External Debt Dynamics in an Endogenous Growth Model

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Abstract

Economies experience periods of countercyclical borrowing in which the stock of external private debt is negatively correlated with output growth rates but also undergo periods of procyclical borrowing in which debt shows a positive correlation with growth. We find that a group of middle-income countries spend around one-half of the time in each of the two states. We construct an open economy model with endogenous growth and stochastic productivity shocks to investigate this evidence, exhibiting a stochastic balanced growth path. We prove the existence of an invariant distribution for the debt-capital ratio and characterize its dynamical properties, stating the conditions under which the normalized debt stock is sustainable. In this economy, periods with high debt levels are consistent with decreasing and increasing debt patterns depending on the aggregate growth rate. The model is calibrated to Argentinian data, and the fit is surprisingly good. Our results also allow us to rationalize the variability of the correlation between the trade balance and output growth since the global approach used in the paper allows us to unravel the underlying dynamics of the stock of private debt.

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1 Introduction

This paper examines the dynamics and sustainability of private external debt in small open economies. These economies often experience periods of countercyclical borrowing (CCB), where private external debt levels are high when output growth rates are low and vice versa. However, economies also undergo periods of procyclical borrowing (PCB); during these periods, we jointly observe high (low) debt levels and high (low) growth rates. We argue that two factors mainly drive these observations. First, productivity shocks affecting aggregate income lead individuals to use private borrowing and saving to maintain a constant consumption profile. This is the well-known consumption smoothing effect. However, in a growing economy, the stock of external debt is not only influenced by the economy's growth rate; the debt level also affects the growth rate. Thus, when a positive shock occurs, households smooth consumption, and therefore, consumption grows less than output; the economy builds up capital, decreasing the stock of debt relative to the size of the economy. Conversely, when productivity is low, consumption grows faster than the economy, which requires an increase in debt stock. This reduces the rate of capital accumulation and increases the debt-capital ratio. We call this the endogenous growth effect. We argue that these two effects shape the dynamic behavior of the debt stock, determining the periods of CCB and PCB and the long-term sustainability of private debt.

This paper has several methodological and theoretical contributions. First, we consider a stochastic endogenous growth model for a small open economy and provide empirically verifiable conditions under which the stock of external private debt is globally and stochastically sustainable. We generalize the notion of current account sustainability that is often studied in standard undergraduate textbooks (see, for instance, chapter two of Schmitt-Grohe, et. al. (2022)). In our setting, the shock's probability distribution, persistence, and volatility determine the strength of the consumption smoothing and endogenous growth effects. We say a productivity shock is high if it exceeds the borrowing cost. If high productivity shocks are sufficiently likely, we prove that the debt stock is endogenously sustainable because the invariant distribution of the debt-capital ratio is bounded above with full probability. This approach aligns with the results of Bohn (1995), which depart from the present value framework typically used to address the issue of debt sustainability (see D'Erasmo and Mendoza (2016) for a detailed discussion). Our work offers a novel approach to analyzing debt sustainability by showing the global stability of a stochastic balanced growth path in an unbounded economy. Moreover, unlike Bohn (1995), our study examines an economy with incomplete markets.

To achieve these results, the paper must address the challenges of computing and characterizing the global dynamics of a stochastic endogenous growth economy. For this purpose, the stock of debt is normalized by the physical capital stock. Our economy displays a globally stable stochastic balance growth path (SBGP) along which debt-to-capital stays in a steady state characterized by an invariant measure. The proposed normalization redefines the system's state variables and transforms the SGBP into an ergodic distribution over the state space. The proof requires carefully chosen assumptions to ensure the existence of a homogeneous, unbounded dynamic programming problem. These assumptions constrain the dynamics of debt and capital. Still, they are empirically testable as they do not depend on the limiting behavior of endogenous variables, a common feature of transversality conditions. Then, suppose the stochastic process satisfies the conditions mentioned above, and additional restrictions on preferences and technology are satisfied[1]. In that case, the debt-capital ratio has an invariant distribution. This result is crucial for the computation of the model economy.

This study also contributes to the literature on small open economies, beginning with Mendoza's work (1991). Earlier research primarily focused on the correlation of the trade balance and output growth rates to analyze PCB. However, empiri-

[1]The proof of existence of an invariant distribution requires first-degree homogeneity of the dynamic programming problem, and thus restricts the set of production technologies and utility functions that can be included in our framework.

cal evidence shows significant variation in this correlation across countries, ranging from strongly negative to slightly positive, as documented by Correia et al. (1995), Neumeyer and Perri (2005), and the empirical evidence shown below. Such variability can be hard to reconcile with the predictions of the standard growth model unless different fundamentals are considered. Therefore, our approach considers the empirical distribution over the entire sample domain rather than focusing on a single statistic. By adopting a global approach, we account for additional dimensions of the data information set that are not included in a simple correlation coefficient. More importantly, this method increases the flexibility in computing the equilibrium solution relative to the available data, providing a more comprehensive understanding of the underlying dynamics.

Hence, our paper departs from earlier studies in several dimensions, especially the computational approach. Our global approach contrasts with the linear approximation methods used in other studies. The limitations of linear approximations in capturing highly nonlinear dynamics are well known. Local approximations around a stationary solution can also be misleading in our framework. In our endogenous growth framework, equilibrium paths are characterized by an invariant distribution, and therefore, there might be multiple potential candidates around which to conduct the linear approximation. This complicates the application of linear methods since there is no straightforward way to construct a reliable linear solution. By constructing a global solution method using dynamic programming theory, we can overcome these difficulties and provide a robust way to model the system's dynamics.

Notwithstanding, some works –e.g., Mendoza (1991)– use dynamic programming to compute equilibrium paths. However, in this type of economy, the endogenous growth effect is absent since there is no growth or it is exogenous. For instance, Aguiar and Gopinath (2007) consider an exogenous growth model where the productivity shock is divided into transitory and trend components to explain the cyclical component of emerging market economies. However, this approach has been challenged due to the excessive volatility of the permanent component required to match sample moments –see García-Cicco, Pancrazi, and Uribe (2010). Our work reconciles these two views since the stochastic component is transitory and permanent in our endogenous growth setting. This implies that the volatility required to match the data is lower than in previous works.

Finally, there is extensive literature on the relationship between sudden stops of private capital flows and current account deficits (e.g., Calvo (1998)) and the interplay between private capital flows and public debt crisis –see Arellano, Atkenson, and Wright (2015), Wright (2006) and Kim and Zhang (2012). Our work differs from these studies regarding the time horizon and the role assigned to the public sector. Our focus is on the sustainability of external private debt in the long run, while previous works have centered on the emergence of external debt crises in the short run. From a normative point of view, our findings are based on a constrained planner framework. We argue that to ensure debt sustainability over the long run; authorities must impose restrictions on the growth rate of debt every period, suggesting continuous oversight. This contrasts with the role assigned to the government by Kim and Zhang (2012), who propose that governments should eventually suspend access to international capital markets.

The paper is organized as follows. In section 2, we present empirical evidence on the dynamics of the private external debt stock for a set of small open economies and report the correlations between trade balance and growth rates. In section 3, we lay out the theoretical framework and derive the paper's main results on the dynamics and sustainability of the debt stock. Section 4 calibrates our model to the Argentinian economy and shows that the fit is surprisingly good. Section 5 concludes.

2 Empirical Evidence

This section presents the empirical evidence collected for a group of emerging and developed economies. Our main contribution is constructing a set of measures of external debt for our set of countries. The details of the data sources are provided in the appendix. Briefly, we consider 14 countries, 10 in Latin America and 4 in Europe. To construct the external debt stocks, we use the International Debt Statistics database from the World Bank, the Quarterly External Debt Statistics from the World Bank and the International Monetary Fund, and the Balance of Payments and International Investment Position (BoP/IIP) report from the International Monetary Fund. The capital stock and GDP are from the Penn World Tables v. 10.01. The data for the Trade Balance comes from the BoP/IIP report and the IMF's Direction of Trade Statistics database.

To have a stationary measure of the stock of external debt, we consider the ratio $b = B/K$, where B is the private stock of debt and K is the stock of capital. To illustrate our main stylized fact, we focus on Argentina between 1959 and 2019. We represent the stock of normalized debt b against the economy's growth rate for Argentina during that period in Figure 1, which we present below. It is clear from this graph that both variables are uncorrelated. There are periods in which the level of debt correlates negatively with the growth rates, which we call state countercyclical borrowing (CCB). These periods are characterized by levels of debt below (above) the median and aggregate growth rates above (below) the median. Other periods, which we call procyclical borrowing (PCB), are characterized by debt and growth rates above (or below) the median; in this case, the economy shows a positive correlation between debt and growth rates.

Argentina Private Leverage D/K ***(X) vs Growth (Y)*** 1950 - 2019

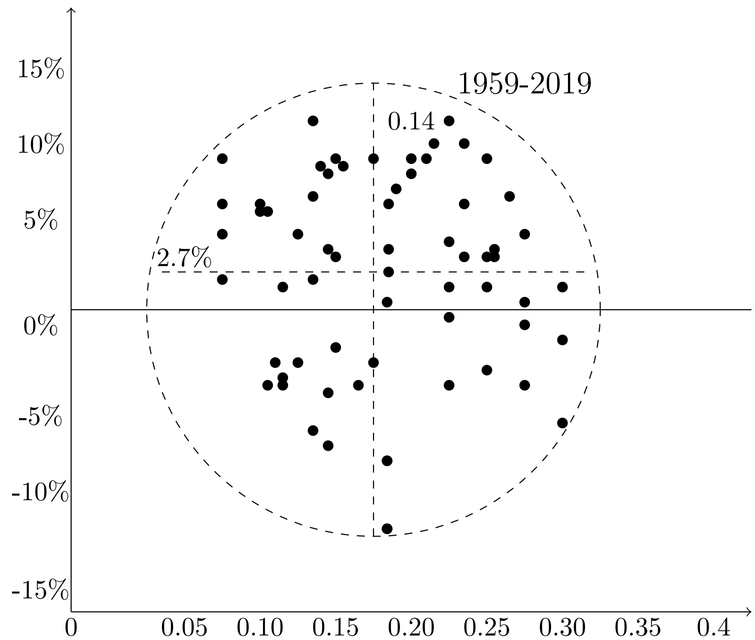

Figure 1: Argentina 1950-2019

When studying open economies, previous studies have usually focused on the correlation of the trade balance with output; see Mendoza (1991) and Correia et al. (1995). Table A1 presents some evidence of this statistic for a set of emerging and developed countries. It is displayed in the last column, and countries are ordered according to their value. This statistic is somewhat related to our focus, the debt-to-capital ratio b, but notice that in one case, the variable of interest is a flow or trade balance, and in another, it is a stock debt. The evidence in that column shows that the correlation of the trade balance with growth rates varies considerably among the countries in our sample, ranging from clearly negative values in Mexico (-0.43) to positive ones, Colombia and Belgium.[2]

Table A1 also shows the share of time each of these economies spent in the PCB region. The PCB and CCB regions are defined concerning the median value of debt and growth rates available for that country. Since one of the variables is a stock and

[2]This is in line with the evidence shown in Neumeyer and Perri (2005)

the other is a flow, we should not expect to observe a clear relationship between the two variables. Table A1 shows that most of the countries spent around 50% of the time in the PCB/CCB region. The median time spent in the PCB region is 49.8% for countries with a negative correlation between the trade balance and GDP growth rates and 51% for those with a positive correlation.

This evidence suggests that considering only one statistic to understand the underlying forces that drive the behavior of real economies may hide more complex dynamics. The problem is that the use of dimensionless statistics may remove a relevant part of the information set of a multidimensional dynamical system. More specifically, we contend that to understand the behavior of open economies, it is essential to study the global dynamics of the stocks of debt rather than reducing the dimensionality of the information set to a single –or a set of– numerical statistic(s). With this purpose, in the next section, we construct an endogenous growth model for a small open economy and focus on the dynamics of the stock of external private debt. We aim to use this simple stochastic growth model to explain the debt dynamics shown in Figure 1. Moreover, we state the conditions under which an invariant distribution exists that guarantee the sustainability of the stock of normalized debt.

3 An Endogenous Growth Model with Debt

In this section, we lay out the basic framework of our analysis. We consider a small open economy where the representative agent can borrow from a perfectly competitive international capital market. In this international market, foreign financial assets B_t, which pay the real interest rate $R > 1$, can be traded with the rest of the world.

The aggregate technology is given by a constant return to scale production function $Y_t = A_t K_t$ with constant marginal returns to capital, which allows for sustained growth. This technology was first introduced by Rebelo (1991) in a class of endogenous growth models and has been extensively used due to its tractability. This type of technology will allow us to prove the existence of an invariant distribution for the stock of normalized debt and will enable us to characterize the global dynamic behavior of the stochastic economy. More precisely, since we want to determine the conditions under which the debt stock is sustainable in real economies, our model must display positive growth rates. Nonetheless, for small open economies, it has been argued that exogenous growth entails excessive volatility –see García-Cicco, Pancrazi, and Uribe (2010). Thus, since one of our work aims is to characterize the global dynamics of the stock of debt, we need to use a production function that allows us to have a homogeneous problem. In that framework, it is possible –see Alavarez and Stokey (1998)– to characterize the global behavior of the model economy.

Consequently, the term A_t represents the level of technology and the productivity shocks to the economy.[3] We assume that it can take a finite number of positive values $A_i \in \mathbb{A}$, where $\mathbb{A} = \{A_1, ...A_N\}$ satisfies the following set of assumptions.

Assumption 1 :

i) $A_i > 0$ *for all* $i \in N$,

ii) *there exists at least one* A_i *and* A_j *in* $\mathbb{A}$ *such that* $A_i < R < A_j$,

iii) $\beta\alpha^\theta < 1$, *where* $\alpha \equiv \max_{1\leq i\leq N}\{A_i\}$.

Note that β is the discount factor, and θ represents the curvature of the utility function of the standard maximization problem described below. The technology level A_t follows a stochastic process represented by a finite state Markov chain with transition probability Π, such that $\Pi(A_i)$ describes the next period's distribution probability over $\mathbb{A}$ when the economy is at state A_i. Thus, the resource constraint of the economy is given by

$$C_t = A_t K_t + RB_t - K_{t+1} - B_{t+1}. \tag{1}$$

[3]Notice that in our framework, there is no distinction between trend and cycle since A_t determines the economy's growth rate.

Notice that we assume $B \leq 0$ and therefore B_t represents net borrowing at period t, and A is the net depreciation of the capital stock. Let us define the space of state variables as $X \equiv [K, B]$ and assume that the agents of this economy have homothetic preferences given by

$$u(C) = \frac{C^\theta}{\theta}. \tag{2}$$

We make the following assumptions,

Assumption 2 :

(i) $X \in [0, +\infty) \times (-\infty, 0]$,

(ii) $-B' \leq -RB$ *and* $K' \leq AK$,

(iii) $\theta \in (0, 1)$.

The first part of Assumption 2 implies that the capital stock has to be positive and that in our small open economy individuals can only be net debtors. Second, in our setting, variables will usually display unbounded growth; this may pose several challenges when applying dynamic programming theorems. From Stokey et al. (1989) and Alvarez and Stokey (1998), it is known that to have a well-defined dynamic programming problem, the states must belong to a cone. Hence, we have to impose some restrictions on the growth rates and set an upper bound on $\|x'\| / \|x\|$, where $x' \in X$ denotes next period's value and $\|\cdot\|$ is the Euclidean norm of a vector. Along these lines, (ii) plays the role of standard transversality conditions setting upper bounds on the (gross) growth rates of debt and capital. The first condition on B requires a partial periodic repayment of debt, and the second is an upper bound on the capital growth rate, which is stronger than a standard transversality condition. These restrictions also guarantee that the space of states is a cone. Finally, the (gross) growth rate in equilibrium may not be above 1, and consumption may be arbitrarily close to 0. In this case, and if $\theta < 0$, U may converge to $-\infty$, a fact that may preclude the existence of an optimal path. We impose $\theta \in (0, 1)$ to rule out this possibility.

The literature on open economies usually includes the labor-leisure margin and adjustment costs to match investment volatility [see, for instance, Mendoza (1991), Aguiar and Gopinath (2007), or García-Cicco, Pancrazi, and Uribe (2010) among others]. In our framework, the labor-leisure margin is absent since it is not necessary to match the stylized facts presented above. Additionally, part *ii*) of Assumption 2 also plays the role of adjustment costs by setting an upper bound to the growth rate of capital. Recall that our focus is on studying the global dynamics of external debt. For that, we need to establish the existence of an invariant distribution and characterize its properties. Although useful for the computation of equilibrium solutions in more general settings, dynamic programming theorems are not powerful enough to establish the existence and dynamic properties of invariant distributions. Hence, we must impose additional assumptions to keep the normalized problem well-behaved. Finally, many works use linear approximations to study transitional dynamics. Nevertheless, these techniques are not helpful for our purposes either since they reduce the problem's dimensionality and construct the equilibrium solution around a unique stationary solution. At the same time, our model displays an invariant distribution. All these difficulties prevent the appropriate characterization of the global dynamics by standard methods and, therefore, impose restrictions on the choice of the theoretical framework for analysis.

3.1 The Stochastic Centralized Equilibrium

Once the economy's structure has been laid out, we proceed to state the optimization problem of the representative agent. Since this problem is optimal, we can state it as the following social planner's problem.

Stochastic Centralized Optimization Problem

$$\begin{aligned} V(K,B,A) &= \max_{K',B'} \quad u(AK+RB-K'-B')+\beta E_A V(K',B',A') \\ & s.t. \quad K'+B' \leq AK+RB \end{aligned} \tag{3}$$

DEFINITION 1 : *A Stochastic Centralized Equilibrium (SCE) is a value function V, and policy functions K', B' and C that solve the Stochastic Centralized Optimization Problem.*

To characterize the equilibrium dynamics of the economy, let us define $H(X, A, \theta)$ as the space of functions that are continuous, homogeneous of degree θ and bounded uniformly in any $x \in X, \|x\| = 1$ (i.e., the unit circle norm) for any $A_i \in \mathbb{A}$. Let T and $G : X \rightarrow X$ represent the Bellman operator and the policy function in the centralized economy such that $G(x) = [B'(x), K'(x)]$. Let $b \equiv B/K$ be our definition of normalized debt and assume that it may be bounded below by some negative constant so that $b \geq \underline{b}$. As shown in the following theorem, the policy function G is homogeneous of degree 1; this allows us to define the following functions. Let the growth rate of the capital stock for any value of b be defined as $g_1(b) \equiv K'(K,B)/K$, and similarly define $g_2(b) \equiv B'(K,B)/K$.

THEOREM 1 : *Under assumptions 1-2,*

A) *The value function V: (i) $V \in H(X, A, \theta)$, (ii) can be computed by successive approximations using T, and (iii) V is differentable on X and strictly concave.*

B) *For any $A_i \in \mathbb{A}$, the policy function G: (i) is homogeneous of degree 1 in x, (ii) $g_2(b, A_i)$ is increasing and continuous in $b \in [\underline{b}, 0]$, and (iii) $g_1(b, A_i)$ is increasing and continuous in b.*

PROOF: See the appendix.

Let $b_+(b, A_i) \equiv g_2(b, A_i)/g_1(b, A_i)$ be the function that represents the next period's debt in terms of today's value b. We have the following result.

Corollary 2 : *For any $A_i \in \mathbb{A}$, b_+ is increasing and continuous in b for $b \geq \underline{b}$.*

3.2 Global Dynamics

Now, we proceed to describe the global dynamics of our economy by proving the existence of an ergodic distribution and defining its characteristics. Before that, notice that ours is a growing economy in which variables are non-stationary and display positive growth rates over time. Therefore, to prove the existence of an ergodic distribution, we need to normalize our economy by dividing total external debt by the capital stock; this ratio will be our state variable b. The study of the dynamics of our system is then carried out in the plane determined by normalized debt b and the growth rate of the economy $\hat{y}$. Besides, to show that in the long run, the joint distribution of $(b, \hat{y})$ is characterized by a non-trivial invariant probability distribution, we have to carry out a global analysis of the transitional dynamics. We use the results on the properties of the value and policy functions derived in the previous section. Once the existence of such an ergodic set is proved and characterized, it is straightforward to provide the conditions under which the stock of debt is sustainable in the long run in the sense that the equilibrium path of the debt-capital ratio is globally stable.

Before studying the dynamics of the stochastic economy, we revise some of the results for the deterministic version of our model. The dynamics of the non-stochastic economy are similar to the standard AK model and are mainly driven by the fixed productivity parameter A. In the AK model, the growth rate of consumption is given by the Euler equation (see equation (4) below), and it is constant along the equilibrium path. The dynamics of the other variables are straightforward since the initial choice of consumption determines the growth rate of capital and output. In our framework, the possibility of borrowing from international markets increases the complexity of the model dynamics, which depends on the values of the initial level of debt B_0 and the technological level A. The return on debt R.[4]

[4]Since the dynamics of the deterministic model with debt is more complex to analyze than the

Now, we turn to the stochastic economy. For clarity, the figures in this section include only two levels of productivity $A_L < R < A_H$. Nonetheless, the theoretical results shown below apply to any finite number of productivity levels satisfying Assumption 1. In our model, the dynamics of consumption is determined by the Euler equation derived from the optimization problem

$$1 = E_t \left[\beta \left(\frac{C_t}{C_{t+1}} \right)^{\frac{1}{1-\theta}} A_{t+1} \right]. \tag{4}$$

This equation shows that higher expected productivity values imply higher consumption growth rates. Let us assume that the economy is at the highest possible value of technology A_H. In a deterministic model, the growth rate of consumption is given by A_H; however, in a stochastic setting, the future expected value of A is lower than the actual level, and therefore, the growth rate of consumption is lower than A_H; this is the consumption smoothing effect. Due to this effect, the economy devotes more resources to increasing capital stock and reducing debt burdens. Whether the stock of normalized debt b decreases or increases depends on the support and probability distribution over $\mathbb{A}$ and the interest rate R. If these are such that the economy grows at a sufficiently high rate, the stock of physical capital grows faster than the stock of debt B and b decreases. Along the same lines, it is straightforward to see that if the economy is at the low productivity level, A_L, the consumption growth rate is higher than A_L since the expected value of A is higher than A_L. Due to consumption smoothing, the capital stock decreases, and the stock of debt increases. Normalized debt will increase if the support and probability distribution of $\mathbb{A}$ and R are such that B grows faster than K. Thus, the growth rates of the economy and the two assets are determined by the support and probability distribution over $\mathbb{A}$ and the exogenous interest rate R.

dynamics of the simple AK model and given that it is not the aim of this paper, the results on the dynamics of the deterministic economy necessary to study the stochastic economy are relegated to the appendix.

The dynamic behavior of our economy is represented in Figures 2 to 4, which we discuss below in detail. Figure 2 represents the dynamics of the debt-capital ratio b for two possible values of A. From Corollary 2, we know that b_+ is increasing in b. We now proceed to present the main results on the global dynamics, but first, we make the following definition.

Definition 2 : *A Stochastic Balanced Growth Path (SBGP) for the Stochastic Centralized Economy is an invariant measure for the stock of normalized debt* b.

In the following two lemmas, we prove the existence of an SBGP and characterize the global dynamics of b.

Lemma 3 : *There exists an invariant distribution that describes the long-run behavior of the state variable* b.

Proof: See Futia (1982).

Lemma 4 : *For each set* $\mathbb{A}$ *there is a transition probability matrix* Π *such that there are two values of the stochastic process* $\bar{A}$ *and* $\underline{A}$ *for which,*

(i) *For any* $A > \bar{A}$ *the function* $b_+(\cdot, A)$ *lies above the 45 degree line in Figure 2.*

(ii) *For any* $A < \underline{A}$ *the function* $b_+(\cdot, A)$ *lies below the 45 degree line in Figure 2.*

Moreover, for such $\mathbb{A}$ *and* Π*, there exists a lower bound* $\underline{b}$ *such that the path of the stock of normalized debt satisfies* $b_t > \underline{b}$ *with probability one for any* $t > 0$ *and* $b_0 > -\infty$.

Proof: See the appendix.

Lemma 3 establishes the existence of an invariant distribution; this result comes from the continuity of $b_+(\cdot, A)$ and the compactness of the state space. Lemma 4 extends the scope of Lemma 3 and characterizes the conditions under which there is a non-trivial invariant distribution. To prove the existence of a SBGP, the homogeneity of the dynamic problem is the key ingredient. This homogeneity comes from the linear technology, the restrictions on the growth rates of the capital stock and external debt, and the individual preferences (i.e., $\theta \in (0,1)$). The restrictions on θ imply a mild degree of curvature on the instantaneous utility function, which is unbounded above but bounded below. Second, to have a well-defined value function, we need to impose conditions on the growth rates of B and K that are stronger than the standard transversality conditions. Since technology is linear, the homogeneity of the value and policy functions is required to impose linear restrictions on debt and capital growth rates. The homogeneity of the policy functions is essential to prove the topological –continuity and compactness– and qualitative –monotonicity– properties of the normalized policy function b_+, which characterize the global stochastic dynamics of the balanced growth paths.

The conditions of Lemma 4 require sufficiently extreme values in $\mathbb{A}$ coupled with sufficiently persistent transition probabilities. These conditions guarantee that there are states at which the economy grows at rates higher and lower than R. In Figure 2, this implies that there is a value A_H such that function $b_+(\cdot, A_H)$ is below the 45-degree line, and a value A_L such that $b_+(\cdot, A_L)$ is above the 45-degree line. The consequence is that a non-trivial invariant distribution determines the dynamics of the state variable b. When the economy is at A_H, the capital growth rate is higher than the growth rate of the stock of debt, and therefore, b increases towards zero (the stock of debt decreases). Conversely, if the economy is at A_L, the capital growth rate is lower than that of debt, and b decreases (debt increases). It is also possible that $b_+(\cdot, A_L)$ crosses the 45-degree line before $\underline{b}$, then the economy has a non-stochastic steady state at b^* for technology level A_L. The existence of this non-stochastic steady state sets an upper bound on b. In any case the dynamics of b is limited to one of the two compact sets $[\underline{b}, 0]$ or $[b^*, 0]$.

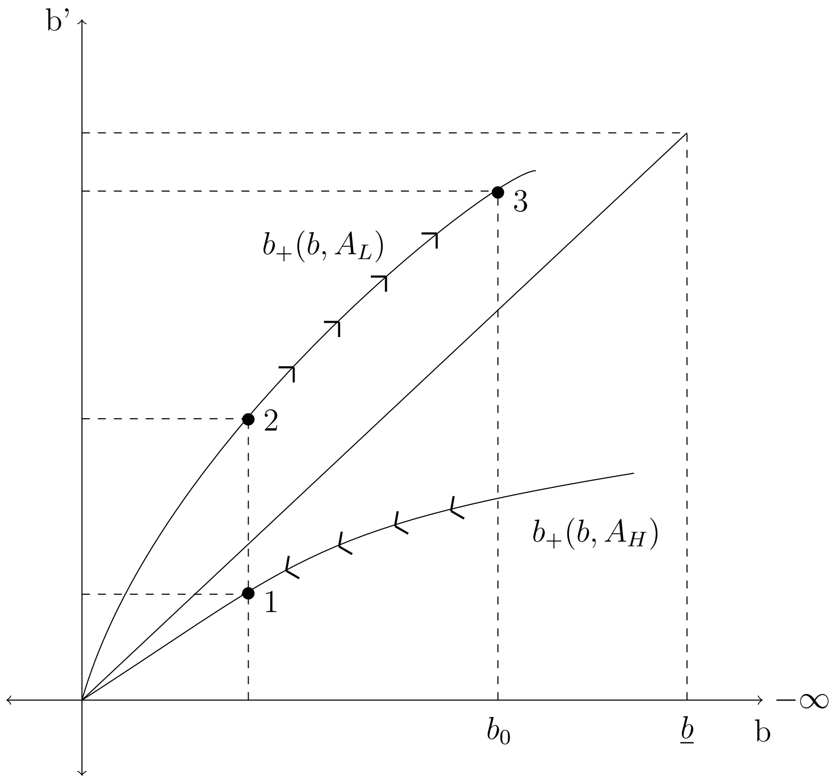

Figure 2: Dynamics of Normalized Debt

In Figure 3, we draw function $g_1(\cdot)$ for two possible values of A; the vertical axis represents growth rates, and the horizontal axis represents net external assets. As shown in Theorem 1, g_1 is increasing in b for any value of A. Also, higher levels of technology entail higher growth rates of the economy, which is reflected in the fact that $g_1(b, A_H)$ is above $g_1(b, A_L)$.

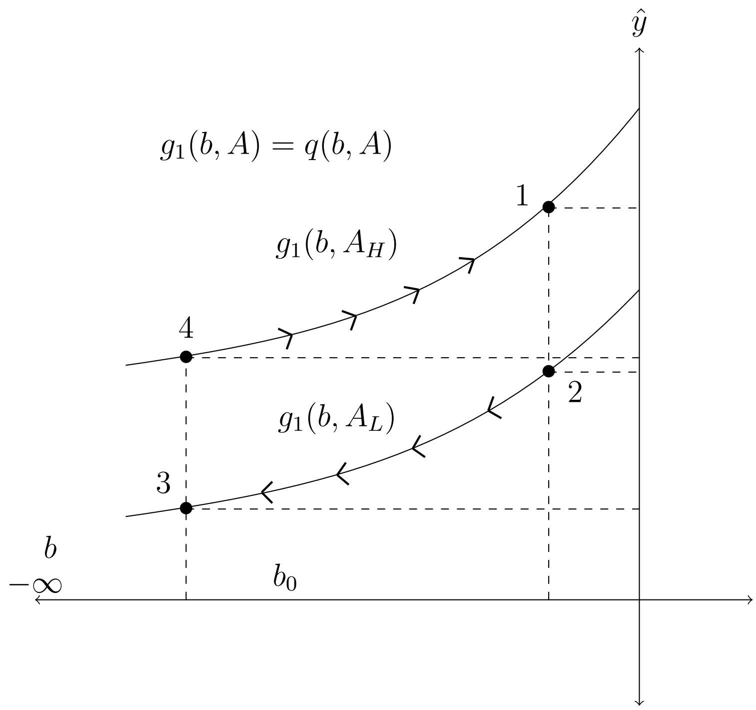

Figure 3: Growth Rate and Debt

Figure 4 includes the previous two figures, Figure 2 is rotated and located at the lower-left quadrant, and Figure 3 is placed in the upper-left quadrant. We also add a new graph in the upper-right corner that describes the dynamics of the system along the exact dimensions of the empirical evidence presented in the previous section[5], stock of debt and growth rates $(-b, \hat{y})$. We have divided this quadrant into four areas defined by the median growth and median borrowing. The upper-left and lower-right areas correspond to counter-cyclical borrowing (CCB), while the other two regions correspond to pro-cyclical borrowing (PCB). When the economy is in the CCB area, normalized debt b levels are low (high), and growth rates are above (below) the median. The PCB region is characterized by low (high) levels of debt and growth rates.

To study the dynamic behavior of our economy, let us assume that initially, the level of debt and productivity is such that the economy is at point 3. Thus, the

[5]Notice that the upper-right quadrant is the mirror image of the upper-left quadrant.

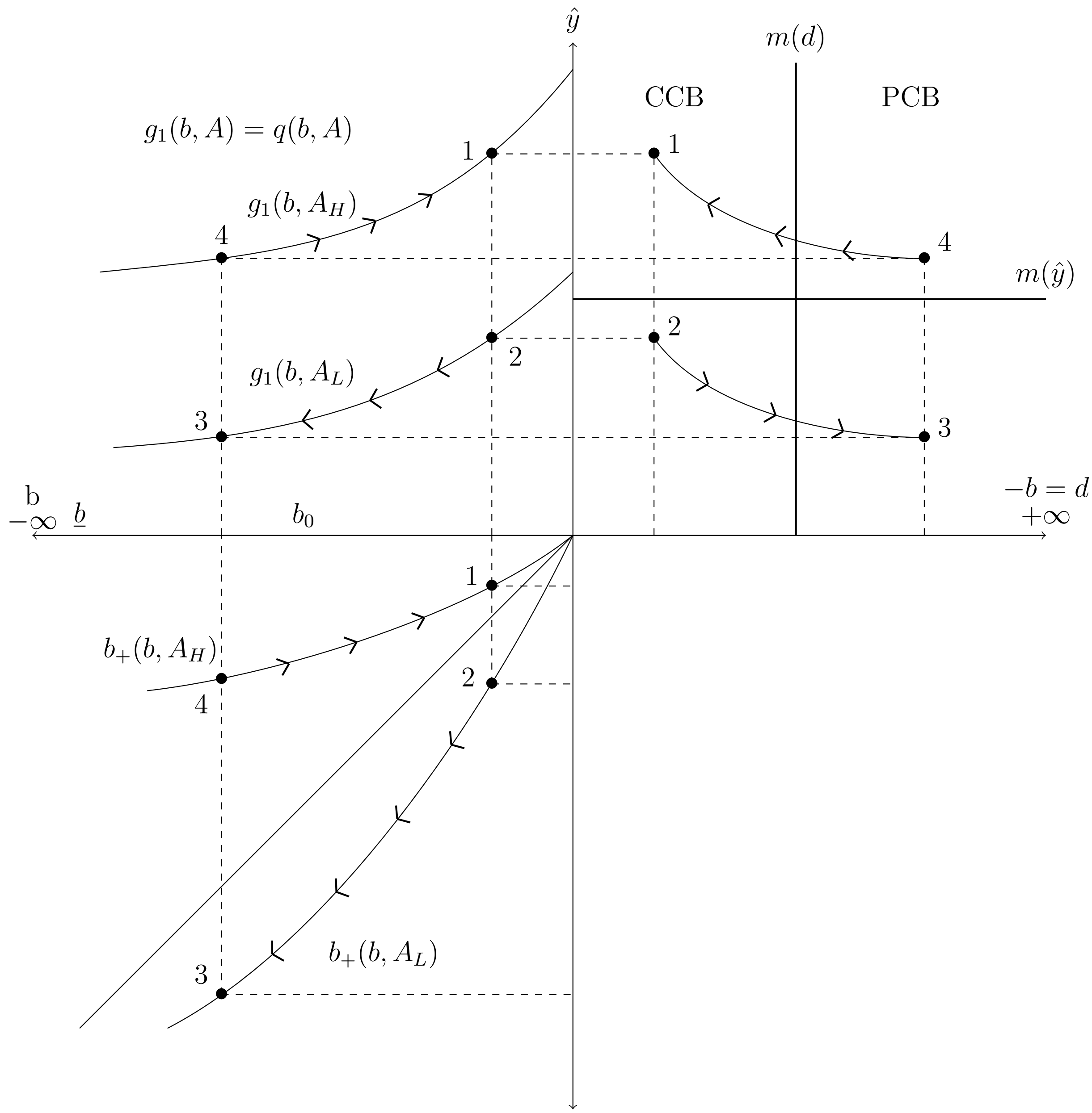

Figure 4: Global Dynamics

amount of debt is high, and the economy experiences low growth rates (CCB region). When a positive shock moves the economy to point four, the economy grows at a rate higher than R, but the consumption growth rate is lower than A_H (i.e., the consumption smoothing effect). The economy is, therefore, accumulating capital and reducing the stock of debt, which is still high relative to capital. However, since K

is growing faster than B both the aggregate growth rate $\hat{y}$ (given by $g_1(b, A_H)$) and b (given by $b_+(\cdot, A_H)$) increase. This is shown in the transition to point 1. Along this transition, the economy moves into the CCB region while the share of resources needed to reduce the stock of debt shrinks; this allows the economy to build up capital faster, increasing the growth rate. When the economy reaches point 1, productivity is shocked, which decreases from A_H to A_L. The economy moves from point 1 to 2 (PCB region), and the dynamics of debt are now described by $b_+(\cdot, A_L)$ instead of $b_+(\cdot, A_H)$. While the productivity level is A_L, the expected productivity of the economy is higher than the actual level. It is then optimal to increase borrowing and reduce capital accumulation to smooth consumption. As a result, B grows faster than K, and therefore, the growth rate of the economy and b decrease. This behavior describes the transition from points 2 to 3, along which the share of resources needed to pay the cost of debt increases. At point 3 (CCB region), a positive shock raises productivity from A_L to A_H, and the debt dynamics are again driven by $b_+(\cdot, A_H)$. This process is repeated over time, starting from any initial condition, and therefore, the dynamic behavior of the normalized economy is characterized by cyclical or recurrent paths that are a consequence of the global stochastic stability of b.

3.3 Debt Sustainability

The results obtained above allow us to address the question of external debt sustainability in the context of our model economy. Nonetheless, it must be stressed that our study only considers the interaction among small agents whose individual behavior cannot affect aggregate outcomes. This implies that the public sector and its potential strategic interactions with other agents are absent from our analysis. Therefore, there are many issues[6] related to public debt sustainability that our framework is not helpful to address –see D'Erasmo and Mendoza (2016) for a survey. So, we can

[6]For instance, fiscal reaction functions, fiscal strategies to restore fiscal solvency, international externalities, sovereign default, or government commitment.

consider our economy as one in which small private agents or small public entities acquire external debt.

As explained above, Figure (4) shows the recurrent behavior of the equilibrium path of our model economy. How much time the economy stays in each of the four areas depicted depends on the volatility and persistence of the shock A. Hence, the shape of the invariant distribution is determined by the properties of the stochastic process, the probability distribution over the set $\mathbb{A}$, and the price of debt R. If the shock is very volatile, the economy will display large vertical movements, and we should observe significant variations in growth rates. In contrast, if the shock is very persistent, we should observe wider variations in the stock of normalized debt. Second, for economies with a low probability of experiencing large shocks to productivity or low values of A_H, the function $b_+(\cdot, A_H)$ may be close to –or even cross– the 45-degree line. This type of probability distribution leads to dynamics in which the economy remains in a region with high debt levels. For lower probabilities or lower values of A_H, the function $b_+(\cdot, A_L)$ –and even $b_+(\cdot, A_H)$– may be entirely below the 45-degree line leading to an explosive path for b and the stock of debt would not be sustainable. Thus, in our setting, the conditions for debt sustainability are given by the support and probability distribution on $\mathbb{A}$. That is, the sustainability of debt is not only defined by growth rates but also by the stochastic properties of the cycle. This differs from previous works in which the sustainability of debt is attained by setting a lower bound on the economy's growth rate (see D'Erasmo et al. (2016) and references therein).

On the other hand, economies with high probabilities of experiencing large productivity shocks should have a function $b_+(\cdot, A_L)$ closer to –or that even crosses– the 45-degree line. In this case, the support of the invariant distribution would be $[b^*, 0)$, and the lower the probability of low-productivity events, the lower the value of b^*. For extreme cases where low-productivity events are very unlikely, the economy may converge to the non-stochastic steady state $b^* = 0$. Finally, the third element in this analysis is the rate of return on debt R. In a world with high interest rates, the probability of high-productivity events (relative to R) is lower, and as discussed above, the support of the invariant distribution will shift to the left, that is, the area with higher debt levels. Alternatively, low levels of R will lead to invariant distributions around regions with lower levels of normalized debt.

In summary, in our framework, the stock of debt is sustainable if the properties of the stochastic process and the interest rate are such that an invariant distribution exists for the measure of normalized debt b. This definition of sustainability implies that the total stock of debt B may grow without bound. The key point is that a given stock of debt B can display arbitrarily high growth rates as long as the economy grows at sufficiently high rates. A high probability for high levels of productivity entails high growth rates for the capital stock, which allows increases in B without leading to explosive paths of the stock of normalized debt. That is, the stock of external debt is endogenously sustainable as long as the conditions on $\mathbb{A}$ and Π made in Assumption 1 and Lemma 4 are satisfied.

Debt sustainability depends on country-specific factors –the probability distribution over $\mathbb{A}$ and external factors –the world interest rate R. More productive countries converge to distributions with low levels of normalized debt. This type of probability distribution means that, on average, investment projects in those countries yield a high return. Foreign investors are thus willing to lend resources to those projects rather than investing in other countries. Then, consider an economy financially constrained[7] In which there is a change in the support of the distribution of $\mathbb{A}$ assigning higher probability to more productive states. We have seen that function $b_+(\cdot, A_H)$ will shift upwards in Figure 4, moving the invariant distribution of b closer to the origin and reducing lending restrictions. On the other hand, a country facing an increase in the probability of low-productive states may have to deal with borrowing

[7]Notice that in our framework $\underline{b}$ is not a constraint since it is determined endogenously (see Lemma 4). Then, financially constrained means an economy that cannot reach debt levels above some exogenous limit imposed by international markets.

restrictions since functions $b_+(\cdot, A_H)$ and $b_+(\cdot, A_L)$ will shift downwards, driving the distribution of normalized debt, and $\underline{b}$, toward lower values of b.

We can also contemplate the consequences of shocks to the world economy. For instance, let us consider an increase in the world's interest rate R. For our small open economy, this implies that there are more states in which the economy's growth rate is lower than R, and therefore, the distribution of b will move to lower values of b. As a result, we should observe more economies financially constrained. Conversely, a positive shock to the world economy materialized by a drop in R would move the distribution of normalized debt closer to zero with fewer economies financially constrained.

4 Calibration

The results above will also allow us to assess the extent to which the model dynamics are consistent with the empirical evidence shown. We calibrate the model to match the Argentinean economy using the frequency of pro-cyclical borrowing as a targeted moment. We then obtain as a non-targeted moment the mean growth of the GDP. We structurally estimate the process of the TFP.

The table contains the parameters obtained after the calibration process together with the benchmark values used in the literature (see Garcia Cicco et al. (2010)).

Table 1: Selected parameters (Part 1)

Model/Parameter	β	θ	$\pi(A_i, A_i)$
Model	0.95	0.7	0.58
Garcia Cicco, et. al. (2010)	0.92	2.0	0.99

As regards table 1, note that the standard calibration for an RBC in the literature (i.e., Garcia Cicco et al. (2010)) implies: a) a higher risk aversion coefficient θ, b) a higher persistence of shocks, which typically is the result of higher autoregressive coefficient. The difference in the discount factor is not significant. As we will see when we simulate our model using the parameters in Garcia Cicco et al. (2010), these differences arise because we are hitting a different stylized fact with respect to the literature (i.e., the correlation between trade balance and GDP as against the pro-cyclicality of external borrowing) and due to the necessity to guarantee the existence of a well-behaved, balanced growth path, which in turn ensures the sustainability of external indebtedness. In table 1, we also include the values for the diagonal of the transition matrix $\pi(A_i, A_i)$ for all A_i, which will be critical to understanding the connection between the global stochastic dynamics and pro-cyclicality.

Table 2 below contains the lower and upper bound for TFP shocks and the interest rate. The values in tables 1 and 2 can be used to verify assumptions $1', 2', 3'$. The first one (i.e., $A_i > 0$ for all i) can be seen immediately as $A_{LB} > 0$. The second one (i.e., $\beta\alpha^\theta < 1$, with $\alpha \equiv max\{A_i, R\}$ for all i) requires some computations: $\beta\alpha^\theta = 0.95(1.07)^{0.7} = 0.996 < 1$. Finally, the third one (i.e., $A_{LB} < R < A_{UB}$) follows from table 2.

Table 2: Selected parameters (part 2)

Parameter	A_{LB}	A_{UB}	R
Model	0.94	1.07	1.01

The table below contains the results of the calibration. The model matches both pro-cyclicality (57.1% vs 56.1%) and mean GDP growth (2.5% vs 3.0%).

Table 3: Simulation results

Statistic	$PC(-b, \hat{y})$	$AVG(\hat{y})$
Data	$57.1%	2.5%
Model	56.1%	3.0%
Garcia Cicco, et. al	$96.0%	-3.3%

PC stands for pro-cyclicality (i.e., the frequency at which simulations or data stays above or below the median in both variables), $-b$ is normalized debt, $\hat{y}$ is the net growth of GDP, and AVG stands for average. The row labeled "Garcia Cicco et al. (2010)" solves the model presented in this paper but using the parameters borrowed from that paper, which we present in table 1.

We will interpret the results using figure 4. We will begin with the targeted moment, the frequency in which Argentina engages in pro-cyclical borrowing. Take, for instance, a low productivity shock, which moves the economy from point 1 (in one of the counter-cyclical regions of the north-west quadrant of the figure, the high growth-low debt one), to point 2 (in the other pro-cyclical region of the south-west quadrant, the low growth-low debt one). This implies a discrete reduction in growth due to a finite number of shocks. The economy jumps from the high to the low growth branch. If persistence is sufficiently high, the phase diagram will take the economy out of the pro-cyclical into the counter-cyclical region, always below the median growth rate. In point 3, the economy reaches one of the counter-cyclical regions, the low growth-high debt. When the high shock hits the economy, it jumps to point 4. We depicted this point in the high growth - high debt pro-cyclical region, which implies that *the economy has visited the four regions in the northwest quadrant after being hit by only two shocks, matching the stylized fact presented in section 2.*

Intuitively, if the level of productivity is high, then capital accumulation is sufficiently fast to sustain a path of above-the-median growth and deleverage. As we assume $R < A_{UB}$, the upper bound on the growth rate of capital $K' \leq A_{UB}K$ slacks more frequently than the upper bound on the growth rate of debt $-B' \leq -RB$, which implies that normalized debt $-b \equiv -B/K$ goes down. Moreover, we showed that the growth rate of capital is increasing in net external assets b: for the same productivity and interest rate levels, more capital and less debt implies a reduction in debt services, which implies that a higher fraction of GDP can be used to increase current and future consumption through capital accumulation. Thus, less debt implies more growth. Moreover, as debt becomes smaller relative to capital, the burden of debt services goes down even further, which explains the curvature of the high-growth branch (i.e., as normalized debt goes down, it is possible to increase the growth rate at a faster pace) even under the same productivity level. As both shocks occur with positive probability when the economy enters into the low productivity region, characterized by $A_{LB} < R$, economic growth is insufficient relative to the growth rate of debt as $K' \leq A_{LB}K$. Thus, according to the same argument, the economy accumulates normalized debt, and the capital growth rate goes down. Finally, as we assume $A_{LB} < R < A_{UB}$ and both shocks happen with positive probability, the economy loops between the high and low growth branches, which gives the recurrence observed in the southeast quadrant.

We turn to the non-targeted moment, the average growth rate. Note that the high/low growth branch, related to the trend behavior in the model, is associated with the high/low TFP shock, which is associated with the economic cycle. Moreover, even with stationary productivity shocks, we can shape the growth rate distribution. As this distribution determines the slope of the GDP trend, we match the non-targeted moment using trend-cycle interactions. In particular, the *level* of shocks determines the *level* of the growth rate. Now, note that the longer we stay in one branch, the further we move from the median growth rate. Thus, we will miss the targeted and non-targeted moments if shocks are too persistent. That is, *the frequency of pro-cyclical debt is deeply connected with the trend growth of the economy.*

The above paragraphs describe the relevance of global stochastic stability (i.e., given that all shocks are observed with positive probability, paths do not diverge) and debt sustainability (i.e., paths orbit around non-stochastic steady states) to generate the *qualitative behavior* presented in figure 4. That is, the circular trajectories reflect the recurrence of stable paths. Now we move to the *quantitative behavior* of the economy. First, Assumption 2-(iii) requires an upper bound for the risk aversion coefficient $\theta \in (0, 1)$ that departs from the macro literature (i.e., Garcia Cicco et al. (2010) set $\theta = 2$). To prove the existence of a well-defined unbounded dynamic programming problem, which is a necessary condition for the existence of a balanced growth path and, thus, of sustainable debt, the value of this parameter must be substantially different with respect to the literature standards. Second, the inertia of the TFP process must also be different. This can be seen in the last column of table 1. If we use the values of $\pi(A_i, A_i)$ and θ in Garcia Cicco et al. (2010), pro-cyclicality is way above the values observed in data (96.0% vs 57.1%) and mean growth is far away from the target (-3.3% vs 2.5%). Thus, to reduce the pro-cyclicality, we must reduce both parameters as the high inertia in the TFP process used in the RBC literature implies too much persistence in A, inducing excessive pro-cyclicality. Third, $\pi(A_i, A_i)$ must be set accordingly. The probability of staying in the diagonal 0.58 is closely related to the pro-cyclical frequency. As we describe above, to move from points 2 to 3 in figure 4, and thus move from the pro-cyclical to the counter-cyclical regions in the north-west quadrant of this figure, it is necessary to stay in the same demarcation curve, which in turn requires persistence of shocks.

5 Conclusions

This paper comprehensively analyzes private external debt dynamics in a small open economy, highlighting the interplay between consumption smoothing and endogenous growth effects. Introducing a stochastic endogenous growth model establishes the conditions for debt sustainability, emphasizing the role of productivity shocks and borrowing costs in shaping debt behavior. The methodological innovations, including the normalization of debt by capital and the use of ergodic distributions, enhance the model's applicability to real-world economies. Ultimately, the findings offer valuable insights into the stability of debt-to-GDP ratios, contributing to a deeper understanding of external debt sustainability in incomplete markets models.

There are several interesting avenues for future research. First, it is possible to study the implications of our results, based on a constrained planner, for a decentralized economy. In an economy subject to sustainability risk, the interest rate faced by the private sector is endogenous and depends on the level of indebtedness. Typically, firms and households fail to internalize the effects of borrowing on future interest rates, leading to overborrowing and threatening debt sustainability. Then, the government can design a set of macroprudential tools to control excessive borrowing. Contrary to standard models, public policy directly affects trend growth without exogenous growth, as we show that economies with less debt grow more.

Extending the endogenous growth model beyond the AK family would be relevant. We would obtain flexibility in the dynamic behavior. The challenge is to derive a well-defined measure of normalized debt for all possible values of the state space in a more general framework. Our results rely heavily on the dynamic program's homogeneity, the production function's structure, and the linearity of restrictions associated with debt and capital, leading to a policy function homogeneous of degree one. This property of the policy function is essential to obtain a tractable debt-to-GDP ratio with stable stochastic global dynamics.

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Appendix A: Additional tables

Table A1: Procyclicality and Trade Balance Correlation

Countries	**Pro-cyclicality**	$\boldsymbol{Corr(TB, \hat{y})}$
MEX	-0.347	51.02%
ESP	-0.058	51.28%
BRA	-0.363	32.65%
ITA	-0.359	27.66%
ARG	-0.188	57.14%
BOL	-0.048	40.82%
CHL	-0.062	54.55%
PER	-0.006	44.90%
Median	**-0.125**	**47.96%**
STD/Mean	**-0.814**	**22.01%**
ECU	0.021	53.06%
PRT	0.105	23.08%
PRY	0.351	59.09%
COL	0.252	48.98%
BEL	0.424	60.53%
Median	**0.252**	**53.06%**
STD/Mean	**0.648**	**27.76%**

Appendix B: Proofs

Proof of Theorem 1

To show the results for the stochastic economy, we first define the non-stochastic version of the centralized problem:

Centralized Optimization Problem

$$V(K,B) = Max_{K',B'} \quad u(AK + RB - K' - B') + \beta V(K', B')$$

Subject to

$$\text{(i) } -B' \leq -RB, \text{ (ii) } K' \leq AK, \text{ (iii) } K' + B' \leq AK + RB$$

As in Stokey et al. (1989), the stochastic version of the proof for finite shocks is a straightforward extension of the proof for the non-stochastic economy. We thus prove the analogous of theorem 1 for the non-stochastic economy.

We begin by quoting a result from Stokey (1994) that applies to non-stochastic economies in $H(\theta, X)$.

Theorem A1: [Optimization problems in $H(\theta, X)$ for non-stochastic economies] Assume:

a) $\theta \in (0, 1]$ and $X \in \mathbb{R}^L$ be a cone.

b) Let Γ be the correspondence defined by restrictions $(i) - (iii)$ of the centralized optimization problem. Then: $\Gamma : X \to X$ is non-empty, compact, valued, continuous, and the graph of Γ, $Gr\Gamma$, is a cone. That is, $\Gamma(0) = 0$ and $y \in \Gamma(x)$ implies $\lambda y \in \Gamma(\lambda x)$ for all $\lambda > 0$ and $x \in X$.

c) $\beta \in (0, 1)$ and there exists $\alpha > 0$ with $\alpha^\theta < \beta^{-1}$ such that $\|y\| \leq \alpha \|x\|$ for all (x, y) in the graph of Γ.

d) $U : Gr\Gamma \to \mathbb{R}$ is homogeneous of degree θ and for some $0 < B < +\infty$ satisfies: $|U(x, y)| \leq B(\|x\| + \|y\|)^\theta$.

Then, we have:

(i) $V \in H(X, \theta)$

(ii) $T : H(X, \theta) \to H(X, \theta)$ and is a contraction of modulus β

(iii) The policy correspondence G is non-empty, compact-valued, upper-hemi-continuous and homogeneous of degree 1.

PROOF: See lemma 2 and theorem 1 in Stokey (1994). ♦

Lemma A1 [Properties of K' and B' in the Centralized Optimization Problem].

1. $K'(K, B)$ is increasing in K for any B *and* $B'(K, B)$ is increasing in K for any B.

2. $K'(K, B)$ is increasing in B for any K *and* $B'(K, B)$ is increasing in B for any K.

3. $B'(K, B)$ and $K'(K, B)$ are continuous.

PROOF: Under assumptions $a)$ to $d)$ of Theorem A1, we know from exercise 9.10.c in Stokey, et. al. (1989) that V is monotonic. Then, an increase in either K or B implies $\Gamma(x') \supseteq \Gamma(x)$, where $x', x \in X$ and $x' \geq x$, which in turn implies $V(x') \geq V(x)$ with $V'(x) \geq 0$. Then, by a standard contradiction argument, we show that the strict concavity of V implies facts $1 - 3$.

For an interior optimal solution

$$U'(C(x)) = \beta V'(K'(x), B'(x)) \geq 0$$

where $C(x) = AK + RB - K'(K,B) - B'(K,B)$ as $x = [K,B]$. Three possibly binding inequality constraints exist: $C \geq 0, -B' \leq -RB$, and $K' \leq AK$. Given our utility function, we know that at the optimum $C > 0$, the other two constraints cannot hold simultaneously. Then we have the following optimality condition:

$$[U'(C(x)) - \beta V_i'(K'(x), B'(x))][H_i(x_i)] = 0,$$

where $H_1(x_1) \equiv AK - K' \geq 0$ and $H_2(x_2) \equiv B' - RB \geq 0$. If $H_1(x_1)$ is binding, we know that $U'(C(x)) \leq \beta V'(K'(x), B'(x))$ and thus any non-optimal allocation must satisfy $U'(C(x)) > \beta V'(K', B')$ for any feasible K', B'. On the contrary, if $H_2(x_2)$ is binding we know that $U'(C(x)) \geq \beta V'(K'(x), B'(x))$ and thus any non-optimal allocation must satisfy $U'(C(x)) < \beta V'(K', B')$ for any feasible K', B'.

We assume $x' \geq x$, with $x' = [K_2, B_1],\ x = [K_1, B_1]$ and $K_2 > K_1$, and proceed by contradiction. There are three cases: (I) $K'(x') < K'(x)$ and $B'(x') \geq B'(x)$, (II) $K'(x') \geq K'(x)$ and $B'(x') < B'(x)$, (III) $K'(x') < K'(x)$ and $B'(x') < B'(x)$. We consider these three cases separately.

(I) Let us assume that our hypothesis is not satisfied and $K'(x') < K'(x)$ and $B'(x') \geq B'(x)$. As already discussed, consumption is strictly interior. Moreover, $K'(K_2, B_1) < K'(K_1, B_1) \leq AK_1 < AK_2$. Thus, the only inequality constraint that may bind is $-B'(K_2, B_1) \leq -RB_1$. This fact in turn implies that any non-optimal allocation satisfies $U'(C(x')) < \beta V'(K', B'(x'))$ or $U'(C(x')) < \beta V'(K'(x'), B')$. As K', B' must be feasible and $\Gamma(x') \supseteq \Gamma(x)$, in (I) and (II) we will use either $K'(x)$ or $B'(x)$ to generate a contradiction. Since $C(x'), K'(x'), B'(x')$ are the optimal decision rules we have

$$\beta V'(K'(x'), B'(x')) > \beta V'(K'(x), B'(x')) > U'(C(x') \geq \beta V'(K'(x'), B'(x')),$$

which is a contradiction. Note that the first inequality comes from $K'(x') < K'(x)$ and the strict concavity of V, the second from non-optimality of $K'(x), B'(x')$ and the third one from the optimality of $K'(x'), B'(x')$ when $H_2(x_2) \geq 0$.

(II) In this case, we may have either $H_1(x_1) = 0$ (capital is binding) or $H_2(x_2) = 0$ (net external assets are binding). In the first case, we have:

$$\beta V'(K'(x'), B'(x')) \leq \beta V'(K'(x), B'(x')) < U'(C(x') \leq \beta V'(K'(x'), B'(x')),$$

where the first inequality comes from $K'(x') \geq K'(x)$, the second from the non-optimality of $K(x)$ and the third from the fact that if $H_1(x_1) = 0$, optimality implies $U'(C(x') \leq \beta V'(K'(x'), B'(x'))$. If $H_2(x_2) \geq 0$ we have:

$$\beta V'(K'(x'), B'(x')) > \beta V'(K'(x'), B'(x)) > U'(C(x') \geq \beta V'(K'(x'), B'(x')),$$

where the fist inequality follows from $B'(x') < B(x)$ and the strict concavity of V, the second from the non-optimality of $K'(x'), B'(x)$ when $H_2(x_2) \geq 0$ and the third from the optimality of $C(x'), K'(x')$ and $B'(x')$.

(III) This case follows from standard arguments in scalar optimization as we know that, if $x' > x$ and $\Gamma(x') \supseteq \Gamma(x)$, due to the concavity of preferences $C(x') > C(x)$ and $K'(x') + B'(x') > K'(x) + B'(x)$ as H_1 and H_2 cannot bind simultaneously. This completes the proof of (1). A similar argument can be used for (2). The proof of (3) follows from $1 - (A) - (iii)$ and $1 - (B) - (i)$. ♦

Theorem 1-NS: [Characterization of the solution of the centralized economy] Suppose that assumptions 1 and 2 hold. Then, we have:
(A) The value function V: (i) $\in H(X, \theta)$, (ii) can be computed by successive approximations using T, (iii) V is differentable and strictly concave.

(B) The policy function G: (i) is homogeneous of degree 1, (ii) $g_2(b)$ is increasing and continuous in any finite b, (iii) $g_1(b)$ is increasing in $b < 0$ with $g_1(0) > 0$ and continuous in any finite b.

PROOF OF THEOREM 1-NS: We proceed in 4 steps to prove the theorem.

• **Part A:** $(A) - (i), (ii)$ and $(B) - (i)$

Note that we impose $\theta \in (0, 1)$. The results in Theorem 1 $(A) - (i), (A) - (ii)$ and $(B) - (i)$ follow directly from Theorem A1. Then, we must show that the centralized optimization problem satisfies assumptions $a) - d)$ of this theorem.

a) Follows directly from assumption 2.

b) Follows directly from the definition of Γ:

$$\begin{aligned} &\Gamma = \{y = [K', B'] \in \mathbb{R}^2 \ such\ that\ (i) - B' \leq -RB, \\ &(ii)\ K' \leq AK,\ (iii)\ K' + B' \leq AK + RB\ for\ each\ x = [K, B] \in \mathbb{R}^2\} \end{aligned} \tag{5}$$

c) From assumption 2 (ii) we have: $(K')^2 + (B')^2 \leq (AK)^2 + (RB)^2 \leq \alpha(K^2 + B^2)$ where $\alpha \equiv max\,\{A, R\}$. Then assumption 1 (iii) implies $\alpha^\theta < \beta^{-1}$ as desired.

d) As noted by Stokey, this is equivalent to showing that $|U(x, y)|$ is bounded for $\|x\| = 1$ and $y \in \Gamma(x)$. This is satisfied by (5) as $\theta \in (0, 1)$ implies that U is bounded below as long as $C \geq 0$ (condition (iii) in (5)) and bounded above by conditions (i) and (ii) in (5) as long as $\|x\| = 1$. ■

• **Part B:** $(A) - (iii)$.

This follows from the contraction mapping proved in $(A) - (ii)$ and the results from theorems 4.8 and 4.10 of Stokey et al. (SLP, 1989). We showed that Γ is non-empty, compact-valued, and continuous (assumption 4.3 is SLP), U is bounded and continuous on $Gr\Gamma$ (assumption 4.4 is SLP), U is, as will be shown below, strictly concave as $\theta \in (0, 1)$ (assumption 4.7 is SLP) and Γ has convex graph from the linear structure inherited from restriction $(i)-(iii)$ of the centralized optimization problem. Then, theorem 4.8 in SLP implies that V is strictly concave, and theorem 4.10 that V is differentiable. Moreover, using the results from $(B)-(i)$, G is continuous.

To guarantee the concavity of the return function, we must proceed in several steps: First, U takes positive values when consumption is interior, as $U(C) = C^{\theta}/\theta$ and $\theta \in (0,1)$. Second, U is strictly concave in C and, as $C = AK + RB - B' - K'$, U is strictly quasi-concave in $[x, y]$ with $x = [K, B], y = [K', B']$. Then, by exercise 4.8 in Stokey et al. (1989, see page 91), U is strictly concave. [8] ■

- **Part C:** $(B)-(ii)$.

We need to show that $\frac{B'(K,B)}{K} \equiv g_2(b)$ is increasing in b. Using Lemma A1 we will show that $g_2(b)$ is increasing in b. First, note that any pair $b_1 > b_2$ there are $[K_1, B_1] > [K_2, B_2]$ such that $b_1 \equiv B_1/K_1 > b_2 \equiv B_2/K_2$ as $K_1 > K_2 > 0$. Then, we need to show that $B'(K_1, B_1) > B'(K_2, B_2)$. To do this, note that from lemma A1, we have:

$$B'(K_1, B_1) \geq B'(K_2, B_1) \geq B'(K_2, B_2)$$

This implies,

$$g_2(b_1) = \frac{B'(K_1,B_1)}{K_1} \geq \frac{B'(K_2,B_2)}{K_1} > \frac{B'(K_2,B_2)}{K_2} = g_2(b_2),$$

where the last (strict) inequality comes from $K_1 > K_2 > 0$ and $B < 0$. As $b_1 > b_2$ g_2 is increasing in b. To see the continuity, it is clear that the continuity at $K = 0$ and $B = -\infty$ can't be verified. Thus, we only check the cases when b is finite. Note that $b \to \hat{b}$ if $K \to a$ and $\hat{b} \equiv B/a$ for some $B > -\infty$ and $a > 0$. Then, we have:

$$lim_{K\to a}\frac{B'(K,B)}{K} = lim_{K\to a}B'\left(1, \frac{B}{K}\right) = B'(1, \frac{B}{a}) = B'\left(1, lim_{K\to a}\frac{B}{K}\right)$$

[8] The exercise is stated for homogeneous functions of degree 1. However, the proof can be extended for this case as the property is only required to show the existence of a constant γ such that $\gamma^{\theta}U(c) = U(\gamma c) = U(c')$ with $U(c) \geq U(c')$. As $\gamma^{\theta}, \gamma \in (0,1)$, the proof is the same.

where $a \in \mathbb{R} \cup +\infty$ and the continuity of B' follows from theorem A1 after noticing that u is strictly concave and thus the u.h.c of G can be extended to continuity using standard results. A similar argument can be used for $B \to a$ when $a \in \mathbb{R} \cup 0$. ■

• **Part D:** $(B) - (iii)$.

We only show that g_1 is increasing since the continuity follows the same argument used in part C. From lemma A1, we know:

$$\tfrac{K'(K,\lambda B)}{K} = g_1(\lambda b) < g_1(b) < \tfrac{K'(\lambda K,\lambda B)}{K} = \lambda g_1(b)$$

where the first inequality follows from lemma A1 and $\lambda > 1$ and the second from the fact that $K, K' > 0$ and $\lambda > 1$. Then, there exists an $\alpha < 0$ such that $\lambda^\alpha g_1(b) = g_1(\lambda b)$ which in turn implies

$$0 < \lambda^\alpha = \frac{g_1(\lambda b)}{g_1(b)} < 1 \tag{6}$$

Since $\lambda b < b$ and the previous equation shows that $g_1(\lambda b) < g_1(b)$, we have that g_1 is increasing in $b < 0$ for any A, R. ■

■ ■

We now prove the stochastic version of the previous theorem.

Proof of Theorem 1

As in theorem 1-NS, we proceed in 4 steps. • Part A: $3-(A)-(i), 3-(A)-(ii)$ and $3-(B)-(i)$, • Part B: $3-(A)-(iii)$, • Part C: $3-(B)-(ii)$ and • Part D: $3-(B)-(iii)$.

Part A: $3-(A)-(i), 3-(A)-(ii)$ and $3-(B)-(i)$.

The proof follows directly from section 9.3 of Stokey—et. al. (1989). A one-to-one mapping exists between assumptions a) - d) in theorem A1 and assumptions 9.18 to 9.20 in Stokey. et al. (1989), after adapting the definition of the feasibility correspondence. Let Γ be the correspondence defined by restrictions $(i)-(iii)$ of the stochastic centralized optimization problem. Then: $\Gamma : X \times \mathbb{A} \to X$. Then we have assumption a) in Th. A1 is equivalent to assumption 9.18, assumptions b) and c) are contained in assumption 9.19 and assumption d) in 9.20. Then, $3-(A)-(i), 3-(A)-(ii)$ and $3-(B)-(i)$ follows from exercise 9.10-a). ■

Part B: $3-(A)-(iii)$.

As can be seen on exercise 9.10.b and 9.10.c in Stokey, et. al. (1989, see page 272 and 273), the result follows from the same arguments used in Part 1-A-(iii) of the proof in theorem 1-NS. ■

Part B: $3-(B)-(ii)$.

The proof of lemma A1 in theorem 1-NS-(B)-(ii) can be easily extended for any $A_i \in \mathbb{A}$ after noticing that $\beta V(K', B', A')$ can be replaced with $\beta E_{A_i} V(K', B', A')$. This can be done without loss of generality because we showed the strict concavity and differentiability of V for the stochastic case in part 3-A-(iii). Using the analogous of lemma A1, the fact that g_2 is increasing and continuous in $b \equiv B/K$ for any $A_i \in \mathbb{A}$ follows from the same arguments used in $1-(B)-(ii)$ ■

Part B: $3-(B)-(iii)$.

Given that lemma A1 also holds for the stochastic case, we can use (6) to show that g_1 is decreasing and continuous in b for any $A_i \in \mathbb{A}$. To show that $g(0) > 0$, note that conditions $(i)-(iii)$ in the non-stochastic centralized problem are equivalent to conditions $(i)-(iii)$ in the stochastic centralized problem. Thus, we can use the same argument used in $1-(B)-(iii)$ to show that $g(0) = 0$ implies a contradiction ■

■ ■

Proof of Lemma 4

To prove **(i)** let us consider $b_+(\underline{b}, A)$. It can be shown that there exists a value $\bar{A}$ sufficiently large such that,

$$dfrac b_+(\underline{b}, \bar{A})\underline{b} < 1. \tag{7}$$

To see this, note that an increase in A_H increases $E(A)$, which by the Euler equation raises the growth rate of consumption η_c. But if η_c increases, then the growth rate of the capital stock or the growth rate of total debt B must rise. When $\eta_c > R$, then it must be that $\eta_K > 0$ since the growth rate of B is bounded by R. Moreover, if η_c is large enough, we may have $\eta_K > \eta_B$ which implies that b decreases. Therefore, it is possible to find a $\bar{A}$ such that (7) is satisfied.

We have shown that $b_+(\underline{b}, \bar{A})$ is below the 45-degree line; the next step is to prove that the function $b_+(\cdot, \bar{A})$ is below the 45-degree line for any value of b. Let us start at $b = \underline{b}$. At this value of debt, $\eta_K > \eta_B$ and therefore $b_+(\underline{b}, \bar{A}) > \underline{b}$. The reason is the same as before: at the productivity level $\underline{A}$, $E(A)$ implies a growth rate for consumption such that the capital stock has to grow at a rate higher than B since its growth rate is bounded by R. If the economy remains at $\bar{A}$, b decreases and eventually will converge towards zero.

To prove **(ii)**, let us consider a level of debt $b = \epsilon < 0$ sufficiently close to zero. As we have shown $b_+(\epsilon, \bar{A}) < \epsilon$, but since the marginal utility of consumption is unbounded for low values of consumption, it is possible to find a value of the technology low enough such that $b_+(\epsilon, \underline{A}) > \epsilon$. Then, there are two possibilities. The first one is that at some point, $b_+(\cdot, \underline{A})$ crosses the 45-degree line. In that case, there is an interior non-trivial steady state b^* such that $b_+(b^*, \underline{A}) = b^*$. The second option is that $b_+(\cdot, \underline{A})$ is above the 45 degree line for any value of b and $b_+(\underline{b}, \underline{A}) > \underline{b}$.

Appendix C: Database methodology

This section provides a detailed description of the databases used in our empirical analysis. We gathered data from 14 countries, 10 in Latin America (Argentina, Bolivia, Brazil, Chile, Colombia, Ecuador, Mexico, Peru, Paraguay and Uruguay) and 4 in Europe (Belgium, Italy, Portugal and Spain).

Generally, we rely on data sets developed by international organizations to ensure consistency across countries. The analysis period varies according to data availability and quality; we include at least 40 years of data, and only one has less than 30 years. We extensively use the Application Programming Interface (API) of the World Bank (WBGAPI) and International Monetary Fund APIs and Data Services. We preserve their notation to facilitate the replication of our results.

External debt stocks

The Guide for Compilers and Users of the External Debt Statistics from the IMF (*The Guide* henceforth) defines the (gross) external debt stock as "the outstanding amount of those actual current, and non-contingent, liabilities that require payment of interest and/or capital by the debtor at some point in the future and that are owed to nonresidents by residents of an economy". These external liabilities are disaggregated[9] by the debtor/institutional sector –public or private– or by its maturity –short-term or long-term. Private debt comprises all of the external obligations of private debtors and is named "Private non-guaranteed External Debt Stock". This definition can also be adjusted by subtracting the obligations of private debtors guaranteed by a public entity.

Short-term debt comprises all of the external obligations that have a maturity of one year or less, while long-term debt comprises those with a maturity of more than one year. Therefore, for any given country at time t, the total external debt stock,

[9]This classification is consistent with the Sixth Edition of the Balance of Payments and International Investment Position Manual (hereafter, BPM6) published by the IMF (2009).

$D_{TOT,t}$, is composed of both public and private-issued debt

$$D_{PUB,t} = D^{ST}_{PUB,t} + D^{LT}_{PUB,t} \quad \text{and} \quad D_{PRIV,t} = D^{ST}_{PRIV,t} + D^{LT}_{PRIV,t}.$$

We are interested in estimating D_{PRIV}. To construct the database for the external debt stocks, we rely on 3 data sources: The International Debt Statistics (IDS) database, developed by the World Bank; the Quarterly External Debt Statistics (QEDS) database, jointly developed by the World Bank and the International Monetary Fund, and the Balance of Payments and International Investment Position (BoP/IIP) Report, developed by the International Monetary Fund. We combine the information from these sources, ensuring consistency across measures.

IDS (WB) & QEDS (WB/IMF)

The International Debt Statistics (IDS) database covers low- and middle-income countries that report their external debt stocks to the World Bank's Debtor Reporting System (DRS). The main characteristic of this database is that it covers data from 1970/75 to the present for all reporting countries. However, only 8 of the countries in our sample are included in this dataset[10]. This source is valuable because these countries are heavily under-reported in the other two data sources. We focus on the following six indicators.

Indicator name	**Ident. Code**
External debt stocks, total	DECT
External debt stocks, long-term	DLXF
External debt stocks, short-term	DSTC
External debt stocks, public and publicly guaranteed (PPG)	DPPG
External debt stocks, private nonguaranteed (PNG)	DPNG
Use of IMF credit	DIMF

The use of IMF credit (DIMF) is expressed as an independent indicator and corresponds entirely to the public sector. Second, DPPG and DPNG refer only

[10]These countries are ARG, BOL, BRA, COL, ECU, MEX, PER, and PRY.

to long-term debt; therefore, short-term external debt lacks sectoral disaggregation. The external debt stocks is $\mathrm{DECT}_t = \mathrm{DLXF}_t + \mathrm{DSTC}_t + \mathrm{DIMF}_t$ where $\mathrm{DLXF}_t = \mathrm{DPPG}_t + \mathrm{DPNG}_t$.

The item DSTC for short-term debt lacks sectoral disaggregation. Given that the share of short-term debt in total debt ($DSTC/DECT$) is above 10% for most countries, we use additional data sources to obtain consistent estimates.[11] Thus, we use the Quarterly External Debt Statistics (QEDS) database, which includes data for low-, middle- and high-income countries that subscribe to the IMF's Special Data Dissemination Standard (SDDS). This database, jointly developed by the World Bank (WB) and the International Monetary Fund (IMF), breaks down external debt stocks by debtor, maturity, instruments, and currency. However, this data set is only available between 2003 and 2006 and quarterly. Hence, we use Q4 estimates and the short time span to differentiate between the public and private components.

The QEDS database has disaggregated data for four institutional sectors.[12] Two of them correspond to the public sector (*General Government* and *Central Bank*) and the other two (*Deposit-taking corporations excluding Central Bank* and *Other Sectors*) correspond to the private sector. Following IMF's criteria, Direct Investment is identified as a separate sector, *Direct Investment: Intercompany Lending*, which we consider part of the private sector. The first four sectors can be separated into long-term and short-term components, but this is not possible for Direct Investment; we follow IMF's conventional criterion and classify it as long-term. Thus, we have 14 indicators.

To construct the sectoral breakdown, we define α_t as the share of private short-term external debt on total short-term external debt at time t,

$$\alpha_t = \frac{\mathrm{DSTC}_{\mathrm{PRIV},t}}{\mathrm{DSTC}_{\mathrm{TOT},t}} = \frac{\mathrm{DSTC}_{\mathrm{CB},t} + \mathrm{DSTC}_{\mathrm{OT},t}}{\mathrm{DSTC}_{\mathrm{GG},t} + \mathrm{DSTC}_{\mathrm{MA},t} + \mathrm{DSTC}_{\mathrm{CB},t} + \mathrm{DSTC}_{\mathrm{OT},t}}$$

[11] Notwithstanding, if we consider short-term debt entirely private, our results do not change substantially.

[12] This follows BPM6's classification of Institutional Sectors, which we will use for IIP analysis.

Indicator	Sector	Maturity	Ident. code
Gross Ext. Debt Pos.	All sectors	All maturities	DECT
	General Government	All maturities	DECT_{GG}
		Long-term	DLXF_{GG}
		Short-term	DSTC_{GG}
	Central Bank	All maturities	DECT_{MA}
		Long-term	DLXF_{MA}
		Short-term	DSTC_{MA}
	Deposit-Taking Corp., exc. CB	All maturities	DECT_{CB}
		Long-term	DLXF_{CB}
		Short-term	DSTC_{CB}
	Other Sectors	All maturities	DECT_{OT}
		Long-term	DLXF_{OT}
		Short-term	DSTC_{OT}
	DI: Intercom Lending	All maturities	DECT_{IL}

We use $\bar{\alpha}$, the average value of α_t for each country, to estimate external debt stocks throughout analysis (1970/75-2019).[13]

$$D_{PRIV,t} = \underbrace{\text{DPNG}_t}_{D^{LT}_{PRIV,t}} + \underbrace{\bar{\alpha} \cdot \text{DSTC}_t}_{D^{ST}_{PRIV,t}}$$

BoP/IIP (IMF)

The International Investment Position (IIP) is defined in *The Guide* as "a statistical statement that shows at a point in time the value and composition of (1) financial assets of residents of an economy that are claims on nonresidents ..., and (2) liabilities of residents of an economy to nonresidents". The IMF published the IIP along with the Balance of Payments. According to *The Guide*, the "gross external debt position equals the debt liabilities in the IIP statement, i.e., total IIP liabilities excluding all equity (equity shares and other equity) and investment fund shares and financial derivatives and employee stock option (ESO) liabilities". With these definitions, we compute the external debt stocks of the remaining countries[14]. The IIP provides the

[13]There are only two countries for which it is significantly different from 1, ARG and MEX.
[14]These are BEL, CHL, ESP, ITA, PRT, and URY.

information necessary to compute a country's gross external debt position at a given year, which is disaggregated by institutional sector and maturity.

The IIP is based on BPM6 functional categories and institutional sectors, ensuring consistency between the balance of payments (BoP) and the IIP. The gray-shaded entries in the following table (see Table A4.1 in *The Guide*) represent debt liabilities covered on the gross external debt position. [15] Several remarks must be made. First,

	Type	**Functional category**		**Ident. code**	**Sector**
IIP	Assets	-	-	-	-
	Liabilities	DI	Equity and investment fund shares	-	-
			Debt instruments	ILDD	Private
		PI	Equity and investment fund shares	-	-
			Debt securities	ILPD	$ILPD_{GG}$ $ILPD_{MA}$ $ILPD_{CB}$ $ILPD_{OT}$
		FD	-	-	-
		OI	Other equity	-	-
			Special drawing rights	ILOSDRFR	Public
			Other debt instruments	ILOOFR	$ILOOFR_{GG}$ $ILOOFR_{MA}$ $ILOOFR_{CB}$ $ILOOFR_{OT}$

DI = direct investment; PI = portfolio investment; FD = financial derivatives (other than reserves) and employee stock options; OI = other investment.

Debt Instruments related to direct investment relationships (ILDD) are classified as private and correspond to the category DI (*Intercom. Lending*). Second, Special Drawing Rights Allocations (ILOSDRFR) are classified as public. Third, Debt Securities in PI (ILPD) and Other Debt Instruments in OI (ILOOFR) have a sectoral breakdown. As for the QEDS database, we use subscripts to identify each sector. However, notice that this sectoral breakdown is not exhaustive, and it is not possible to identify public and private components of DI, CB, and OT. Still, given that public

[15]BPM6's classification includes four sectors: 2 of them correspond to the public sector (*General Government* and *Central Bank*) and the other 2 to the private sector (*Deposit-taking corporations excluding Central Bank* and *Other Sectors*).

DI and public banking account for a small portion of the total, we should expect minor inaccuracies. Thus, the private-issued external debt is defined as

$$D_{PRIV,t} = \text{ILDD}_t + \underbrace{\text{ILPD}_{\text{CB},t} + \text{ILPD}_{\text{OT},t}}_{\text{ILPD}_{\text{PRIV},t}} + \underbrace{\text{ILOOFR}_{\text{CB},t} + \text{ILOOFR}_{\text{OT},t}}_{\text{ILOOFR}_{\text{PRIV},t}}$$

The main drawback of the IIP data is that it has multiple gaps. To address this issue, we fill the missing gaps for a missing sector by subtracting the other sectors from the total. If this is impossible because two institutional sectors are missing, we group them into public (GG and MA) and private (CB and OT) sectors. This methodology allowed us to fill in missing data for 4 of the 6 IIP-reporting countries; the series started in the early 1980s except for Portugal (1993). For the two remaining countries, Chile and Uruguay, we follow Acosta (2020):

- Chile: IIP data is complete for 2003-2019 but partially or incomplete for years before 2000. We use two sets of external debt statistics provided by the Central Bank of Chile: the External Debt 1960-2000 based on the Economic & Social Indicators report and the External Debt by Institutional Sectors. Both are consistent with the IIP classification.

- Uruguay: IIP data is complete for 2002-2019 and partially or incomplete for years before 2002. For those years, we rely on Acosta's estimates, but there is no information on consistency with IIP.

We deflate external debt with the CPI to obtain constant 2017 US dollars (USD) and represent it by $D_{PRIV}\_\text{USD2017}_t$.

Capital stock

The capital stock is obtained from the Penn World Tables v. 10.01 (see Feenstra et al. (2015)). The capital stock is usually divided into structures and equipment.

Structures include residential and non-residential buildings, while Equipment includes Machinery, Transportation Equipment, and Other Assets. Thus, we define Productive Capital as the sum of Machinery and Equipment and Total Capital as the sum of all items. The PWT provides the capital stock and a price deflator, and we use both exchange rates to obtain an estimate of the stock of capital at US 2017 prices.

National Accounts data: GDP & Balance of Trade

GDP and real growth rate

We use PWT's National Accounts detail and focus on q_gdp, representing GDP at constant national 2017 prices. We then compute GDP at constant 2017 USD (q_gdp_USD2017) by dividing q_gdp by the US exchange rate, xr.

Balance of Trade

To obtain trade balance (TB) data, we use two data sets published by the IMF. Our primary data source is the BoP/IIP database, from which we obtain the value of the trade balance of goods and services (the net exports of goods and services). However, given that trade on goods and services is not available for all countries for all years under study, we rely on a complementary data source, the Direction of Trade Statistics (DOTS) database, also developed by the IMF. This database gathers data on merchandise (only goods) and trade statistics (exports and imports). We fill in missing data from the primary data source with DOTS data. Trade balance data is published in current U.S. dollars (USD); thus, the data is converted into constant 2017 U.S. dollars (USD). The TB to GDP ratio in continuous 2017 USD is denoted by TB_gdp_t.

Prociclicality analysis

The variables relevant for the procyclicality analysis are

1. D_{PRIV}_USD2017$_t$, private sector external debt at constant 2017 U.S. dollars.

2. N_USD2017$_t$, capital stock at constant 2017 U.S. dollars.

3. q_gdp_USD2017$_t$, Gross Domestic Product at constant 2017 U.S. dollars.

4. g_t, growth rate of the Gross Domestic Product

5. TB_gdp_t, trade balance to GDP ratio

The debt-to-capital ratio that is defined as $\frac{D_t}{K_t} = \frac{D_{PRIV}\text{\_USD2017}_t}{N\text{\_USD2017}_t}$. We then compare $\frac{D_t}{K_t}$ and g_t with its median values. If the variable's value is above the median, it is in the PCB region; in any other case, it is CCB. We then compute the correlation between TB_gdp_t and g_t, a more typical procyclicality measure.