., when the Government is indifferent, defaults). Now, using this notation, we rewrite the right-hand side of equation (9):

$$
\begin{aligned}
&\mathbb{E}_y[V(b_+(b,B,y;h^n),h^n(B,y),y';h^n)] = \sum_{y'\in[Y_{LB},Y_{UB}]} \pi(y',y) \\
&[\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),h^n(B,y),y';h^n) > V^{def}(y')\right\} V^c(b_+(b,B,y;h^n),h^n(B,y),y';h^n) \\
&\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),h^n(B,y),y';h^n) \leq V^{def}(y')\right\} V^{def}(y')].
\end{aligned} \tag{20}
$$

Note that V^{def} and the indicator functions $\mathbb{I}$ in Equation (20) do not change with b_+. The latter follows from the absence of partial default, and the latter from the price-taking assumption, because individual decisions cannot affect the Government's default choice, as otherwise they will affect the equilibrium interest rate. This is not the case in models with public debt and centralized default as in Arellano (2008). Then, we can rewrite Equation (9) as:

$$
\begin{aligned}
&u'\left(F(b,B,y) - b_+(b,B,y;h^n)\right) \geq \\
&R(B,y;h^n) \sum_{y'\in[Y_{LB},Y_{UB}]} \pi(y',y)\mathbb{I}\left\{V^c(b_+(b,B,y;h^n),h^n(B,y),y') > V^{def}(y')\right\} \\
&u'\left(F(b_+(b,B,y;h^n),h^n(B,y),y') - b_+(b_+(b,B,y;h^n),h^n(B,y),y';h^n)\right),
\end{aligned} \tag{21}
$$

where the second and the third lines capture the interaction between the default and the marginal utility of consumption. That is, even though there are incomplete markets and the interest rate can be factored out of the expectation in the Euler equation (i.e., it does not depend on y' even outside the recursive equilibrium), the number of future exogenous states in which the Government repays affects consumption decisions today through the Euler equation. It can be shown that $R(B,y;h^n)$ is decreasing in B for all h^n, provided that h^0 is appropriately chosen such that an equilibrium exists. We defer all technical details related to the existence proof of a stationary equilibrium to a separate paper. The monotonicity of R for admissible equilibrium laws of motion h^n is essential to show the monotonicity of individual policy functions, b_+ and c, which in turn will help us to show the existence of a stable steady state.

Proofs

To prove Lemma 1, we need an additional assumption that is standard in the literature: the Government will never default on its assets. Following Aguiar and Amador (2019) (see Assumption 4 in that paper), we impose this restriction.

Proof of Lemma 1. Any sequence of consumption is valued by $U = \sum \beta^t u(c_t(\omega))\mu_{y_0}(\omega)$ and u bounded above and unbounded below. Under these assumptions, it is standard to show (see Duffie et al. (1994) page 765) that any utility maximizing sequence $\hat{c}_t(\omega) > \underline{c}$ with $\underline{c} > 0$ almost everywhere in Ω. This fact implies that marginal utility is uniformly bounded above. Under assumption 2 of this paper, Assumption 4 in Aguiar and Amador (2019) and Lemma 1 in Braido (2013) implies that there exists $\rho \in (0, 1)$ with: $b_+ \leq R^*/(1-\rho)$ almost everywhere in Ω, where R^* is the risk-free gross rate. The lower bound on b follows from the restrictions on problem 8. Thus, $b_+ \in [B_{LB}, B_{UB}]$. Because there is a uniform lower bound in consumption, $c \geq C_{LB} \equiv \underline{c}$.

Given these results, it is easy to show that $R(B)$ is bounded above. Suppose not. We will deal only with the case $b = B$ (i.e., only equilibrium candidates) and $R(B) = +\infty$ when $B = b < 0$. Under Assumption 4 in Aguiar and Amador (2019), all default thresholds are negative. Then, $R(B) = +\infty$ when $B = b > 0$ is not possible, as the Government would never choose to default with assets. Then, equations (19) and (21) implies that $u'(c(y, b, B; h)) \geq 0$, which satisfies $c, c_+ \geq \underline{c}$. If $R(B)$ is unbounded and if the expected value of the marginal utility of future consumption is finite, then consumption today must be zero, which is a contradiction as consumption is uniformly bounded away from zero. Then, as $c_+ \geq \underline{c}$, the right-hand side of the Euler equation (21) is equal to 0 as the marginal utility of future consumption is zero. As we must have $c = F(b, B, y) - b_+ \geq \underline{c} > 0$ and $F(b, B, y) = bR(B) + y$, b_+ must be unbounded below, which contradicts the uniform lower bound on b_+ in equation (18) (i.e., $b_+ \geq -\bar{b}$). The lower bound on $R(B)$ is given by R^*, which is standard under risk-neutral pricing. Thus, $R(B) \in [R_{LB}, R_{UB}]$. Finally, the upper bound on c is given by: $Y_{UB} + R_{UB}B_{UB} - B_{LB}$. Thus, $c \in [C_{LB}, C_{UB}]$

□

Proof of Lemma 2. We begin by defining an appropriate space of functions for c and h. Let C be the space of candidate functions for h. As in in Coleman (1991), we require:

$$C(\mathbb{B} \times Y) = \left\{ \begin{array}{c} 0 \leq C(B,y) \leq F(B,y) \\ 0 \leq C(B',y) - C(B,y) \leq F(B',y) - F(B,y) \; if \; B' \geq B \end{array} \right\} \tag{22}$$

Where $F(B, y) = y + R(B)B$ with $B \in [B_{LB}, B_{UB}] \equiv \mathbb{B}$ from lemma 1. Further, equation (22) implies that both c and b_+ are (weakly) increasing in B for each $y \in Y$. We will now define an operator on $C(\mathbb{B} \times Y)$, A, and we will show that $Ac \in C(\mathbb{B} \times Y)$. This is sufficient to demonstrate that h^n is increasing for all n. This is the first step to derive an existence proof, which we present in a separate paper. The equation below defines operator A, implicitly from the Euler Equation:

$$u'(Ac(B,y))) = \beta E\left[u'\left(c\left(F(B,y) - Ac(B,y), y'\right)\right) R\left(F(B,y) - Ac(B,y), y'\right)\right], \tag{23}$$

where Ac defined the *Coleman-Reffett* operator and it may not be equal to c. Equation (23) simply defines the candidates for the fixed point $Ac = c$.

To show the first part of Lemma 2, we must prove that A maps $C(\mathbb{B} \times Y)$ into itself.

Take $c \in C(\mathbb{B} \times Y)$. Let $B'(B, y) = y + R(B)B - Ac(B, y)$. Thus, for any $\hat{c}, \tilde{c} \in C(\mathbb{B} \times Y)$, with $\hat{c} \leq \tilde{c}$, we must show that $A\hat{c} \leq A\tilde{c}$ and $\hat{B}' \leq \tilde{B}'$. In order to do so, notice that:

$$\begin{gathered} u'(A\hat{c}(B,y))) = \beta E\left[u'\left(\hat{c}\left(\hat{B}', y'\right)\right) R\left(\hat{B}', y'\right)\right] \geq \beta E\left[u'\left(\tilde{c}\left(\hat{B}', y'\right)\right) R\left(\hat{B}', y'\right)\right] \geq \\ \beta E\left[u'\left(\tilde{c}\left(\tilde{B}', y'\right)\right) R\left(\hat{B}', y'\right)\right] \geq \beta E\left[u'\left(\tilde{c}\left(\tilde{B}', y'\right)\right) R\left(\tilde{B}', y'\right)\right] = u'(A\tilde{c}(B,y))), \end{gathered}$$

where the first inequality follows from $\hat{c} \leq \tilde{c}$, the second from the optimality of consumption, and the third because R is decreasing in B as any candidate function for consumption is increasing in B. Note that the last inequality implies $\hat{B}' \leq \tilde{B}'$ and the first and the last terms together imply $A\hat{c} \leq A\tilde{c}$ as desired. Thus, $AC(\mathbb{B} \times Y) \subseteq C(\mathbb{B} \times Y)$, which in turn implies that h^n, $b_+(b, B, y; h^n)$ and $c(b, B, y; h^n)$ are increasing for any $(B, y) \in \mathbb{B} \times Y$ with $b = B$ and any $n = 0, 1, ...$

It remains to show that either c or b_+ is strictly increasing. Suppose not. Then, for some $y \in Y$ and $\tilde{B}, B \in \mathbb{B}$, with $\tilde{B} > B$, we have $b_+(B, y) = b_+(\tilde{B}, y)$ and $c(B, y) = c(\tilde{B}, y)$. From equation (8) we know that:

$$V(B, y) = u(c(B, y)) + \beta E[V(b_+(B, y), y')] \text{ with } c(B, y) + b_+(B, y) = y + R(B)B$$

Suppose that $B > 0$. This is without loss of generality as, from lemma 1, $B_{UB} > 0$. Thus, $c(\tilde{B}, y) + b_+(\tilde{B}, y) < y + R(\tilde{B})\tilde{B}$. This inequality implies that there is a basket $\tilde{c}(\tilde{B}, y) > c(\tilde{B}, y)$ which is also feasible and:

$$u(\tilde{c}(\tilde{B}, y)) + \beta E[V(b_+(\tilde{B}, y), y')] > V(\tilde{B}, y)$$

The strict inequality implies a contradiction and it follows that $b_+(B, y) = b_+(\tilde{B}, y)$ *or* $c(B, y) = c(\tilde{B}, y)$ but not both. As y and $\tilde{B}, B$ are arbitrary, we can extend the result for any $y \in Y$ and any strictly ordered pair $\tilde{B}, B \in \mathbb{B}$. □

To show the stability of the equilibrium, we prove that there is an ergodic invariant measure characterizing the steady state of the model. Ergodicity requires recurrence and connectedness (i.e., irreducibility). From Meyn and Tweedie (1993), a discontinuous Markov process has an invariant measure if it has a regeneration point called *atom*. The atom is a point that the process hits with positive probability, starting from any initial condition in an appropriate state space. In our case, it is the default state $(0, y^{def})$, which resets the leveraging cycle and creates an orbit. As hits to the default threshold happen in a finite time, the orbit has a finite radius, generating recurrent sets. As the process is monotonic and does not diverge to infinity due to the finiteness of the orbits, it is stable as it never leaves the state space, which at the same time is the support of the invariant measure. Under standard results, after demonstrating recurrence, we can prove the connectedness of the process, thereby obtaining joint ergodicity and global stability.

To prove the existence of an invariant measure, thus, we first need to define the equilibrium state space of the Markov process. We use Definition 2 to describe Z_1 and Z.

Equation (17) implies that we can construct Z using φ. We will assume the existence of a stationary equilibrium, which implies that there is at least one measure function φ. We present the conditions for the uniqueness of this equilibrium in a separate, more technical paper. Thus, given an element in Z_1 and $y_+ \in Y$ we can find at most 1 vector $(c, R, B_+, y_+, c_+, R_+)$ associated with it. That is, iterating this procedure, it is possible to construct a *finite time path from the Stationary Equilibrium* using φ. Using these paths, we will demonstrate that this equilibrium is also ergodic, despite being discontinuous.

Let us start by formally defining an "accessible atom", which can be thought as a point that is non-negligible from a probabilistic perspective and gets "hit" frequently. Let $P^n_\varphi(z, A)$ be the probability that the Markov chain goes from z to any point in A in n steps with A being measurable, let ψ be some measure, and $B(Z)$ be the Borel sigma algebra generated by Z. Then the set $A \in B(Z)$ is *non-negligible* if $\psi(A) > 0$. A chain is called *irreducible* if, starting from any initial condition, the chain hits all non-negligible sets with positive probability in finite time (i.e. $\psi(A) > 0 \rightarrow P^n_\varphi(z, A) > 0$.) Intuitively, irreducibility is a notion of connectedness for the Markov process as it implies non-negligible sets are visited with positive probability in finite time.

We are now in position to define an atom and state an important intermediate result.

Definition 3 (Accessible Atom). *A set $\alpha \in B(Z)$ is an atom for (Z, P_φ) if there exists a probability measure μ such that $P_\varphi(z, A) = \mu(A)$ with $z \in \alpha$ for all $A \in B(Z)$. The atom is accessible if $\psi(\alpha) > 0$.*

Intuitively an atom is a set containing points in which the chain behave like an i.i.d. process. Any singleton $\{\alpha\}$ is an atom. Note that there is a trade off: if the atom is a singleton, the i.i.d. requirement is trivial but, taking into account that the state space is uncountable, the accessibility clause becomes an issue as it is not clear how to choose ψ. The same happens with irreducibility: when the state space is finite, it suffices to ask for a transition matrix with positive values in all its positions. In the general case, we need to define carefully what is a meaningful set. Fortunately, when the state space Z is a product space between a finite set (Y) and an uncountable subset of ^{3}, containing (B, c, R), there is a well know results that help us find an accessible atom in an irreducible chain (for proof, see Proposition 5.1.1 in Meyn and Tweedie (1993).)

Lemma 3 (Irreducibility and accesible atoms). *Suppose that $P^n_\varphi(z, \alpha) > 0$ for all $z \in Z$. Then α is an accessible atom and (Z, P_φ) is a $P_\varphi(\alpha, .)-$irreducible.*

Proposition 3 follows directly from standard results in Meyn and Tweedie (1993) [12]. Note the relevance of the atom, $\alpha = z_* = (0, y_{LB}, y^{def}, R^*)$ for the stochastic stability of the process: we

[12] If in In proposition 5.1.1 we assume that the atom is a singleton, we still have to deal with the reference measure ψ. Typically, ψ is set to be the "maximal" measure. Fortunately, if the chain is irreducible with respect to some measure, say $P_\varphi(\alpha, .)$, then it can be "expanded" to ψ (e.g, see Meyn and Tweedie (1993), Proposition 4.2.2)

define a meaningful set to be the one that can be hit by the chain starting from it. In this sense, it is similar to a saddle path point in a phase diagram in non-stochastic models where endogenous variables can only take 1 initial condition that leads to convergence to the steady state state.

To apply Proposition 3, the finiteness of Y in assumption 1 and the definition of the Markov kernel P_φ in equation (17) are essential. As we are considering a point, in order to show that $P^\tau_\varphi(z, \{z_*\}) > 0$, it suffices to find a finite sequence $\{y_0, ..., y_\tau\}$ such that the economy defaults when $y_\tau = y_{LB}$.

The effect of an atom in the recurrence structure of the chain is essential to define an invariant measure (i.e. a measure μ which satisfy $\mu = \int P_\varphi(z, A)\mu(dz)$). Suppose that the atom is hit for the first time with positive probability in period $\tau_{z_*} < \infty$ starting from z_0. Then, it is possible to define a (not necessarily probability) measure μ which gives the expected number of visits to a particular set in $B(Z)$, called it A, before τ_{z_*}. Then $\mu(A)$ gives the sum of the probabilities of hitting A *avoiding* the atom. In period $\tau_{z_*} - 1$ when "forward" μ 1 period (i.e. by applying the Markov operator to it, $\int P_\varphi(z, A)\mu(dz)$) the expected number of visits to A avoiding the atom is the same as the chain will hit z_* in period $t = \tau_\alpha$. Thus, μ must not change or equivalently $\mu = \int P_\varphi(z, A)\mu(dz)$. That is, μ is an invariant measure. Provided that $\tau_{z_*} < \infty$, it is possible to normalize μ to be a probability measure. Further, as the accessibility of the atom comes together with the irreducibility of the chain (see lemma 3), the invariant measure is unique as the chain does not break into different "unconnected islands". Finally, the Krein-Milman theorem guarantees the ergodicity of the chain provided its uniqueness (see Futia (1982)).

We first show that (Z, P_φ) satisfy the conditions of proposition 3.

Lemma 4 (Accessible atom in the default model)**.** *Let the atom be* $z_* = (0, y_{LB}, y^{def}, R^*)$*. Then, under assumptions 1, 2 and 3, for any* $(B_0, y_0) \in Z_1$*,* $P_\varphi^{\tau(B_0,y_0)}(z, \{z_*\}) > 0$ *and* $\tau(B_0, y_0) < \infty$*.*

Proof. To show that $B^N(y_{LB}) < \bar{B}(y^{def}) < 0$ note that under assumption 3, $B^N(y_{LB}) < 0$. Then, equation 14 and lemma 2, implies that there is at least 1 y^{def} with such a property.

We now show that starting from any initial condition, the chain hits the atom. We will first show that for any $y \neq y_{LB}$ and any B_0, there is a positive probability path $\{y_0, y_1, ..., y_\tau\} = \{y_0, y_{LB} ..., y_{LB}\}$, and an associated sequence $z(y^t)$ for which the economy defaults when $y_\tau = y_{LB}$.

If $b_{+,*}(B_0, y_0) < B_0$ using lemma 2 we know that: $b_{+,*}(B_0, y_{LB}) < b_{+,*}(B_0, y_0) = B_1$. Then, $B_2 = b_{+,*}(B_1, y_{LB}) < b_{+,*}(B_0, y_{LB}) < B_1$. Then, $B_3 = b_{+,*}(B_2, y_{LB}) < b_{+,*}(B_1, y_{LB})$. Continuing with this logic, as $\bar{B}(y^{def})$ is finite, $B_{\tau+1} < \bar{B}(y^{def})$ as the chain would have converged to $B^N(y_{LB})$ in the absence of default. By the definition of h, then the planner chooses to default at period τ, which in turn implies that the economy hits the atom in this time period.

If $b_{+,*}(B_0, y_0) \geq B_0$, choose $y_t = y_0$ until $B^{y_0} = b_{+,*}(B^{y_0}, y_0)$. Because of remark **??**, we know that this point exist for every $y \in Y$ and is finite. Thus, the chain hits B^{y_0} in finite time. Call this period $s = t$ and thus $B^{y_0} = B_{s+1}$. Choose $y_{s+1} = y_{LB}$. Then, $B_{s+2} = b_{+,*}(B_{s+1}, y_{LB}) < B_{s+1}$ and $B_{s+3} = b_{+,*}(B_{s+2}, y_{LB}) < b_{+,*}(B_{s+1}, y_{LB}) = B_{s+2}$. Continuing with this logic, the chain will hit $\bar{B}(y^{def})$ and thus the atom in finite time.

If $y_0 = y_{LB}$, choose $y_1 = y$ with $y \neq y_{LB}$ and repeat the previous reasoning. □

Now using lemma 4, we show that the chain has a unique invariant measure.

Proof of theorem 1. Note that lemma 4 imply that $P^{\tau}_{\varphi}(z_*, \{z_*\}) > 0$ with $\tau < \infty$. The results in Remark 4.2.1, proposition 4.2.2, theorem 8.2.1 and theorem 10.2.1 in Meyn and Tweedie (1993) imply that (Z, P_{φ}) has an unique invariant measure. As $\tau < \infty$ for any initial condition in Z, theorem 10.2.2 in Meyn and Tweedie (1993) implies that the invariant measure is a probability measure. As it is unique, the Krein-Milman theorem (See Futia (1982)) implies that this measure is ergodic.

□

Supplementary Appendix

Supplementary Appendix for Section 2

The equilibrium, as characterized in the two-period model without internalization, is obtained as follows.

Algebraic details

Operating on the pricing function

$$\ln(R^*) - \ln(R(B_1)) = -\lambda y^d - \lambda b_1 R(B_1)$$

$$r^* - r(B_1) = -\lambda y^d - \lambda B_1 R(B_1)$$

$$r(B_1) = \frac{r^* + \lambda y^d + \lambda B_1}{1 - \lambda B_1}$$

Operating the FOC

$$\alpha e^{-\alpha(\omega + b_1)} = \beta R(B_1)\alpha \int_R e^{-(\lambda+\alpha)y + \alpha b_1 R(B_1)} dy$$

we can factorize the exponent and solve the integral (for instance, using Barrow's method):

$$\alpha e^{-\alpha(\omega + b_1)} = \beta R(B_1)\alpha e^{(\alpha b_1 R(B_1))} \lambda \int_R e^{(-(\lambda+\alpha)y)} dy$$

Note that the Barrow method is such that

$$\int_A^K e^{-(\lambda+\alpha)y} dy = \frac{-\lambda}{\lambda+\alpha} e^{-(\lambda+\alpha)K} - \frac{-\lambda}{\lambda+\alpha} e^{-(\lambda+\alpha)A}$$

but as here the upper limit is ∞,

$$\lambda \int_A^\infty e^{-(\lambda+\alpha)y} dy = \frac{\lambda}{\lambda+\alpha} e^{-(\lambda+\alpha)(y^d + b_1 R(B_1))}$$

Then

$$\alpha e^{-\alpha(\omega + b_1)} = \beta R(B_1)\alpha e^{(\alpha b_1 R(B_1))} \frac{\lambda}{\lambda+\alpha} e^{-(\lambda+\alpha)(y^d + b_1 R(B_1))}$$

Simplify

$$e^{-\alpha(\omega+b_1)} = \frac{\lambda\beta R(B_1)}{\lambda+\alpha} e^{-(\lambda+\alpha)y^d - \lambda b_1 R(B_1)}$$

$$e^{(\lambda+\alpha)y^d - \alpha\omega} = \frac{\lambda\beta R(B_1)}{\lambda+\alpha} e^{\alpha B_1 - \lambda B_1 R(B_1)}$$

Taking logs

$$(\lambda+\alpha)y^d - \alpha\omega = \ln\left(\frac{\lambda\beta}{\lambda+\alpha}\right) + \ln(R(B_1)) + B_1(\alpha - \lambda R(B_1))$$

$$(\lambda+\alpha)y^d - \alpha\omega = \ln\left(\frac{\lambda\beta}{\lambda+\alpha}\right) + r(B_1) + B_1(\alpha - \lambda - \lambda r(B_1))$$

Supplementary Appendix for Section 3.1

Lemma 5 (Unstable paths)**.** *Under assumptions 1 and 2, if $R(B)$ is decreasing in B, then there exist some $\hat{B} < 0$ such that for any $y \in Y$ with $-\hat{B} > y$ and $\beta R(\hat{B}) > 1$, $B \leq \hat{B}$ implies that $b_+(B, y)$ converges to $\bar{B}$ for any weakly decreasing path $[y \downarrow, ..., y_T]$.*

Proof of Lemma 5. Under the assumptions of these lemma, (9) implies $u(c) > E[u'(c_+)]$, where the dependence on $b, B, y; h$ was omitted for simplicity. Then, as $-\hat{B} > y$ and consumption is uniformly bounded away from zero because of lemma 1, $b_+ - \hat{B} < y + (R(\hat{B}) - 1)\hat{B} < 0$. Equivalently, $b_+(\hat{B}, y) < \hat{B}$. Because of 2, b_+ is increasing in B and by assumption R is decreasing in B. Thus, $\beta R(b_+(b_+(\hat{B}, y), y)) > \beta R(b_+(\hat{B}, y)) > 1$. As problem (8) is a standard savings problem and y follows a weakly decreasing path, we know that $u(c_+) > E[u'(c_{++})]$. Taking expectations on the second inequality and using the first we get $u(c) > E[u'(c_{++})]$. Continue with these logic and noting that u' is bounded below by zero, we get $lim_{T\to\infty} E\left[u'(c_T)\right] = 0$ (A1). Because of lemma 1, we know that $c_T + b_{T+1} \leq Y_{UB} + R_{UB}B_{UB}$, which then implies that $b_{T+1} \to \bar{B}$. As y and $\hat{B}$ were arbitrary and A1 was obtained after taking expectations for every period, the convergence is in finite time, which in turn implies that the weakly decreasing path has positive probability.

□

The intuition for lemma 5 follows from the conditions $-\hat{B} > y$, a debt to GDP ratio bigger than 100%, and $\beta R(\hat{B}) > 1$, a sufficiently high interest rate. By noting that a weakly decreasing path represents a persistent recession, we can say that sufficiently high debt levels coupled with a poor growth prospect lead to a default. Depending on the level of GDP during default, $y^{def}(y)$, its welfare effects can vary significantly. We will see the values of $y^{def}(y)$ are critical to show the existence of equilibrium and to generate an ergodic representation. Notice that there is a clear connection between $\hat{B}$, $\bar{B}$ and the type of recession required to induce default. Clearly, $\bar{B}(y)/y$ for any $y \in Y$ defines an upper bound for the debt to GDP ratio. If in any period t, $\hat{B}/y_t$ is close to $\bar{B}(y_t)/y_t$, then it takes a short and mild recession to cause a default.