Ergodic Stationary Equilibrium in Open Economies with Collateral Constraints: Long run implications for Balance of Payments Crises *

Damian Pierri[a,†] Kevin Reffett[b,‡]

[a] Department of Economics-Universidad Autonoma de Madrid, and ESCP Business School
[b]Department of Economics, Arizona State University
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Abstract

Using generalized Markov equilibrium (GME) methods to characterize the sequential competitive equilibrium (SCE), we prove the existence of ergodic stationary equilibrium in the sudden stops model of Bianchi ([12]), which is the workhorse model of financial crises in emerging markets in the international economics literature. Our ergodicity result holds even in the presence of multiple and discontinuous SCE equilibria. We show that by adding an (endogenous) state variable to the GME representation of SCE, we are able to extend the representation of memory relative to simple (minimal state space) recursive equilibrium, which allows us to construct an ergodic GME selector from the set of SCE. Efficient numerical methods are shown to confirm the robustness of our framework in analyzing global stability amid recurring balance of payments crises. We then apply our results to the critical question of characterizing long-run implications of balance-of-payment crises, and we show in the ergodic stationary equilibrium, sudden stops do not disrupt the economy's stochastic steady state. In particular, we identify conditions where crisis-induced deleveraging resets debt cycles, generating a recurrent and, thus, stable long-run stochastic dynamic structure.

Keywords: Financial Crises, Sudden Stops, Small Open Economies, Ergodicity, Recursive Equilibrium, Generalized Markov Equilibria

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†damian.pierri@gmail.com, www.damianpierri.com.
‡kevin.reffett@asu.edu

1 Introduction

The literature on sudden-stop models of financial crises in emerging markets is extensive and growing. The workhorse model in this literature is discussed in Bianchi ([12]) and Schmitt-Grohé and Uribe ([63], [64]), among others. The authors present a stochastic two-sector endowment small-open economy where agents face an occasionally binding price-dependent equilibrium collateral constraint.[1] The presence of a relative price of nontradables to tradables in equilibrium collateral constraints introduce a well-known "pecuniary externality" into the structure of the dynamic equilibrium, and it is the critical feature which allows these models to explain the collapse of consumption and prices during financial crises, as well as creating a potential role for the government macroprudential policies that prevent such financial collapses and Fisherian deflations. The presence of these equilibrium collateral constraints is also known to introduce significant complications in the structure of dynamic equilibrium, yet little is known about the existence of long-run stationary structure of dynamic stochastic equilibrium in these models. In particular, little is known about either the existence of sequential equilibria or how to characterize the set of stationary equilibrium associated with SCE stochastic dynamics.

The existence of recursive equilibrium is proven in Pierri and Reffett ([53]). In a recent series of papers by Schmitt-Grohé and Uribe ([62], [63]), and in addition in Pierri and Reffett ([53]), the authors have argued that the presence of pecuniary externalities is a potential source of multiple sequential competitive equilibria (SCE) as well as sunspot equilibria. Multiple equilibria greatly complicate the characterization of stationary equilibrium dynamics as SCE are typically not continuous relative to initial conditions (and recursive equilibrium is not continuous on their minimal state space). This challenges characterizing the set of long-run stationary distributions associated with the model. Given equilibrium discontinuities, questions about the existence of stationary equilibria are an issue. Further, and very important for the estimation of these models (e.g., as in Benigno et al. ([9]), the lack of ergodic selectors from the set of SCE also creates a challenge for interpreting the estimated parameters of econometric versions of these models. Even simulations from calibrated models may differ in stochastic properties from those generated using an ergodic stationary equilibrium.

In this paper, we contribute to the literature in many ways. From a theoretical point of view, the paper makes many contributions. First, we prove the existence of a Sequential Competitive Equilibrium (SCE), as well as show how to construct a Generalized Markov Equilibrium (GME) representation of SCE for the workhorse model of Bianchi ([12]). Second, and perhaps most central to the paper, we prove the existence of *ergodic* GME selectors from the set of SCE. In particular, using a novel approach to characterizing stationary equilibria based on Generalized Markov Equilibrium (GME) methods, we formally characterize and compare the stochastic equilibrium dynamics across *different classes* of stationary equilibria. . That is, we prove the existence of such ergodic GME selectors from the set of SCE, and we demonstrate that the memory structure in the GME representations is critical for proving the existence of ergodic stochastic equilibrium dynamics. Since much of the quantitative literature (e.g., Bianchi, ([12])) focuses on *short-memory* (minimal state space) recursive equilibrium, these findings are particularly significant both from a theoretical and applied point of view.

Our new ergodicity result is unexpected, as given multiple equilibria, stochastic equilibrium dynamics are in general *discontinuous* as a function of initial conditions. This implies standard methods for proving ergodicity, which relies on the continuity of SCE (or recursive equilibria), fail here. Thus, our result on the existence of ergodic stationary equilibrium is surprising. Although this paper focuses on GME methods in the context of the canonical two-sector endowment sudden-stop models, the broader methodological theme of this paper is *much more* general. It advocates using GME approaches in various stochastic equilibrium models with endogenous equilibrium collateral constraints and multiple equilibria.

The critical factors in our identifying ergodic GME selections are the occasionally binding equilibrium price-dependent collateral constraints that arise naturally in this class of models and the flexibility of the

[1] A small sampling of additional work in this literature includes the early work of Mendoza ([43], [44], [45]), Bianchi ([12]), Bianchi and Mendoza ([13], [15], [16]), Benigno, Chen, Otrok, Rebucci, and Young (e.g., see [6], [7], [8], [10]), Bianchi, Liu, and Mendoza ([14]),Seoane and Yurdagul ([59]), Deveraux and Yu ([25]), and Schmitt-Grohe and Uribe ([62], [63], [64]), Chi, Schmitt-Grohe, and Uribe ([21]), Ottonello, Perez, and Vassaco ([50], [51]), Bengui and Bianchi ([11]), Rojas and Saffie ([57]), Pierri, Mira and Montes-Rojas ([52]), Drechsel ([28]), Drechsel and Kim ([29]), Arce [4], Arce, Bengui and Bianchi ([5]), among many others.

GME equilibrium framework. By embedding some qualitative features of simple savings models into the GME, the collateral constraint helps to characterize the regeneration properties of stochastic dynamics, which are crucial for proving the existence of ergodic GME selectors. Mathematically, these selectors are identified by constructing an "atom" in the equilibrium Markov process, where it regenerates. This regeneration property is closely linked to the global stability of stochastic equilibrium dynamics. The ergodicity of SCE selectors thus becomes a natural tool for analyzing short- and long-term stochastic dynamics across multiple stochastic paths.

Our theoretical findings also allow us to explore the *differences* between the minimal state space recursive equilibrium, the canonical definition used in the literature, and the GME representations of SCE in characterizing stochastic equilibrium dynamics. To do this, we derive a computable framework to approximate and analyze the short- and long-term properties of models with price-dependent inequality constraints and multiple equilibria. Despite expanding the state space, we efficiently compute ergodic GME selections by utilizing a block-recursive structure within the equations system defining the sequential equilibrium. This results in a numerically efficient global framework characterizing both the short- and long-term stochastic dynamics in incomplete market models that feature financial frictions, multiple equilibria, and price-dependent inequality constraints.

We are also able to demonstrate that it is possible to use GME methods to compare stationary, non-stationary, and ergodic equilibrium selections and their long-term stochastic properties. Using numerical methods, we find that ergodic selections have a smoother consumption path than other GME selectors. Since all GME equilibria represent the same type of financial friction and market incompleteness, we conclude that these differences arise from another factor. Specifically, only under ergodic GME selections do financial crises lack permanent effects. The excessive consumption volatility seen in stationary non-ergodic equilibria, including the canonical version used in the literature, is caused by unstable paths that move the economy away from the steady state. In this sense, the permanent effects of balance-of-payment crises reduce long-term consumption smoothing, leading to lasting impacts on welfare.

Our findings also have practical significance for understanding the stochastic structure of the stationary equilibrium of these economies. We show this fact by applying our results to the study of the long-run structure of the balance of payments crises that arise in the models. Sustained access to international capital markets enables risk sharing and consumption smoothing, which is generally considered welfare-enhancing under standard assumptions. However, a balance of payments crisis disrupts foreign credit availability, significantly impacting residents' well-being. While the short-term effects of such crises are well understood, we argue that their long-term consequences are not. This gap exists because modern literature on the balance of payments crises typically relies on a canonical recursive equilibrium framework with minimal state space to analyze the stationary distribution of endogenous variables. We claim that this framework must be revised to study the long-run dynamics.

Our GME approach allows us to characterize the long-run stochastic structure of balance of payments crises in emerging economies. We construct selections directly from the sequential equilibrium incorporating the appropriate amount of memory from the sequential into the recursive equilibrium via the additional state variable. This added flexibility allows us to manage stochastic paths, ensuring recurrence and, thus, stability. Although balance of payments crises may increase volatility in consumption and external debt, we prove this is insufficient to destabilize the global economy. Consequently, even when such crises repeatedly hit an economy, using the appropriate definition of recursive equilibrium, we can prove that stochastic paths remain within the model's steady state. The deleveraging that follows a crisis regenerates the debt cycle, providing the economy with enough recurrence to maintain stability. Since these crises do not alter the steady state, they do not have permanent effects.

Two significant challenges arise when examining the long-run effects of balance of payment crises in these models. First, as mentioned earlier, is the presence of multiple equilibria and discontinuous selections. This fact precludes standard results from proving the existence of stationary equilibrium, as they rely on its continuity. We should mention that this fact would also complicate the econometric treatment of these models as standard methods, like the simulated method of moments, require the stationarity and the ergodicity of the equilibrium selection.

Second, and less frequently discussed yet still economically significant, is the relationship between ergodicity and global stochastic stability, which is robust to discontinuous equilibria. While it is possible to prove that an equilibrium is ergodic if it is globally stable, the canonical equilibrium concept, with its limited state space, lacks the flexibility needed to guarantee global stability. This last representation implies that a crisis could destabilize the economy, moving it away from its steady state, which generates lasting effects contradicting the common assumption that the frequency of crises does not impact the long-run distribution of the economy.

From a formal standpoint, we introduce a new equilibrium concept with an expanded state space to demonstrate how the balance of payments crises shape the economy's stochastic steady state. We argue that carefully selecting an additional state variable can control economic paths and link the equilibrium's global stochastic stability with the model's ergodic steady state. This connection between ergodicity and stability directly impacts model calibration and structural estimation. By restricting the curvature of the utility function, we prove that a point in the set of deep parameters defines an equilibrium and an associated frequency of crises, represented by a distribution of finite hitting times in the stochastic process. We show that this distribution is closely tied to a specific steady state of the economy, which is also an ergodic distribution. Therefore, repeated balance of payments crises do not have long-term effects, as they do not impact the economy's stochastic steady state.

1.1 This Paper and the Existing Literature

The literature argues that inefficiencies in models of balance of payment crises come from the higher-order moments of the ergodic distribution. The presence of overborrowing generates that the stationary distribution of debt has fatter left tails compared to the efficient equilibrium (See Bianchi, [12]). At the same time, this class of models typically has multiple equilibria (see, for instance, Schmitt-Grohé and Uribe,[63]). Thus, to calibrate and simulate the model, we need a solid theoretical structure for the presence of multiple equilibria that accurately replicate the long-run distribution of endogenous variables. Our results show how to construct an ergodic selection, allowing us to simulate the model safely.

The literature often deals with local dynamics or with numerical simulations. However, to our knowledge, very few papers characterize global stochastic dynamics theoretically using numerically efficient methods. We prove the stochastic steady state is globally stable despite the frequent balance of payment crises. Thus, these events do not have permanent effects because they do not change the economy's steady state. That is, any point in the set of deep parameters of the economy defines a frequency of crises which, at the same time, generates at least one ergodic distribution (one for each possible selector).

Relative to the work of Bianchi ([12]), we extend his results in several directions. We study stochastic equilibrium dynamics within the context of a GME representations of SCE. The GME representations allow us to address many new and exciting features. For example, we can compare the stochastic properties of different types of stationary equilibrium within the context of the same model and even deal with a non-stationary representation.

Relative to the work of Schmitt-Grohé and Uribe (e.g., [63], [64])), we take a very different quantitative approach to studying the properties of stochastic dynamic equilibria by using GME methods. In Schmitt-Grohé and Uribe ([63], [64]), their approach to characterizing the existence of multiple equilibria is built upon the deterministic versions of the model, and in particular "local" sequential equilibrium behavior, near a (deterministic) steady-state.[2] Our results are never "local" or "deterministic", rather always global and stochastic. Our approach also builds a theory of stationary stochastic equilibrium from any arbitrary initial condition, providing a systematic approach to understanding the interplay between modeling multiple equilibrium, memory, and the associated stationary equilibrium from the vantage point of selectors from the set of GME representations of SCE. What is critical in our approach is the presence of "hit" times for the collateral constraint, which is the formal representation of a balance of payment crises. Then, they are used to regenerate the equilibrium Markov process.

Our GME approach is general and can potentially be applied to other models of Sudden Stops found in the existing literature, such as models with elastic labor supply and production. Versions of the models with production include the early papers of Mendoza ([44]) and Mendoza and Smith ([46])), as well as

[2] These papers then can characterize stochastic SCE dynamics using techniques building "stationary sunspot" approaches (e.g., see related work in Woodford ([66]) and Schmitt-Grohé ([60]), for example).

the series of interesting papers by Benigno, Chen, Otrok, Rebucci, and Young (e.g., see [7], [8]), [10]).[3]

This paper also contributes to the literature on self-generation methods and GME methods via strategic dynamic programming approaches to dynamic stochastic general equilibrium models. Self-generation techniques were first introduced in repeated games in Abreu, Pearce, and Stacchetti ([1], [2]) and are related to the implementation of the methods for studying the existence of Markovian equilibrium in dynamic stochastic models found in Blume ([18]) and Duffie, Geanokopolis, Mas-Colell, and McLennan ([27]). Approaches to making operational these methods in the context of GME representations and enlarge state spaces stems from the work of Kydland and Prescott ([40]), Feng, Miao, Peralta-Alva, and Santos ([31]), among others.[4]A novel aspect of this paper is that by exploiting the structure of equilibrium price-dependent collateral constraints, we can propose a systematic approach to SCE selections, based upon ergodic GME selectors.

Since Kydland and Prescott ([35]), it is well-known that some recursive problems require an additional state variable to represent their sequential equilibrium recursively. In this sense, expanding the state space helps prove the existence of a GME representation of sequential equilibrium on an enlarged (endogenous) state space. In this paper, we are interestee in using the enlarged set of state variables to shparpen the characterization of the stochastic equilibrium dynamics associated with a sequential equilibrium, and in particular the recurrent stochastic structure of collateral constrained equilibrium regimes. Conveniently, we are able to obtain this sharper characterization of global stochastic equilibrium dynamics by adding a single (endogenous) state variable to the minimal state space to obain a canonical representation our generalized Markovian equilibrium. We use that structure to prove the existence of a recurrent structure in the stochastic equilibrium dynamical system associated with the recursive equilibrium.

It is also important to mention that at least since the paper of Blume ([18]), when studying questions of equilibrium stochastic stability, it has long been recognized the trade off between the multiplicity of sequential equilibria and the continuity properties of the associated recursive representation. In other words, the presence of multiple equilibria can generate *discontinuous* recursive equilibria. By enlarging the state space, it is sometimes possible to obtain a continuous Markov equilibrium (see Pierri, [49]); but unfortunately, there is no general theory about how to do this (see Kubler and Schmedders, ([38]), for a counter-example). In the context of models with price-dependent collateral constraints, we show that the GME methods we develop are critical in the presence of this type of discontinuities, which may affect the existence of an ergodic equilibrium.

Given the approach to characterizing dynamic equilibria in this paper, we also address many of the interesting questions raised in recent work that discussed the critical difference between local versus global methods for these models relative to solving equilibrium functional equations (e.g., as discussed in the new work of De Groot, Durdu and Mendoza ([26]) and Mendoza and Villallazo ([47])), but also focuses on the implications of GME methods for characterizing, computing, and simulating ergodic, stationary, and non-stationary equilibria.

The remainder of the paper is as follows: Section 2 discusses the model and proves the existence of SCE. Section 3 defines and characterizes the GME. Section 4 proves the existence of ergodic equilibrium selectors studies stationarity equilibria. Section 5 discusses global dynamics, solves the model, and computes and simulates ergodic, stationary, and non-stationary equilibrium. Section 6 concludes and presents directions for future research.

2 The Model and Sequential Competitive Equilibrium

We consider the endowment version of the two-sector small open economy model studied in Bianchi ([12]) and Schmitt-Grohe and Uribe ([62], [63], [64]). The model is an open economy with a fixed interest rate. Time is discrete over an infinite horizon and indexed by $t \in \{0, 1, 2, ...\}$. There is a representative agent and two sectors of perishable goods, a tradable consumption good y_t^T and a non-tradable consumption good y_t^N. Each household has a strictly positive amount of each good in each period. Upon receiving their current period endowments, households sell endowments at current market prices and choose to

[3]These extensions are studied in Pierri and Reffett ([54],[55]).

[4]See Kubler and Schmedders ([39]), and Cao ([20]).

consume both goods. The consumption of tradeable and non-tradeable is denoted, respectively, by c_t^T and c_t^N. The relative price of non-tradeable relative to the numeraire tradeable in period t is given by p_t.

Household preferences are defined over infinite sequences of dated consumption vectors of tradeable and non-tradeable goods $c_t = (c_t^T, c_t^N) \in X \supset \mathbf{R}_+^2$ where X is the commodity space for consumption of tradeable and non-tradeable in each period, and are assumed to be time separable with subjective discount factor $\beta \in (0,1)$. These preferences are represented by a nested utility function, which is a composition of two functions: a utility over composite consumption $U : \mathbf{R} \longrightarrow \mathbf{R}$, and an aggregator $A : X \longrightarrow \mathbf{R}$ over tradeable and non-tradeable consumption $c_t = (c_t^T, c_t^N)$, where the preferences $U(A(c))$ gives the instantaneous utility of the vector of consumption $c \in X$ in any period. Then, lifetime discounted expected utility preferences of a typical household are given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t)) \tag{1}$$

where the mathematical expectation operator here is taken over the stochastic structure of uncertainty concerning the date 0 information.[5]

Uncertainty in the economy is modeled as an iid stochastic process governing tradeable endowments $y = \{y_t^T\}_t$, where each element of the sequence y has distribution given by the measure $\chi(\cdot)$. $\{y_t^N\}_t$ is non-stochastic and $y_t^T \in Y$, where the shock space Y is a finite set. Then, using standard results, it is possible to define an infinite horizon stochastic process for shocks. [6]

Households face a sequence of budget constraints. Given a candidate price sequence $p = \{p_t\}_{t=0}^{\infty}$, and denoting the net debt position for a typical household with debt borrowed at date t but maturing at date $t+1$ by d_{t+1}, the budget constraint for a household in any period t is given by:

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + \frac{d_{t+1}}{R} \tag{2}$$

where $R = 1 + r$, as this is a small open economy, is taken as given. The sequential budget constraints here follow the timing convention used in Schmitt-Grohé and Uribe ([63], [64]), and assume consumption and income decisions are taken at the beginning of the period, and interest is then paid/earned over that same period. We adopt this timing only because it proves to be convenient in characterizing the structure of dynamic equilibrium.[7] In addition, a typical household also faces a period-by-period flow collateral constraint on debt given by:

$$d_{t+1} \leq \kappa (y_t^T + p_t y_t^N) \tag{3}$$

where $\kappa > 0$ [8]

2.1 Existence of Sequential Competitive Equilibrium (SCE)

We now consider sufficient conditions for the existence of a SCE. In a SCE, the representative household takes as parametric an interest rate R, a level of initial debt $d_0 \in D$ (where $D \subset \mathbf{R}$ is a compact set of debt states, which will be constructed in a moment), a stochastic process governing $y^T = \{y_t^T\}_{t=0}^{\infty}$

[5] A typical functional form for the consumption aggregator$A(c)$ in the literature is the Armington/CES aggregator

$$A(c_t^T, c_t^N) = [a c_t^{T\, 1-\frac{1}{\xi}} + (1-a) c_t^{N\, 1-\frac{1}{\xi}}]^{\frac{1}{1-\frac{1}{\xi}}}$$

with $\xi > 0$, $a \in (0,1)$, which is increasing, strictly concave, and supermodular on X when $X = \mathbf{R}_+^2$ with its product order.

[6] e.g., see Stokey, Lucas and Prescott ([65], Chapter 7)).

[7] Bianchi ([12]) uses a slightly different timing convention, but it turns out this timing convention is without loss of generality in our case (see, for example, Adda and Cooper ([3]) For a detailed discussion of this matter).

Also, in this paper, we do not address the critical question of "future" vs "current" wealth collateral constraints, as discussed in Brooks and Davis ([17]) and Ottonello et al. ([51]). Some cases of "future" wealth collateral constraints can be put into our framework as discussed in Pierri and Reffett ([**?**]).

[8] As is well-known in this literature, one can write down more fundamental versions of this model where this debt constraint emerges as an equilibrium object from the primitives of the underlying economy. For our purposes, we follow the literature and impose this form of a price-dependent collateral constraint.

conditional on an initial level of tradeable endowment $y_0^T \in Y$ with $y_0^T > 0$, a constant level of non-tradeable endowment $y_t^N = y^N$, and a history-dependent sequence of measurable prices $p = \{p_t(y^t)\}_{t=0}^{\infty}$, where we use the notation $y^t = \{y_0^T, y_1^T, .., y_t^T\}$ to denote the history of tradeable endowment realizations up to period t. Then, the household solves the following problem:

$$V^*(s_0, p, R) = \max E_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t)) \tag{4}$$

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + \frac{d_{t+1}}{R}; \quad d_{t+1} \leq \kappa(y_t^T + p_t y_t^N),\ t \in \{0, 1, 2, ...\} \tag{5}$$

where the initial states are $s_0 = (d_0, y_0^T)$, and $y_0^T \in Y$. We denote the optimal policy sequences for consumption and debt, achieving the maximum on (4) by

$$c^*(s_0, p, R) = \{c_t^*(s_0, p, R)\}_{t=0}^{\infty};\ d^*(s_0, p, R) = \{d_t^*(s_0, p, R)\}_{t=0}^{\infty}\} \tag{6}$$

Under standard assumption, it is possible to show that this problem also has a dual (see the supplementary appendix). However, for this paper, a primal characterization is more tractable. Then, first-order conditions are given by:

$$[c^N(y^t)][\{\beta^t U'(A(c(y^t))\}\{-A_1(c^T(y^t))p(y^t) + A_2(c^N(y^t))\}] = 0, \quad y^t - a.e \tag{7}$$

$$\begin{gathered} [\kappa\{y^t + p(y^t)y^N\} - d(y^t)][U'(A(c(y^t))A_1(c^T(y^t)) \\ -E_t(U'(A(c(y^t y_{t+1}))A_1(c^T(y^t y_{t+1}))] = 0, \quad y^t - a.e., \end{gathered} \tag{8}$$

where A_i denotes the partial derivative of A with respect to the $i-th$ coordinate. Before defining an equilibrium, we must describe feasibility. As it is standard in the literature, we only require the market of non-tradable goods to clear: $c_t^N = y^N$. Then, a SCE for this economy is then defined as follows: [9]

Definition 1 *A Sequential Competitive equilibrium (SCE) is a collection of progressively measurable random variables for consumption $c^*(s_0, p^*(s_0, R), R)$, debt $d^*(s_0, p^*(s_0, R), R)$, and relative prices of nontradeable to tradeable consumption $p^*(s_0, R)$ such that:*

1) the representative agent chooses $c^*(s_0, p^*(s_0, R), R)$, $d^*(s_0, p^*(s_0, R), R)$ to solve (4) given s_0 at $p^*(s_0, R)$ such that $V^*(s_0, p^*(s_0, R), R)$ is finite and equations (7-8) hold,

2) markets clear $c_t^{N*}(s^t) = y^N$ where y^t holds $\mu_{y_0^T}$-a.e.

Before discussing our sufficient conditions for the existence of SCE, we need to mention a few key technical issues that arise when studying the structure of SCE in stochastic infinite horizon debt models that characterize open economies. We need to impose sufficient structure on the model such that we can obtain enough boundedness (actually, compactness) for the stochastic equilibrium dynamics of the model. The existing literature in small open economies has not addressed how to provide lower and upper bounds for stochastic equilibrium paths for endogenous variables based on primitive conditions. Yet, it is well-known that these issues can be delicate. To obtain such lower and upper bounds, what is typically done in the general equilibrium literature (see, for instance, Braido ([19])) is to impose bounds on the marginal utility of consumption. These bounds involve modifying standard preferences, but *do not affect the stochastic dynamic behavior* of the model when compared to the more "traditional" CES preferences

[9] In the appendix, where we prove the existence of sequential competitive equilibrium, we had the formalities of measurability of SCE processes are more rigorous. See the details there.

(as we shall discuss in the quantitative section of the paper in section 5). [10] They are critical, though, to ensure the necessary compactness, as well as provide the basis for the existence of a stationary and compact state space in both the SCE and recursive (Markovian) formulations of the dynamic equilibrium of the model.[11] In our case, on top of compactness, these bounds are essential to guarantee that a crisis happens in a finite time, which simultaneously helps us prove the existence of an invariant probability measure even in the presence of discontinuities.

We then state our assumptions on the primitive data of the model as follows:

Assumption 1: The functions $U(x)$ and $x = A(c)$ satisfy the following conditions:

- *a.i) Inada on Composite consumption $A(c)$. $lim_{x \longrightarrow \infty} U'(x) = 0$* or
- *a.ii) Satiation (only if $\beta R \geq 1$). $\exists x \in X$ such that $\quad \forall y \in B_\epsilon(x) \quad U(y) \leq U(x) \quad \epsilon > 0$,*
- *a.iii) Non-negativity.* Let $A(.,.)$ map $X \mapsto \mathbb{R}_+$, where X is the consumption space.
- *b) Domain.* $\mathbb{R}^2_+ \subset X, X$ open
- *c) Smoothness, monotonicity, and curvature of the aggregator.* $A(c) \in C^2$. That is, $A(c)$ is continuously differentiable at least twice. Moreover, $A(c)$ is jointly strictly concave and strictly increasing.
- *d) Marginal Rate of Substitution between non-tradable and tradable consumption.* The marginal rate of substitution is separable: $A_2(c)/A_1(c) = A_2(c_2)/A_1(c_1)$. [12]
- *e) Lower bound on the marginal utility of tradable.* $lim_{c_1 \longrightarrow \infty} A_1(c_1, c_2) = cl_1 > 0$ for all $c_2 \in X \mid_{c_2}$.
- *f) Upper bound on the marginal utility of tradable.* $lim_{c_1 \longrightarrow 0} A_1(c_1, c_2) = cu_1 < \infty$ for all $c_2 \in X \mid_{c_2}$.
- *g) Smoothness, monotonicity, and curvature of the utility function.* $U(z) : X \longrightarrow \mathbb{R}$ is strictly increasing, strictly concave and C^2-smooth.

Assumption 1.a.ii can be used to prove the existence of any SCE only in the case where $\beta R \geq 1$ is typically missing in the literature. It simply says that lifetime preferences must have an asymptotic "satiation point" $x \in X$. As this preference structure significantly affects the underlying stochastic process, attracting equilibrium paths to the satiation point, we defer the treatment of this case to future research. Thus, we focus on the traditional case $\beta R < 1$. We can dispense with Assumption 1.a.ii if the discount factor is sufficiently low.

Assumptions 1.b, 1.c, 1.e, and 1.f ensure sequential equilibrium prices are bounded above and bounded below away from zero. This, in turn, will imply that in an SCE, debt is bounded above due to the collateral constraint in (3) and bounded below as wealth will be finite at every possible node. The supplementary material for section 2 in the online appendix contains examples of aggregators satisfying these restrictions.

Assumption 1.g is standard in the general equilibrium literature (see for instance Braido ([19]))

We now state our first critical result for constructing the existence of SCE: [13]

Lemma 2 *Suppose $\beta R < 1$. Under assumptions 1-a.i, 1-a.iii, 1-b to 1-g, **if** a SCE (c, d, p) exists, then: (a) [Compactness provided existence] $[c(y^t), d(y^t), p(y^t)] \in K_1 \times K_2 \times K_3 \subset \Re^4 \quad y^t -$ a.e. and uniformly in $[y_0, d_0] \in Y \times K_2$, where $K_1 \times K_2 \times K_3$ is compact, (b) [Necessity of first-order conditions] the SCE must satisfy* (7),(8) .[14]

[10] That is when studying the quantitative stochastic properties of these models in calibrated settings, compactness of the state space is often present without these types of "boundary" conditions on preferences. Compactness is imposed directly on debt. We replace restrictions on endogenous variables with assumptions on the model's primitives.

[11] The compactness needed is in the sense that realizations of equilibrium, random variables are all contained in a compact subset of a finite-dimensional space.

[12] Assumption 1c) and 1d) imply that $A_2(c_2)/A_1(c_1)$ is increasing in c_1 given $c_2 = y^N$.

[13] All the proofs of the Lemmas and Theorems in the paper are in the Appendix.

[14] Note that, because of the compactness of the equilibrium, the transversality condition $lim_{t \longrightarrow \infty} \beta^t E_t(U'(A(c^*_t))A_1(c^*_t)) = 0$ is also satisfied.

Notice that this lemma does not guarantee the existence of the SCE, which will be considered in Theorem 3 below. We also should mention if we allow the cardinality of Y to be arbitrarily large, proving the existence of this type of equilibrium can be rather challenging (See, for instance, Mas-Collel and Zame, [42]). Once the almost everywhere compactness of any "suitable candidate for equilibria" (c, d, p) is proved using lemma 2, we show the existence of the SCE assuming that Y is a finite set and that the model has sufficient discounting β. MasColell and Zame ([42]) needed to state by assumption the uniform compactness of the SCE for the case of uncountable shocks.

The next theorem gives sufficient conditions for the existence of the SCE.

Theorem 3 *Suppose $\beta R < 1$ and the Y is a finite set. Under Assumptions 1-a.i,1-a.iii, 1-b -1-g, then: (a) [Existence] there exist a SCE, (b) [Sufficiency of first-order conditions] any $[c(y^t), d(y^t), p(y^t)]$ that satisfy (7),(8) and $c_t^N(s^t) = y^N$ where y^t holds $\mu_{y_0^T}$-a.e is a SCE.* [15]

3 Generalized Markov Equilibrium (GME)

The GME approach will have significant implications from numerical and qualitative perspectives. GME representations of SCE differ from the standard definition in the literature, and they are deeply connected to recursive representations in Duffie et al. ([27]) or Feng, et. al. ([31]). For expository purposes, the supplementary appendix for this section contains the standard definition of equilibrium in the literature.

As discussed in Santos and Peralta-Alva ([58]), the starting point of any simulation experiment for a stochastic dynamic equilibrium model is an appropriate *stochastic steady state* notion. The most frequent stochastic steady notion used in the literature is an *invariant measure* (IM) for *some* recursive representation of the sequential equilibrium. Heuristically, an IM gives a sense of *probabilistic time invariance*. That is, if $\{x_t\}$ is a sequence of random variables generated from some Markov process and x_t is distributed according to an IM μ, then x_τ will be distributed according to μ for $\tau > t$.

This section introduces the notion of GME, which has an expanded state space compared to the standard definition in the literature, with minimal state space. This equilibrium is a slightly modified version of Feng et al.'s. ([31]). We carefully select the additional state variable to guarantee the existence of an appropriate stochastic steady state. Moreover, a GME is defined using the set of equations characterizing the SCE. Thus, the additional state variable and the direct connection with the SCE bring more memory into the model at the cost of allowing additional sources of equilibrium multiplicity. This paper proves that it is possible to refine the equilibrium set in a GME by picking a selection that ensures the existence of an IM. In this sense, we impose long-run restrictions to refine the equilibrium set.

Provided a stationary recursive representation [16] an IM implies that simulations obtained from the SCE can be approximated by a time-invariant and finite set of functions, abstracting from numerical errors. [17] This is possible using a *law of large numbers*, which in turn requires the IM to be *ergodic*. From a practical perspective, ergodicity ensures roughly speaking that "averages converge". The ergodicity of the IM guarantees that the Cesaro average of any simulation starting from any initial condition will converge to an *expected value* computed using the stochastic steady-state distribution. This last fact connects the model with observed (time-independent) stylized facts.

GME representations also offer a valuable advantage from a qualitative standpoint. Introducing an additional state variable gives the recursive equilibrium an enhanced "memory" compared to the traditional minimal state space equilibrium. This added memory is crucial in stabilizing global stochastic

[15] Any equilibrium is compact Because of the bounds on marginal utility in assumption 1t. Then, the transversality condition $lim_{t\longrightarrow\infty}\ \beta^t E_t(U'(A(c_t^*))A_1(c_t^*)) = 0$ can be added to the set of sufficient conditions without loss of generality.

[16] In some models, such as some OLG economies, it is impossible to define a determined system of equations that directly connects the SCE with a recursive representation using only one additional state variable with respect to the minimal state space case. Feng and Hoelle ([30]) address these issues. As we deal with a model with a representative agent, we do not face those challenges.

[17] As pointed out in Santos and Peralta Alva ([58]), truncation and interpolation errors could accumulate over time if they are not "controlled".

dynamics, even amidst recurring balance of payment crises. The GME framework introduces the concept of a regeneration pointâ"a pivotal anchor around which the system's dynamics revolve, consistently returning to it after each crisis. This feature ensures that consumption and debt dynamics remain well-behaved, preventing divergent paths and allowing us to prove the ergodicity of the equilibrium.

3.1 A convenient recursive representation

In this subsection, we define and characterize the set of Generalized Markov Equilibria (GME). As this equilibrium has a bigger state space when compared with the standard definition in the literature (see the supplementary appendix for the previous section), it is more flexible (see Kubler and Schmedders [38] for a discussion). We use that flexibility to stabilize the dynamical system associated with the model that we presented before: if we restrict attention to a finite set of shocks, it is easy to characterize a "regeneration point" for the global stochastic dynamics in the model using the GME. This is not possible using the standard definition of equilibrium, which is required to be continuous, and thus, it cannot jump to the regeneration point. The flexibility of the GME allows the dynamical system to revert to this point every time the model hits the collateral constraint, which is the first step to finding a robust, recurrent, and stable structure in multiple equilibria that typically generate discontinuous selections.

As can be seen in Section 2, any SCE can be characterized using a set of primal first-order conditions (as inequalities) that do not depend on Lagrangian multipliers. The usefulness of this representation will be clear in this subsection. Moreover, the existence of well-behaved envelopes for the value function, as discussed in the supplementary appendix for section 2, also follows standard results. Then, the SCE can be characterized recursively. By the following equations:

$$[-A1(c^T)p + A_2(c^N)] = 0 \tag{9}$$

$$[\kappa\{y^T + py^N\} - d_+][U'\{A_1(y^T + d_+R^{-1} - d)\} - E(m_+)] = 0 \tag{10}$$

$$d_+ \leq \kappa\{y^T + py^N\}. \tag{11}$$

Note that we are adding a variable to the canonical characterization of the SCE: $m = \frac{\partial\ V}{\partial\ d}$, which is the envelope of the value function in the household's problem. This variable enlarges the state space with respect to the canonical one: instead of characterizing dynamics in terms of $[d, y^T]$, we will study the dynamical system using $[d, y^T, m]$. Equations (9) to (11) coupled with the definition of the envelope map any point in this (enlarged) state space to $[d, y^T, d_+]$, giving the desired control over potential divergent paths. Given the compactness of the equilibrium set, the results in Feng et al. ([31]) imply that equations (9) and (10) can be used to derive a correspondence, Φ, the so-called *equilibrium correspondence*, which contains the entire set of GME representations of SCE, where $\Phi : Z \times \ Y \longmapsto \ Z$ with $z = [d \quad y^T \quad y^N \quad c^T \quad c^N \quad p \quad m]$ with $z \in Z$, $y^T \in Y$ and Z compact. We now discuss the relationship between the state space $[d, y^T, m]$ and the state variable in the equilibrium correspondence Z.

In the unconstrained case, we can solve the system in blocks, as it is easy to see that, in this case, the policy function for debt can be characterized in terms of $[d, y^T]$. Then, we use the definition of the envelope, which pins down d_+, the market clearing condition for non-tradable consumption, which pins down c^N, the equilibrium budget equation, which pins down c^T, and (9) to compute p.

When the (collateral) constraint hits, we know that $U'\{A_1(y^T + R^{-1}\kappa\{y^T + py^N\} - d)\} \geq \ E(m_+)$ and the equilibrium for any given period can be computed using the following set of equations:

$$p = \frac{A_2(y^N)}{A_1(y^T + R^{-1}\kappa\{y^T + py^N\} - d)} \tag{12}$$

$$c^T = y^T + R^{-1}\kappa\{y^T + py^N\} - d \tag{13}$$

$$c^N = y^N \tag{14}$$

$$m_+ = U' A_1(c_+^T) \tag{15}$$

$$U'\{A_1(y^T + R^{-1}\kappa\{y^T + py^N\} - d)\} \geq \; E(m_+) \tag{16}$$

Given $[d, y^T]$, the remaining variables in Z can be computed using (12) to (15) as long as $U'\{A_1(c^T)\} \geq E(m_+)$. As the recursive equilibrium notion in Feng et. al. ([31]) is computed "backwards" (i.e. given "z_+" we obtain z), the Euler equation imposes a looser restriction on the system when compared to the non-binding case. This fact turns out to be very useful to prove the ergodicity of the process as we can carefully select $[d, y^T]$ every time the collateral constraint binds, allowing us to construct a regeneration point. For the general case we can compute the GME as follows: given a hit to the collateral constraint and $[d, y^T]$, equation (12) generates a well-defined system with finite roots as it is stated in Schmitt-Grohé, S. and Uribe, M ([63], [64]). Then, $E(m_+)$ coupled with (16) and equation (15) backward one period allows us to recover m. We can then use the budget equation (13) to compute c^T and the market clearing condition for non-tradable consumption, equation (14), to recover c^N.

We can compute not only a stationary but also an ergodic equilibrium using the procedure described above. We are now in a position to define a GME formally.

Definition 4 *Generalized Markov Equilibrium (GME). Let Z be the compact set which contains any state z_s that solves* (9), (10) *and* (11) *backwards with $s = 0, 1, \ldots$. The equilibrium correspondence Φ mapping $Z \times \; Y \to Z$ can be defined as follows: let $L = 0$ be the system formed by equations* (9), (10) *and* (11). *Note that any vector $(z_s, z_{s+1}(y_{LB}^T), \ldots, z_{s+1}(y_{UB}^T))$, where $y_{s+1}^T \in \{y_{LB}^T, \ldots, y_{UB}^T\}$, satisfies $L(z_s, z_{s+1}(y_{LB}^T), \ldots, z_{s+1}(y_{UB}^T)) = 0$. Then, $z_{s+1}(y_{s+1}^T) = \varphi\left(z_s, y_{s+1}^T\right)$, where $\varphi \in \Phi$ is a selection of the equilibrium correspondence. We say that φ is a GME. Any φ that is independent of time is a stationary GME. Let (Z, P_φ) defines a stationary Markov process with kernel P_φ. If (Z, P_φ) has an ergodic invariant measure, we say that φ is ergodic.*

Given the compactness of Z, as there are a finite number of shocks, the measurability requirements for P_φ follow from Feng, et. al. ([31]). In the next section, we will characterize the dynamics of the measurable dynamical system (Z, P_φ).

4 Stationarity and Ergodicity under a finite set of shocks

In this section, we will derive the paper's main result: there is an ergodic selection of the GME, which is globally stable, even though it may be discontinuous. These results imply that repeated balance of payment crises do not affect the steady state of the economy, which guarantees that they do not have long-run effects. Moreover, we discuss in detail the limitations of the canonical definition of equilibrium with minimal state space. For completeness, we present this definition in the online appendix (see the supplementary material for section 3). We first discuss the existence of an appropriate selection and the limitations of the canonical definition of equilibrium. Then, we move to the connection between the economy's steady state, the ergodic invariant measure, and the global stability of the system. Finally, we prove the main result.

4.1 Selection mechanism, GME and the canonical definition of equlibruim

The GME's existence follows directly from the compact SCE (see Feng et al., [31]). Then, we can focus on deriving the stochastic steady state of the model. To this purpose, we show that Φ has an ergodic selection. Formally, if we restrict the number of possible distinct values that y^T can take to be finite, we can prove the existence of an ergodic probability measure associated with a selection φ of Φ. In this framework, equation (12) can generate a *point* that the process hits with positive probability, starting from any initial condition, following the dynamics induced by the Euler equation (16). This point will be called *atom* and solves the system of equations given by (12) to (16) for wealth d_*, with the Euler equation holding with equality and for the lowest possible y^T. The supplementary material for this section, in the online appendix, contains the formal definition and characteristics of the atom of a stochastic process.

We can construct the atom using the GME because this equilibrium concept has an expanded state space with respect to the canonical definition of recursive equilibrium[18], and it is robust to discontinuous selections. In the standard definition of equilibrium, with minimal state space, ergodicity requires continuity. However, even if we allow discontinuous selections, the canonical definition is not sufficiently flexible to generate globally stable dynamics. As discussed below, this last property is essential to prove ergodicity in discontinuous environments. More to the point, in the GME, allowing for discontinuous equilibria is critical as it allows the process to jump to the atom, which in turn helps us to prove ergodicity. Thus, to ensure that balance-of-payments crises do not affect the long-run distribution of endogenous variables, we need to modify the definition of recursive equilibrium and allow for expanded state space and discontinuities.

Now, we discuss how the additional state variable m helps the dynamical system to revert to the atom. Every time the system hits the collateral constraint, it can be characterized by equations (12) to (16). With minimal state space, given y^T, d determines a continuation value for debt d_+, which satisfies the mentioned system. Notice that d will probably differ from d_* every time the system is constrained, but the equilibrium laws of motion may still be discontinuous. To show the existence of an ergodic invariant using standard results, we need to show compactness and the continuity of the endogenous laws of motion in the dynamical system. Thus, using minimal state space is not enough. With an expanded state space, given y^T, (d, m), using equations (13) and (15), *jointly* determine d and d_+. Thus, by appropriately choosing m_+, *which determines the selection* φ [19], we can choose d to be equal to d_*, the atom, every time the system hits the collateral constraint as equation (12) only depends on d and d_+, given y^T. The backward nature of the GME (i.e., given m_+, choose d, d_+) is critical here. The standard minimal state space equilibrium operates forward (i.e., given d, choose d_+) and thus cannot guarantee that the system reverts to the atom every time it hits the collateral constraint.

To construct the ergodic selection, we use the block-recursive nature of the system formed by equations (9), (10), and (11). When the system is not constrained, we can solve the problem's inter-temporal and intra-temporal parts separately. Because the market clearing condition for non-tradable consumption implies that the only general equilibrium price is in the intra-temporal part, we solve the inter-temporal part first, which determines debt and tradable consumption, using a standard partial equilibrium concave savings problem. For some pairs (d, y^T), d_+ will be above the indebtedness levels the collateral constraints allow. We collapse all these states to a point (d_*, y^T_{LB}) by applying the GME to the system formed by equations (12)-(16). The paragraph above explains how we can collapse the constrained states to a singleton. Once we partition (d, y^T) into constrained and unconstrained states, the GME selects the point regenerating the system, giving it the required stability. As the atom is described by the constrained system with the Euler equation holding with equality, we can use the same savings problem to solve the

[18] For expository purposes, the supplementary appendix contains the formal definition of the canonical recursive equilibrium notion used in the literature.

[19] The selection φ of the equilibrium correspondence Φ defines a (possibly discontinuous) function that we can use to iterate the system forward. This function maps (d, y^T, d_+, y^T_+) to d_{++}, the continuation value for d_+, such that it satisfies equations (13), forward one period, (15) and (16). However, the selection is constructed backward as we choose $m_+(y^T_+)$ from a compact set, which is possible due to the assumptions on preferences, with bounded marginal utility, and the envelope definition in equation (15). The supplementary material for section 5 in the online appendix describes an algorithm to compute this selection efficiently.

entire model. We use two mappings once we identify the region in the expanded state space y^T, d, m in which the collateral constraint binds. First, equations (12) and (15) implies that y^T, d, m maps to y^T, d, d_+. Then, the Euler equation implies that, for each y_+^T, $d_{++}(y_+^T)$ maps to y^T, d, d_+. Then, for the subset of y^T, d, d_+ for which the constraint binds, we choose $d_{++}(y_+^T)$ such that the Euler equation is satisfied at $y_{lb}, d_*, d(d_*, y_{lb})$, where $d(d_*, y_{lb})$ is the savings function of the unconstrained equilibrium.

4.2 Ergodic invariant measure, stability, and steady state

The power of an atom for characterizing the behavior of the Markov chain is well-known.[20] However, it is helpful to illustrate its effect in the recurrent structure of the chain, which is critical to define an invariant measure (i.e., a measure μ which satisfies $\mu = \int P_\varphi(z, A)\mu(dz))$ that we use to characterize the steady state of the model. Suppose that the atom α [21] is hit for the first time with positive probability in period $\tau_\alpha < \infty$ starting from z_0. Then, it is possible to define a (not necessarily probability) measure μ, which gives the expected number of visits to a particular Borel set, called A, before τ_α. Stated differently, $\mu(A)$ gives the sum of the probabilities of hitting A *avoiding* α. Imagine the system is in period $\tau_\alpha - 1$ starting from z_0. Remarkably, when we "forward" μ one period (i.e., by applying the Markov operator to it, $\int P_\varphi(z, A)\mu(dz))$ the expected number of visits to A avoiding α is the same as the chain will hit α in period $n = \tau_\alpha$. Thus, μ must not change or equivalently $\mu = \int P_\varphi(z, A)\mu(dz)$. That is, μ is an invariant measure. Provided that $\tau_\alpha < \infty$, it is possible to normalize μ to be a probability measure, which we will call π (this property is called "positivity"). Further, as the accessibility of the atom comes together with the connectedness of the chain (see proposition 17 in the supplementary appendix to this section), it is not surprising that the invariant measure is unique as the chain does not break into different "unconnected islands". Finally, the Krein-Milman theorem ensures the ergodicity of the chain provided its uniqueness (see Futia, ([32])).

Now, to connect the existence of an invariant measure with the frequency of crises, let τ_α be the time when the process hits the collateral constraint. Then, $\mu(A)$ gives the cumulative probability of hitting A *avoiding* a crisis. Thus, the frequency of a Sudden Stop affects the stationary distribution μ. Frequent crises imply more volatility. To understand this relationship for equilibrium paths in the steady state, note that every time the process hits the collateral constraint, it reverts to α. Moreover, at the atom, we have $P_\varphi(\alpha, A) = \nu(A)$, where this property follows trivially as α is a singleton. Then, $z_{\tau_\alpha+1}$ is *independent* of the past, which implies that it loses all the inertia inherited from the Markovian structure of the process. Thus, the equilibrium stochastic dynamics behave unconditionally concerning the past, increasing its variability.

Note that for any given set of endowments, parameters, and preferences that define a sequential equilibrium and a selection φ, we will have a unique distribution hitting times τ_α. As discussed above, the frequency of crises associated with this distribution is essential to construct the unique ergodic distribution, which defines the stochastic steady state of the model. Thus, *the occurrence of frequent crises not only does not alter the model's steady state but also is required to show its existence and qualitative properties.*

To relate these facts with the empirical performance of the model, we need a Law of Large Numbers. As the measure is ergodic, it is well-known that $\sum_{t=0}^{T}(z_t/T) \to E_\mu(z)$ almost everywhere, where $\to$ means T tends to infinite. [22] In other words, the existence of an ergodic measure ensures that a sample mean computed by an increasing large time series of simulated data will hit the steady state of the model, represented by the mean $E_\mu(z)$, for a large fraction of possible paths $\{z_t\}_{t=0}^{\infty}$.

We now turn to the connection between the dynamical system's global stability, the invariant measure's uniqueness, and the process's ergodicity. The process will hit the atom in a finite time. Thus, it creates

[20]Meyn and Tweedie mention the importance of an atom for general state space Markov chains relative to countable state space Markov chains (e.g., [48], p96). A discussion of the importance of the existence of an atom in the context of the Markov chain theory is outside the scope of this paper. Still, a systematic discussion of this fact is presented in Meyn and Tweedie (Chapters 8, 10, and 17).

[21]In the previous subsection, we refer to d_* as a critical element in the state space as it is the atomic level of debt. Formally, the atom is given by: $\alpha \equiv [d_*, y_{lb}^T, y^N, c_\alpha^T, c^N, p_\alpha, m_\alpha]$.

[22]e.g., see Stokey, Lucas and Prescott, ([65]), chapters 11 and 12)

an orbit that endows the dynamical system with a recurrent structure, which implies that each atom will have an invariant probability measure. To deal with the uniqueness of this measure, which in turn ensures ergodicity, we will construct a stable state space. The process will hit in finite time any meaningful (i.e., with positive measure) subset of this state space. This property, called irreducibility, will ensure the uniqueness and ergodicity of the invariant measure. Irreducibility is a notion of contentedness (i.e., the process does not have islands) of the dynamical system generated by an ergodic selection $\varphi \in \Phi$. Coupled with the recurrent structure associated with the atom, it gives the model a notion of global stability. As positive probability paths starting from any initial condition will eventually, in finite time, orbit around the regeneration point, the atom, and the process does not converge to isolated subsets, the state space is stable. In this space, it is possible to construct a stationary or invariant measure based on the probability distribution of hits to the collateral constraint. The invariant measure, in turn, characterizes every meaningful set in terms of its probability, and, as it is time-invariant, it represents a notion of a stochastic steady state. Finally, we can use this measure to construct an expected value of any coordinate of Z, which can be approximated using a Law of Large Numbers for ergodic processes. Then, the process's global stability and the invariant measure's ergodicity are deeply connected through the presence of a regeneration point and the frequency of hits to the collateral constraints.

In our framework, the existence of multiple equilibria and discontinuous selections is far from a problem to show the existence of a stochastic steady state. The presence of two regimes, one defined for equations (9) and (10) when the collateral constraint does not bind, and the other given by equations (12) to (16), together with the possible multiple solutions to the equation (12), suggests the presence of multiple possibly discontinuous Markov equilibria. If we allow for discontinuous selections of the equilibrium correspondence in the GME, we can construct a transition function that "jumps" to the atom every time the collateral constraint is hit, generating a "crises". Thus, the presence of multiple equilibria, which is in part a consequence of the long-term memory of the GME, and its implications for the smoothness of the selections $\varphi \in \Phi$ increases the predictive power of the model in the sense that it allows a better match of long-run empirical regularities due to the ergodicity of the equilibrium.

4.3 Main theorem

We now state the main theorem in this paper. The results in Meyn and Tweedie ([48]) give us the tools to prove all the intermediate steps required to go from the existence of a stationary selection of the equilibrium correspondence to its ergodicity.[23]

Theorem 5 *There exists an $\varepsilon > 0$ such that $y_{lb} \in (0, \varepsilon)$, a compact set $J_1 \subseteq Z$ with $\Phi : J_1 \times Y \longrightarrow J_1$ and a selection $\varphi \sim \Phi$ such that the process defined by (J_1, P_φ) has an unique ergodic probability measure.*

Proof: See the appendix.

5 Applications

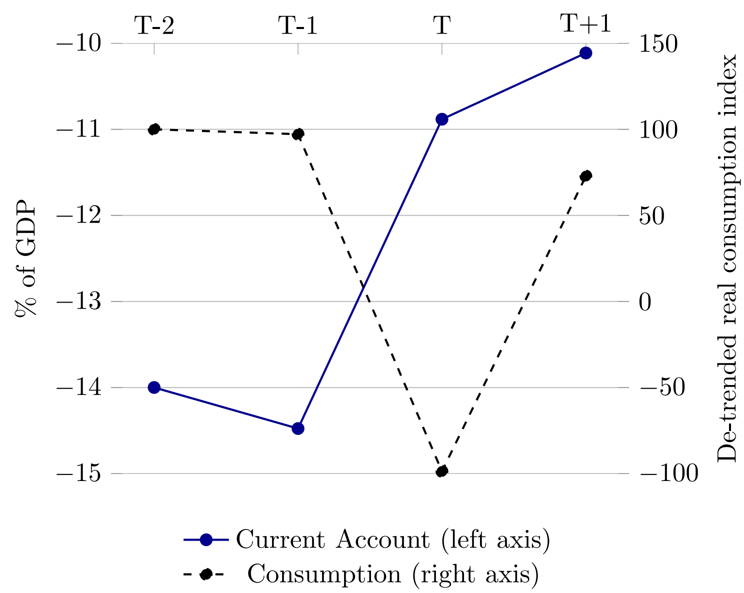
In this section, we characterize global dynamics using the flexibility of GME representations and test the long-run implications of balance-of-payment crises using the ergodic model. Moreover, we explore the differences concerning other equilibrium notions. The stationary non-ergodic equilibrium, which contains the standard definition in the literature, displays somewhat different dynamics. In particular, consumption is more volatile. As the degree of market incompleteness is the same in both equilibria, we conclude that this is due to the permanent effects of the crisis, as Sudden Stops affect the stochastic steady state in this type of equilibrium, which does not happen in the ergodic selection.

[23] See Meyn and Tweedie ([48]), chapters 5, 8, and 10 for a detailed discussion of the implications of the existence of an atom for the existence of an invariant probability measure.

5.1 Global dynamics

This subsection shows that an ergodic GME, summarized by equations (9) to (11), can generate Sudden Stops *without* relying on non-anticipated changes in the model's parameters nor MIT shocks.[24] At the same time, these events can be divided into 3 phases: i) an increase in the current account deficit characterizes the pre-sudden-stop, ii) the Sudden Stop consists of a sharp drop in consumption and a current account reversal, and iii) the post-sudden-stop. Figure 1, borrowed from Pierri, et. al. ([52]), illustrates this situation.

Figure 1: Anatomy of a Sudden Stop: Portugal 2008



There are 3 phases of a sudden stop: pre-phase, crises, and post-phase. The dotted line represents the current account divided by GDP at current prices. The left index is measured in percentage points. The right index, in full line, contains a consumption index constructed using the de-trended (by Hodrick and Prescott, HP) series at constant prices. The index's base is 100 at "T-2", where "T" is the date of the sudden stop (2008).

To model a Sudden Stop, we must show the existence of an abrupt reversal in foreign borrowing. By "abrupt," we mean that it must be an intra-period event. This is the *same* effect caused by a sharp reduction in κ, which is an exogenous event (i.e., an event that can not be explained within the theoretical framework embodied in the model). Note that changing this parameter will affect the stationary distribution of endogenous variables if it exists, which, contrary to standard assumptions, implies that balance of payment crises have a permanent effect. We will show that the GME can generate a Sudden Stop without relying on a change in κ or any other parameter, which is a necessary condition to ensure that balance of payment crises do not have permanent effects.

Let "T" in figure 1 be the time of the sudden stop. Before that, in phase i), we are in the unconstrained regime. In the case of Portugal, we observe a small decrease in consumption (3%) and an increase in the current account deficit. By setting $y^T_{T-2} = y^T_{T-1} = y_{lb}$, as the unconstrained regime can be thought of as a standard partial equilibrium concave savings problem, we can match the observed path [25]. Then, suppose that the collateral constraint is binding for some debt level d and $y^T = y_{lb}$. This happens in period T without loss of generality [26]. Then, equation (16) implies:

[24] The literature generates a crisis by changing the degree of access to international capital markets, measured by κ.

[25] The proof follows immediately from Lemma 6 and 9 in the appendix of section 4

[26] Lemma 7 in the appendix containing the proofs of section 4 implies that starting from any initial condition, the collateral constraint binds in finite time with positive probability.

$$U'\{A_1(y_{lb} + R^{-1}\kappa\{y_{lb} + py^N\} - d)\} \geq \; E(U'\{A_1(y^T_+ - \kappa\{y_{lb} + py^N\} + R^{-1}\overline{d})\}), \tag{17}$$

where $\overline{d}$ defines an ergodic selection $\varphi \sim \Phi$. We omit the dependence of $\overline{d}$ on y^T_+ as we are applying an integral over this last variable.

One of the most relevant aspects of a GME is that it is computed "backwards". We want to find d, d_+, given m_+. We replace this last variable in the right-hand side of the Euler equation using equation (15). It turns out that this fact can be used, combined with the primal version of the Euler equation (17), to obtain a sudden stop without requiring a jump in the exogenous parameters of the model. Let c, d_+ be today's consumption and tomorrow's debt level in (17) respectively. Note that, as we are in the constrained regime and P, the real exchange rate given by intra-temporal optimality equation (9), is increasing in tradable consumption, any $c_* < c$ will also satisfy (17), where c_* is the level of consumption at the atom. Take P_* with $y_{lb} + py^N > y_{lb} + p_*y^N$, where $P(c^T) \equiv p > P(c^T_*) \equiv p_*$. If there are multiple equilibria, there are at least two possible branches for the equilibrium selection. Taking the lowest one, we know that p is decreasing in d as the branch has a negative slope (see Schmitt-Grohe and Uribe [63] for a detailed discussion on the shape of these branches for the standard CES preferences) which in turn implies $d < d_*$. Thus, $c = y_{lb} + R^{-1}\kappa\{y_{lb} + py^N\} - d > y_{lb} + R^{-1}\kappa\{y_{lb} + p_*y^N\} - d_* = c_*$ as desired. As equation (17) holds with inequality, this implies that the level of debt tomorrow is binding at p_*, that is $d_{+,*} = \kappa\{y_{lb} + p_*y^N\}$. Thus, when the process hits the collateral constraint at (c, p, d_+), it jumps to $(c_*, p_*, d_{+,*})$, with $c_* < c$, $p_* < p$, and $d_{+,*} < d_+$, generating a sharp depreciation of the exchange rate, a consumption drop and deleverage without a changing any parameter in the model. That is, a hit to the collateral constraint generates an endogenous crisis.

It is standard in the literature to generate a balance of payments crisis using a reduction in κ, which will affect the model's steady state. The rationale is simple: a Sudden Stop reduces the willingness to lend to a given country. As κ determines the maximum borrowing capacity of an open economy when the collateral constraint is binding, a reduction in it represents a tightening in the level of financial frictions, which induces deleverage and recession. As we showed in previous paragraphs, in the GME, it is unnecessary to modify κ, which implies that we can endogenously generate this type of crisis in the model without affecting its steady state.

The discussion in the preceding paragraphs suggest that it is possible to set the level of the next period borrowing such that it is binding for the new level of consumption c_* as equation (17) is allowed to hold with strict inequality. Remarkably note that there exist a $\kappa' < \kappa$ such that $\kappa'\{y_{lb} + py^N\} = d'_+ = \kappa\{y_{lb} + p_*y^N\}$ which implies that $\kappa\{y_{lb} + p_*y^N\}$ *is a level of debt associated with a Sudden Stop* if the pair (d_*, p_*) implies a sharp contraction in consumption and a reversal in the current account. The first fact was already shown. We will show the second below. Note that the order of magnitude of this "recession" depends on the level of consumption reached in phase i) before the sudden stop, which, given the unconstrained nature of this phase and the smoothness of consumption in that regime is similar to c and bigger than c_*. Thus, *memory in the form of a sequence* $y^T_{T-n}, ..., y^T_{T-1}$, *matters in order to capture the quantitative properties of the Sudden Stop*, and the GME can capture it. Technically, the compactness of the SCE ensures that we can adjust the severity of the crises along with a finite lower bound on consumption, both proved in lemma 2

It remains to show that the jump from (d, p) to (d_*, p_*) generates a current account reversal. Let $d_+, d_{*,+}$ be the level of next period debt associated with c, c_* respectively. To generate the mentioned reversal, we must have $d_{*,+} - d_* < d_+ - d$. As $d_{*,+} = y_{lb} + p_*y^N < d_+ = y_{lb} + py^N$ and $d < d_*$, the desired result follows. To sum up, the sudden stop generates a *reduction in consumption and a current account reversal* as desired. Finally, as this event happens for a particular trajectory of exogenous shocks $y_0, y_1, ..., y_T$ in finite time, the event has positive probability. Still, it can be considered a *"rare event"* as noted by Mendoza and Smith (([46])).

It is sometimes observed that after a sharp depreciation, it follows a recovery in consumption coupled with a real appreciation and a current account improvement. This is the case of Portugal 2008 in phase iii). A GME can replicate these facts as a path in the unconstrained regime. In particular, the model associates an increase in the national income with this phase iii). That is, we must observe an increase in $y^T_{T+1} - d_+$. This shock implies an increase in consumption, which generates real appreciation due to the intra-temporal optimality condition $p(c^T)$.[27] Moreover, the concavity of the utility function implies $d_{++} < d_+$, as it is possible to smooth consumption in an unconstrained economy. Thus, the observed current account improvement follows.

5.2 Empirical Procedure, Algorithms and Simulations

We now turn to the quantitative implications of the results presented in section 4.

We first show how to compute the ergodic selection in the GME. It can simulate a recurrent, and thus ergodic, behavior as the stochastic paths visit the atom in a crisis. Following the theoretical results in section 4, each ergodic selection has a unique invariant measure. Then, we solve the model for a parameter set borrowed from the empirical literature and compute the effects of a change in the interest rate in the model's long run.

A non-stationary equilibrium depends on time. In this case, there is no notion of invariance and thus no stochastic steady state. Therefore, analyzing if crises affect the model's long run is impossible. In the case of stationary non-ergodic selections, simulations will depend on initial conditions. However, as the model is compact, it is possible to define a measure associated with the limit of Cesaro averages that can be interpreted as a steady state. In a stationary non-ergodic selection, any different initial condition implies a different distribution of hitting times; thus, the frequency of crises also depends on the starting position of the economy. In this case, each crisis defines a different steady state. In the case of an ergodic equilibrium, the frequency of crises is independent of the initial condition; thus, the steady state is unique and not affected by Sudden Stops. Stated differently, the stochastic steady state is unaffected by crises, so these events do not have permanent effects.

We investigate numerically the difference between an ergodic, a stationary, and a non-stationary equilibria. It is possible to remove the ergodic component of any selection in a GME simply by visiting a different point every time the collateral constraint binds. However, this selection is still time-invariant and, thus, stationarity. Moreover, as any GME has an expanded state space, it contains the standard definition of recursive equilibrium, which has minimal state space. We compute the difference between simulations generated by an ergodic and a stationary GME equilibrium. We found that ergodic simulations generate smoother consumption paths due to the lack of permanent effects of balance of payment crises. Finally, we compute a non-stationary GME. This equilibrium can generate large fluctuations in macroeconomic fundamentals (i.e., current account) using standard preferences and the same structure of shocks.

5.2.1 Numerical procedure

We describe how to compute an ergodic selection and use it to simulate the long (a path of length N) and short-run (of length $T < N$) behavior of the model. The online appendix contains additional details.

Let (d, d_+) denote the current and future debt levels. The presence of the collateral constraint implies that $d_+ \neq d(d, y)$, where $d(.,.)$ denotes the policy function in the unconstrained regime. Thus, the ergodic selection of the GME φ depends on $\varphi(d, y, y_+)$ for every $(d, y, y_+) \in K_2 \times Y \times Y$ if the collateral does not bind and on $\varphi(d_+, d, y, y_+)$ for every $(d_+, d, y, y_+) \in K_2 \times K_2 \times Y \times Y$ if it binds. This fact implies that the GME is computationally efficient. Once $d(.,.)$ is available, it can be computed fast.

The algorithm in the online appendix, called GME ergodic algorithm, generates a sequence $\{p_t, d_{t+1}\}_{t=0}^T$ which *depends* on $p_{T+1}(y_{T+1})$ and $d_{T+1}(y_T)$ for a given point in the set of deep parameters Θ. These variables are pinned down by picking an ergodic selection for the GME. The characteristic of the ergodic selection is that the process reverts to the atom when the collateral binds. If we only need a stationary GME, we may allow $d_+ \neq d(d, y)$ even if the collateral does not bind. Thus, we add memory to the selection as d_+ may not depend on (d, y). Notice that this last stationary equilibrium will be different

[27] See Lemmas 9 and 10 in the appendix for section 4.

with respect to the canonical minimal state space one. It is possible to restrict the stationary GME to match the canonical minimal state space equilibrium, and at the same time, both equilibrium types will not be ergodic. Any sequence generated from the minimal state space equilibrium depends *only* on the point in the parameter space Θ and the draw from the stochastic process which generates tradable output, for a given d_0, y_0. That is, $\varphi \in \Phi$ *does not necessarily satisfied* $d_{T+2} = d(d(d_T, y_T), y_{T+1})$ as it is the case for the minimal state space algorithm.

The online appendix also contains the numerical procedure to compute a stationary GME. In this equilibrium, the path $\{p_t, d_{t+1}\}_{t=0}^{T}$ is more flexible as the transition function in the GME can be computed pointwise as in the SCE. This is the *numerical implication of expanding the memory in the recursive equilibrium as the transition function* φ *is computed exactly as in the SCE* (i.e., pointwise for each element in the draw from the exogenous Markov process determining tradable GDP (Y, q)).

The online appendix also describes a non-stationary GME Algorithm. In this case the sequence $\{p_t, d_{t+1}\}_{t=0}^{T}$ *depends* on the histories of the form $p_y(y^t)$, $d_t(y^{t-1})$ with $y^t = y_0, ..., y_t$. Thus, there is a trade-off: *we gain flexibility concerning the stationary/ergodic GME to incorporate more "memory" from the SCE, but in return, we can not claim that these paths are connected with the steady state of the model* and that they are independent of time.

5.2.2 Results

We now solve the model. The table below contains the parameters borrowed from Pierri, et. al. ([52]).

Table 1: Parameters

Parameter	κ	β	σ	ξ	a	z_l	z_h	$p(z_l)$	R	R'
Value	0.3	0.99	2.0	0.5	0.5	0.5	1.5	0.2	1.05	1.025

We now show the results of simulating the ergodic and the stationary process. We present the effects of a reduction in the interest rates in both cases for consumption (Table 2) and debt (Table 3).

Table 2: Summary of Simulations Statistics for Consumption

Statistics	Mean(R)	STD(R)	Mean(R')	STD(R')
Ergodic	1.0497	0.450	1.0501	0.453
Stationary	1.01	0.52	1.04	0.53

Table 3: Summary of Simulations Statistics for Debt

Statistics	Mean(R)	STD(R)	Mean(R')	STD(R')
Ergodic	0.230	0.086	0.235	0.094
Stationary	0.27	0.06	0.29	0.07

The statistics are reported at distinct truncation levels, as there are some cases for which two decimal positions are insufficient to differentiate between simulations. A reduction in the interest rate generates the expected changes in ergodic and stationary simulations: an increase in consumption and, thus, a reduction in the savings rate, which implies more debt. However, there are two differences between ergodic and stationary selections. In particular, stationary simulations overestimate i) the elasticity of average consumption and debt concerning the interest rate and ii) the volatility of consumption for the same interest rate, which implies that debt is less volatile. For the first fact, the intuition goes as follows:

as the atom is not strongly affected by the change in the interest rate, only through its effect on the unconstrained policy function and ergodic simulations are generated as a sequence of recurrent sets that have a regeneration point in the atom, average observed endogenous variables are not severely affected by the change in the interest rate. For the second fact, note that in the ergodic simulation, debt is regenerated to a very low level, as by construction, the atom is defined as hitting the collateral constraint with equality. Thus, de-leveraging is more significant in an ergodic crisis, which implies that the economy has more time to accumulate debt and to smooth consumption before hitting the collateral constraint again. Moreover, in stationary simulations, global stability is not guaranteed to hold. Thus, simulations may contain divergent paths even for the same initial condition.

The takeaway from the above results is that the atom is not severely affected by the change in the parameter we introduce in the simulation exercises. The most direct way to change the ergodic distribution is to affect the regeneration point. In this case, the atom does not change significantly after the interest rate shock, even though the unconstrained policy function and the value of the collateral are both affected. In particular, the atom is given by:

$$d(d_*, y_{lb}; R) = \kappa \left[y_{lb} + y^N \left(\frac{A_2(y^N)}{A_1(y_{lb} + (d(d_*, y_{lb}; R)/R) - d_*)} \right) \right]$$

After the change in the interest rate, both the left and right-hand side of the equation above rises as $d(d_*, y_{lb}; R)$ goes up and $d(d_*, y_{lb}; R)/R$ goes down. As κ and especially y_{lb} are small, the right-hand side does not change significantly, implying a slight reduction in the atom.

Table 4: Non Stationary Crises

Non Stationary Simulations	$t-1$	t	$t+1$
Current Account / GDP	-10%	-8%	$+5\%$

Table 4 reflects the flexibility contained in the model. By changing selections, the SCE can replicate a sharp reversion in the current account, as it is frequently observed in data, without requiring a change in the parameters (i.e. a reduction of κ or y_{lb}).

6 Extensions and Concluding Remarks

This section of the paper briefly explores how our findings can be extended to more generalized model versions. Traditional analyses often assume that balance-of-payments crises leave the economy's steady-state unaffected. This assumption carries a significant implication: sudden stops are believed to have no lasting effects. However, the equilibrium concepts commonly employed in the literature do not inherently ensure this outcome.

In response, we introduce an equilibrium framework that preserves the model's steady state despite recurring balance-of-payments crisesâ"an ergodic selection of the Generalized Markov Equilibrium (GME). Additionally, we present alternative recursive representations that account for the potential long-term effects of such events, including stationary GMEs. Central to our approach is modeling crises as shocks to the collateral constraint. Any solution to the resulting system of binding equations can depend on the history of shocks, which may vary from one crisis to another. The economy's initial conditions also shape these crises. This leads to an important implication: in a stationary, non-ergodic equilibriumâ"such as the canonical minimal state space representationâ"the frequency of crises influences the long-term values produced by simulations, ultimately altering the model's steady state.

To address the challenges mentioned above, we propose a theoretical framework for efficiently computing ergodic GME selections, ensuring that the models' steady state remains unaffected by recurrent balance of payment crises such as sudden stops. Notably, the expanded state space required to ensure stability and ergodicity does not compromise the model's computability.

One possible direction per future work is to include production in the standard Sudden Stop model (e.g., as in Benigno et al. ([8]). Here, the authors allow endogenous labor supply firms to produce tradeable and non-tradeable consumption goods (but with a fixed capital stock). If the preferences are Greenwood-Huffman-Hercowitz's (GHH) preferences, nothing in our arguments changes. If preferences over leisure are more general, the arguments in this paper have to be changed a great deal, but the methods of the paper can still be extended to such economies.

Another exciting and essential variation of this model is the case when collateral constraints on debt are based on *future* wealth of the household, not the current wealth. The importance of these issues has been discussed in Ottonello, Perez, and Varraso ([51]), and in a related context in Brooks and Dovis ([17]). The stochastic dynamics of the model appear to be much more complicated to characterize, so it is unclear how to extend the ergodicity results for using GME representations of SCE (even if they exist).

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Appendix

Proofs for section 2

We now turn to the characterization of the SCE. To get the paper self-contained, we write the primal version of the first-order conditions here as they will be useful in proving the long-run properties of the equilibrium. We keep the notation $c_1 \equiv c^T, c_2 \equiv c^N$ in line with Assumption 1.
For any given sequence of prices $p = \{p_t\}_{t=0}^{\infty}$, from the first order conditions in dual form [28], we can obtain the following two expressions, in primal form, critical for characterizing the stochastic structure of binding collateral constraints:

$$[c_2(y^t)][\{\beta^t U'(A(c(y^t))\}\{-A_1(c_1(y^t))p(y^t) + A_2(c_2(y^t))\}] = 0, \quad y^t - a.e \tag{18}$$

$$\begin{aligned}[\kappa\{y^t + p(y^t)y^N\} - d(y^t)][U'(A(c(y^t))A_1(c_1(y^t)) \\ -E_t(U'(A(c(y^t y_{t+1}))A_1(c_1(y^t y_{t+1}))] = 0, \quad y^t - a.e\end{aligned} \tag{19}$$

where E_t can be obtained using the $y_t th$ row of the transition matrix for Markov shocks as the cardinality of this shock set Y is finite. In the i.i.d case, we have $E_t = E$.
We make a few remarks on (18) and (19). First, the characterization of optimal solutions uses the primal formulation of the the problem, and hence is written in terms of the "complementary slackness" version of the Karush-Kuhn-Tucker (KKT) equations. In particular, in either equation (18) or (19), the first bracket contains the inequality constraint, $c_2 \geq 0$ in equation (18) or $\kappa\{y^t + p(y^t)y^N\} - d(y^t) \geq 0$ in equation (19), and the second bracket consists of the derivative of the objective function with respect to the control, $c_2(y^t)$ in equation (18) and $d(y^t)$ in (19). Additionally, note we have eliminated the control, c_1, and the restriction in equation (2) from the KKT system. Formally, $c_1(y^t)$ must be replaced with $-p(y^t)c_2(y^t) - d(y^{t-1}) + y^t + p(y^t)y^N + \frac{d(y^t)}{R}$. This last issue is relevant for the dual formulation of the problem as it allows us to avoid dealing with the Lagrange multiplier associated with (2). It turns out that the dual representation is more difficult to characterize in terms of its dynamic behavior when compared with the primal version.
Now, to close the characterization of the model, we need a terminal condition on the right-hand side of the Euler equation $\beta^t E_t(U'(A(c(y^t y_{t+1}))A_1(c_1(y^t y_{t+1}))$, which results after iterating equation (19) (see Constantinides and Duffie ([22]) for a discussion). Under assumptions 1-a.i, 1-a.iii, 1-e, and 1-f, this requirement will be satisfied. The relevance of these assumptions, their relationship with the restriction we place on βR and the sufficient conditions for the compactness of the SCE will be proved in lemma 2, as stated in the body of the paper. Note that the results below are necessary conditions. The associated sufficient conditions for existence will be proved in Theorem 3.

Proof of Lemma 2

Proof. I) Let $Y = [y_{min}, y_{max}]$. From assumption 1-g), lemma 1 in Braido ([19]) implies that $-d(y^t) < \frac{R}{1-\rho} \equiv k_{2,min}$ uniformly in $y^t \in \Omega$ with ρ sufficiently close to 1. From assumption 1-a3), 1-b), 1-c) and 1-d), equation (18) implies $p(y^t) = \frac{A_2(c_2(y^t))}{A_1(c_1(y^t))}$. Definition 1-2), equation (18), assumption 1-e), 1-f) then imply $p(y^t) \in [\frac{y^N}{cu_1}, \frac{y^N}{cl_1}] \equiv K_3 \equiv [k_{3,min}, k_{3,max}] \quad y^t - a.e.$. Then, the collateral constraint implies $d(y^t) \leq \kappa(y_{max} + k_{3,max}y^N) \equiv k_{2,max}$. Then, $d(y^t) \in [k_{2,min}, k_{2,max}] \equiv K_2 \quad y^t - a.e.$. Finally, using K_2 and K_3, K_1 can be derived using equation (2) and Definition 1-2). Using these results, it is straightforward to verify that $lim_{t \longrightarrow \infty} \beta^t E_t(U'(A(c_t^*))A_1(c_t^*)) = 0$. Note that the integral in E_t is taken with respect to either the density associated with $\chi(y_t)$ or the $y_t - jth$ row in the transition matrix in case Y has finite cardinality. Thus, this result also holds $y^t - a.e.$. Then, the arguments in Lemma 2 in Kubler and Schmedders ([39]) hold, which implies that equations (18) and (19) together with the terminal condition on $\beta^t E_t(U'(A(c_{t+1}))A_1(c_1(t+1)))$ is necessary to define 1, as desired. ■

[28] Even though they are standard, for completeness, we present them in the online appendix, see equations 22-25.

Proof of Theorem 3

Proof. As in Kubler and Schmedders ([39]), we will start with a truncated economy $t = 0, ..., T$ and then extend the argument by induction. We need to rewrite the conditions in Definition 1 to show the theorem. Any SCE satisfies conditions A and B, as defined below.

Condition A

$$Max_{d(y^t)} \sum_t \sum_{y^t} U(c_1(y^t), y^N)\mu(y^t)$$

$$s.t$$

$$c_1(y^t) = y_t^T - d_t + min\left\{d_{t+1}/R, R^{(-1)}\kappa(y_t^T + p(y^t)y^N)\right\}$$

and Condition B

$$p(y^t) = \frac{A_2(y^N)}{A_1(c_1(y^t))}$$

Because of the linearity of the restriction, we can substitute c_1 into the objective function in condition A. Thus, because of lemma 2, the maximization problem is only restricted by the fact that $d(y^t) \in K_2$ for all y^t. Thus, $\{d_{t+1}(y^t)\}_{y^t} \in K_2^{(Y)^T}$, where $(Y)^T$ is the number of possible nodes. Note that given p, this is a strongly concave problem restricted by continuous correspondence. By Berge's maximum theorem, we know that $d_{t+1}(d_0, y^t, p)$ will be a continuous function of $p \in K_3^{(Y)^T}$ for all $d_0 \in K_2$ and $y^t \in (Y)^T$. Then we can use Condition B to define a continuous operator for $p((Y)^T)$, P, from a compact sect $K_3^{(y)^T}$ to itself, which has a fixed point. To see this, note that in the unconstrained case, the system of equations generated by conditions A and B is block recursive (i.e., we first solve for d_{t+1} give d_t then compute p_t for $t = 0, 1, ..T$). Thus, the proof is immediate. When the collateral constraints bind, condition B generates a finite number of roots $p_*(y^t, R_n)$, where R_n stands for the n-th root (this is extensively discussed in Schmitt-Grohé, and Uribe, [63]). That is, we know that n-th root satisfies $p_*(y^t, R_n) = P(p_*(y^t, R_n))$. Due to the compactness of equilibrium, it is possible to locally generate an arbitrarily large sequence of prices $p_j(y^t, R_n)$ which iteratively (i.e., $p_{j+1}(y^t, R_n) = P(p_j(y^t, R_n))$) converges to $p_*(y^t, R_n)$ for any n. This is possible as a continuous function is either increasing or decreasing locally. Finally, we can embed this sequence into $min\left\{d_{t+1}/R, R^{(-1)}\kappa(y_t^T + p(y^t)y^N)\right\}$, where we obtain d_{t+1} from the unconstrained version of the problem.

To show the sufficiency of equations (18), (19), and the transversality condition, note that Condition A defines a strongly concave problem with a unique solution. Thus, the set of solutions to Condition A given $p(y^T)$ using Berge's theorem is equivalent to the solutions found using (18), (19), and the transversality condition. To see this, let us define the correspondence:

$$\Gamma(y_t^T, d_t) = \left\{d_{t+1} \in K_2 : K_{LB,2} \leq d_{t+1} \leq R^{-1}\kappa(y_t^T + p_t y^N)\right\},$$

where $K_2 = [K_{LB,2}, K_{UB,2}]$, and the return function $U(c_1(y^t), y^N) \equiv F(y_t^T - d_t + R^{-1}d_{t+1})$. Then, condition A can be seen as a maximization problem with control given by the sequence of iteratively feasible debt: $\left\{d_{t+1}(y_0^T, d_0)\right\}$ $d_{t+1} \in \Gamma(y_t^T, d_t)$ for a given sequence of prices $\{p_t\}$. This problem can be characterized using the sufficient conditions in Kamihigashi ([34]): F is continuous in $K_2 \times K_2$ for any y_t^T, given lemma 2 is bounded, $F_{-d_t} \geq 0$ and it is strictly concave. To see this last property, note that the Hessian of F in $K_2 \times K_2$ is zero, and the elements in the diagonal are both negative. Thus, this function is concave. However, for any given d_t, there is a unique d_{t+1} for each c_1 and U is strictly concave in c_1. Thus, (18), (19) and the transversality condition are sufficient for a maximal $\left\{d_{t+1}(y_0^T, d_0)\right\}$. Then, the arguments above imply that adding condition B suffices to show the existence of an SCE. ∎

Proofs for Section 4

Proof of Theorem 5

We will prove the theorem using several preliminary lemmas. First, it will be shown that equation (19), when it holds with equality, generates a sequence of increasing levels of debt d_+ for any d as long as $y^T = y_{lb}$ if y_{lb} is sufficiently small. Then, using this result, we show that starting from any initial condition, the collateral constraint will bind in a finite time. This lemma will be helpful to show the existence of an accessible atom in the third lemma. Then, the fourth lemma shows the existence of a unique invariant ergodic measure.

From now on, we will assume that assumption 1 holds. Additionally, assume that in any SCE we have $d_{t+1} \leq H$ with $H \equiv \kappa(y_{ub} + P_{ub}y^N)$. Lemma 6 shows that H will not bind in equilibrium once the collateral constraint is imposed.

Lemma 6 *In equilibrium when $R\beta < 1$, we have $d_{++} > d_+ > d$ for any $d \in [\underline{d}, \overline{d})$* [29] *if the collateral constraint does not bind, $y^T = y^T_+ = y_{lb}$ and $y_{lb} \in (0, \epsilon)$ with $\epsilon > 0$.*

Proof. Note that in equilibrium, WLOG, it is possible to write $U'(A(c_1, c_2))A_1(x_1) \equiv u'(y^T - d + R^{-1}d_+)$. Then, under the assumptions stated in the lemma, it is clear that the equation (19) can be written as:

$$u'(y^T - d + R^{-1}d_+) = R\beta \sum_{y^T_+} u'(y^T_+ - d_+ + R^{-1}d_{++})q(y^T_+)$$

Where $d_{++} \in K_2$. Suppose, to generate a contradiction, $d \in K_2$ and $d_+ \leq d$. Then, as $R > 1$ for ϵ sufficiently small, we have $u'(y^T - d + R^{-1}d_+) \longrightarrow u'_{ub}$, where u'_{ub} can be constructed using the definition of u' together with assumptions $1 - a1$ and $1 - c$ and $1 - f$. Then, as $R\beta < 1$, $u'_{ub} > R\beta \sum_{y^T_+} u'(y^T_+ - d_+ + R^{-1}d_{++})p(y^T_+)$ which implies that d_+ is not optimal. Then, we must have $d_+ > d$ as desired. Replacing d with d_+, we get $d_{++} > d_+$. ■

Lemma 7 *For any $z \in Z$ the sequence $\{z, \phi_1, \phi_2, ...\}$, generated by (Z, P_φ) and with $\{\phi_1, \phi_2, ...\}(z)$, will hit the collateral constraint in finite time.*

Proof. Take any $y^T_0 \in Y$, $d_0 \in K_2$ and a sequence with τ elements in $Y \times Y \times ... \times Y$ with $\{y_0, y_{lb}, ..., y_{lb}\}$. Then, the results in section 3 imply that, as long as the collateral constraint does not bind, $d_{\tau+1}(y^T_\tau, y_{lb}, ..., y_{lb}, y_0; d_0) = d'(y^T_\tau, d)$ and $P_\tau(y^T_\tau, y_{lb}, ..., y_{lb}, y_0; d_0) = P(d'(y^T_\tau, d))$, where the equalities follow from applying iteratively backward the minimal state space policy function on equations (9) and (10) together with the envelope theorem, both derived in section 3. Note that the dependence of P on y^N has been omitted. Further, if the collateral constraint does not bind, we know from section 3 that $d'(y^T_\tau, d)$ / $P(d'(y^T_\tau, d))$ is decreasing /increasing in y^T_τ for each d. Further, $d'(y^T_\tau, d)$ / $P(d'(y^T_\tau, d))$ is increasing /decreasing in d for each y^T_τ. Lemma 9 below formally proves these claims. Then, using Lemma 6 we know that $\{d_1, ..., d_{\tau+1}\}$ / $\{P_0, ..., P_{\tau-1}\}$ is a (strictly) increasing / decreasing sequence which in turn implies that $g_t(y^T_t, y_{lb}, ..., y_{lb}, y_0; d_0) \equiv \kappa(y^T_t + P_t y^N) - d_{t+1}$ is a strictly decreasing sequence in t. To complete the proof, we must show that: i) there exists a $y^T_\tau \in Y$ such that $g_\tau \leq 0$ and ii) $\tau < \infty$.

i) Suppose the collateral constraint does not bind. Then, $d_{t+1} \longrightarrow H$. By the definition of H and the fact that $d_{t+1} = d'(y_{lb}, d_t)$, we know that $|H - d_{t+1}| = H - d_{t+1} < \varepsilon$ for $t \geq N_\varepsilon$. For any given κ, we

[29] The set $[\underline{d}, \overline{d})$ is equal to the compact set containing net external debt K_2, after substracting its upper bound, $\overline{d}$.

can take $\varepsilon \equiv \kappa(y_{ub} + P_{ub}y^N) - \kappa(y_{lb} + P_{ub}y^N) = \kappa(y_{ub} - y_{lb})$. Then, $d'(y_{lb}, d_t) > \kappa(y_{lb} + P_{ub})$, which is a contradiction. Then, the collateral constraint binds. That is, $g_\tau \leq 0$ for $g_\tau(y_{lb}, y_{lb}, ..., y_{lb}, y_0; d_0)$

ii) Simply take $\tau = N_\varepsilon$.

Note that H can be defined for any $y^T > y_{lb}$ which in turn implies that ε can be assumed to be arbitrarily small as desired. As the initial conditions were arbitrary, the proof is completed. ∎

We need to make additional notations before stating and proving the following lemma.

Let $z \in Z$. Then, any solution to the system defined by equations (9), (10), and (11) (which of course, include solutions to (12) to (16) for the constrained case) will be denoted $z(d, y^T) \equiv z = [d \quad y^T \quad y^N \quad c_1 \quad c_2 \quad p \quad m]$.

Lemma 8 *Let $J_1 \equiv Z$, where Z was defined in section 3.1. There is a point $d_* \in K_2$ with $d_{ub} > d_* > d_{lb}$ and a selection $\varphi \in \Phi$, where Φ is the equilibrium correspondence, which contains all Generalized Markov Equilibria, such that for any $(y_0^T, d_0) \in Y \times K_2$, there is a sequence $\{\phi_0, \phi_1, \phi_2, ...\}$, generated by (J_1, P_φ), which satisfies $\phi_\tau = z(d_*, y_{lb}) \in J_1$ with $\tau < \infty$.*

First, some notation. Let $d'(d_0, y_{lb})$ be the policy function obtained from solving the unconstrained case. It can be seen from standard results that $d'(d_0, y_{lb})$ is independent of prices, and thus, we can take this policy function for any c^T. Then, $\varphi \in \Phi$ satisfies:

$$d'(d_*, y_{lb}) = \kappa \left[y_{lb} + y^N \left(\frac{A_2(y^N)}{A_1(y_{lb} + (d'(d_*, y_{lb})/R) - d_*)} \right) \right] \tag{20}$$

$$U' \left\{ A_1 \left(y_{lb} + (d'(d_*, y_{lb})/R) - d_* \right) \right\} = \beta R E_\varphi[-d'(d_*, y_{lb})] \tag{21}$$

Where φ is defined by taking any vector $d''(y') \in K_2$ for any $y' \in Y$ such that:

$$U' \left\{ A_1 \left(y_{lb} + (d'(d_*, y_{lb})/R) - d_* \right) \right\} = \beta R \sum_{y'} [U' \left\{ A_1 \left(y' + (d''(y')/R) - d'(d_*, y_{lb}) \right) \right\}] q(y')$$

Before proving Lemma 8, we need preliminary results: lemmas 9, 10, and 11.

Lemma 9 *Let $y^D \equiv y^T - d$ and $c_1\left(y^D\right)$ be a tradable optimal unconstrained consumption. Then, in any unconstrained equilibrium, $y^D > \widetilde{y}^D$ implies i) $c_1\left(y^D\right) > c_1\left(\widetilde{y}^D\right)$ and ii) $d'\left(y^D\right) < d'\left(\widetilde{y}^D\right)$*

Proof. i) Suppose not. Then, $y^D > \widetilde{y}^D$ and $c_1\left(y^D\right) \leq c_1\left(\widetilde{y}^D\right)$. Then, the budget constraint in any unconstrained equilibrium implies $d'\left(y^D\right) < d'\left(\widetilde{y}^D\right)$. As $c_1\left(y^D\right)$ and $d'\left(\widetilde{y}^D\right)$ are optimal, we have:

$$U'\left(A_1(y^D)\right) \geq U'\left(A_1(\widetilde{y}^D)\right) = \beta R E\left(-d'\left(\widetilde{y}^D\right)\right) > \beta R E\left(-d'\left(y^D\right)\right)$$

Which implies a contradiction as $c_1\left(y^D\right)$ and $d'\left(y^D\right)$ are assumed to be optimal.

ii) Let $y^D > \widetilde{y}^D$ and $c_1\left(y^D\right) > c_1\left(\widetilde{y}^D\right)$. Assume, in way of contradiction $d'\left(y^D\right) \geq d'\left(\widetilde{y}^D\right)$. Then:

$$U'\left(A_1(y^D)\right) < U'\left(A_1(\widetilde{y}^D)\right) = \beta R E\left(-d'\left(\widetilde{y}^D\right)\right) \leq \beta R E\left(-d'\left(y^D\right)\right)$$

Which implies a contradiction as $c_1\left(y^D\right)$ and $d'\left(y^D\right)$ are assumed to be optimal. ∎

Lemma 10 *In any unconstrained equilibrium, there is a decreasing sequence of debt with increasing tradable consumption for the same level of tradable output,* y^T

Proof. Let $d'(d_t, y_t^T) = d_{t+1}$. Thus, $U'\left(A_1(y_t^T - d_t + (d_{t+1}/R))\right) = \beta RE\left(-d_{t+1}\right)$. Take $\widetilde{d}_t < d_t$. Then $U'\left(A_1(y_t^T - \widetilde{d}_t + (d_{t+1}/R))\right) < \beta RE\left(-d_{t+1}\right)$. Then, there exist $\widetilde{d}_{t+1} < d_{t+1}$ with $(\widetilde{d}_{t+1}/R) - \widetilde{d}_t > (d_{t+1}/R) - d_t > 0$ such that $U'\left(A_1(y_t^T - \widetilde{d}_t + (\widetilde{d}_{t+1}/R))\right) = \beta RE\left(-\widetilde{d}_{t+1}\right)$. By letting $d_{t+1} = \widetilde{d}_t$, $\widetilde{d}_{t+1} = d^*_{t+2}$ and $\widetilde{d}_t = d^*_{t+1}$, $d_t = d^*_t$, we obtain the decreasing sequence $\left\{d^*_{t+i}\right\}_i$ which generates a increasing consumption sequence $\left\{c^*_{1,t+i}\right\}_i$ because $(\widetilde{d}_{t+1}/R) - \widetilde{d}_t > (d_{t+1}/R) - d_t > 0$ and $y^T_{t+i} = y^T$ for all i. ■

Lemma 11 *The selection* $\varphi \in \Phi$ *exists.*

Proof. There exist $d_* \in K_2$ such that:

$$d'(d_*, y_{lb}) = \kappa\left[y_{lb} + y^N\left(\frac{A_2(y^N)}{A_1(y_{lb} + (d'(d_*, y_{lb})/R) - d_*)}\right)\right]$$

This is possible as the LHS / RHS of this equation is increasing/decreasing in d, both sides are continuous functions of d according to the results in section 3.2 and

$$d'(0, y_{lb}) < \kappa\left[y_{lb} + y^N\left(\frac{A_2(y^N)}{A_1(y_{lb} + (d'(0, y_{lb})/R))}\right)\right]$$

The collateral constraint does not bind if the debt is non-positive. ■

Proof of Lemma **8**.

Proof. Take any $y_0^T, d_0 \in J$ and a sequence $\{y_0^T, y_{lb}, ..., y_{lb}\}$ of $\tau + 1$ elements. The results in Lemma 6 imply that, as long as the collateral constraint does not bind in period $\tau < \infty$ (this assumption can be imposed WLOG due to Lemma 7), there is a constant sequence of tradable consumption $\{c_{1,0}, ..., c_{1,\tau-1}\}$ with $c_1(y_0^T, d_0) = c_{1,t}$ for $0 \leq t \leq \tau - 1$ which can be implemented as a SCE. Further, as it is shown in Lemma 9, it is possible to choose $c_1(y_0^T, d_0)$ to be decreasing in d_0 and increasing in y_0^T. Equipped with these paths, we will deal with a fraction of all possible initial conditions in $Y \times K_2$. To deal with the rest of the space, if necessary, we will use lemma 6 and the inequality in the Euler equation to deal with the constrained case.

Now take $y_0^T = y_{lb}$ and $d_0 = d_*$ with $\kappa(y_{lb} + P(x_1(y_{lb}, d_*))y^N) = d'(y_{lb}, d_*)$. The existence of d_* follows from Lemma 11. This point defines the "atom." We must show that a positive probability sequence exists starting from any initial condition, which hits the atom in finite time. Intuitively, the atom will be defined as the level of wealth, d, for which the collateral constraint binds with equality for the lowest possible level of tradable income. Thus, the strategy of the proof is to show that regardless of the initial condition, it is possible to construct a positive probability path that will hit the constraint: a) later, in period τ, with $0 < \tau < \infty$, b1) with a higher tradable consumption level and with more debt, both today or b2) with more debt tomorrow (and not necessarily with more consumption today). Call this last point $z(d_\tau, y_\tau)$. First, note that as the collateral constraint hits the constraint with an equality in the atom, we will have more debt for any other hitting path $z(d_\tau, y_\tau)$, but not necessarily more consumption. This fact implies that we are generating the deleverage necessary to stabilize the dynamical system and simultaneously replicate a sudden stop, typically characterized by an abrupt reduction in external indebtedness. To see this, note that the Euler inequality implies that $z(d_*, y_{lb})$ *also* satisfies the system of equations that defines the GME for the constrained case due to the strict inequality in the primal formulation of this equation. That is, the qualitative properties of $z(d_\tau, y_\tau)$ imply, due to the inequality in the Euler equation in the primal characterization of the sequential equilibria coupled with the backward nature of the definition of any GME, that $z(d_\tau, y_\tau)$ and $z(d_*, y_{lb})$ are *both* a solution to the constrained system which defines the GME.

We will now show that the chain will hit the atom, $z(d_*, y_{lb})$, starting from any initial condition in $J = Y \times K_2$. We will proceed in 2 regions:

i) $d_0 < d_*$ and $y_0^T = y_{lb}$,

ii) $d_0 > d_*$ and $y_0^T > y_{lb}$.

The case $d_0 \geq d_*$ and $y_0^T = y_{lb}$ is trivial as the collateral constraint binds.

The case $d_0 \leq d_*$ and $y_0^T > y_{lb}$, when $d_1 < d_0 < d_*$ can be treated as case i) by setting $y_1^T = y_{lb}$. If $d_0 < d_1 < d_*$, we can set $y_1^T = y_2^T = ... = y_s^T = y_{ub}$ until we get $d_s < d_{s+1} < d_0 < d_*$ to set $y_{s+1}^T = y_{ub}$ and use case i).

Intuitively, region i) ensures, due to Lemma 9, that the collateral constraint will not bind at $t = 0$ and that initial tradable consumption is bigger than the atom level. Using Lemma 6 and 7 we will construct a positive probability sequence with increasing debt, which ensures the "reversion" to the atom when the path hits the constraint.

Region ii) has two possible sub-regions. ii.a) $y_0^T - d_0 > y_{lb}^T - d_*$, in which case again due to Lemma 9 we will have bigger initial consumption and smaller debt when compared with the atom level. Thus, we can construct a sequence with increasing debt in an unconstrained environment using the arguments in region i) until we exceed the atom's level and revert to it when the path hits the collateral constraint. ii.b) $y_0^T - d_0 < y_{lb}^T - d_*$, in which case the consumption level is smaller when compared to the atom level due to Lemma 9. Thus, we must construct an increasing consumption sequence to revert to the atom. This case has multiple possibilities, and we sketch the remainder of the proof. In particular, the constraint can be below the level implied by the atom $\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) < \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < d'(y_0^T, d_0)$ (ii.b.1) or above $\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) > \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < d'(y_0^T, d_0)$ (ii.b.2). In region ii.b.1, we can generate an increase in consumption in a constrained environment if p is not sufficiently sensitive or elastic (ii.b.1.1). Contrarily, if prices are sufficiently elastic, an increase in consumption will take place in an unconstrained environment (ii.b.1.2). Finally, in region ii.b.2, when the constraint is above concerning the atom level, as debt is also bigger we may have a constrained or an unconstrained regime. In these cases, we can use the results in regions ii.b.1 to increase consumption until we reach the atom consumption.

We now turn to prove the discussion in the preceding paragraph formally.

i) For $d_0 < d_*$ and $y_0^T = y_{lb}$, we have $\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < \kappa(y_{lb} + P(c_1(y_{lb}, d_0))y^N)$ and $d'(y_{lb}, d_0) < d'(y_{lb}, d_*)$ from Lemma 9 which, using Lemma 7, implies that for the sequence $\{\phi_0, ..., \phi_t\}$ the chain will hit the collateral constraint in $t = \tau > 0$ with $d_\tau > d_*$.

We claim that the system of equations given by (12) to (16) can also be solved by $z(d_*, y_{lb})$ and thus we have that $\{\phi_0, ..., z(d_*, y_{lb})\}$ is an equilibrium trajectory. To prove this claim, note that from the definition of an equilibrium correspondence, we have that any d in z with $z \in Z$ that solves $U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_1(y_{lb}, d_0))y^N) - d)\} \geq E(m_+)$ is a predecessor of $z_+(y_+^T)$ for any $y_+^T \in Y$. As $U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) - d_*)\} > E(m_+)$, equations (12) to (16) imply that $\{\phi_0, ..., z(d_*, y_{lb})\}$ is an equilibrium trajectory as desired.

ii) For $d_0 > d_*$ and $y_0^T > y_{lb}$.

Region ii.a): $y_0^T - d_0 > y_{lb} - d_*$. As y^D is bigger in this case when compared with the atom level and we have more tradable output, $d'(y_0^T, d_0) < \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < \kappa(y_0^T + P(c_1(y_{lb}, d_0))y^N)$ and we can use the arguments in region i).

Region ii.b): $y_0^T - d_0 < y_{lb} - d_*$.

Region ii.b.1): $\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) < \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < d'(y_0^T, d_0)$. That is, the collateral hits at $t = 0$ with $c_1(y_{lb}, d_0) > c_1(y_0^T, d_0)$, which implies that we must generate an increasing sequence of consumption from the constrained regime. Note that when we increase tradable consumption, depending on the sensibility of p concerning c^T, we may enter into a constrained (ii.b.1.1) or an unconstrained regime (ii.b.1.2).

Region ii.b.1.1): $1 \geq \kappa y^N R^{-1} p'$, where p' is the derivative of (12) with respect to d_+ [30]. As $c_1(y_{lb}, d_*) > c_1(y_0^T, d_0)$ and

$$\begin{aligned} \kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) &< \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) \\ &< d'(y_0^T, d_0) \end{aligned}$$

we have:

$$\begin{aligned} U'\{A_1(y_0^T + R^{-1}\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) - d_0)\} &> U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) - d_*)\} \\ &= E(m_+; -\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N)) \\ &> E(m_+; -\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N)) \end{aligned}$$

The above inequality implies that any path in this region with $P(c_1(y_t^T, d_t)) < P(c_1(y_{lb}, d_*))$ is optimal. By setting $\left\{y_0^T, y_{lb}, ..., y_{lb}\right\}$ we can construct an increasing sequence $\left\{P(c_1(y_0^T, d_0)), P(c_1(y_{lb}, d_1)), ..., P(c_1(y_{lb}, d_\tau))\right\}$ $\longrightarrow P(c_1(y_{lb}, d_*))$ as desired.

Region ii.b.1.2): $1 < \kappa y^N R^{-1} p'$. In this case, an increase in consumption takes us to the unconstrained region. Using Lemma 10, we can generate a path $\left\{y_0^T, y_1^T, ..., y_\tau^T\right\}$, with $y_t^T > y_{lb}$; $t = 1, ..., \tau$, of increasing consumption and decreasing debt until we get $c_1(y_{lb}, d_*) < c_1(y_\tau^T, d_\tau)$. Then, set $\left\{y_\tau^T, y_{lb}, ..., y_{lb}\right\}$ The argument in region i) ensure that there is a finite time, $\tau + \tau_1$, such that the path hits the atom.

Region ii.b.2): $\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) > \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < d'(y_0^T, d_0)$. In this case, we can be either unconstrained or constrained. We can use the same arguments as in region ii.b.1) for the former. For the latter, note that by setting $\left\{y_0^T, y_{lb}, ..., y_{lb}\right\}$ we enter into the ii.b.1) region for $t > 0$ as $\kappa(y_{lb} + P(c_1(y_{lb}, d_t))y^N) < \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N)$ as $d_t > d_*$. ■

Lemma 12 *The results in Lemmas 6 to 8 imply that there exists a selection $\varphi \sim \Phi$ and a Markov process (J_1, P_φ) that has an accessible atom, $z(d_*, y_{lb})$, and is $P_\varphi(z(d_*, y_{lb}), .)-$irreducible*

Proof. It follows directly from standard results in probability theory. For expository purposes, we present the relevant result in the supplementary material for section 4. In particular, Definition 16 and Proposition 17 in the online appendix contain the definition of an accessible atom and $P_\varphi(z(d_*, y_{lb}), .)-$irreducibility. ■

Lemma 13 *Let (J_1, P_φ) be the process defined in Lemma 8. If the collateral constraint hits at time $\tau > 0$ with $d_\tau > d_*$ and $c_{1,\tau} > c_1(y_{lb}, d_*)$ or with $d_{\tau+1} > d(d_*, y_{lb})$, then $\{\phi_0, \phi_1, ..., \phi_{\tau-1}, z(d_*, y_{lb})\}$ is an equilibrium trajectory.*

Proof. If the collateral constraint binds for consumption $c_{1,\tau}$ and debt d_τ, then it must satisfy $U'\{A_1(y_\tau^T + R^{-1}\kappa(y_\tau^T + P(c_{1,\tau})y^N) - d_\tau)\} \geq E(m_+)$. The conditions in the remark imply that $U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_{1,\tau})y^N) - d_*)\} > E(m_+)$ as desired. Next note that

$$\begin{aligned} U'\{A_1(y_\tau^T + R^{-1}d_{\tau+1} - d_\tau)\} &\geq U'\{A_1(y_\tau^T + R^{-1}d(d_*, y_\tau^T) - d_*)\} \\ &= E(m_+(d(d_*, y_\tau^T))) \geq E(m_+(d_{\tau+1})). \end{aligned}$$

■

[30]For concreteness, in regions ii.b.1.1) and ii.b.1.2) we present the threshold values for p' based on the aggregator$A(c)$ typically used in the literature:

$$A(c_t^T, c_t^N) = [a c_t^{T\ 1-\frac{1}{\xi}} + (1-a) c_t^{N\ 1-\frac{1}{\xi}}]^{\frac{1}{1-\frac{1}{\xi}}}$$

with $\xi > 0$, $a \in (0, 1)$.

Lemma 14 *Let* (J_1, P_φ) *be the Markov process in Lemma 8. Then,* (J_1, P_φ) *has a unique, ergodic, invariant probability measure.*

Proof. Note that Lemma 8 imply that $P_\varphi^\tau(z(d_*, y_{lb}), \{z(d_*, y_{lb})\}) > 0$ with $\tau < \infty$. Given the results in Remark 4.2.1, proposition 4.2.2, theorem 8.2.1 and theorem 10.2.1 in Meyn and Tweedie ([48]) imply that (J_1, P_φ) has an unique invariant measure. As $\tau < \infty$ for any initial condition in J_1, theorem 10.2.2 in Meyn and Tweedie ([48]) implies that the invariant measure is a probability measure. As it is unique, the Krein-Milman theorem (See Futia, [32]) implies that this measure is ergodic. ■

Online Appendix

This section contains additional details that complement the body of the paper.

Supplementary material for section 2

In the body of the paper, we characterize equilibrium using primal methods. Below, for completeness, we include the standard dual characterization. Moreover, we provide details and concrete examples of the preferences satisfying assumption 1.

Dual characterization of the optimization problem

Under Assumption 1 on preferences, the households' sequential optimization problem satisfies standard convexity, continuity conditions, and continuous differentiability assumptions on preferences, and by an appeal to well-known duality arguments in the literature (e.g., Rincon-Zapatero and Santos ([56]), theorem 3.1), we can show that there exists a well-defined standard Lagrangian formulation for the sequential primal problem in (4) with (summable) dual variables $\beta^t \lambda_t$ and $\beta^t \lambda_t \mu_t$ associated with the sequence of constraints (2) and (3), respectively, as well as standard envelope theorems from date 0 household states. Noting our constrained system satisfies sequential linear independence constraint qualifications, strong duality holds between the resulting Lagrangian formulation and the primal program in (4) and the infinite-dimensional system of KKT multipliers are well-defined and unique. We can then formulate a system of First-order conditions for this problem in a sequential competitive equilibrium using the Lagrangian dual as follows: the optimal stochastic processes $c^*(s_0, p, R)$ and $d^*(s_0, p, R)$ satisfy

$$\lambda_t^* = U'(A(c_t^*))A_1(c_t^*) \tag{22}$$

$$p_t = \frac{A_2(c_t^*)}{A_1(c_t^*)} \tag{23}$$

$$[\frac{1}{R} - \mu_t^*]\lambda_t^* = \beta E_t \lambda_{t+1}^* \tag{24}$$

$$\mu_t^*[d_{t+1}^* - \kappa(y_t^T + p_t y_t^N)] = 0, \mu_t^* \geq 0 \tag{25}$$

Equations (22) to (25) characterize optimization.

Details on preferences

This paper characterizes the sequential competitive equilibrium when $\beta R < 1$ with Y finite. We do not discuss existence issues for the case that $\beta R \geq 1$. In a related paper, Pierri and Reffett ([**?**])) identify sufficient conditions for extending the results to this case (including setting with more general shocks and continuous shock spaces). The critical complication for introducing $\beta R \geq 1$ is obtaining compactness when studying the long-run behavior of such a model relative to stationary equilibrium. It turns out that if we endow the model with a satiation point, the equilibrium has a degenerate steady state as consumption converges to a Dirac measure, a.e. It is possible, as in Hansen and Sargent ([37]), to allow for a generalization of this last type of equilibrium by assuming that the satiation point (called "bliss point") is a random variable. Although it can be useful in some applications (i.e., asset pricing with no trading, etc.), this type of equilibrium has restrictive dynamics. Considering the question, we defer this case's discussion to a separate paper.
An example of the restrictions on preferences implied by Assumption 1 can be seen in the table below.

Pref.	Does SCE Exist?	$c > 0$?	MU bounded?	Are Homothetic?
CD	Unknown	YES	NO	YES
LOG	Unknown	YES	NO	YES
CES	Unknown	YES	NO	YES
Mod. CD	Theorem 1	YES	YES, above	NO
Mod. LOG	Theorem 1	YES	YES, above	NO
Mod. CES	Theorem 1	YES	YES, above	NO
Mod. CES 2	Theorem 1	YES	YES, AF zero	NO

Table 1: Restriction on preferences

The abbreviation "MU" stands for "marginal utility" and "AF" for "away from." Below, we provide a concrete parametrization of preferences for each of the cases presented in Table 1. Note that Theorem 3 requires MU to be bounded above and away from zero. The requirement that MU is bounded above breaks the homotheticity of preferences (i.e., Inada conditions), but the assumption of bounded away from zero allows for the homothetic case.

More to the point, the proposed utility functions can be made arbitrarily close to their homothetic counterpart and the numerical section in this paper. This suggests that assuming boundedness, instead of imposing restrictions on marginal utility and using standard CES preferences, works in practice. We need bounds on marginal utility to ensure the equilibrium set's compactness to complete the existence proof of the SCE. As the numerical results in this paper extensively use the quantitative implications of this last type of equilibrium, our simulations in this paper in section 5 are well behaved under the standard CES preferences.[31]

Examples of utility functions in table 1

This section contains concrete examples of utility functions that ensure that marginal utility is bounded.

$$Mod. \quad CD \quad (c_1+\gamma)^{\alpha}\,(c_2+\gamma_1)^{\beta}$$
$$u : X \to \Re, \quad X \supseteq \Re^2_+, \quad \gamma, \gamma_1 > 0, \quad c \in X \Rightarrow c + [\gamma, \gamma_1] > 0$$

The "Mod. CD" preferences are defined over a consumption set that includes the "zero"-vector, and γ ensures that marginal utility remains bounded above the entire consumption set, X. The "Mod. LOG" and "Mod. CES" is similar. Just replace $(c_1+\gamma)^{\alpha}\ (c_2+\gamma_1)^{\beta}$ by $ln\,(c_1+\gamma)\ +\ ln\,(c_2+\gamma_1)$ and by $(a\,(c_1+\gamma)^{\alpha} + a\,(c_2+\gamma_1)^{\alpha})^{(1/a)}$ respectively with $a > 0$. The "Mod. CES 2" are rather different as they are intended to keep MU bounded away from zero. In particular,

$$Mod. \quad CES\ 2 \quad (a_1\,(c_1)^{(1-\alpha)} + (1-\alpha)a_2c_1 + b_1\,(c_2)^{(1-\alpha)} + (1-\alpha)b_2c_2)^{(1/(1-\alpha))}$$
$$u : X \to \Re, \quad X \supseteq \Re^2_+, \quad \alpha > 1, \quad a_1, a_2, b_1, b_2 > 0$$

Combining "Mod. CES" with "Mod. CES 2" ensures that MU remains bounded above and away from zero, which guarantees that p is positive and finite. These are sufficient conditions for the existence of an a.e. compact equilibrium.

[31] Non-homotheticity of preferences have been used recently in many strands of the macroeconomics literature. For example, Rojas and Saffie [57] recently explored the importance of non-homothetic preferences in models of sudden stops. In the context of precautionary savings models, Straub ([?]) found that consumption is linear in permanent income under homothetic preferences,t, which is at odds with the data. He extends the canonical precautionary savings model with heterogeneous agents to include non-homothetic preferences.

Supplementary Material for section 3

This section contains the standard, minimal state space, and definition of recursive equilibrium. The maximization problem faced by the agent can then be written as:

$$\begin{gathered} V(b,B,y) = \; max_{[b'^T,c^N]}\; u(A(c^T,c^N)) + \beta V(b',B',y') \\ \text{Subject to} \\ c^T + p(B,y)c^N + d = y^T + p(B,y)y^N_t + \frac{d'}{R} \\ d'^T + p(B,y)y^N) \\ B' = \Gamma(B,y). \end{gathered}$$

Where Γ is the perceived law of motion, b is the individual endogenous state, B is the aggregate endogenous state, and y is the tradable shock. We can now define a *Minimal State Space Recursive Equilibrium.*

Definition 15 *Minimal State Space Recursive Equilibrium (MSSRE). A MSSRE is defined by a pricing function $p(B,y)$, a perceived law of motion $B' = \Gamma(B,y)$, policy functions $\hat{b}', \hat{c}^T, \hat{c}^N$, and a value function V such that; i) the policy functions $\hat{b}', \hat{c}^T, \hat{c}^N$ solve the recursive maximization problem given $p(B,y)$ and $B' = \Gamma(B,y)$, ii) there are rational expectations $b' = B' = \Gamma(B,y)$, iii) market clear $y^N = \hat{c}^N$.*

Supplementary Material for Section 4

Let us start by formally defining an "accessible atom", which can be thought of as a point that is non-negligible from a probabilistic perspective and gets "hit" frequently. Let $\varphi \sim \Phi$ be a selection of the equilibrium correspondence defined in the previous subsection. The compactness of $Y \times Z$ and Z guarantees the measurability of φ. [32] Further, $P_\varphi(z,A) \equiv \{p(y^T \in Y : \varphi(z,y^T) \in A)\}$ defines a Markov operator (i.e. $P_\varphi(.,A)$ is measurable and $P_\varphi(z,.)$ is a probability measure) and (Z, P_φ) a Markov process where y^T is assumed to be iid with probability $p(y^T)$. [33] Let $P^n_\varphi(z,A)$ be the probability that the Markov chain goes from z to any point in A in n steps with A being measurable, let ψ be some measure, and $B(Z)$ be the Borel sigma-algebra generated by Z. Then the set $A \in B(Z)$ is *non-negligible* if $\psi(A) > 0$. A chain is called *irreducible* if starting from any initial condition, the chain hits all non-negligible sets with positive probability in finite time (i.e., $\psi(A) > 0 \longrightarrow P^n_\varphi(z,A) > 0$). [34] Intuitively, irreducibility is a notion of connectedness for the Markov process as it implies non-negligible sets are visited with positive probability in finite time.
We can now define an atom and state a significant intermediate result.

Definition 16 *Accessible Atom. A set $\alpha \in B(Z)$ is an atom for (Z, P_φ) if there exists a probability measure ν such that $P_\varphi(z,A) = \nu(A)$ with $z \in \alpha$ for all $A \in B(Z)$. The atom is accessible if $\psi(\alpha) > 0$.*

Intuitively, an atom is a set containing points where the chain behaves like an iid process. Any singleton $\{\alpha\}$ is an atom. Note that there is a trade-off: if the atom is a singleton, the iid requirement is trivial, but considering that the state space is uncountable, the accessibility clause becomes an issue as it is unclear how to choose ψ. For instance, the typical Lebesgue measure is useless as any singleton has zero measure. The same happens with irreducibility: when the state space is finite, it suffices to ask for a transition matrix with positive values in all its positions. In general, we need to define a meaningful set carefully, as it is impossible to list all of them. Fortunately, when the state space Z is a product space between a finite set (Y) and an uncountable subset of $\Re^m$ there is a well know results that please help us find an accessible atom in an irreducible chain (for proof, see Proposition 5.1.1 in Meyn and Tweedie.)

[32] for example, Stokey, Lucas, and Prescott, ([65]), Th. 7.6, p. 184).
[33] For example, see Grandmont and Hildenbrand ([33]).
[34] e.g., see Meyn and Tweedie (([48], proposition 1)

Proposition 17 *Suppose that $P_{\varphi}^{n}(z,\alpha)>0$ for all $z\in Z$. Then α is an accessible atom and (Z,P_{φ}) is a $P_{\varphi}(\alpha,.)-$irreducible.*

Proposition 17 follows directly from standard results in Meyn and Tweedie ([48]). The reference measure may differ from ψ, called "maximal". Fortunately, if the chain is fundamental concerning some measure, say $P_{\varphi}(\alpha,.)$, then it can be "expanded" to ψ (e.g, see Meyn and Tweedie, ([48], Proposition 4.2.2).
To apply Proposition 17 to our model, the finiteness of Y and the definition of the Markov kernel P will be critical. As we are considering a point, to show that $P_{\varphi}^{n}(z,\alpha)>0$, it suffices to find a finite sequence $\{y_0,...,y_n\}$ such that $\{\alpha\}$ is a solution to equations (12) to (16), the system associated with a binding collateral constraint. We want to associate the atom with an economic crisis. This will allow us to connect the invariant measure with a sudden stop (SS) or, equivalently, to relate the model's steady state with the frequency of crises. Typically, the literature associates a hit to the collateral constraint with a crisis. We are extending these results and connecting the frequency of crises with the stochastic steady state.

Supplementary Material for section 5.2 (numerical procedure)

We now present the ergodic, stationary, and non-stationary algorithms together with some details of the procedure involved.

GME Ergodic Algorithm

Step 1: Computation

- Fix the vector of parameters from section 2.1, $[\kappa,\beta,\sigma,\xi,a]\equiv\Theta\gg\overrightarrow{0}$ with $U(c_1,c_2)=\frac{A(c_1,c_2)^{(1-\sigma)}}{1-\sigma}$ and $A(c_1,c_2)=(a(c_1)^{(1-1/\xi)}+(1-a)(c_2)^{(1-1/\xi)})^{\frac{1}{1-1/\xi}}$.
- Fix $Y\times K_1\times K_2$
- Compute $d(d,y)$ from the unconstrained problem That is. *ignoring* the collateral constraint.
- Compute d_* from equation (20) in the appendix
- Compute $\varphi\in\Phi$ from equation (21) in the appendix

Step 2.1: Stationary simulation

- Take a "draw" of length $T+1$ from (Y,q), the exogenous Markov process which generates tradable output.
- Fix $[d_{T-1},y_{T-1}]$ from $Y\times K_2$, obtain d_T from $d(d,y)$. Verify if the collateral constraint binds and adjust d_T if necessary. Then compute $d_{T+1}(y_T,d_{T-1},y_{T-1})$ from $\varphi\in\Phi$ for every $y_T\in Y$. Compute the rest of the endogenous variables from equations (12)-(16).
- Take $[p_T(y_T),d_T(y_{T-1})]$ as given from the previous • and compute $[p_{T-1}(y_{T-1}),d_{T-1}(y_{T-2})]$ from equations (12)-(16). Note that d_{T-1} is in the preimage of $d'(d,y)$ with $d_T=d(d_{T-1},y_{T-1})$.
- Repeat the above • until you get $[p_0(y_0),d_0]$, where d_0 is allowed to be independent of y_0 as they are both initial conditions of the system.

Step 2.2: Ergodic simulation

- Take a "draw" of length $N+1$ from (Y,q), the exogenous Markov process which generates tradable output.

- Compute $[p_j(y_j), d_j(y_{j-1})]$ for $j = N+1, N, N-1$ as in the stationary simulation procedure
  - If d_N binds, $y_{N-1} - d_{N-1} + d_N/R > y_{LB} - d_* + d(d_*, y_{LB})/R$ and $d_N > d(d_*, y_{LB})$, the process hits the atom and reverts to it.
- Continue until $[p_0(y_0), d_0]$

The numerical procedure is based on the policy function of the unconstrained problem, $d'(.,.)$. Then the atom d_* is computed using $d'(.,.)$ in (20). Note that if the process hits the constraint in period t, then $d(d_t, y_t) > d(d_*, y_{lb})$. Thus, to find a regeneration point, we need

$$U'\left\{c_1\left(y_t - d_t + d(d_t, y_t)/R\right)\right\} - \beta R \sum_{y'} U'\left\{c_1\left(y' - d'(d_t, y_t) + d''(d_t, y_t, d', y')\right)\right\} \equiv f(d''(y'); d, d', y_t) \geq 0$$

$$U'\left\{c_1\left(y_{lb} - d_* + d'(d_*, y_{lb})/R\right)\right\} - \beta R \sum_{y'} U'\left\{c_1\left(y' - d'(d_*, y_{lb}) + d''(d_*, y_{lb}, d', y')\right)\right\} \geq 0$$

Where, for exposition purposes, we denote $d_+ = d'$ and $d_{++} = d''$ if the collateral constraint does not bind or if it binds with equality. Note that in section 4 we require d'' to be independent of d, d', y. This fact implies that we can easily find d'', but some additional conditions must be satisfied by paths to revert to the atom (i.e., consumption and debt must be greater than the atomic level). These restrictions affect the frequency at which the atom is hit. Thus, to improve the recurrent structure of sets and gain computational efficiency, we modified the selection from the GME. When we allow d'' to depend on an expanded state space, we can find a stationary Euler equation for each d, y as d' has at most two solutions when the collateral constraint is binding.
Numerically, this selection is easily implemented, and the algorithm is fast as we do not need to compute every selection of the GME.

GME Non-Stationary Algorithm

Step 1

- Fix the vector of parameters from section 2.1, $[\kappa, \beta, \sigma, \xi, a] \equiv \Theta \gg \overrightarrow{0}$
- Fix $Y \times K_1 \times K_2$ in order to define a compact set for (y, p, d) respectively.
- Fix a $[p_{T+2}(y_{T+2}), d_{T+2}(y_{T+1})]$ for each $(y_{T+1}, y_{T+2}) \in Y \times Y$. This will define a selection $\varphi \in \Phi$ using the Euler equation in time T.

Step 2

- Take a "draw" of length $T+1$ from (Y, q), the exogenous Markov process which generates tradable output.
- Fix $[p_{T+1}(y_{T+1}), d_{T+1}(y_T)]$ from $K_1 \times K_2$ and compute $[p_T(y_T), d_T(y_{T-1})]$ from equations (12)-(16)
- Take $[p_T(y_T), d_T(y_{T-1})]$ as given from the previous • and compute $[p_{T-1}(y_{T-1}), d_{T-1}(y_{T-2})]$ from equations (12)-(16). Include $[p_{T+1}(y_{T+1}), d_{T+1}(y_T)]$ in the Euler equation. This step implies a change in the selection from Φ and, thus, breaks the stationarity of the process.
- Repeat the above • until you get $[p_0(y_0), d_0]$, where d_0 is allowed to be independent of y_0 as they are both initial conditions of the system.