Dynamic Complementarities and Multiple Equilibria in Open Economies with Collateral Constraints*

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Abstract

Using constructive order-theoretic techniques, we characterize the set of minimal state-space recursive competitive equilibria in the canonical small open economy model of sudden stops with price-dependent collateral constraints. We provide a computable theory of both the existence of equilibria and equilibrium comparative statics in important deep parameters of the model (i.e., parameters on equilibrium collateral constraints). We show multiplicity of recursive equilibria arises in these models because of the presence of dynamic equilibrium complementarities in states where equilibrium collateral constraints bind. Finally, we also give sufficient conditions for the uniqueness of recursive equilibria. The paper is the first constructive approach to characterizing dynamic equilibrium in an importance class of incomplete markets general equilibrium models with price-dependent inequality constraints and infinite horizons, something that has been elusive to the literature.

Keywords: *Monotone methods, Small Open Economies, Collateral Constraints, Multiple Equilibria.*

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1 Introduction

The recent literature on financial crises in emerging markets is voluminous.[1] The workhorse model in this literature is the stochastic two-sector endowment small-open economy where agents face an occasionally binding price-dependent equilibrium collateral constraint.[2] The presence of a relative price of nontradables to tradables in equilibrium collateral constraints introduce a well-known "pecuniary externality" into the structure of the dynamic equilibrium of these models, and it is the critical feature which allows these models to explain the collapse of consumption and prices during financial crises, as well as creating a potential role for government macroprudential policies that prevent such financial collapses and Fisherian deflations.

However, the presence of these equilibrium collateral constraints also has the potential to introduce significant complications in the structure of dynamic equilibrium, yet little is known about the structure of dynamic equilibrium in these models. In a recent series of papers by Schmitt-Grohé and Uribe ([72], [73]), the authors have argued that the presence of pecuniary externalities is a potential source for multiple sequential competitive equilibria (SCE) as well as sunspot equilibria. Although their arguments for the possibility of multiple dynamic equilibria are intriguing, in many ways, they are incomplete. First, it is never shown there exists *any* dynamic equilibrium in these models.[3] Additionally, their analysis arguing for the possibility of multiple equilibria only applies to the *deterministic* version of these *stochastic* models, and they only identify possible multiplicities of SCE near the model's deterministic steady-state *if SCE exist* in the deterministic version of the model. Of course, the presence of multiple dynamic equilibria not only complicates the characterization of the equilibrium dynamics from a theoretical and numerical perspective, and it also introduces an entirely new set of challenges for researchers using these models to study the design of optimal macroprudential policies that seek to mitigate overborrowing and financial collapses. So aside from the lack of results on the existence of dynamic equilibrium, there is also a lack of understanding of what are sufficient conditions for the *uniqueness* of dynamic equilibrium in these models, and if such conditions are even plausible in applications of these models for the quantitative analysis of optimal macroprudential policies.

The aim of this paper is straightforward: to characterize completely the set of recursive competitive equilibrium (RCE) in the canonical class of sudden stop models, as well as provide sufficient conditions for the uniqueness of RCE in these models. The paper also provides the first attempt to show existence of dynamic equilibrium in a macroeconomic model with incomplete markets, infinite horizons, and occasionally binding price-dependent collateral constraints using constructive methods. The methods here are designed to handle the presence of multiple equilibria and discontinuous selections. In our analysis, we focus on RCE as they are the class of SCE that is typically studied in much of the applied work using the baseline sudden stops model. We show the set of RCE forms a (nonempty) complete lattice. We also provide a powerful collection of simple iterative methods that compute RCE via successive approximation algorithms, as well as using a new collection of "generalized iterative methods" that allows us to compute "local RCE bounds" starting from *any initial guess* at RCE. Appealing to the monotonicity of our approach, we can also provide a new set of (computable) RCE comparative statics results relative to important deep parameters of these economies (e.g., parameters of equilibrium collateral constraints). Here, aside from providing the usual comparisons of the least and greatest RCE in deep parameters, very importantly, we provide a new generalized iterative method that offers an equilibrium comparison of *any* RCE. In studying the structure of RCE in this class of models, we precisely characterize how multiple RCEs arise in these models. We show that pecuniary externalities create the possibility of an implicit equilibrium *dynamic complementarity* between household decision rules on tradables consumption and per-capita levels of equilibrium tradables consumption in states where equilibrium collateral

[1] A small sampling of work in this literature includes the early work of Mendoza ([46], [47]), Bianchi ([17]), Bianchi and Mendoza ([18]), Benigno, Chen, Otrok, Rebucci, and Young (e.g., see [10], [11], [12]). More recent papers include the papers of Deveraux and Yu ([32]), and Schmitt-Grohe and Uribe ([72], [73], [74]), Chi, Schmitt-Grohe, and Uribe ([23]), Ottonello, Perez, and Vassaco ([56], [57]), Bengui and Bianchi ([15]), Rojas and Saffie ([66]), Pierri, Mira and Montes-Rojas ([59]), Drechsel ([34]), Drechsel and Kim ([35]), Arce [4], Arce, Bengui and Bianchi ([5]), among many others.

[2] For the prototype of this model, see Bianchi ([17]) and Schmitt-Grohé and Uribe ([73], [74]), for example.

[3] Actually, to the best of our knowledge, we are not aware of any result on the existence of dynamic equilibrium for the prototype sudden stops model in this literature or any of its extensions.

constraints bind. We refer to this form of implicit equilibrium dynamic complementarity in these models as a *pecuniary complementarity.*

A succinct preview of the results in the paper goes as follows: we first construct a recursive representation of SCE in these models. To construct such a recursive representation, we identify a novel complication that arises in these models, as the collateral constraint is the price-dependent (hence, an equilibrium object), we need to parameterize the aggregate economy in the model in a manner such that *in all states,* the equilibrium collateral constraint allows households to choose a level of tradables and nontradables consumption that is *positive.* That is, is identify a necessary "strict interiority" condition of feasible consumption for household decision problems that must be met in an RCE (or SCE, for that matter). This strict positivity of consumption issue turns out to be a critical issue that has not been systematically addressed in the existing literature. After constructing such a recursive representation of SCE and characterizing household decision problems in RCE, using equilibrium versions of household Euler inequalities, we then develop a new two-step monotone fixed point method for characterizing the set of RCE and show that the set of RCE forms a nonempty complete lattice in the space of tradables consumption policies. We then identify sufficient conditions for the uniqueness of RCE, and the sufficient conditions are *strong* relative to applications.

Our two-step fixed point method addresses a critical complication that arises when constructing RCE; namely, we have to identify equilibrium states that "partition" the RCE into two regimes: unconstrained vs. constrained states. Moreover, we must locate this partition for two consecutive periods in the same operator, as the collateral constraint might bind jointly or separately in any of these pairs of periods. To accomplish this characterization of equilibria, in the first step of our equilibrium construction, we parameterize the equilibrium collateral constraint at a fixed function for per-capita tradables consumption and then compute using a monotone contraction map the (unique) *strict positive contingent* RCE for tradables consumption over "unconstrained" states. This first-step fixed point defines a second-step (monotone) operator. The domain of this second step is the function space that we used to parameterize the equilibrium collateral constraint, which in turn helps us compute a set of strictly positive fixed points for tradables consumption. The positivity of consumption, at the same time, makes the equilibrium collateral constraint consistent with the contingent RCE in the first step, which implements the set of RCE collateral constraints and then defines the *set* of RCE. We note, in our characterization of RCE, the equilibrium collateral constraints themselves form a complete lattice (hence, have a least and greatest element). This dramatically complicates the application of optimal macroprudential policy design.

Next, because of the order continuity properties of the equilibrium operator, we are also able to develop iterative methods that can be used to construct elements of the set of RCE when starting iterations from *any* element of the candidate element of the set of RCE functions. These new iterative fixed point methods are based on not only classical applications of the Tarski-Kantorovich principles, but also apply recent new results in the extend the classical Tarski-Kantorovich principle (which characterizes both the "least" and "greatest" fixed point of a monotone mapping) to the case of constructing "local fixed point bounds" to *any* initial iterate of a monotone mapping. [4] We then build a computable approach to characterizing the set of equilibrium comparative statics in important deep parameters such as those present in the model's collateral constraints for *any* RCE (not merely least and greatest RCE as is typically done in the equilibrium comparative statics literature based upon monotone methods).[5] These new generalized iteration methods can also be used to check whether a particular RCE is "order stable" in the sense of the so-called correspondence principle of Samuelson ([67]) and Echenique ([36]).

An essential contribution of the paper relative to the existing literature on multiple equilibria in these models is that we show that in a RCE, the presence of pecuniary externalities creates a very natural *global dynamic equilibrium complementarity* between aggregate levels of tradables consumptions and the corresponding household decision rules on optimal tradables consumption, and it is this dynamic complementarity that leads naturally to the presence of multiple RCEs. This dynamic complementarity

[4] These new results on generalized Tarski-Kantorovich principles are found in the recent paper of Olszewski ([55]) for order continuous mappings, and extended to ascending order upper hemicontinuous correspondences in Balbus, Olszewski, Reffett, and Wozny ([7]).

[5] See the recent paper of Balbus, Olszewski, Reffett, and Wozny ([8]) for discussion of iterative monotone comparative statics.

arises in equilibrium states where the equilibrium collateral constraint binds. In particular, we prove that when aggregate levels of perceived borrowing and tradables consumption are relatively "low" (respectively, "high"), for example, as the price of nontradables to tradables consumption is also relatively "low" (respectively, "high") as the equilibrium collateral constraint is relatively "tight" (respectively, "loose"). This creates an equilibrium (dynamic) complementarity via the household's Euler inequality that is precisely how least (respectively, greatest) RCE can become self-fulfilling with levels of low aggregate borrowing (respectively, high aggregate borrowing).[6] This intuition is formalized mathematically in this paper, by the construction of a monotone RCE operator, defined implicitly in the model's equilibrium Euler inequality, and make precise and complements the arguments on multiple equilibria in the work of Schmitt-Grohé and Uribe ([72], [73]) and Bianchi and Mendoza ([21]).

To obtain monotonicity for our RCE operator using an equilibrium versions of the household's Euler inequality, we characterize a new equilibrium single-crossing condition that naturally arises in this class of models via the existence of "pecuniary externalities" that affect the entire equilibrium structure of the model when price-dependent equilibrium collateral constraints bind. We refer to this new form of dynamic complementarity as a *pecuniary complementarity*, and we show it provides a *global* source of multiple RCE. Our approach provides a rigorous method for explaining the "self-fulfilling" nature of multiple RCE for stochastic small open economies with equilibrium price-dependent collateral constraints, and our results, therefore, provide a precise formal justification for the source of multiplicity suggested in the papers of Schmitt-Grohé and Uribe ([73], [74]). More specifically, using order continuous (monotone operator) techniques, we prove that when constructing RCE, if agents perceive the aggregate laws of motion of equilibrium states governing future per-capita aggregate borrowing will be "low" (resp., "high") in the future, RCE household collateral constraints will be tighter (resp., looser) as the relative price of tradables will be lower (resp., high) because perceived RCE prices for nontradable endowments will be lower (resp, higher). These expectations can be *globally self-generating* for some parameterized versions of consumption aggregators (including versions of Armington aggregators that have been used in the applied literature), which in turn implies the existence of a least (resp., greatest) borrowing RCE where consumption/debt will be lower (resp., higher).

1.1 This Paper and the Existing Literature

The literature on sudden stops models of emerging market financial crises begins with the important papers by Mendoza ([46], [47]), Mendoza and Smith ([48]), Bianchi and Mendoza ([18]), and Bianchi ([17]), but has continued with many recent papers exploring the different dimensions of Sudden Stops and emerging market financial crises.[7] This paper contributes to this existing literature along several dimensions.

Relative to the work of Bianchi ([17]) and many other papers in this literature using recursive methods to characterize sudden stops and optimal macroprudential policies numerically, we provide the first results on the existence and characterization of the set of RCE for this class of sudden stops models. Additionally, from a computational perspective, we provide a new collection of "time-iteration" algorithms that work based on an order continuous (hence monotone) operators and are able to immediately show how to compute least and greatest RCE (as well as their equilibrium comparative statics in important deep parameters of the model) by simple successive approximations on our equilibrium operator.

In Bianchi's work (and many other papers in this literature using so-called "time iteration" methods), the authors essentially work in the "dual" of the systems of Euler *equalities* which include Kuhn-Tucker multipliers on equilibrium versions of both household budget constraints and collateral constraints where one locates states where Kuhn-Tucker multipliers associated with the collateral constraints bind numerically and by essentially by searching over the state space to obtain numerical approximates to the (RCE) policies.[8] We work in the "primal" systems of Euler inequalities associated with the household's dynamic

[6]See Schmitt-Grohé and Uribe ([72], [73]) for a discussion of this sort of pattern of equilibrium that is found in numerical studies of these models.

[7]For a discussion of this extensive recent theoretical literature, see Mendoza and Rojas ([49]), Bianchi and Mendoza ([21]), Schmitt-Grohe and Uribe ([73]), [15], and Davis, Deverous, and Yu ([30]).

[8]We should note that proving the existence of the recursive dual for the household's optimization problem is not trivial,

programming problem and study the existence of equilibrium problem in the *space of equilibrium policy functions*. As we show in sections 3 and 4 of the paper, working with an operator that directly iterates between "tomorrow's" and "today's" equilibrium policy function to a stationary solution (i.e., a fixed point) seems very natural.

We then provide a new "generalized" time-iteration method based on lim-inf and lim-sup iterations of our monotone equilibrium operators. These methods are discussed in the recent work of Olszewski ([55]) and Balbus, Olszewski, Reffett, and Wozny ([7], [8]) where the authors show how to use iterative methods to construct "local" lower and upper RCE "bounds" for a "lim-inf" and "lim-sup" generalized iterative schemes relative to *any* initial guess at RCE for order continuous operators, as well as construct comparative statics on the order limits of these generalized iterations. Such generalized iterative methods could prove useful in studying the design of macroprudential policy instruments useful in dealing with binding collateral constraints in particular RCE. All our iterative schemes (either "standard" or "generalized" time iteration methods) are shown to be constructive as we are also able to identify *globally stable* equilibrium selectors (i.e., RCE that satisfies the well-known "correspondence principle" discussed in the seminal work of Samuelson ([67]) and Echenique ([36]). As in a world with multiple equilibria, characterizations of optimal macroprudential policies amount (in large part) to characterizations of *equilibrium comparative statics* of equilibria, identifying RCEs that are stable under perturbations seems useful for policy design experiments.

Relative to the work of Schmitt-Grohé and Uribe (e.g., [73], [74])) and Bianchi and Mendoza ([21]) on multiple equilibria in these models, we first provide a global characterization of multiple RCE in the stochastic versions of this model. As in their work, we show that the source of RCE multiplicities stems from the presence of the pecuniary externalities associated with the equilibrium collateral constraint, but unlike their work, we show how the presence of the equilibrium collateral constraints create a dynamic complementarity in the equilibrium version of the household's Euler inequality, and show that the nature of the equilibrium multiplicity is global.[9] That is, in Schmitt-Grohé and Uribe ([73], [74]), their approach to characterizing the existence of multiple equilibria is built upon the *deterministic* versions of the model, and in particular sequential equilibrium behavior "local" near a (deterministic) steady-state. Our results are never "local" or "deterministic" rather are global and based on functional equations built from an equilibrium version of the households Euler inequality and equilibrium collateral constraint. In particular, we show that the sources of multiple RCE stem from the existence of an *implicit equilibrium dynamic complementarity* generated between equilibrium collateral constraints between further equilibrium per-capita tradables consumption and current equilibrium per-capital tradables consumption.

Also, relative to all this literature above, our paper systematically addresses the critical issue of guaranteeing in a dynamic equilibrium strictly positive equilibrium consumption paths. This is a well-known problem, and has been discussed in papers such as Bianchi and Mendoza ([20]) and Schmitt-Grohé and Uribe ([73], [74]), but the literature has failed to provide any systematic approach to this challenging issue. The complication per strict positivity of consumption is rather simple to explain: when constructing the dynamic equilibria existence problem, agents take as given candidate economy-wide equilibrium paths for tradables consumption, which in turn determines the equilibrium paths for relative prices of nontradables to tradables consumption and hence equilibrium collateral constraints. The "space" of such candidate paths for economy-wide tradables consumption must be such that strictly positive household consumption choices are *feasible* (i.e., there cannot exist equilibrium states for the economy such that relative values of household endowments are so low that, given the existing household debt levels and current wealth, consumption is forced to be zero or negative). So, when defining the dynamic equilibrium existence problem for these models, this condition on strict positivity of consumption paths has to be built into the equilibrium fixed point problem. In this paper, we do this explicitly, and we verify the strict positivity of consumption paths in a RCE by constructing an appropriate strictly positive uniform lower bound on equilibrium tradables consumption, and then using monotonicity of our RCE operator

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as one has to be very careful to make certain parameterizations of the equilibrium collateral constraints allow for a strict interior point in the recursive dual formulation. We discuss this question carefully in section 3 of the paper as we need to use the recursive dual to obtain envelope theorems of the value function via the work of Rincon-Zapatero and Santos ([65]).

[9] Their papers also characterize stochastic SCE dynamics using a technique building local "stationary sunspot" approaches near steady state (e.g., see related work in Woodford ([79]) and Schmitt-Grohé ([69]), for example). Our work is different, and builds a global theory of multiple RCE from any initial state of the model.

to map this point "up" and iterate then to a least RCE which is strictly positive over its minimal state space (including debt holdings). What is critical here is that in our construction, this means the actual "maximal" debt that is sustainable in this least RCE is *endogenous.*

Our paper is also related to an emerging literature that seeks to characterize dynamic models with equilibrium borrowing constraints and/or occasionally-binding constraints. Relative to this literature, we provide a new set of methodological tools for characterizing the RCE and SCE in models with equilibrium price-dependent collateral constraints. Our multistep fixed point approach to RE can be shown to be useful in other related dynamic equilibrium models where stationary equilibrium partitions into states where collateral constraints are "slack." versus, "binding", and include models of credit cycles in the spirit of Kiyotaki and Moore ([41]),[10] models of financial frictions and production with collateral constraints such as Moll ([52]), or models of self-fulling credit cycles such as Azariadis, Kaas, and Wen ([6]).

Finally, our work is also directly related to the extensive literature on the equilibrium comparative statics in dynamic economies using monotone methods. [11] This paper is, perhaps, most closely related to a recent paper by Datta, Reffett, and Wozny ([29]) who propose a multistep monotone-map method that proves especially suited for dynamic models with multiple RCE Our paper extends some of the ideas in this paper for a class of multi-step monotone-map methods to dynamic models with equilibrium price-dependent collateral constraints and equilibrium "regime" shifts. In addition, our paper is related to Mirman, Morand, and Reffett ([51]), Acemoglu and Jensen ([1]), and Datta, Reffett, and Wozny ([29]) as it provides sufficient conditions for monotone dynamic equilibrium comparative statics in the deep parameters of the economy.

The remainder of the paper is organized as follows: Section 2 presents the baseline model and provides a preliminary characterization of household decision problems in a SCE. In section 3, we construct a recursive representation of the sequential economy, construct household decision problems, as well as household Euler inequalities. Section 4 proves the existence of RCE via monotone methods and discusses how pecuniary externalities in these models generate multiple RCEs. Section 5 presents sufficient conditions for the uniqueness of RCE. Section 6 discusses (a) how to relax some conditions on asymptotic marginal utilities for some consumption aggregators in these models, (b) extensions to more general shock spaces, and (c) discusses the structure of dynamic pecuniary externalities in these models. Section 7 of the paper concludes. Most of the proofs in the paper are included in the appendix.

2 The Model

The model we study in this paper is the canonical endowment version of the two-sector small open economy with a fixed interest rate studied in Bianchi ([17]).[12] Time is discrete over an infinite horizon and indexed by $t \in \{0, 1, 2, ...\}$. There is a representative agent and two sectors of perishable consumption goods in the economy, a tradable consumption good y_t^T and a non-tradable consumption good y_t^{NT}. Each household has a strictly positive amount of each good in each period. Upon receiving their current period endowments, households sell endowments at current market prices and choose consumption of both goods where the consumption of tradable and non-tradable is denoted, respectively, by c_t^T and c_t^{NT}. It turns out to be useful to take as the numeraire the tradable good, so the relative price of non-tradable goods to tradable goods in period t is denoted by p_t.

Uncertainty is modeled as an iid stochastic process governing the endowments of tradable goods $y = \{y_t^T\}_t$ where each element of sequence y has distribution given by the measure $\chi(\cdot)$. Further, we assume the realizations for tradable endowments in any period have $y_t^T \in Y$ where the shock space

[10] See the survey of Gertler and Kiyotaki [39] for a nice discussion of this large literature, as well as Kiyotaki and Moore ([42]) for recent work along these lines.

[11] This literature starts with the papers of Coleman ([24], [25]) and Reffett (([63]), [64]), and extended in Datta, Mirman, and Reffett ([28]), Morand and Reffett ([53]), and Mirman, Morand, and Reffett ([51]), Curatola and Faia ([27]), Pierri ([58]), Bernstein, Plante, Richter, and Throckmorton ([16]), and Ferraro and Pierri ([37]), among many others.

[12] Although we limit our discussion in this paper to stochastic representative agent versions of the endowment version of the sudden stops models, our methods can be extended to production versions of these models (e.g., Beningo et al ([11], [12]) and Bengui and Bianchi ([15]) without much difficulty.

Y is finite set.[13] In addition, as allowing nontradables endowments to be stochastic plays no role in the characterization of stochastic equilibrium dynamics in these models (or this paper), as is custom in the literature, we assume the sequence of non-tradable endowments $\{y_t^{NT}\}_t$ is deterministic and fixed with $y_t^{NT} = y^{NT} > 0$ for all t . By an application of standard results in the literature on stochastic processes, these assumptions imply we can define a stochastic process for tradables endowments denoted by $(Y^{\infty}, \Omega, \mu_{y_0^T})$ with takes realizations in each period in Y, where Y^{∞} is the space of infinite sequences in Y, and we assume $y_0^T \in Y$ is the initial condition for this stochastic process for tradables endowment.[14]

Household preferences are defined over infinite sequences of dated consumption vectors of tradable and non-tradable goods and denoted by $\{c_t\}_{t=0}^{\infty} = \{(c_t^T, c_t^{NT})\}_{t=0}^{\infty} \in X^{\infty}$ where $X = \mathbf{R}_+^2$ is the commodity space for consumption of tradables and nontradables in each period. Preferences are assumed to be time separable with subjective discount factor $\beta \in (0,1)$. Household preferences over consumption goods in any period are then represented by a nested utility function, which is a composition of two functions: a utility over composite consumption $U : \mathbf{R}_+ \longrightarrow \mathbf{R}$, and a consumption aggregator $A : X \longrightarrow \mathbf{R}$ over tradable and non-tradable consumption $c = (c^T, c^{NT})$ where the preferences $u(c) = U(A(c))$ gives the instantaneous utility of the vector of consumption $c \in X \subset \mathbf{R}_+^2$ in any period. Then, the household has lifetime preferences given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t)) \tag{1}$$

where the mathematical expectation operator is taken over the stochastic structure of uncertainty with respect to the date zero information.

We impose the following basic assumption on preferences in the paper:

Assumption 1: (i) *The functions* $U(c_a)$ *and* $c_a = A(c)$ are both continuous, strictly increasing, C^2, strictly concave, with $A(c)$ supermodular, (ii) $lim_{c_a} U'(c_a) = 0$, (iii) *A:* $X \mapsto \mathbf{R}_+$*, for* $c^{NT} > 0$, $lim_{c^T \longrightarrow \infty} A_1(c^T, c^{NT}) = a > 0$, $lim_{c^T \longrightarrow 0} A_1(c^T, c^{NT}) = \infty$.

We make two remarks on our assumptions on preferences. Initially, we should mention that our assumptions are completely standard, excepting the possibly Assumption 1.iii. [15] Per Assumption 1.iii, the issue of bounding debt *in equilibrium* in these models has not been formally addressed in the existing literature. Bounding debt is a well-known critical issue in dynamic models with incomplete markets and infinite horizons. This issue in equilibrium in sudden-stop models can be addressed in at least one of two ways. The first way is to impose a condition on asymptotic marginal utility in a manner that bounds consumption as in Assumption 1.iii. This is a technical condition that requires $U(A(c))$ to have a marginal utility that is asymptotically invariant for arbitrarily large levels of tradables consumption good. This can be used directly to construct a natural upper bound on debt and tradable consumption in any RCE.[16] We should mention, given this upper bound (on tradables consumption and debt), computing the "greatest" RCE for tradables consumption simple via successive approximations via the monotonicity of our RCE operator from this upper bound will be very simple. We should this in section 4 of the paper. In addition, we should mention Assumption 1.iii *does not change* in any meaningful way nature of the stochastic RCE dynamics at all in numerical simulations of these models [17] An alternative approach to

[13]All the results of the paper hold for more general endowment processes for tradables consumption, including the case that endowments follow a first order Markov process with stationary transition $\chi(y^T, y^{T\prime}))$ on a continuous shock space with the exception that the set of RCE will be on being a sigma complete lattice (as opposed to a complete lattice). We discuss these extensions in the last section of the paper.

[14]Here, we follow, for example, Stokey, Lucas, and Prescott ([75], Chapter 7)).

[15]This assumption does not restrict our ability to allow for standard parametric formulations of $U(A(c))$ that have been used in applied work in the literature up to an affine transformation of the standard $A(c)$ used in the literature (i.e., CES aggregators $A(c)$). For example, if we let $U(A(c)) = u(A(c) + ac_T)$ where $a > 0$, $u : \mathbf{R}_+ \to \mathbf{R}$ is power utility, and $A(c)$ is an Armington aggregator, then $U(A(c))$ will satisfy assumption 1, and not really change much in terms of the stochastic dynamics of these models.

[16]In the existing literature, the typical preference specification is applied work assumes $A(c)$ is such that $a = 0$ in Assumption 1.iii.

[17]See our companion paper studying the sequential equilibrium in these models in Pierri and Reffett ([60]).

imposing something like Assumption 1.iii to bound our economy that is sometimes available for *some* consumption aggregators $A(c)$, and we discuss this in the last section of the paper. In this discussion, we show that if one can find an "greatest fixed point" for equilibrium tradables consumption in states where the collateral constraint binds, there is no need for Assumption 1.iii. For some common functional forms (and parameterizations of those functional forms) for $A(c)$ that have been used in the applied literature, this alternative condition works, and there is no need for this condition. For the more general case of $A(c)$ strictly increasing, strictly concave and supermodular, Assumption 1.iii is sufficient to bound these economies.

Second, it bears mentioning also that the aggregator $A(c)$ plays a critical role in creating the possibility of multiple RCEs. A typical functional form for the consumption aggregator $A(c)$ in the literature is the Armington/CES aggregator

$$c_t = A(c_t^T, c_t^N) = [ac_t^{T\ 1-\frac{1}{\xi}} + (1-a)c_t^{N\ 1-\frac{1}{\xi}}]^{\frac{1}{1-\frac{1}{\xi}}}$$

with $\xi > 0$, $a \in (0,1)$, which is increasing, strictly concave, and supermodular on X when $X = \mathbf{R}_+^2$ endowed with the componentwise partial ordering. We shall show how, for such an aggregator, for many choices of ξ, this aggregator will create highly nonlinear pecuniary externalities and robust global multiplicities of RCE via highly nonlinear. Alternatively, one can pick other versions of $A(c)$ (e.g., log aggregators), where although pecuniary externalities exist, RCE will be unique. So, the choice of $A(c)$ will play a critical role in the study of multiple RCE in these models.

The households in this economy are assumed to face a standard sequence of budgets constraints over their lifetime and make sequential choices for consumption and debt. In particular, given a candidate price sequence $p = \{p_t\}_{t=0}^{\infty}$. Denoting the net debt position for a typical household with debt borrowed at date t but maturing at date $t+1$ by d_{t+1}, the budget constraint for a household in any period t is given by:

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + \frac{d_{t+1}}{R} \tag{2}$$

where agents are allowed to borrow or lend at a fixed interest rate $R = 1 + r$ and, as this is a small open economy, R is taken as given. In this paper, we shall only consider the case that $\beta R < 1$. That is, we follow the timing convention used in Schmitt-Grohé and Uribe ([73], [74]), and assume consumption and income decisions are taken at the beginning of the period, and interest is then paid/earned over that same period. [18] We adopt this timing only because it proves to be convenient in characterizing the structure of dynamic equilibrium.[19]

Finally in addition to the budget constraint in (34), the a critical feature of these models is that each household also faces a period by period flow collateral constraint on debt given by:

$$d_{t+1} \leq \kappa(y_t^T + p_t y_t^N) \tag{3}$$

where $\kappa > 0$ [20] It is in the presence of this equilibrium collateral constraint that this model is capable of generating the interesting equilibrium dynamics in this model, and it also this constraint that creates the possibility of multiple dynamic equilibrium

Then, in a SCE, a typical household takes as parametric a fixed interest rate R, a level of initial debt $d_0 \in \mathbf{D}$ where $\mathbf{D} \subset \mathbf{R}$ is a compact[21], a stochastic process for endowments $y^T = \{y_t^T\}_{t=0}^{\infty}$ given initial endowment $y_0^T \in Y$ with $y_0^T > 0$, a constant level of non-tradable endowment $y_t^N = y^N$, and a history-dependent sequence of prices $p = \{p_t(y^t)\}_{t=0}^{\infty}$ where each $\infty > p_t(y^t) > 0$ for every history y^t, where we use the notation $y^t = \{y_0^T, y_1^T, .., y_t^T\}$ to denote the history of tradable endowment realizations

[18]In this paper, the assumption that $\beta R < 1$ greatly simplifies our proofs (in particular, our construction of the set of RCE). This assumption can be relaxed and is discussed in Pierri and Reffett ([60]).

[19]Bianchi ([17]) uses a slightly different timing convention. Still, it turns out this timing convention is without loss of generality in our case (see, for example, Adda and Cooper ([2]) for a detailed discussion of this matter).

[20]Additionally, we should mention that an interesting case of this model is when the collateral constraint depends on future income streams. This case is outside the scope of this paper, but it was studied in Pierri and Reffett ([60]).

[21]Under Assumption 1, one can guarantee the existence of a maximal tradable consumption (and hence, maximal debt). That allows one to construct the compact set $\mathbf{D}$.

up to period t. The household then maximizes lifetime utility choosing sequences for consumption and debt in equation (1) subject to constraints equations (2) and (3) for each period. That is, formally, the household's sequential problem in any candidate SCE can be stated as follows: given $R \geq 0$, d_0, $\beta R < 1$, and stochastic processes for p and y, the household solves the following:

$$V^*(s_0) = \max E_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t)) \tag{4}$$

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + \frac{d_{t+1}}{R}; \quad d_{t+1} \leq \kappa(y_t^T + p_t y_t^N),\ t \in \{0, 1, 2, ...\}$$

where the initial states are $s_0 = (d_0, y_0^T)$, and $y_0^T \in Y$. We denote the optimal policy sequences for consumption and debt achieving the maximum in the problem (4) by

$$c^*(s_0) = \{c_t^*(s_0)\}_{t=0}^{\infty};\ d^*(s_0) = \{d_t^*(s_0)\}_{t=0}^{\infty}\} \tag{5}$$

Under Assumption 1, it can be shown that: (a) the value function $V^*(s_0)$ is finite, and (b) the optimal policy sequences for consumption and debt given by $c^*(s_0)$ and $d^*(s_0)$ are well-defined functions, (c) using the results in Rincon-Zapatero and Santos ([65]), theorem 3.1), as the household sequential optimization problem satisfies standard convexity and continuity conditions as well as standard sequential constraint qualifications (i.e., linear independence constraint qualifications), given the continuous differentiability assumptions on preferences, (i) there exists a well-defined standard Lagrangian formulation for the sequential primal problem in equation (4) with (summable) dual variables $\beta^t \lambda_t$ and $\beta^t \mu_t$ associated with the sequence of constraints in equations **(2)** and **(3),** respectively**.**[22] Then, the system of necessary and sufficient first-order conditions for the Lagrangian dual version of the household problem in equation (4) in a SCE is then given as follows:

$$\lambda_t^* = U'(A(c_t^*))A_1(c_t^*) \tag{6}$$

$$p_t = \frac{A_2(c_t^*)}{A_1(c_t^*)} \tag{7}$$

$$[\frac{1}{R} - \mu_t^*]\lambda_t^* = \beta E_t \lambda_{t+1}^* \tag{8}$$

$$\mu_t^*[d_{t+1}^* - \kappa(y_t^T + p_t y_t^N)] = 0, \mu_t^* \geq 0 \tag{9}$$

along with sequential budget constraints in (2) and collateral constraints in (3) where the set of sequences of Kuhn Tucker multipliers $\lambda(s_0) = \{\lambda_t(s_0)\}_{t=1}^{\infty}$ and $\mu(s_0) = \{\mu_t(s_0)\}_{t=1}^{\infty}$ are well-defined and unique.

3 Recursive Competitive Equilibrium

To construct a RCE, we first define a minimal state space for the household decision problem. This state space will consist of two components each period: (a) the individual state of a typical representative household, and (b) the aggregate state of the aggregate economy used to compute prices. Per (a), a typical household will enter any period in a RCE with an individual level of debt $d \in \mathbf{D} \subset \mathbf{R}$ as well as an endowment of tradable and non-tradable denoted by the vector $y = (y^T, y^N)$, where $y \in \mathbf{Y} \times \{y^{NT}\} \subset \mathbf{R}_{++}^2$ and $\mathbf{D}$ can be considered a compact set under assumption 1. So the individual state of the household will be characterized by the vector $(d, y) \in \mathbf{D} \times \mathbf{Y} = \mathbf{S}$ where $\mathbf{S}$ is compact. In addition, for (b) the household will face an aggregate economy that has per-capita aggregate measures of each of these individual state variables. That is, the aggregate state variable is a vector $S = (D, Y) \in \mathbf{D} \times \mathbf{Y} = \mathbf{S}$ where $D \in \mathbf{D}$ is the per-capita level of aggregate debt, Y^T (resp„ Y^N) are the per-capita endowment

[22]It is important to note this statement also requires one to restrict the possible paths for prices $p = \{p_t(y^t)\}_{t=0}^{\infty}$ to have each $p_t(y^t) \in [p_l, p^u]$ where $p_l > 0$. When we specialize the SCE to the case of RCE, we can be explicit on how to use Assumption 1 to create such bounds. For the case of more general SCE and parameterizing the sequence $p = \{p_t(y^t)\}_{t=0}^{\infty}$, see Pierri and Reffett ([60]) for a discussion.

draws for tradable (resp,., non-tradable) endowments with vector $Y = (y^T, y^{NT}) \in \mathbf{Y} \times \{\mathbf{y}^{NT}\} \subset \mathbf{R}^2_{++}$. Therefore, the state of a household entering any given period in a RCE will be denoted by $s = (d, y, S) \in \mathbf{D} \times \mathbf{Y} \times \mathbf{S}$. Then, finally by $s^e = (d, y, d, y) \in \mathbf{D} \times \mathbf{Y} = \mathbf{S}^e$, we shall denote equilibrium state space for a RCE (i.e., the "diagonal" of the state variable of the household $s = (d, y, D, Y) \in \mathbf{S} \times \mathbf{S}$ when $d = D$, $y = Y$). This then becomes the minimal state space in a RCE.

To develop a recursive representation of the household's sequential optimization problem in equation (4), we need to construct a recursive representation of the aggregate economy that agents will take as given when solving their individual decision problems. Anticipating the structure of RCE, the relative price for nontradables to tradables consumption (denoted by $p(C^T)$) will be equal to the equilibrium marginal rate of substitution between the two types of consumption goods:

$$\frac{U_2(C^T, y^{NT})}{U_1(C^T, y^{NT})} = \frac{A_2(C^T)}{A_1(y^{NT})} = p(C^T) \tag{10}$$

where C^T is the per-capita aggregate level of tradables consumption. As we shall impose in any RCE that nontradables consumption $c^{NT} = y^{NT} = Y^{NT}$ where Y^{NT} is the per-capita endowment of nontradables; for the rest of the paper, we shall suppress the dependence of the price $p(C^T)$ on the constant Y^{NT}. Under the supermodularity and concavity conditions in Assumption 1 on $A(c)$, the relative price $p(C^T)$ is increasing in C^T. Let the collection of candidate per-capita tradables consumption $C^T : \mathbf{S} \to [0, c^{\max}] \subset \mathbf{R}_+$ that parameterizes the price $p(C^T)(S)$ in equation (10) be defined as follows:

$$\begin{aligned} \mathbf{C}^f(\mathbf{S}) = \{C^T(S) | 0 \leq C^T(S) \leq c_{\max},\ C^T \text{ is continuous, increasing in } Y, \\ \text{decreasing in } D \text{ such that } (1 + \frac{\kappa}{R})y^T - d_{\max} + \frac{\kappa}{R} p(C^T(S))y^{NT} > 0\} \end{aligned} \tag{11}$$

where $c_{\max}$ and $d_{\max}$exist under Assumption 1. [23] We endow the space $\mathbf{C}^f$ with the standard pointwise partial order $\geq$. The space $(\mathbf{C}^f, \geq)$ is a a join lattice. [24]

Next, using an equilibrium version of the household's budget constraint as well as equilibrium collateral constraint each parameterized by $C^T \in \mathbf{C}^f(\mathbf{S})$, we can now construct the implied law of motion for per-capita debt D for the aggregate economy as follows:

$$D' = \Phi(C^T)(S) = \inf[R\{C^T(S) - Y + D\}, \kappa\{y^T + p(C^T(S))y^{NT}\}],\ C^T \in \mathbf{C}^f(\mathbf{S}) \tag{12}$$

where R is the current interest rate. Here, the "inf" operation is taken pointwise by state over the two equilibrium regimes in a candidate RCE governed by $C^T \in \mathbf{C}^f(\mathbf{S})$. The first term of this infimum represents aggregate debt in states where collateral constraints do not bind, while the second term represents aggregate debt in states where the collateral constraint binds. As D is the only endogenous aggregate state in this economy, given the stochastic process of the endowment shocks, we now have a full characterization of the stochastic transition structure of the aggregate economy in any candidate RCE $C^T \in \mathbf{C}^f(\mathbf{S})$. Using equations (10) and (48), we can also now generate a recursive representation of sequential price system $p = \{p\{S_t\}\}$ by constructing candidate recursive representations of the laws of motion for the per-capita aggregate debt D which in conjunction with realizations of stochastic endowments $\{Y_t\}$ will allow us to generate realizations of sequential paths for candidate RCE prices $p = \{p(S_t)\}_{r=}^{\infty}$

We can now construct a recursive representation of a household's sequential optimization problem in (4) any candidate for a RCE governed by an element $C^T \in \mathbf{C}^f(\mathbf{S})$. A household enters any period in the state $s = (d, y, S)$, faces a fixed interest rate $R > 0$ such that $\beta R < 1$, for any function $C^T(S) \in \mathbf{C}^f(\mathbf{S})$

[23] The strict positivity constraint $(1 + \frac{\kappa}{R})y^T - d_{\max} + \frac{\kappa}{R} p(C^T(S))y^{NT} > 0$ in the definition of $\mathbf{C}^f(\mathbf{S})$ in equation (11) requires the elements $C^T \in \mathbf{C}^f(\mathbf{S})$ be such that in every aggregate state $S \in \mathbf{S}$, the household's feasible correspondence will have a *strict interior point*. More on this in a moment.

[24] That is, if C_1^T and C_2^T are any elements of $\mathbf{C}^f(\mathbf{S})$, the join $C_1^T \vee C_2^T = \sup\{C_1^T(S), C_2^T(S)\}$) is socially feasible, satisfies the monotonicity conditions in the definition of $\mathbf{C}^f(\mathbf{S})$, and critically we have $(1 + \frac{\kappa}{R})y^T - d_{\max} + \frac{\kappa}{R} p(C_1^T \vee C_2^T(S)))y^N > 0$ (as $p(C^T)$ is increasing). But this is not the case for the meet $C_1^T \vee C_2^T = \inf\{C_1^T(S), C_2^T(S)\}$ will be in $\mathbf{C}^f(\mathbf{S})$. Namely, although the meet will also be socially feasible, satisfy the requisite monotonicity conditions, we could have $(1 + \frac{\kappa}{R})y^T - d_{\max} + \frac{\kappa}{R} p(C_1^T \wedge C_2^T(S)))y^N \leq 0$.

(and the implied law of motion for per-capita debt given by equation (48)), the household chooses consumption of tradables and nontradables, as well as debt in each state given from the correspondence:

$$G(C^T)(s) = \{c \in \mathbf{R}^2_+, d' \in \mathbf{D} |\ (49a) \text{ and } (50) \text{ hold}\}$$

where

$$c^T + p(C^T(S))c^N \leq y - d + p(C^T(S))y^{NT} + \frac{d'}{R} \tag{13a}$$

$$d' \leq \kappa(y^T + p(C^T(S))y^{NT}) \tag{14}$$

As $C^T(S) \in \mathbf{C}^f(\mathbf{S})$ is continuous, $G(C^T)(s)$ is a continuous correspondence in $s = (d, y, S) \in \mathbf{S} \times \mathbf{S}$. Notice, the strict positivity constraint $(1+\frac{\kappa}{R})y^T - d_{\max} + \frac{\kappa}{R}p(C^T(S))y^{NT} > 0$ in the definition of $C^T(S) \in \mathbf{C}^f(\mathbf{S})$ guarantees that in every aggregate state $S \in \mathbf{S}$, the household's feasible correspondence will have a strict interior point. Aside from the obvious economic reason, the condition is needed to guarantee a *strictly positive optimal* (or equilibrium) *tradables policy*, it is also needed to check Slater conditions required when constructing a dual Lagrangian representation of the household's primal optimization problem (so the set of Kuhn Tucker multipliers of this dual is well-defined). It is a straightforward to see $G(C^T)(s)$ is additionally convex-valued.

Under Assumption 1, we now use standard dynamic programming argument in the literature to construct a unique value function $V^*(C^T)(s)$ solving a Bellman equation for each $C^T \in \mathbf{C}^f$, $C^T(S) > 0$ for all $S \in \mathbf{S}$:

$$V^*(C^T))(s) = \max_{x=(c,d')\in G(C^T)(s)} U(A(c)) + \beta \int V^*(d', y', Y', \Phi(C^T)(S); C^T)\chi(dy') \tag{15}$$

with optimal policies associated with given by:

$$\begin{aligned} x^*(C^T)(s) = \arg \max_{x=(c,d')\in G(C^T)(s)} & U(A(c)) \\ & + \beta \int V^*(d', y', Y', \Phi(C^T)(S); C^T)\chi(dy') \end{aligned} \tag{16}$$

where by a standard application of Berge's maximum theorem to the right side of the Bellman equation in (15) along with standard dynamic programming constructions, the unique value function $V^*(C^T)(s)$ is jointly continuous in $s \in \mathbf{S} \times \mathbf{S}$, and the optimal policies for consumption $c^*(C^T)(s) = (c^{T*}(C^T)(s), c^{NT*}(C^T)(s))$ and debt $d'^*(C^T)(s))$ exist. Further, by the strict concavity of the primitive data under Assumption 1 for each $C^T \in \mathbf{C}^f$, the optimal policies are unique, $V^*(C^T)(s)$ is strictly concave and decreasing in d for each (y, S), and increasing in y, each (d, S).

The first order theory for the optimal policy function $c^*(C^T)(s)$ defined in (16) can be constructed appealing to the duality and envelope theorem results in Rincon-Zapatero and Santos (([65]), Proposition 3.1 and Theorem 3.1). That is, a recursive Lagrangian dual formulation of (15) for each $C^T \in \mathbf{C}^f$ is given by:

$$v^*(C^T)(s) = \inf_{\lambda,\mu\geq 0} \max_{c,d'\in\mathbf{C}\times\mathbf{D}} L(c, d', \lambda, \mu; s, v^*; C^T) \tag{17}$$

where $c \in \mathbf{C} = \{c \in \mathbf{R}^2_+ | c^T \in [0, c^{\max}], c^N \in \mathbf{R}_+\}$ and $d' \in \mathbf{D}$ where

$$\begin{aligned} L(c, d', \lambda, \mu; s, v^*, C^T) = {} & U(c^T, c^N) + \beta \int v^*(d', y', \Phi(S; C^T), y; C^T)\chi(dy') \\ & + \lambda\{y^T - p(C^T(S))y^{NT} - \frac{d'}{R} - c^T + p(C^T(S))c^N\} \\ & + \lambda\mu\{\kappa(y^T + p(C^T(S))y^{NT}) - d'\} \end{aligned} \tag{18}$$

where under Assumption 1, the recursive Lagrangian dual formulation of the primal constrained dynamic program problem in (15) admits a system of unique system of *stationary* Kuhn Tucker multipliers, $\lambda^*(C^T)(s)$ and $\mu^*(C^T)(s)$ associated with the infinite horizon sequential dual program that dualizes the household's sequential primal optimization problem in (4) from all initial conditions with the associated unique (stationary) Kuhn Tucker multipliers $\{(\lambda^*(C^T)(s), \mu^*(C^T)(s))\ ;(c^{T*}(C^T)(s), c^{N*}(C^T)(s), g^*(C^T)(s))\}$ in (18) the unique stationary saddlepoints of (17) with the envelope theorem for (17) in d given by:

$$\begin{aligned} \partial_d v^*(C^T)(s) &= \partial_d V^*(C^T)(s) \\ &= U_1(A(c^*(C^T))(s))A_1(c^{T*}(C^T)(s)) \end{aligned} \tag{19}$$

where $c^*(C^T)(s) = (c^{T*}(C^T)(s), c^{N*}(C^T)(s))$ is the vector of unique optimal solutions for consumption goods for the primal dynamic program in (15).

Using this envelope theorem in (19), the system of first order conditions (necessary and sufficient) for our primal representation of the dynamic programming problem in (15) are the following:[25]

$$p(C^T(S)) = \frac{A_1(c^*(C^T)(s))}{A_2(c^*(C^T)(s))} \tag{20}$$

$$\begin{aligned} -\{\frac{1}{R}\}U_1(A(c^*(C^T)(s)))A_1(c^*(C^T)(s)) + \beta \int \lambda^*(d'^*(C^T)(s), y', \Phi(S;C^T), y'; C^T) \leq 0 \\ = 0 \ \text{ if } d'^*(s) \leq \kappa(y^T + p(C^T(S))y^{NT}) \end{aligned} \tag{21}$$

$$d'^*(s) - \kappa(y^T + p(C^T(S))y^{NT}) \leq 0 \tag{22}$$

where the law of motion on individual debt in (21) is given by:

$$d'^*(C^T)(s) = \inf\{R\{c^{T*}(C^T)(s) - y^T - p(C^T(S))y^{NT} + d\}, \kappa\{y^T + p(C^T(S))y^{NT}\} \tag{23}$$

and the law of motion on per-capita debt D' in (21) is given by $\Phi(S;C^T)$ in equation (48).

We can now state the formal definition of RCE:

Definition 1 *A minimal state space RCE in this economy is a function for per-capita tradables $C^{T*}(s)$, a household value function $V^*(C^{T*})(s^e)$ that solves the functional equation in (15) at $C^{T*} \in \mathbf{C}^f(\mathbf{S})$, with optimal solution for consumption $c^*(C^{T*})(s) = (\ c^{T*}(C^{T*})(s), c^{NT*}(C^{T*})(s))$ and optimal debt policy $d'^*(C^{T*})(s))$ defined in (16) such that when in each state s^e,we have tradables $c^{T*}(C^{T*})(s^e)=C^{T*}(s^e) > 0$ and nontradables $c^{NT*}(C^{T*}(s^e)) = y^{NT}$ with the associated nontradables relative price $p(C^{T*}(s^e)) > 0$ and finite.*

4 Existence of Recursive Competitive Equilibrium

We now prove the existence of RCE under Assumption 1. A key challenge to constructing a RCE in dynamic models with occasionally binding collateral constraints is that our approach needs to be able to characterizing the "regime change" in equilibrium policy functions over states where collateral constraints bind vs. do not bind. To address the technical complication, we develop a *two-step* fixed point method based upon an operator. Roughly speaking, to do this, we first construct an operator $A(c, C^T) = A(A_{uc}(c, C^T), A_c(C^T))$ with arguments (c, C^T) where c represents equilibrium tradables consumption in equilibrium states $s^e \in \mathbf{S}^e$ where households are *not* collateral constrained, C^T parameterizes the equilibrium collateral constraint at relative prices $p(C^T)$, $A_{uc}(c, C^T)$ is an operator which computes "contingent" RCE equilibrium tradables consumption in equilibrium states where households are *not*

[25] Notice although we work primarily with the primal first-order theory for the RCE, one can easily work with the Lagrangian dual representation given the results in Rincon-Zapatero and Santos ([65] regarding envelope theorems. That could be useful, for example, if one uses our RCE constructed to build a theory of optimal further macroprudential policies with multiple equilibria. See Pierri and Reffett [62] for discussions of this fact.

collateral constrained given a collateral constraint parameterized by $p(C^T)$ for fixed C^T, and $A_c(C^T)$ is the level of tradables consumption in states where households are collateral constrained (and this mapping is completely determined by C^T). [26] Then, in the first step of our equilibrium construction, we prove there exists a *unique* strictly fixed point $c^*(C^T)$ for the operator $A_{uc}(c; C^T)$ for each fixed C^T (and we interpret $c^*(C^T)$ as a "contingent" equilibrium tradables consumption over states where the collateral constraint does *not* bind given collateral constraint fixed at C^T (which determines completely the equilibrium states in the current period where collateral constraints *do bind*). Then, in the second step, we then uses this first step fixed point $c^*(C^T)$ to define our actual RCE operator which is the mapping $A^*(C^T) = A(c^*(C^T), C^T)$, and we show this operator has a complee lattice of strictly positive fixed points each RCE equilibrium tradables consumption, and these RCE tradables induced a RCE over all the variables of the model (e.g., prices and laws of motion on debt, etc.).

It also bears mentioning that when construction our RCE operator $A^*(C^T) = A(c^*(C^T), C^T)$ shows both (a) how to explicitly constructs the states where collateral constraints bind and do not bind any *particular* RCE, and (b) provides a natural way of modeling equilibrium dynamic complementarities in those models. Per the later point, in the first step, we shall show as C^T increases, collateral constrained tradables consumption $A_c(C^T)$ increases pointwise over the state space (as the equilibrium collateral constraints get relaxed with pointwise increases in C^T as the value of nontradeables endowments $p(C^T)$ rises. This relaxing of the collateral constraint then allows *unconstrained* trades consumption in the unique "contingent equilibrium" $c^*(C^T)$ to increase. This increase, in turn, will imply in in the second step of our construction, our RCE operator $A^*(c^*(C^T), A_c(C^T))$ will very naturally increasing in C^T. This form of "pecuniary complementarity" in equilibrium is precisely what allows for the possibility of multiple equilibria with some RCE having both lower levels of debt and tradables consumption, and while other RCE are associated with higher levels of debt and consumption.

4.1 Constructing a RCE operator

We first construct the mapping $A(c, C^T)$ mentioned above. Anticipating the structure of RCE over multiple equilibrium regimes, and noting that maximal collateral constrainted tradable consumption is always greater than unconstrained collateral constrained tradable consumption in an RCE,. we assume the structure of the (unknown) RCE tradable consumption over the two equilibrium regimes is given by the mapping $C(c, C^T)(s)$ defined as follows:

$$C(c, C^T)(s) = \inf\{c(d, y), C_c(S, C^T(S))\} \tag{24}$$

where we shall assume (i) the function $c(d, y) \in \mathbf{C}^p(\mathbf{S})$ representing the RCE tradables consumption in states where households are not collateral constrained is an element of the space:

$$\begin{aligned} \mathbf{C}^p(\mathbf{S}) = \{c(d,y) = \tilde{c}(d,y,d,y) | 0 \leq c(d,y) \leq y^T - d + (d_{\max}/R) \\ c(d,y) \text{ decreasing in } d, \text{ increasing in } y, \\ \text{s.t. } (*) \ -d' = R(y^T - d - c(d,y,d,y)) \text{ decreasing in } d, \text{ increasing in } y\} \end{aligned} \tag{25}$$

with $d_{\max}$ is the maximal level of debt which can constructed from c_{nax} under Assumption 1 (hence, $y^T - d + (d_{\max}/R) \leq c_{\max}$), (ii) $C^T(S) \in \mathbf{C}^f(\mathbf{S})$ defined in (11) is used to parameterize the relative price $p(C)$ in the equilibrium collateral constraint, and (iii) $C_c^T(S, C^T(S))$ is the RCE tradables consumption for the households when households are collateral constrained and given by:

$$C_c(S, C^T(S)) = (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S)Y^{NT}$$

Also, denote by $\mathbf{C}^p_{++}(\mathbf{S})$ the set of $c \in \mathbf{C}^p$ such that $c(d, y) > 0$.

[26] For this informal discussion here, the domains of the operators $A_{uc}(c, C^T), A_c(C^T)$, $A(A_{uc}(c, C^T)$, and $A^*(C^T) = A(c^*(C^T), C^T)$ will suppressed from the discussion. Of course, in our actual construction of the mappings in this section, we make all the domains of our operators explicit, and discuss their properties.

In the appendix, we prove the space $\mathbf{C}^p(\mathbf{S})$ is: (a) an equicontinuous collection of continuous functions (hence, compact) in $\mathbf{C}^f(\mathbf{S})$; (b) a nonempty subcomplete sublattice of $\mathbf{C}^f(\mathbf{S})$ in its relative pointwise partial order. Further, the space $\mathbf{C}^p_{++}(\mathbf{S})$ can be made into a complete metric space with an appropriate metric. See Li and Stachurski ([43]). Finally, we endow the space $\mathbf{C}^p(\mathbf{S})$ with the pointwise partial order. Notice for any function $c \in \mathbf{C}^p$, the implied policy function for debt $d'(d, y)$ will be *decreasing* in y, and *increasing* in d (i.e., the mapping -$d'(d, y)$ works like "savings" in this model when households are not collateral-constrained to smooth endowment shocks when tradables consumption is not collateral constrained).

We now define an operator $A(c, C^T)(s^e)$ for $(c, C^T) \in \mathbf{C}^p \times \mathbf{C}^f$ which will become the foundation of our two step equilibrium analysis. To define this mapping, we first rewrite the household's Euler inequality in (21) in any equilibrium as follows: imposing equilibrium restrictions that $d' = D'$ and $y' = Y'$, and for for any $c(d, y) \in \mathbf{C}^p_{++}$ and any $C^T(S) \in \mathbf{C}^f$, we can substitute $C(c, C^T)(s)$ defined in equation (24) for consumption "tomorrow," and define a new mapping Z^*_{uc} given by:

$$Z^*_{uc}(x, s^e, c, C^T) = \frac{U_1(x, y^{NT})}{R} - \beta \int U_1(\inf\{c(\Phi(x)(d, y), y'), C_c(\Phi(x)(d, y), y', C^T(\Phi(x)(d, y), y'))\chi(dy') \tag{26}$$

where $s^e = (d, y, d, y)$ is the equilibrium state of a household in a RCE and the mapping $d'(s^e) = \Phi(x)(d, y) = \inf[R\{x - Y + D\}, \kappa\{y^T + p(x)y^{NT}\}]$.[27] For any $c \in \mathbf{C}^p_{++}$, $C^T \in \mathbf{C}^f(\mathbf{S})$, under Assumption 1 the mapping $Z^*_{uc}(x, s^e, c, C^T)$ is strictly decreasing and continuous in x. We can then compute the function $x^*_{uc}(c, C^T(S))(s^e)$ implicitly in this expression at the unique root that makes:

$$Z^*_{uc}(x^*_{uc}(c, C^T(d, y))(s^e), s^e; c, C^T) = 0 \tag{27}$$

which is well-defined as a function as Z^*_{uc} is strictly decreasing and continuous under Assumption 1 noting $c(d, y) \in \mathbf{C}^p_{++}$ and $C^T(d, y) \in \mathbf{C}^f(\mathbf{S}))$ (hence, both continuous in their arguments). Also, as Z^*_{uc} decreasing and the continuous d, and increasing and continuous pointwise in evaluation map eval(c, $C^T)(s^e)$, the unique root $x^*_{uc}(c, C^T)(s^e)$ is continuous in s^e, and continuous (in the topology of pointwise convergence) in (c, C^T) for each s^e. Further, by a standard comparative statics argument, under Assumption 1, the root $x^*_{uc}(c, C^T)(s^e)$ is increasing in (c, C^T, y) and decreasing in d. Finally, for $c \in \mathbf{C}^p_{++}$, by the Inada conditions in Assumption 1, the root $x^*_{uc}(c, C^T)(s^e) > 0$.

We can now use $x^*_{uc}(c, C^T)(s^e)$ to construct an the operator $A(c, C^T)(s^e)$ from which we will use to construct RCE. To do this, the implied debt level associated with the implied tradables consumption level $x^*_{uc}(c, C^T)(s^e)$ obtain in equation (27) is given by:

$$d_{x^*_{uc}}(c, C^T)(s^e) = R\{x^*_{uc}(c, C^T)(s^e) - y^T + d\} \tag{28}$$

If $d_{x^*_{uc}}(c, C^T)(s^e)$ satisfies

$$d_{x^*_{uc}}(c, C^T)(s^e) \leq \kappa\{Y^T + p(C^T(d, y))Y^{NT})$$

the household is not be debt-constrained state s^e given the mapping $C(c, C^T)(s)$[28] and we let our RCE operator be defined at this state s^e to be:

$$A_{uc}(c; C^T)(s^e) = x^*_{uc}(c, C^T)(s^e) \tag{29}$$

Alternatively, if $d_{xu}c^*(c, C^T)(s^e)$ has $d_{xu}c^*(c, C^T)(s^e) > \kappa\{Y^T + p(C^T(d, y))Y^NT)$, the collateral constraint binds, and the implied household tradables the collateral-constrained tradables consumption gives consumption level:

$$A_c(C^T)(s^e) = (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(d, y))y^{NT}. \tag{30}$$

[27] Notice, the mapping $\Phi(x)(d, y)$ in equation (26) is the mapping $\Phi(C^T)(S)$ in equation (48) where in a RCE, we set $x = C^T$, and $d = D$ and $y = Y$.

[28] Notice, when the debt $x^*_{uc}(c, C^T(S))(s^e) = \kappa\{Y^T + p(C^T(S))Y^N)$, the collateral constraint is saturated, but not binding. In this case, the implied KKT multiplier on the collateral constraint would be 0.

Then, define the operator $A(c, C^T)(s^e)$ for $(c, C^T) \in \mathbf{C}^p \times \mathbf{C}^f$ at state s^e for each $C^T \in \mathbf{C}^f$: when $d = D$, $y = Y$

$$\begin{aligned} A(c, C^T)(s^e) &= A(A_{uc}c, C^T), A_c(C^T)) = \inf\{A_{uc}(c, C^T)(s^e), A_c(C^T)(s^e)\} \text{ when } c(d, y) > 0 \quad (31) \\ &= 0 \text{ else} \end{aligned}$$

where the operator $A(c, C^T)(s^e)$ on $(c, C^T) \in \mathbf{C}^o \times \mathbf{C}^f$ has exactly the form of the operator we discussed at the beginning of this section.

4.2 Existence of RCE

We now construct the set of RCE using the operator $A(c, C^T)$ in equation (31). In the first step, we characterizing the fixed point set of the mapping $A_{uc}(c; C^T)(s^e)$ in equation (29) for each $C^T(S) \in \mathbf{C}^f(\mathbf{S})$. Lemma 2 shows that the mapping $A_{uc}(c; C^T)(s^e)$ is an order continuous operator in $c \in \mathbf{C}^p$, and a monotone contraction on the complete metric space $c \in \mathbf{C}^p_{++}(\mathbf{S})$ (where the metric we use is the one introduced in Li and Stachurski ([43]) for Euler equation operators). Hence, $A_{uc}(c; C^T)(s^e)$ will have a unique strictly positive fixed point $c^*(C^T)(s^e) \in \mathbf{C}^p_{++}$ for each $C^T(S) \in \mathbf{C}^f(\mathbf{S})$.

Lemma 2 *Under Assumption 1, the first step operator $A_{uc}(c; C^T)(s^e)$ has a unique strictly positive fixed point $c^*(C^T)(s^e) \in \mathbf{C}^p_{++}(\mathbf{S})$ for each $C^T(S) \in \mathbf{C}^f(\mathbf{S})$ and can be computed (for example) as $\inf_n A^n_{uc}(c_{\max}; C^T)(s^e) = c^*(C^T)(s^e)$. Furthermore, the fixed point mapping $c^*(C^T)(s^e)$ is order continuous (hence, monotone increasing) on $\mathbf{C}^f(\mathbf{S})$.*

We now use the fixed point mapping $c^*(C^T)(s^e)$ in Lemma 2 to define our RCE operator. Before we do this, we first define two function spaces that will be used to build the domain of our RCE operator. The first function is the space $\mathbf{C}^{f*}(\mathbf{S}) \subset \mathbf{C}^f(\mathbf{S})$ where we shall show RCE exist:

$$\begin{gathered} \mathbf{C}^{f*}(\mathbf{S}) = \{C^T(S) \in \mathbf{C}^f(\mathbf{S}) |\ C^T(S) \in [C_m(S), c_{\max}]\ , \qquad (32) \\ (^{**})\ D'(D, Y) = \kappa(Y^T + p(C^T(S))Y^N \text{ is increasing in } Y,\ \text{decreasing in } D,\ C_m(S) >> 0\} \end{gathered}$$

where $C_m(S) \in \mathbf{C}^{f*}(\mathbf{S})$ is a *strictly positive* function represents the "lower bound" for tradables consumption in a RCE.[29] We can then define the space $\mathbf{C}^*$ to be the domain for the RCE operator:

$$\mathbf{C}^*(\mathbf{S}^e) = \{C(s^e) = \inf\{c^*(d, y, C^T(d, y), C_c(d, Y, C^T(d, y)) \in \mathbf{C}^p_{++} \times \mathbf{C}^{f*} \text{ when } d = D,\ y = Y\text{"}\} \quad (33)$$

Notice that the only endogenous state variable is debt and in equilibrium, $d = D$ is one dimensional, and the shocks are discrete, the space $\mathbf{C}^*(\mathbf{S})$ is equicontinuous. This leads to the following conclusion.

Lemma 3 *The space $\mathbf{C}^*(\mathbf{S}^e)$ is a nonempty complete lattice under pointwise partial orders.*

We remark that the elements of $\mathbf{C}^{f*}(\mathbf{S})$ have the exact same structure for tradables consumptions as the elements in $\mathbf{C}^p(\mathbf{S})$ defined in equation (25) over $S \in \mathbf{S}$, but as they represent tradables consumption in states where the collateral constraint binds, they have the associated debt dynamics satisfying condition (**) (i.e., are decreasing in output and increasing in debt.). That means the primary difference between RCE over uncollateral constrained states vs. collateral-constrained states is that in collateral constrained states, debt dynamics *reverse in order* relative to those in uncollateral constrained states (i.e., compared to the debt dynamics associated with the elements $c(d, y) \in \mathbf{C}^p$).

[29] We show in the appendix how to construct $C_m(S)$.

Next, using the fixed point mapping $c^*(C^T)(s^e)$, but restricting $C^T(S) \in \mathbf{C}^{f*}(\mathbf{S})$ (and hence, $C(c^*(C^T),$ $C^T)(s^e) \in \mathbf{C}^*(\mathbf{S}^e)$), we construct our second-step operator $A^*(C)(s)$ on the space $\mathbf{C}^*$:

$$A^*(C)(s^e)) = \inf\{c^*(C^T)(s^e),\ A_c(C^T)(s^e)\} \tag{34}$$

$A^*(C)(s^e))$ is our RCE operator whose strictly positive fixed points will be RCE for this economy. We now have the following lemma:

Lemma 4 *$A^*(C)(s^e))$ is order continuous on $\mathbf{C}^*(\mathbf{S}^e)$.*

We now state our main existence result that both characterizes the set of RCE for this economy, as well as provides a first result on computing the least and greatest RCE via successive approximations on our operator:

Theorem 5 *Under Assumption 1, the set of strictly positive RCE tradables consumption forms a (nonempty) complete lattice in $\mathbf{C}^*(\mathbf{S})$. Further, the least RCE $C^*_\wedge(S)$ (resp., greatest RCE $C^*_\vee(S)$) can be computed by successive approximations*

$$0 < \sup_n A^{*n}(0)(s^e) \to C^*_\wedge(S) \ (\textit{resp.}, \ (\inf_n A^{*n}(c_{\max})(s^e) \to C^*_\vee(s^e) < c_{\max})$$

from initial element $C_0 = 0$ (resp., $C_0 = c_{\max}$).

In Theorem 5, we should mention we characterize how to compute extremal RCE (i.e., least and greatest RCE) via *simple successive approximations* from the least and greatest elements of the space $\mathbf{C}^*(\mathbf{S})$. This is an application of the standard Tarski-Kantorovich principle to our RCE existence problem. Later in the paper, in the additional results and extensions section, we shall generalize this result. In particular, we shall apply the generalized iteration results in Olszewski ([55]) and Balbus, Olszewski, Reffett, and Wozny ([7]) on the computation of tight "lower" and "upper" RCE (fixed point) bounds for the operator $A^*(C)(s^e)$ local to iterations from *any* initial element C_0 in $\mathbf{C}^*(\mathbf{S})$.

4.3 RCE Comparative Statics

We now can provide an important equilibrium comparative statics theorem for the set of RCE. We focus in this section on ordered changes in the discount rate β, the interest rate R, and the parameter governing the tightness of the equilibrium collateral constraint κ. We first present a theorem that characterizes how ordered changes in important parameters of the model translate into ordered changes in the least and great RCE in Theorem 5. Theorem 6 below shows how least and great RCE change relative to ordered changes in the discount rate β, the parameter κ in the collateral constraint (which recall is the fraction of current wealth that the households can borrow against), and the global interest rate R. In the second set of results, we general these equilibrium comparative statics results to *any* RCE. In the later case, we are also able to address issues related to the "correspondence principle" and the order stability of particular RCE.

4.3.1 Comparative statics of least and greatest RCE

We first consider what happens to least and greatest RCE when (a) the discount rate and/or world interest rate falls, or (b) the parameter κ in the equilibrium collateral constraint rises (meaning the collateral constraint is more "slack"). We now introduce the notation for these parameters into the definition of our RCE operator $A^*(C, \beta, \kappa, R)(s^e)$ in equation (34). Then, the next theorem provides a simple method for verifying and computing the ordered changes in the least and greatest RCE relative to the ordered changes mentioned in the parameters (β, κ, R) :

Theorem 6 *Under Assumption 1, the least (resp., greatest) RCE $C^*_\wedge(\beta, \kappa, R)(s^e)$ (resp, $C^*_\vee(\beta, \kappa, R)(s^e)$) are each increasing in κ, and decreasing in (β, R). The equilibrium comparative statics can be computed by the simple successive approximations*

$$\sup_n A^{*n}(0; \beta, \kappa, R)(s^e) \to C^*_\wedge(\beta, \kappa, R)(s^e)$$
$$\inf_n A^{*n}(c_{\max}; \beta, \kappa, R)(s^e) \to C^*_\vee(\beta, \kappa, R)(s^e)$$

Theorem 6 provides sharp RCE comparative statics on the "low" and "high" borrowing equilibria for this class of economies. These equilibrium comparative statics results are obtained by an application of the standard Tarski-Kantorovich principles applied to the operator $A^*(C; \beta, \kappa, R)$. (e.g., see Dugundji and Granas [33] for a discussion).

4.3.2 Iterative Monotone Comparative Statics and the Correspondence Principle

We can also consider an extension of our main RCE comparative statics theorem in Theorem 6 to the case of comparative statics of *any* RCE. That is, for *any* RCE tradables consumption $C^*(p) \in \Psi^*(p)$ at the parameter setting $p = (\beta, \kappa, R)$, if one considers the exact same comparative statics question of Theorem 6 (i.e., if one lowers the world interest rate from R to R', and/or agents are less patient, so the discount rate falls from β to β', the equilibrium collateral constraint becomes more "slack" so the parameter rise from κ to κ'), can we construct a RCE tradables consumption that rises? Theorem 6 shows the "extremal" RCE (i.e., least and greatest RCE) increase if these parameter changes are made but is silent on any other RCE. We also have to be very careful in our interpretation of any such comparative statics result for any RCE (as we know, by the correspondence principle of Samuelson ([67]) and Echenique ([36]) that not all RCE will be "order stable" (i.e., some particular RCE could actually have the reverse comparative statics claimed in Theorem 6).

We now show one can prove a related result can be obtained for any RCE $C^*(p) \in \Psi^*(p)$ when p changes as described above to $p' = (\beta', \kappa', R')$. We should also mention, our new results comes with the caveat that the original RCE $C^*(p)$ might not be "order stable" (i.e., might not satisfying the "correspondence principle" of Samuelson ([67])) under the new parameters $p' = (\beta', \kappa', R')$). That is, the correspondence principle of Samuelson (and others) says that if a particular equilibrium is stable if a small perturbation in parameters lead to a dynamic adjustment of the economy that converges to a new equilibrium. The most relevant version of the correspondence principle for our purposes is that presented in Echenique [36] where he proves under some regularity conditions on $C^*(p)$ in p, the stability of dynamical systems generated by (generalized) adaptive dynamics initiated by the perturbation of the existing equilibrium $C^*(p)$ govern the validity of comparative statics predictions of the particular equilibrium when the parameters are changed. So a critical part of our next theorem is the *second part* of our result which provides sufficient conditions a particular RCE will be order stable relative to our parameter changes (and the new majorizing RCE is actually the old RCE that has dynamically adjusted to the higher RCE). In particular, our generalized iterations provide a characterization of when there exist dynamical adjustment paths for *any particular* $C^*(p)$ at p such that if this parameter is changed to

p' the particular RCE under study will dynamically rise (via our generalized iterations) to a new higher RCE. [30]

With this discussion in mind, we now provide an explicit iterative procedure whose order limit verifies the existence of a new RCE $C^*(\beta', \kappa', R') \in \Psi^*(\beta', \kappa', R')$ which has $C^*(\beta, \kappa, R) \leq C^*(\beta', \kappa', R')$ in a "tight" sense (i.e., there does not exist other majorizing equilibria at the new parameters that are lower in order yet still "higher" than the original RCE $C^*(\beta, \kappa, R)$).[31] To do this, we apply the recent results in a series of recent papers by Olszewski ([55]) and Balbus et al ([7] and [8]) which provide a new of "iterative monotone comparative statics" for order continuous operators equation in complete lattices.[32]

Fix the parameters of the model at their initial setting $p = (\beta, \kappa, R)$ for the moment, and consider defining a pair of generalized iterative methods from *any* initial fixed point $C_0(p) \in \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$ based our (parameterized) equilibrium operator $A_p^*(C_0)$ from *any* initial iterate $C_0 \in \mathbf{C}^*$. That is, compute the *lower-inf* iterations

$$C_\wedge^{k+1,\gamma}(p) = A_p^*(\inf\{C_\wedge^{k,\gamma}, ..., C_\wedge^{k-\gamma,\gamma}\}; p)$$

and the *upper-sup* generalized iterations:

$$C_\vee^{k+1,\gamma}(p) = A_p^*(\sup\{C_\vee^{k,\gamma}, ..., C_\vee^{k-\gamma,\gamma}\}; p$$

where γ control how "backward" looking the adaptive dynamics are, and for $l > k$, and we assume for both the lower and upper iterations that $C_\wedge^{k-l,\gamma} = C_\wedge^{0,\gamma}$ and $C_0 = C_\wedge^{0,\gamma} = C_\vee^{0.\gamma}$ for all $\gamma \in \mathbb{N}$. For any fixed γ, compute the lower and upper order limits (i.e., the lim-inf and lim-sup iterations):

$$\liminf_k C_\wedge^{k,\gamma}(p) = C_\wedge^\gamma(C_0)(p)$$

and

$$\limsup_k C_\vee^{k,\gamma}(p) = C_\vee^\gamma(C_0)(p)$$

where each limit exists in $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$ as $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$ is a complete lattice.

We now notate the order limits to be $C_\wedge^\gamma(C_0)(p)$ and $C_\vee^\gamma(C_0)(p)$ for each depend on the initial $C_0 \in \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$ for each γ:

$$C_\wedge^*(C_0)(p) = \lim_\gamma C_\wedge^\gamma(C_0)(p) \tag{35}$$

and

$$C_\vee^*(C_0)(p) = \lim_\gamma C_\vee^\gamma(C_0)(p) \tag{36}$$

where each order limit again exists in $\mathbf{C}^*$ for both lower and upper iterations as the sequence $\{C_\wedge^\gamma(C_0)(p)\}_{\gamma=1}^\infty$ (resp., $\{C_\wedge^\gamma(C_0)(p)\}_{\gamma=1}^\infty$) is *decreasing* (resp., *increasing*) in γ by construction and $\mathbf{C}^*$ is a complete lattice.

The following table contains a useful graphical exhibition of sequences $C_\wedge^{k+1,\gamma}$, limits $\liminf_k C_\wedge^{k+1,\gamma}$ and profile $C_\wedge^*$ as well as the inequalities and convergences between them for a fixed p (which we suppress from the diagram). An analogous graphical exhibition applies to $C_\vee^{k,\gamma}$, $\liminf_k C_\vee^{k,\gamma}$ and $C_\vee^*$.

$$\begin{array}{ccccccccc}
C_\vee^1(C_0) = \liminf_k C_\wedge^{k,1} & & C_\wedge^{4,1} & & C_\wedge^{3,1} & & C_\wedge^{2,1} & & C_\wedge^{1,1} \\
\rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\
C_\vee^2(C_0) = \liminf_k C_\wedge^{k,2} & & C_\wedge^{4,2} & & C_\wedge^{3,2} & & C_\wedge^{2,2} & \leq & C_\wedge^{1,2} \\
\rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\
C_\vee^3(C_0) = \liminf_k C_\wedge^{k,3} & & \underline{a}^{4,3} & & \underline{a}^{3,3} & \leq & \underline{a}^{2,3} & \leq & \underline{a}^{1,3} \\
\rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\
C_\vee^4(C_0) = \liminf_k C_\wedge^{k,4} & & C_\wedge^{4,4} & \leq & C_\wedge^{3,4} & \leq & C_\wedge^{2,4} & \leq & C_\wedge^{1,4} \\
\downarrow \gamma & & & & & & & & \\
C_\wedge^*(C_0) & & & & & & & &
\end{array}$$

[30] See Balbus et al ([8], section 4) for a detailed discussion of the correspondence principle the context of this section of the paper. So we also include in our result in this section a condition that guarantees that the correspondence principle holds for a equilibrium $C^*(p)$.

[31] The point here is verifying the existence of a "tight" higher RCE at the new parameters is that we always know the greatest RCE at the new parameters will majorize $C^*(\beta, \kappa, R)$. What we would like is a "least upper bound" on the set of majorizing RCE at the new parameter. This is precisely what our iterative monotone comparative statics result will guarantee.

[32] The results in that paper apply in our context (e.g., see Balbus, et al ([8], Proposition 3).

We now state the following direct implication of this construction as a Lemma.

Lemma 7 *Fix $p = (\beta, \kappa, R)$. Then, for each $C_0 \in \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$, the lower (resp., upper) generalized iterations $\{C_\wedge^\gamma(C_0)(p)\}_{\gamma=1}^\infty$ (resp., $\{C_\vee^\gamma(C_0)(p)\}_{\gamma=1}^\infty$) converge to a fixed point $C_\wedge^*(C_0)(p)$ (resp., $C_\vee^*(C_0)(p)$) each is a RCE with $C_\wedge^*(C_0)(p) \leq C_\vee^*(C_0)(p)$.*

Lemma 7 can be viewed as a generalization of the Tarski-Kantorovich principle applied to the initial iterate $C_0 \in \mathbf{C}^*$ relative to the generalized iterations $\{C_\wedge^\gamma(C_0)(p)\}_{\gamma=1}^\infty$ (resp, $\{C_\wedge^\gamma(C_0)(p)\}_{\gamma=1}^\infty$) with each element of the sequence defined in equations (35) and (36. [33] In particular, the Lemma shows that from any initial iterate $C_0 \in \mathbf{C}^*$, one can construct an lower and upper RCE bound relative where each RCE bound is an element of the lower (resp., upper) generalized iterations are constructed via a lim-inf (resp., lim-sup) operator indexed by $\gamma \in \{0, 1, 2, ...\}$. The convergence of these lower (resp., upper) iterations to fixed points is guaranteed by the order continuity of the mapping $A^*(C)$ applied to monotone sequences $\{C_\wedge^\gamma(C_0)(p)\}_{\gamma=1}^\infty$ (resp, $\{C_\wedge^\gamma(C_0)(p)\}_{\gamma=1}^\infty$).

We can now use Lemma 7 to produce a new version of our comparative statics Theorem 6 relative to *any* RCE $C_p^* = C^*(\beta, \kappa, R) \in \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$ for the same changes in the deep parameters $p = (\beta, \kappa, R)$ to $p' = (\beta', \kappa', R')$ in Theorem 6. What is our main theorem in this section below will show that using parameterized versions of our generalized iterations in Lemma 7, as for every γ, we will have

$$C_\wedge^\gamma(C_0)(p) \leq C_\wedge^\gamma(C_0)(p')(\text{ resp, } C_\wedge^\gamma(C_0)(p) \leq C_\wedge^\gamma(C_0)(p'))$$

then, when taking the order limits of the lower and upper, the parameterized generalized iterations in Lemma 7, we will obtain obtain

$$C_\wedge^*(C_0)(p) \leq C_\wedge^*(C_0)(p')(\text{ resp., } C_\wedge^*(C_0)(p) \leq C_\wedge^*(C_0)(p'))$$

A particularly interesting case of this result is when we start iterations off at any *old* RCE $C_0 = C^*(p)$, and initiate the generalized iterations for the operator $A_{p'}^*(C)$ at p'. When we do this, we obtain the following extension of Theorem 6.

Theorem 8 *Say $(\beta', R') \leq (\beta, R)$ and $\kappa' \geq \kappa$. Then for any RCE $C_p^* \in \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$, at $p = (\beta, \kappa, R)$, for $C_0 = C_p^*$, the lower (resp. upper) generalized iterations have order limits $C_\wedge^*(C_p^*)(\beta', \kappa', K')$ and $C_\vee^*(C_p^*)(\beta', \kappa', R')$ in equation (35) and (36) that satisfy:*

$$C^*(\beta, \kappa, R) = C_p^* \leq C_\wedge^*(C_p^*)(\beta', \kappa', K') \leq C_\vee^*(C_p^*)(\beta', \kappa', R')$$

where $C_\wedge^*(C_p^*)(\beta', \kappa', K')$ and $C_\vee^*(C_p^*)(\beta', \kappa', R')$ are RCE in $\Psi^*(\beta', \kappa', R')$. Further, if $C_\wedge^*(C_p^*)(\beta', \kappa', K') = C_\vee^*(C_p^*)(\beta', \kappa', R')$, then the RCE $C_p^* = C^*(\beta, \kappa, R)$ is order stable relative to ordered changes in p.

We make a remark on the interpretation of the last part of the theorem relative to the correspondence principle results in [36]. What the theorem says is if the lower and upper generalized iterations converge to the same order limit from initial iterations starting from the "old" equilibrium (i.e., any equilibrium at the initial set of parameters p), the "old" equilibrium is stable from the perspective of dynamic adjustment processes if our lower and upper generalized iterations converge to a "new" equilibrium that is higher than the "old" RCE. The theorem does not say that all RCEs have this property; instead, it says that for any RCE, we can have a "tight" RCE that majorizes the "old" RCE, and this majorizing RCE can be obtained only using the lower generalized iterations.

[33] The Tarski-Kantorovich principle says that if $f : X \to X$ is an order continuous transformation of X where X is a countable chain complete partially ordered set such that $\wedge X \leq f(\wedge X) \leq f(\vee X) \leq \vee X$, then the least (resp, greatest) fixed point of $f(x)$ can be computed as $\sup\{f^n(\wedge X)\}$ (resp., $\inf\{f^m(\vee X)\}$).

5 Uniqueness of Recursive Competitive Equilibrium

We now provide sufficient conditions for the uniqueness of RCE. We shall begin by discussing how equilibrium multiplicities occur in these models (and in particular, the critical role of the relative price of nontradables to tradables, and the structure of the consumption aggregator). In discussing these issues, we focus on how "large" pecuniary externalities can create a potential role for RCE multiplicities. We provide examples of consumption aggregators such that the implied, relative price of nontradables and tradables are not too "large" to generate multiple solutions for tradables consumption in states where equilibrium collateral constraints bind. Next, we then use this discussion of pecuniary externalities to propose sufficient conditions for the uniqueness of RCE, and we prove that for economies that satisfy these additional conditions on consumption aggregators, RCE are unique.

5.1 Multiple RCE and Pecuniary Externalities

We will say that there are *static* pecuniary externalities if *current* price changes, through general equilibrium effects, affect the availability of credit to the private sector in any *current* state of the *current* period. Pecuniary externalities are *dynamic* if agents expect that this relationship holds in the *continuation* period in any *continuation* state. This section will focus primarily on static pecuniary externalities. We defer to section 6.3 to discuss dynamic pecuniary externalities. We explain in this section why, under general preferences allowed for in Assumption 1, pecuniary externalities can generate multiple equilibria. Moreover, both types of externalities may hold simultaneously in a RCE, interacting with each other and generating severe coordination problems in the decentralized equilibrium.

Pecuniary externalities may generate multiple equilibria, which require a selection mechanism. Moreover, this mechanism must preserve stationarity in a recursive environment. This section first discusses the connection between multiplicity and static pecuniary externalities. Understanding this connection is essential to properly motivating our results. We show that the economy's stochastic and global nature can generate multiple equilibria under very general conditions. Thus, deriving methods to handle multiple equilibria in economies with pecuniary externalities is essential. To illustrate this point, we show that economies in expansion (i.e., when we observe a high shock) are more exposed to coordination problems than recessions. Thus, cyclical aggregate functions generate crises characterized by multiple equilibria.

It turns out that the intratemporal elasticity of substitution is a key element behind externalities and multiple equilibria. By controlling this elasticity, not only can we eliminate both these features of the model but also the presence of the spiraling downward collapse in prices, consumption, and debt often referred to as *Fisherian deflation* (see Bianchi ([17]) among others). We say the economy faces a Fisherian deflation if we observe at least two consecutive deleveraging periods after a crisis. As in this model, leverage is affected by prices due to the general equilibrium nature of the collateral constraint; a Fisherian deflation arises in the presence of pecuniary externalities. We further show that if pecuniary externalities are strong enough, we may observe multiple equilibria. That is, *if credit conditions are too sensitive to prices, then they may induce a coordination problem due to the presence of more than one possible equilibrium.* Thus, balance of payment crises and multiple equilibria are deeply connected. Along the same lines, if pecuniary externalities are sufficiently mild, it is possible to observe a Fisherian deflation in models with a unique continuous equilibrium. Coordination problems may not arise without affecting the possibility of observing a collapse and recession as pecuniary externalities are still present. If this relationship is non-linear, multiplicities and spiraling recessions occur.

5.1.1 Multiple equilibria under static pecuniary externalities: the case of CES preferences

This type of multiplicity is frequently studied in the literature (see Schmitt-Grohé and Uribe ([73])) and it is concerned with the possibility of having more than one solution to equation (7) for a given pair of states (d, y^T) when the collateral constraint is binding. Note that this equation defines a stationary system in debt today d for any level of tradable income y^T.

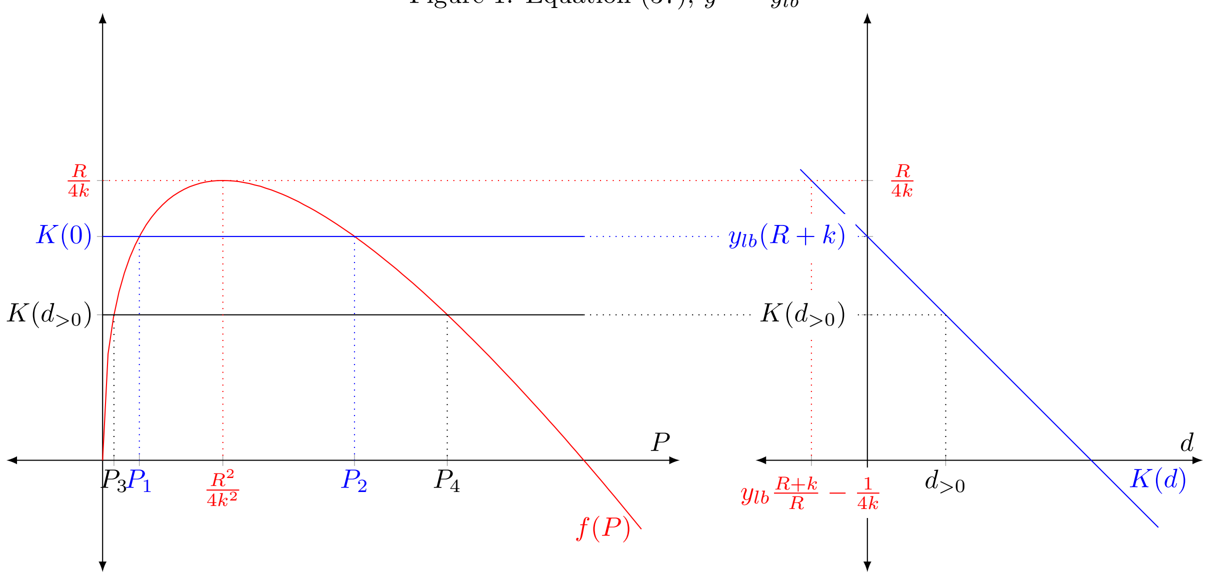
As we want to establish a direct relationship with the literature, we will use standard CES preferences as, for instance, in Bianchi ([17]). Let $U(A(x)) = (A^{1-\sigma} - 1)/(1-\sigma)$, $A(c) = (a(c^T)^{1-1/\xi} + (1-a)(c^N)^{1-1/\xi})^{1/(1-1/\xi)}$ with $\sigma = 1/\xi = 2$ and $a = 1/2$. We will assume that Z is compact. Then, equation (7) becomes:

$$f(P) \equiv P^{1/2} - \kappa P y^N = y_{lb}/y^N + \kappa y_{lb} - d/y^N \equiv K(d), \tag{37}$$

where we have set $y^T = y_{lb}$ and $(1-a)/a = Ry^N = 1$. The left-hand side of (37) is a function of P and the right-hand side of d. Let $f(P)$ and $K(d)$ be the former and the latter, respectively. As we have assumed $Ry^N = 1$, f is increasing for $0 < P < R^2/(4\kappa^2)$ and decreasing otherwise (for $P > 0$, of course). Figure 1 illustrates equation (37) for the described parametrization. The mapping f reaches a maximum at $R^2/(4\kappa^2) \equiv P^*$. Moreover, intertemporal optimality implies that P is a monotonic function of (aggregate) consumption. Then, obtaining the roots in equation (37) written in terms of debt implies finding multiple equilibria for our operator $A_c(C^T)(s^e)$ defined in equation (30) as a function of (aggregate) consumption, and hence multiple RCE for our operator $A^*(C^T)(s^e)$ in equation (34).

We can also look at the multiplicity of RCE regarding exchange rates. The K locus depends on d. The f locus depends on P, depicted in the "x-axis". The $K(0)$ line represents the smallest possible value for d in the constrained regime (i.e., $d = 0$). Between $K(0)$ and $R/4\kappa$, the regime is not collateral-constrained. Below $K(0)$ and over the locus formed by f lie all the candidate pairs (d, P) for the constrained regime.

Figure 1: Equation (37), $y^T = y_{lb}$

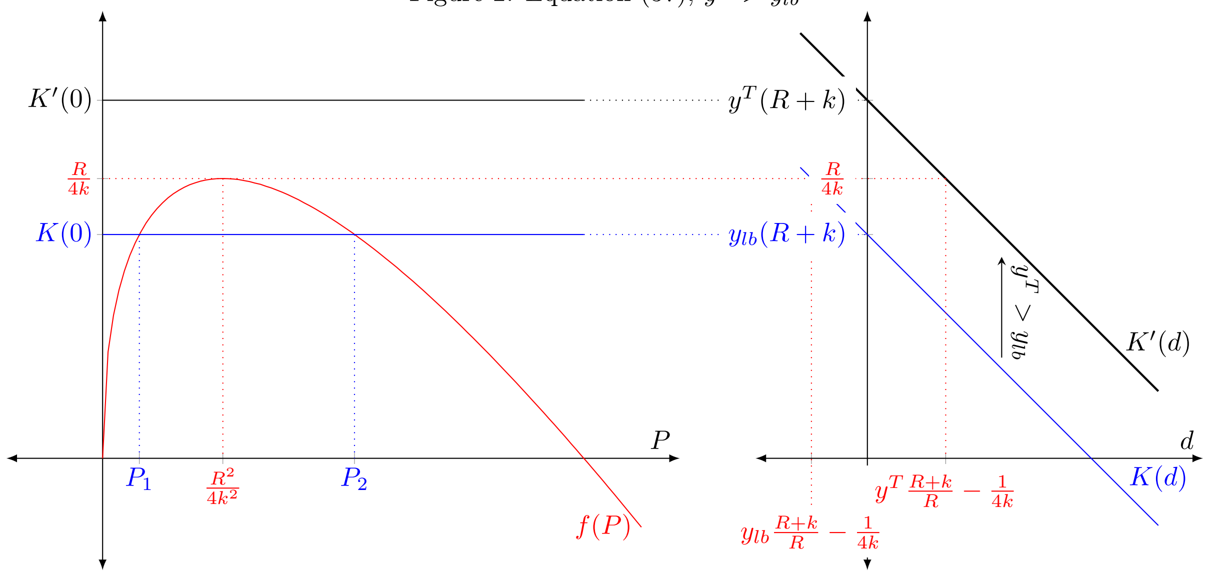


The K locus depends on d. The f locus depends on P. The $K(0)$ line represents the smallest possible value for d in the constrained regime. The regime is not collateral-constrained between $K(0)$ and $R/2\kappa$. Below $K(0)$ and over the locus formed by f lie all the candidate pairs (d, P) for the constrained regime.

Note that for $d = 0$, there are two possible exchange rate levels, $P1$ and $P2$, and a change in d with $d > 0$ can either increase or decrease P. This is depicted in points $P3$ and $P4$ in the same figure. Moreover, an increase in y^T implies that the $K(0)$ locus must jump upwards while the $f(P)$ locus remains constant as it is independent of tradable output by construction. Figure 2 illustrates this situation. Note that the collateral constraint doesn't bind when the agent saves (i.e., $d < 0$) as endowments and prices are positive. Thus, after the depicted increase in y^T, the region of possible multiple prices for a positive level of debt now includes the whole f locus.

Figures 1 and 2 illustrate the implications of the stochastic structure in the presence of static pecuniary externalities: *as we increase the shock level from y_{lb} to y^T, there is an increase in the admissible (positive) debt levels which can generate multiple equilibria.* This fact follows immediately from the definition of K.

Figure 2: Equation (37), $y^T > y_{lb}$



An increase in y^T shifts the K line. Thus, the whole locus f contains all the candidate pairs (d, P) for the constrained regime.

5.1.2 Static pecuniary externalities with different preferences

This section presents two examples of different intratemporal preferences to study the interaction between static pecuniary externalities and multiple equilibria. Even though the absence of a static pecuniary externality is sufficient to obtain a unique equilibrium, it is not necessary. *We may have a unique equilibrium with static pecuniary externalities.*

To keep the paper self-contained, let us define the intratemporal elasticity of substitution:

$$\zeta_{IES} \equiv \partial ln(c^T/c^N)/\partial ln(MRS(c^N, c^T)) = \partial ln(c^T/c^N)/\partial ln(p).$$

If $U(A(x)) = (A^{1-\sigma} - 1)/(1-\sigma)$, $A(c) = (a(c^T)^{1-1/\xi} + (1-a)(c^N)^{1-1/\xi})^{1/(1-1/\xi)}$, $\zeta_{IES} = \xi$.

The discussion in the preceding section implies that as long $\xi = 0.5$, we will generally have i) static pecuniary externalities, ii) multiple equilibria, iii) a spiraling (i.e., observed in at least two consecutive periods) balance of payment crises characterized by deleveraging and a real depreciation (i.e., a reduction in d and p). The critical aspect behind i) and iii), ii) was extensively covered in the previous section, is captured by the collateral constraint in equilibrium: $d_+ = \kappa(y^N p(c^T) + y^T)$, where the dependence of p on c^T follows from equation (7). A balance of payment crisis implies a reduction in c^T, which pushes down debt through the collateral constraint and a depreciation (i.e., a reduction in p).

Now consider imposing a quasi-linear intratemporal structure of preferences. That is, assume that:

$$\tilde{A}(c) = c^T + \frac{(c^N)^{1-1/\xi}}{1-1/\xi}, \tag{38}$$

Under equation (38), p, characterized by equation (7), becomes $p = (y^N)^{-1/\xi}$ and a binding collateral constraint implies $d_+ = \kappa((y^N)^{1-1/\xi} + y^T)$, which not only has one root but also breaks the spiraling recession as it rules out the static pecuniary externality (i.e., the dependence of p on c^T).

The intuition behind the uniqueness is straightforward: the marginal rate of substitution, equation (7), is independent[34] of c^T. As GDP is expressed in tradable goods, which equals $py^N + y^T$, it is exogenous. The intratemporal behavior of tradable consumption is also exogenous. Thus, there is neither static pecuniary externality nor spiraling recession. Of course, inter-temporarily tradable consumption is driven by the standard consumption smoothing channel that comes from the Euler equation. Moreover, uniqueness is guaranteed because the collateral constraint only has one root. Finally, let $d(d, y^T)$ be the policy function for debt in the unconstrained problem. As debt tomorrow satisfies $d_+ = min\{\kappa((y^N)^{1-1/\xi} + y^T), d(d, y^T)\}$, the equilibrium is continuous. Thus, *with quasi-linear preferences, we don't have static pecuniary externalities, and the equilibrium is unique.* The absence of static pecuniary externalities is sufficient for uniqueness. We now turn to necessity.

Assume that $\bar{A}(c) = ln(c^T) + ln(c^N)$. That is, $\zeta_{IES} = \xi = 1$ and $p = c^T/c^N$. In this case, we have:

$$p = \frac{(1+\kappa)y^T - d}{(1-\kappa)y^N} \tag{39}$$

Equation (39) implies the necessity of 2 additional restrictions: i) $p \geq 0$ and ii) $py^N + y^T \geq 0$. The former implies $(1+\kappa)y^T \geq d$ and the latter $2y^T \geq d$. We can eliminate the two restrictions by assuming $0 < \kappa \leq 1$. Now the collateral constraint is given by: $d_+ \leq \kappa(1-\kappa)^{-1}[2y^T - d]$. Thus, we have the following restrictions:

$$d_+ = min\{d(y^T, d), \kappa(1-\kappa)^{-1}[2y^T - d], (1+\kappa)y^T\} \tag{40}$$

Equation (40) implies that if $0.5 < \kappa \leq 1$, then $d_+ = min\{d(y^T, d), (1+\kappa)y^T\}$, which implies uniqueness, continuity, and the absence of pecuniary externalities. However, if $0 < \kappa \leq 0.5$, we have $d_+ = min\{d(y^T, d), \kappa(1-\kappa)^{-1}[2y^T - d]\}$, which implies uniqueness, but the model displays pecuniary externalities and spiraling recessions. To verify this claim, we must have:

$$d_{++} = \kappa(1-\kappa)^{-1}[2y^T - (\kappa(1-\kappa)^{-1}[2y^T - d])] \tag{41}$$

$$\kappa(1-\kappa)^{-1}[2y^T - (\kappa(1-\kappa)^{-1}[2y^T - d])] < \kappa(1-\kappa)^{-1}[2y^T - d] \tag{42}$$

Equation (41) implies that the collateral constraint is binding for two consecutive periods, and equation (42) that there is deleveraging. While the latter follows directly if $0 < d < y^T$, the former requires a more subtle argument. Note that to verify equation (41) we need to have $d(y^T, x) > \kappa(1-\kappa)^{-1}[2y^T - x]$, where x is a potential value for debt, for at least 1 level tradable output y^T. As $d(y^T, x)$ follows from a standard savings problem, we must have $d(y_{ub}, 0) < 0$. Let $K_2 \equiv [K_2^{lb}, K_2^{ub}]$ be the compact set containing d. As $d(y, .)$ is increasing in x for any y, we have $d(y_{ub}, K_2^{lb}) < 0$. Then, $\kappa(1-\kappa)^{-1}[2y_{ub} - x]$ is linear and decreasing in x with $\kappa(1-\kappa)^{-1}[2y_{ub} - K_2^{lb}] > 0$. Then, there exist x^* with $d(y_{ub}, x^*) = \kappa(1-\kappa)^{-1}[2y_{ub} - x^*]$. The last 2 inequalities imply that we have: $d(y_{ub}, x) > \kappa(1-\kappa)^{-1}[2y_{ub} - x]$ for $x \in (x^*, K_2^{ub}]$ as desired. Thus, *we show that with log preferences, we have a unique continuous equilibrium and static pecuniary externalities. That is, uniqueness is not equivalent to the absence of static pecuniary externalities.*

5.2 Uniqueness of RCE

With the discussion of the previous section in place, we can now turn to sufficient conditions for the uniqueness of RCE. To obtain a unique RCE, minimally, we will need to guarantee that the price $p(C^T)$ in (10) is such that static pecuniary externalities are "not too large." To formalize "not too large" in the

[34] With quasi-linear preferences, the elasticity of substitution is not constant but is 0 for compensated price changes.

form of a sufficient condition for uniqueness, note that any state equilibrium state s^e where the collateral constraint binds, our RCE operator $A^*(C^T)(s^e)$ is defined to be:

$$A_c(C^T)(s) = (1 + \frac{\kappa}{R})y^T - D + \frac{\kappa}{R}p(C^T(d,y))y^{NT} - d \tag{43}$$

where recall $A_c(C^T)(s)$ defines the operator $A^*(C)(s^e)$ in equation (34) in states where collateral constraints bind. In this expression, say we are in a fixed point $C^*(s^e)$ for the operator $A^*(C)(s^e)$. Then, we will have using equation (43) at this fixed point $C^*(s^e)$ in an equilibrium state s^e where $d = D$, $y = Y$, solve the following equation for $x = x^*(s^e)$:

$$Z^*_{cc}(x^*(s^e),\ s^e, x^*(s^e)) = x - (1 + \frac{\kappa}{R})y^T - d + \frac{\kappa}{R}p(x^*(s^e))y^{NT} - d$$

and define the correspondence $X^*(s^e)$ to be:

$$X^*(s^e) = \{x \geq 0 | Z_{cc}(x, s^e, x) = 0\}$$

Notice, when the collateral constraint binds, the correspondence $X^*(s^e)$ gives all the values of the set of RCE $C^*(s^e) \in X^*(s^e)$ pointwise at any state s^e.

We now make a few remarks about the correspondence $X^*(s^e)$. First, under Assumption 1, the correspondence $X^*(s^e)$ can easily be shown to be well-defined (e.g., is nonempty and compact-valued by an application of the intermediate value theorem). Further, it has least and greatest element (as the correspondence $X^*(s^e) \subset \mathbf{R}_+$ is chain-valued additional and hence has a least and greatest element). This implies in equilibrium states s^e where the collateral constraint binds, as Schmitt-Grohé and Uribe ([73]) suggest, in general, we will have (globally) "low borrowing" (associated with "least" RE tradable consumption) and "high borrowing" (associated with "greatest" RE tradable consumption) in equilibrium states where the equilibrium collateral constraint binds and the correspondence $X^*(s^e)$ is *not a single-valued.* That is, if the relative price $p(\cdot)$ is such that that mapping

$$\begin{aligned} Z_{cc}(x^*(s^e), x^*(s^e), s^e) = x^*(s^e) - (1 + \frac{\kappa}{R})y^T \\ -d + \frac{\kappa}{R}p(x^*(s^e))y^{NT} - d = 0 \end{aligned} \tag{44}$$

does *not* have *unique* root $x^*(s^e) = C^{T*}(s) \in X^*(s^e)$ for *any* s^e when the collateral constraints binds, there will *necessarily* be multiple *equilibria.*[35]

We now use the intuition of the above argument to answer the additional question of when this class of models has a *unique* RCE. For this, we reconsider Lemma 2 and Theorem 5 under the following additional assumption:

Assumption 2: The consumption aggregator $A(c)$ is such that for the associated $p(x) = \frac{U_1(x,y^{NT})}{U_2(x,y^{NT})}$, the mapping $Z_{cc}(x, s^e, x) = 0$ in equation (44) has a unique root $x^*(s^e)$ for a given κ and R.

We now can show with the addition of Assumption 2 in Theorem 5, the RCE is unique.

Theorem 9 *Under Assumption 1 and 2, there is a unique RCE* $C^*(s^e) = \inf\{c^*(s^e), x^*(s^e)\}$ *where* $x^*(s)$ *is the unique root of equation (44), with*

$$A^{*n}_{uc}(c_0)(d, y) = c^*(s^e)$$

for any $c_0 \in \mathbf{C}^p_{++}$.

[35] Schmitt-Grohé and Uribe ([73]) give a *local* sufficient condition near the deterministic steady-state for this to be the case for the case that the utility aggregator $A(c^T, c^{NT})$ is an Armington aggregator and near a steady state. But clearly, their idea about the source of multiplicity applies in *any* equilibrium state s^e. That is, generally $Z_{cc}(x, x; y, \kappa/R, d)$ is not either strictly increasing or decreasing in x at each s^e under Assumption 1 (hence, roots are unique)

A few remarks on this theorem. First, when using our two-step construction in theorem 5 under the added condition in Assumption 2, we can first compute the *unique* maximal collateral consumption in a RCE as the unique solution $x^*(s^e)$ in

$$Z_{cc}(x^*(d,y),x^*(d,y),d,y) = x^*(d,y) - (1+\frac{\kappa}{R})y^T \\ -d + \frac{\kappa}{R}p(x^*(d,y))y^{NT} - d = 0 \tag{45}$$

Then, the second step of the RCE construction is *not needed.* That is, substituting $x^*(s^e) = A_c(C^T)(s^e)$ in the definition operator $A^*(C)(s^e)$ with

$$C(s^e) = \inf\{c(d,y), x^*(d,y)\} \tag{46}$$

we can now construct the RCE using *only* the first step operator $A^*_{uc}(c)$ (which is a monotone contraction on $\mathbf{C}^p_{++}(\mathbf{S}^e)$ with unique strictly positive fixed point $c^*(d,y) > 0$, so the unique RCE is the strictly positive function $C^*(s^e) = \inf\{c^*(d,y), x^*(s^e) \in \mathbf{C}^*(\mathbf{S}^e)$.

Second, the sufficient condition for uniqueness can be satisfied as discussed in Section (e.g., $A(c)$ is quasi-linear or log). We emphasize that when quantitative versions of this model are studied in the existing literature, $A(c)$ is the Armington/CES aggregator. In this case, for typical parameterizations of this mapping, Assumption 2 will *not* be satisfied, and multiple RCE will exist (e.g., see the discussion in Schmitt-Grohé and Uribe ([73])).

6 Additional Results and Extensions

6.1 Relaxing Assumptions on Consumption Aggregators

In section 5.1.1 we assume that preferences are given by $U(A(x)) = (A^{1-\sigma} - 1)/(1-\sigma)$, $A(c) = (a(c^T)^{1-1/\xi} + (1-a)(c^N)^{1-1/\xi})^{1/(1-1/\xi)}$ with $a = 1/2$ and $\sigma = 1/\xi = 2$. Now, we partially relax this last restriction. In particular, we let $\sigma = 1/\xi$ and $\xi > 0$. We must assume that the relative risk aversion coefficient σ equals the reciprocal of the intratemporal elasticity of substitution ξ to keep the Euler equation tractable. Using σ to calibrate the model is frequent in applications. Thus, this section is relevant as we can study multiplicity in a more general numerically relevant environment.

If we assume $\kappa = R^{-1}$ and $y^N = 1$, equation (7) when the collateral constraint is binding becomes:

$$h(P) \equiv P^\xi - (1+P)y^T = 1 - d \tag{47}$$

We will call a solution to (47) a root. As P increases in aggregate consumption, we can map any element in the minimal state space (y^T, d) to aggregate consumption using the solutions to (47).

We split the analysis into four possible cases based on the combination of values for the intratemporal elasticity (EIS, ξ) and tradable output y^T. In turn, as we need to characterize the state space, we classify the debt values as low, mid, and high. The table below illustrates the results of the analysis.

State Space (y^T, d) / EIS		Low $\xi < 1$	High $\xi > 1$
Low y^T	Low d	No Root	1 Root
	Mid d	2 Roots	2 Roots
	High d	1 Root	No Root
High y^T	Low d	No Root	1 Root
	Mid d	2 Roots	2 Roots
	High d	1 Root	No Root

Table 1: Existence of solutions to equation (7) when the collateral constraint is binding

It is easy to see that $h'^{\xi-1} - y^T,\ h'^{*}) = 0$ with $P^* = (y^*/\xi)^{1/(\xi-1)}$ and $h(0) = -y^T$. Then, when $\xi > 1$, h is convex with a minimum at $P = P^*$, and with $\xi < 1$, it is concave with a maximum at $h(P^*)$.

We called the latter "high" and the former "low" ξ. Further, we say that income y^T is "low" / "high" if y^T is close to y_{LB}/y_{UB}. Finally, as the collateral constraint cannot bind if households hold net assets, root to equation (47) must lie in the following set: $P_r = \{0 < P < \infty,\ h(P) = 1 - d\ with\ h(P) \leq 1\}$.

When $\xi > 1$, h decreases for $0 < P < P^*$ and increases thereafter. Then, for low values of debt, there is only one root as $h(0) = -y^T < 0$ and $h(P)$ crosses $1-d$ only for $P > P^*$. Then, for intermediate values of debt, $h(P^*) \leq 1-d \leq -y^T$, there are two roots. Note that the region with multiple equilibria increases for low-income values as $-y^T$ is closer to 1 than when y^T is "high." Finally, there is no root for high debt values, $1 - d < h(P^*)$. Thus, we must set the maximum level of debt accordingly: $d_{Max} \equiv 1 - h(P^*)$. In this case, the maximal P satisfies $h(P_{Max}) = 1$, which is finite and defines the maximal aggregate consumption in the constrained case.

When $\xi < 1$, the case is similar to the one described in section 5.1.1, and thus, we skip the details. A sufficient condition exists to eliminate the "no Root" region for "low" debt values. As $h(P^*)$ is decreasing in ξ, it suffices to set an upper bound for this parameter, ξ_{UB}, such that $h(P^*) = (1-\xi)(y^T/\xi)^{\xi/(\xi-1)} - y^T \geq 1$, with $(1-\xi_{UB})(y_{UB}/\xi_{UB})^{\xi_{UB}/(\xi_{UB}-1)} - y_{UB} = 1$ as $h(P^*)$ is decreasing in y^T. Then, the only admissible values of ξ are $0 < \xi \leq \xi_{UB}$. This implies that in section 5.1.1, we need to set y^T such that $(0.5)(2y^T) - y^T \geq 1$, which is satisfied for all y^T.

6.2 General shock spaces

We now mention how the main results in the paper change if the shocks in the model to endowments are more general. We are concerned here with the situation of the model where the shocks are: (a) iid but are defined on a continuous (compact) support, or (b) are first-order Markov shocks (for either the discrete or continuous compact support case). In both case (a) and (b), if shocks are on continuous shock spaces, issues of measurability will become important when characterizing the appropriate version of the space $\mathbf{C}^*(\mathbf{D}^e\times\mathbf{Y}^e)$ where existence of RCE arguments will take place. In any case, although some differences in the result does occur, nothing in the constructions in the paper change, and the results will remain similar to those on Theorems 5, 6, and 8 with one notable change.

Let us first consider how to generalize the main theorems on existence and characterization of the set of RCE for the case (a) of iid shocks on a continuous compact support. In this case, when defining the candidate set of RCE $\mathbf{C}^*(\mathbf{D}^e\times\mathbf{Y}^e)$ in equation (33), we will now also need to require each element $C(S^e) \in \mathbf{C}^*$ to also be (Borel) measurable. Let's denote the version of the space $\mathbf{C}^*(\mathbf{D}^e\times\mathbf{Y}^e)$ by $\mathbf{C}^*_{\uparrow m}(\mathbf{D}^e\times\mathbf{Y}^e)$[36]), if we give the space $\mathbf{C}^*_{\uparrow m}(\mathbf{D}^e\times\mathbf{Y}^e)$ the pointwise partial order $\geq_E$, then in Lemma 3, the space $(\mathbf{C}^*_{\uparrow m}(\mathbf{D}^e\times\mathbf{Y}^e), \geq_E)$ is no longer a complete lattice (rather, it is just $\sigma-$complete). [37] Still, it can easily be shown that our operator $A^*(C)$ constructed exactly as before is well-defined and order continuous. That is, for the "first step" operator $A(c; C^T)$ for each $C^T \in \mathbf{C}^*_{\uparrow m}$, this operator still has a unique strictly positive measurable fixed point $c^*(C^T)(s)$. This unique fixed point is again order continuous in C^T on $C^*_{\uparrow m}$, so can again used to define our (second step) RCE operator $A^*(C)$ and study it's set of fixed points in $\mathbf{C}^*_{\uparrow m}(\mathbf{D}^e\times\mathbf{Y}^e)$. As $\mathbf{C}^*_{\uparrow m}(\mathbf{D}^e\times\mathbf{Y}^e)$ is $\sigma-$complete, the least and greatest RCE can be computed as in Theorem 31 via a standard application of the Tarski-Kantorovich theorem. But the one critical thing that changes in Theorem 5 for this new setting is now the set of RCE is only a $\sigma-$complete lattice in $\mathbf{C}^*_m(\mathbf{D}^e\times\mathbf{Y}^e)$ (e.g., by Theorem 1 in [9])). Per Theorem 6, nothing changes, nor does any change in the uniqueness of RCE under Assumptions 1 and 2 in Theorem 9. Finally, the iterative monotone comparative static result in Theorem 8 does not change either (as the operator $A^*(C)$ is order continuous in a $\sigma-$complete lattice $\mathbf{C}^*_{\uparrow m}(\mathbf{D}^e\times\mathbf{Y}^e)$).

Next, if the shocks are Markovian, the operator $A^*(C)$ can again defined exactly as it is defined in the iid shock case, but the major change in the construction of RCE is that RCE are no longer monotone increasing in endowment shocks. So, if from the definition of the space $\mathbf{C}^*_{\uparrow m}(\mathbf{D}^e\times\mathbf{Y}^e)$) now only require tradables consumption to be bounded and measurable in the endowments y, and if we denote

[36]Here the subscript " $\uparrow m$" on the space $\mathbf{C}^*_{\uparrow m}(\mathbf{D}^e\times\mathbf{Y}^e)$ is to indicate that the elements this new version of the space $\mathbf{C}^*(\mathbf{D}^e\times\mathbf{Y}^e)$ are monotone increasing and measurable on y.

[37]We say a lattice X is $\sigma-$complete if for every countable subset $X_c \subset X$, $\vee X_c$ and $\wedge X_c$ are in X. If we take a space of measurable functions with a least and greatest element measurable and give it a pointwise partial order; the space is generally only $\sigma-$complete (i.e., it's only closed under pointwise sup and inf operations for countable subsets).

the resulting space by $\mathbf{C}_m^*(\mathbf{D}^e\times\mathbf{Y}^e)$, and if we give the space $\mathbf{C}_m^*(\mathbf{D}^e\times\mathbf{Y}^e)$ the pointwise order $\geq_E$, then $(\mathbf{C}_m^*(\mathbf{D}^e\times\mathbf{Y}^e),\geq_E)$ remains a $\sigma-$complete lattice in the new version of Lemma 3. As the operator $A^*(C)$ can be shown to remain order continuous on $\mathbf{C}_m^*(\mathbf{D}^e\times\mathbf{Y}^e)$, not much changes in the construction of the set of RCE from the iid shock case. If the shock space is finite (resp, continuous, and compact), the resulting version Theorem 5 has the set of RCE a complete lattice (resp., $\sigma-$complete lattice) in $\mathbf{C}_m^*(\mathbf{D}^e\times\mathbf{Y}^e)$. The comparison of least and greatest RCE in Theorem 6 remains the same, and the uniqueness of RCE under Assumptions 1 and 2 in Theorem 9 still holds. Also, the iterative monotone comparative static result in Theorem 8 does not change.

6.3 Dynamic Pecuniary externalities

In this section, we characterize the presence of dynamic pecuniary externalities. Even if agents are not credit-constrained in the present, the possibility of observing a crisis in the future (i.e., a hit to the collateral constraint) affects current decisions. If these crises are expected to induce multiple equilibria in the future, coordination problems affect the current market conditions. If agents coordinate in a high consumption-high leverage equilibrium, this situation self-fulfills into the present. As the transmission of future market conditions into the present depends on the Euler equation and the presence of pecuniary externalities, most models with price-dependent collateral constraints and incomplete markets have these types of equilibrium dynamic complementarities.

To discuss the presence of dynamic pecuniary externalities, we need elements that we use to characterize the RCE. To keep this section self-contained, we reproduce below some elements of the dynamic programming representation of the household's decision problem. Let $\mathbf{C}^f, S$ be the set containing all possible aggregate or per-capita consumption levels and aggregate states D, Y respectively. For any element $C^T \in \mathbf{C}^f$, we can identify the implied law of motion for per-capita debt D in a RCE by using equilibrium versions of the household's budget constraints and collateral constraints: i.e., the per-capital debt evolves according to:

$$D' = \Phi(S;C^T) = \inf[R\{C^T(S) - Y + D\}, \kappa\{y^T + p(C^T(S))y^N\}],\ C^T \in \mathbf{C}^f \tag{48}$$

As D is the only endogenous aggregate state in this economy, in conjunction with the primitives of the stochastic process on the endowment shocks Y, we now have a full characterization of the stochastic transition structure of the aggregate economy.

When entering the period in state $s = (d, y, S)$, the household's feasible correspondence is given by:

$$G(s;C^T) = \{c \in \mathbf{R}_+^2, d' \in \mathbf{D}|\ (49a) \text{ and } (50) \text{ hold}\}$$

where

$$c^T + p(C^T(S))c^N \leq y - d + p(C^T(S))y^N + \frac{d'}{R} \tag{49a}$$

and

$$d' \leq \kappa(y^T + p(C^T(S))y^N) \tag{50}$$

The aggregate economy is characterized by a law of motion on per-capita debt. D' using (48). Then, a recursive representation of the household's sequential decision problem can be constructed as a unique value function $V^*(s;C^T)$ solving a Bellman equation for each $C^T(S) \in \mathbf{C}^f$:

$$V^*(s;C^T)) = \max_{x=(c^T,c^N,d')\in G(s;C^T)} U(c^T,c^N) + \beta \int V^*(d',y',Y',\Phi(S;C^T);C^T)\chi(dy') \tag{51}$$

The unique optimal policy function associated with the solution to (51) is given by:

$$c^*(s;C^T(S)) = \arg \max_{x=(c^T,c^N,d')\in G(s;C^T)} U(c^T,c^N) + \beta \int V^*(d',y',Y',\Phi(S;C^T);C^T)\chi(dy'), \tag{52}$$

where the vector of consumption policies $c^*(s; C^T) = (c^{T*}(s; C^T), c^{NT*}(s; C^T))$ is jointly continuous in s, and the value function $V^*(s, S; C^T)$ is continuous in s, strictly concave and decreasing in d for each (y, S), and increasing in y, each (d, S).

We know that static pecuniary externalities may generate multiple *present* consumption levels to be compatible with the same position in the state space. Equation (52) shows that *future* consumption levels may also generate multiplicity. For some C^T, $\Phi(S; C^T)$ may bind and generate multiple solutions. This is possible under the same conditions that generate static pecuniary externalities. However, C^T values depend on agents *beliefs*. we will observe an economy with more tradable GDP in the future and thus with a higher borrowing capacity, self-generating expected high consumption environment today. That is, as there is a complementarity between the marginal utility of consumption today and tomorrow, market conditions in the future are transmitted to the present as p is also increasing in c^*. Of course, the contrary happens if C^T is expected to be low. This is the *dynamic pecuniary externality*. That is, future expected aggregate consumption C^T affects present individual consumption c^* though expectations that are represented by $\Phi(S; C^T)$ in equation (52).

7 Conclusions

This paper proposes a new multistep monotone map method for characterizing the set of RCE in the prototype class of sudden stops models which has been the focus of a great deal of work in the applied literature that seeks to model emerging market financial crises. Further, the paper is the first paper that presents a constructive approach to characterizing minimal state space recursive equilibria in infinite horizon incomplete markets models with price-dependent collateral constraints. As our methods are built on a novel application of order continuous (monotone) operator theory, and this new approach presents constructive methods for characterizing both existence of RCE, as well as when characterizing the nature of RCE comparative statics. Our approach also (by construction) characterizes that partitioning of the minimal states where equilibrium collateral constraints bind, and when they do not. In addition, an interesting additional implication of our approach is that we show that the actually state-space, where any RCE exists, depends critically on the particular RCE being characterized. That is, in RCE where equilibrium collateral constraints are "tight," the level of maximal sustainable debt is lower than in RCE where equilibrium collateral constraints are relatively "loose."

The paper is also potentially important as the methods seem amenable to extending to other infinite horizon models with incomplete markets and occasionally binding price-dependent inequality constraints. The key issue to extend the approach to other models with incomplete markets and collateral constraints are that we must have an Euler inequality associated with household dynamic decision problems. Our approach is also aimed at characterizing dynamic equilibria in settings where multiple equilibria and discontinuous equilibrium selections are endemic to the models under study. The approach is constructive and uses equilibrium versions of the household's Euler inequalities to characterize RCE over states where equilibrium collateral constraints do not bind and revert to collateral constrained consumption (which is determined by the collateral constraint itself) over states where collateral constraints bind. This approach seems to extend to other settings where collateral constraints are themselves equilibrium objects.

It also bears mentioning that our methods extend to other models of sudden stops in the literature. For example, models with non-homothetic preferences (e.g., Rojas and Saffie ([66])) and production (e.g., Benigno et. al ([11], [12])) can be handled with our approach. Further, alternative monotone methods can be developed to deliver similar results for models with heterogeneous agents (e.g., see Pierri and Reffett ([61]).) Finally, generalized Markov methods can be developed to study the structure of more general sequential equilibria than those that are (minimal state space) RCE (e.g., see Pierri and Reffett ([60]).

Many questions remain unanswered in the macroprudential policy literature, and one crucial area of exploration is how these methods can enhance the understanding of optimal policy design in an arbitrarily decentralized equilibrium set. The macroprudential policy has been a focal point of recent research [38], a significant challenge arises in defining "optimal policy" in a world of multiple equilibria. In the macroprudential literature, today's optimal tax rate hinges on the marginal utility of tomorrow's consumption. Given that collateral constraints may bind in the future, multiple equilibria become an intrinsic aspect of the problem. Consequently, the optimal tax can vary depending on whether the economy operates in a high- or low-borrowing equilibrium.

Bianchi ([17]) offers a critical perspective, arguing that optimal taxes represent the private agents' "uninternalized " marginal cost of borrowing normalized by the expected marginal utility. These costs increase with the current level of debt when the credit constraint is binding; the tax does not generally influence the level of borrowing. While these insights are correct, we claim that the details of this sort of argument are incomplete in an important way (especially in the presence of multiple equilibria). First, even under perfect commitment by the fiscal agent, taxes set today but payable tomorrow depend on tomorrow's marginal utility (which is endogenous in equilibrium). This tax, in turn, impacts the structure of some RCEs, but it is not clear which RCE? Further, per counterfactuals and the evaluation of optimal policy under multiple RCE, particular equilibria in economies "without policy intervention" of course change under the new policy intervention, but need not be *stable* under the policy perturbation. This is not generally the case of RCE are *unique.*

By developing a formal RCE operator that allows the collateral constraint to bind across *any two consecutive periods*, we show that *constrained* states influence optimal taxes *even in cases of a unique equilibrium.* So "designing" optimal policy in the presence of multiple equilibria seems much more delicate than when RCE are unique. One interesting fact about using our generalized iteration approach in the previous section is. This also allows us to develop a theory of macroprudential policy relative to "stable" equilibrium selections.

Finally, although from a technical standpoint, our results rely on the primal characterization of equilibrium, there is no issue developing recursive dual versions of our monotone methods (i.e., we can work in spaces of Lagrangian multipliers for the budget constraint that are equal in all states to the marginal utility of consumption). The point is we can easily map our primal methods directly into recursive dual methods and isolate exactly the *equilibrium states* where collateral constraints hit (and Kuhn Tucker multipliers for collateral constraints are positive). This means we can map all our *equilibrium constructions* into dual time iteration methods that are typically used to study macroprudential policies (i.e., see for example, the methods used in Bengui and Bianchi ([15]), for example).

Using the equilibrium construction, we can then develop equilibrium versions of constrained planning problems and approach the question of optimal taxes are determined solely by the planner's multipliers *along equilibrium paths* for the decentralized RCE. Since these multipliers can be derived from a near-standard optimization problem, our approach directly applies to a macroprudential policy framework. We explore these issues in Pierri and Reffett ([62]).

[38] See Bianchi ([17]), Bianchi and Mendoza ([20]), Bianch, Liu, and Mendoza ([19]), Bengui and Bianchi ([15]), and Benigno et al ([11], [12], [14]), among many others for discussion and additional references.

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Appendix: Proofs

Proof of Lemma 2

Proof. As a brief outline of our approach to the proof of this key lemma, the proof of part takes place in five steps. We first show the operator $A_{uc}(c; C^T(S))(d,y)$ for each $C^T(S) \in \mathbf{C}^f(\mathbf{S})$, $A_{uc}(c; C^T(S))(d,y) : \mathbf{C}^p(\mathbf{S}) \to \mathbf{C}^p(\mathbf{S})$ is well-defined. Second, we show the mapping $A_{uc}(c, C^T(S))(d,y)$ is jointly monotone on $\mathbf{C}^p \times \mathbf{C}^f$, and order continuous in $c \in \mathbf{C}^p$ for each $C^T \in \mathbf{C}^f$. Third, we show the greatest fixed point of $A_{uc}(c; C^T(S))(d,y)$ (denoted for now by $c^*(C^T(S))(d,y))$ is strictly positive, can be computed by successive approximations from an initial $c_0 = c_{\max}$ for each $C^T \in \mathbf{C}^f$. [39] Fourth, we show the greatest fixed point is increasing in $C^T(S)$ on $\mathbf{C}^f$. Finally, in the fifth step, we show $c^*(C^T(S), \beta, R, \kappa)(d,y)$ is the *unique* strictly positive fixed point in $\mathbf{C}^p$ of $A(c; C^T(S))(d,y)$ for each $C^T \in C^f$.[40]

Step 1: $A_{uc}(c; C^T(S))(d,y) : \mathbf{C}^p \to \mathbf{C}^p$. Fix $C^T \in \mathbf{C}^f$, $c \in \mathbf{C}^p$, and $s = (d, y, S)$.

We first prove the operator $A_{uc}(c; C^T(S))(d,y)$ is well-defined. To see this, observe when $c(d,y) \in \mathbf{C}^p$, $c(d,y) = 0$ for any state (d,y), $C(d', y'; c, C^T) = 0$, we define $x^*_{uc}(s^e, c, C^T) = A_{uc}(c; C^T(S))(d,y) = 0$. So consider the case when $c \in \mathbf{C}^p$, $c(s^e) > 0$. As $(c, C^T) \in \mathbf{C}^p \times \mathbf{C}^f$, the mapping $Z^*_{uc}(x, s^e; c, C^T)$ in equation (26) is strictly decreasing and continuous in x, for any $(d, y, S; c, C^T)$. Compute an implicit mapping $x^*_{uc}(d, y, S; c, C^T)$ in the following equation:

$$Z^*_{uc}(x^*_{uc}(d, y, S; c, C^T), s; c, C^T) = 0$$

If the root $x^*_{uc}(d, y, S; c, C^T)$ exists, it will be unique as Z^*_{uc} is strictly decreasing and continuous in x under Assumption 1. When $x \to 0$, $Z^*_{uc}(x, s^e, c, C^T) \to \infty$ by the Inada condition in Assumption 2. Further, as x gets sufficiently large, $C((R(x - y^T + d), y', R(x - y^T + d, y') \to 0$, hence $Z^*_{uc} \to -\infty$. Then, by the intermediate value theorem, $x^*_{uc}(d, y, S, c, C^T)$ exists (hence, $x^*_{uc}(d, y, S; c, C^T)$ is a function for all pair (c, C^T)).

Next, we show $x^*_{uc}(d, y, S; c, C^T) \in \mathbf{C}^p$. Again, when $c(d,y) \in \mathbf{C}^p$, $c(d,y) = 0$ in any state (d,y), $\Rightarrow C(c, C^T) = 0$; hence, we define $x^*_{uc}(d, y, S; c, C^T) = 0 \in \mathbf{C}^p$. Therefore, consider the case when $c \in \mathbf{C}^p$, $c(s^e) > 0$. As $C^T \in C^f$, for fixed $c \in \mathbf{C}^p$, Z^*_{uc} in (26) is (strictly) decreasing in d, (strictly) increasing in y, and strictly decreasing in x; hence, at such $s = (d, y, S)$, the root $x^*_{uc}(d, y, S; c, C^T)$ is decreasing in d, and increasing in y. Further, when $d_2 \geq d_1$ and $y_1 \geq y_2$, by the concavity of utility in Assumption 1, we have from the definition of the $x^*_{uc}(d, y, S, c, C^T)$ in Z^*_{uc} the following inequality

$$\frac{U_1(x^*_{uc}(d_1, y_1, S; c, C^T), y^N)}{R} \leq \int \beta U_1(C(R(x^*_{uc}(d_2, y_2, S; c, C^T) - y_2^T + d_2), y', R(x^*_{uc}(d_2, y_2, S; c, C^T) - y_2^T + d_2), y')\chi(dy')$$

hence, for the root $x^*_{uc}(d, y, S; c, C^T)$ must make the right side of the above expression fall at $x^*_{uc}(d_2, y_2, S; c, C^T)$ in a new solution, which implies:

$$x^*_{uc}(d_1, y_1, S; c, C^{T*}) - y_1^T + d_1 \geq x^*_{uc}(d_2, y_2, S; c, C^{T*}) - y_2^T + d_2$$

[39] Keep in mind, the first step operator on the space $\mathbf{C}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e; C^T)$ has a trivial fixed point at $c^* = 0$ for all $C^T \in \mathbf{C}^f$. This will not be a problem as in the fourth step, we shall show that the first step mapping restricted to the domain $c \in \mathbf{C}^p_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma(d,y))$ with the norm

$$\rho(c_1, c_2) = \parallel u'_\varsigma \circ c_1 - u' \circ c_2 \parallel$$

where $\parallel u' \circ c_1 - u' \circ c_2 \parallel < \infty$ is a monotone contraction. It does turn out the first step operator is a monotone concave operator on all of $\mathbf{C}^p(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma(d,y))$, so by a standard argument in the monotone map literature (e.g., Datta et al. ([28], Morand,and Reffett ([53]), and Mirman, Morand and Reffett ([51]), the first step operator only has two fixed points (the strictly positive one, and $c^* = 0$ for all $C^T \in \mathbf{C}^f$).

[40] Note the operator $A_{uc}(c, C^T(S))(d,y)$ therefore can have many fixed points (actually, a complete lattice of fixed points), but has a unique strictly positive fixed point (which is the only non-trivial fixed point from the vantage point of constructing RCE). That is, all the "trivial" fixed points take place on the boundary of the space $\mathbf{C}^p$.

or

$$y_1^T - d_1 - x_{uc}^*(d_1, y_1, S; c, C^{T*}) \leq y_2^T - d_2 - x_{uc}^*(d_2, y_2, S; c, C^T)$$

Therefore, for each $C^T \in C^f$, $A_{uc}(c; C^T(S))(d, y) = x_{uc}^*(d, y, S; c, C^T) \in C^p$.

Step 2: *$A_{uc}(c, C^T(S))(d, y)$ is monotone (increasing) on $\mathbf{C}^p \times \mathbf{C}^f$.* Take $x_1 = (c_1, C_1^T)$ and $x_2 = (c_2, C_2^T) \in \mathbf{C}^p \times \mathbf{C}^f$, with $x_1 \leq x_2$ under the pointwise partial order on the product space $\mathbf{C}^p \times \mathbf{C}^f$. First, consider the case $0 \leq x_1 \leq x_2$, where in some state (d, y, S), either $0 = c_1(d, y)$ or $0 = C_1^T(S)$, Then, by definition of the operator $A(c, C^T(S))(d, y)$, $A(c_1, C_1^T)(d, y, S) = 0 \leq A(c_2, C_2^T)(d, y, S)$. So, now consider the case where $0 < x_1(d, y) \leq x_2(d, y)$, so in all states, $0 < c_1(d, y)$ and $0 < C_1^T(S)$. Then, we have from the definition of x_{uc}^* in Z_{uc}^* the following inequality:

$$\begin{aligned}
&\frac{U_1(x_{uc}^*(d, y, S; c_1, C_1^T), y^N)}{R} = \\
&\quad \int \beta U_1(C_1(R(x_{uc}^*(d, y, S; c_1, C_1^T) - y^T + d), y', R(x_{uc}^*(d, y, S; c_1, C_1^T) - y_2^T + d_2), y')\chi(dy') \\
&\qquad \geq \int \beta U_1(C_2(R(x_{uc}^*(d, y, S; c_1, C_1^T) - y^T + d), y', R(x_{uc}^*(d, y, S; c_1, C_1^T) - y_2^T + d_2), y')\chi(dy')
\end{aligned}$$

where for $i = 1, 2$, the subscript on continuation consumption is used to denote.

$$C_i(c, C^T)(d, y, S) = \inf\{c_i(d, y), C_c^T(D, Y, C_i^T(S))\}$$

where recall $C_c^T = (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(D, Y)Y^N$. Therefore, as Z_e^* is strictly falling in x, we have

$$x_{uc}^*(d, y, S; c_1, C_1^T) \leq x_{uc}^*(d, y, S; c_2, C_2^T)$$

Then, if the implied debt at $d_{x_{uc}}(d, y, S; c_1, C_1^T) \leq \kappa(Y^T - D + p(C_1^T)Y^N$, then

$$\begin{aligned}
A_{uc}(c_1, C_1^T(S))(d, y) &= x_{uc}^*(d, y, S; c_1, C_1^T) \\
&\leq x_{uc}^*(d, y, S; c_2, C_2^T) \\
&\leq A_{uc}(c_2, C_2^T(S))(d, y)
\end{aligned}$$

where the second line uses the fact that $d_{x_{uc}}(d, y, S; c_1, C_1^T) \leq \kappa(Y^T - D + p(C_1^T)Y^N \leq \kappa(Y^T - D + p(C_2^T)Y^N$, so $x_{uc}^*(d, y, S; c_1, C_1^T) \leq x_{uc}^*(d, y, S; c_2, C_2^T)$ is feasible (i.e., the unconstrained level of tradables consumption at (c_2, C_2^T) cannot be lower). So the operator $A_{uc}(c, C^T(S))(d, y)$ is monotone on $\mathbf{C}^p \times \mathbf{C}^f$.

Next, we prove $A(c; C^T(S))(d, y) : \mathbf{C}^p \to \mathbf{C}^p$ is order continuous on $\mathbf{C}^p$ for each fixed $C^T(S) \in \mathbf{C}^f$.

First, some definitions. Let X be a countably chain complete partially ordered set,[41] and $X_c = (x_n)_{n \in \mathbf{N}} \subset X$, $x_n \in X$, be a countable chain. We say a operator $A : X \to X$ for is *order continuous* if for any $X_c \subset X$, $A(x)$ (a) sup-preserving: $A(\vee X_c) = \vee A(X_c)$ and (b) inf-preserving: $A(\wedge X_c) = \wedge A(X_c)$.

We remark, order continuous operators are necessarily isotone (e.g., Dugundji and Granas ([33], p. 15)).

We now show for each $C^T(S) \in \mathbf{C}^f$, $A_{uc}(c; C^T(S))(d, y)$ preserves sup operations; a similar argument works for preserving inf operations. Fix the state (d, y), and $C^T(S) \in \mathbf{C}^f$, and denote by $C_c = (c_n(d, y))_{n \in \mathbf{N}}$, $c_n(d, y) \in \mathbf{C}^p$ any countable chain in $\mathbf{C}^p$. Define $\vee C_c(d, y) \in \mathbf{C}^p$ and $\vee A_{uc}(C_c; C^T(S))(d, y) \in \mathbf{C}^{p*}$, which both exist in $\mathbf{C}^p$ (resp, $\mathbf{C}^{p*}$) are both complete lattices (hence, countably chain complete). If in any state (d, y), $\vee C_c(d, y, S) = 0$, then $\vee A_{uc}(C_c; C^T(S))(d, y) = A_{uc}(\vee C_c; C^T(S)) = 0$. Therefore, assume for every state (d, y, S), $\vee C_c(d, y) > 0$. Then, we have the following inequalities for continuation

[41] Let X be a partially ordered set. We say X is countably chain complete if for all countable subset X_c that are a chain (i.e., for no two elements $x_1, x_2 \in X_c$, x_1 and x_2 are ordered), $\vee X_c \in X$ and $\wedge X_c \in X$.

tradables consumption $C(c_n; C^T) = \inf\{c_n(d,y), C_c^T(D,Y,C^T(S))\}$

$$\begin{aligned}
C(\vee C_c) &= C(\vee c^n; C^T) \\
&= \inf\{\vee c_n(d,y), (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))y^N\}) \\
&= \vee \inf_n(\{c_n(d,y), (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))Y^N\}) \\
&= \vee C(c^n; C^T) = \vee C(C_c; C^T(S))
\end{aligned}$$

where in the second line $\vee c_n(d,y)$ is computed, and then the infimum over two continuous functions $(\vee c_n(d,y), (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))y^N)$ is taken over a compact set $(d,y,S) \in \mathbf{D}\times\mathbf{Y}\times\mathbf{S}$, and hence continuous by Berge's theorem, in the third line, $\inf_n$ is computed pointwise over $(d,y) \in D\times Y$ (a compact set and hence continuous) at each $n \in \mathbf{N}$, and this collection is then increasing pointwise in n as C_c is a countable chain) and the sup is taken over $n \in \mathbf{N}$. Then, the remaining equalities follow from p continuous, and the fact that sup and inf operations over two continuous functions are each continuous over the compact set $(d,y,S) \in \mathbf{D}\times\mathbf{Y}\times\mathbf{S}$ by Berge's maximum theorem.

Using these facts, and substituting into the definition of $Z_{uc}^*(x,d,y,S;c,C^T)$, we have for the root $x_{uc}^*(d,y,S,c,C^T)$ the following equalities:

$$\begin{aligned}
Z_{uc}^*(x_{uc}^*(d,y,S;\vee c_n, C^T), d,y,S;\vee c_n, C^T) &= \vee Z_{uc}^*(x_{uc}^*(d,y,S;\vee c_n, C^T), d,y,S;c_n, C^T) \\
&= \vee Z_{uc}^*(x_{uc}^*(d,y,S;c_n, C^T), d,y,S;c_n, C^T) \\
&= Z_{uc}^*(\vee x_{uc}^*(d,y,S;c_n, C^T), d,y,S;c_n, C^T)
\end{aligned}$$

where the first equality follows from $U_1(c,y^N)$ continuous and $C(\vee c_n, C^T) = \vee C(c_n, C^T)$, the second line follows from Z_{uc}^* continuous (pointwise) in (x, c_n) for fixed C^T, the third line follows from Z_{uc}^* continuous in x. Then, noting that for any state where collateral constraints do not bind, we have $x_{uc}^*(d,y,S,c_n,C^T) \leq A_c(C^T)(S)$, our operator $A_{uc}(c, C^T(S))(d,y)$ is for each $n \in \mathbf{N}$ defined as:

$$A(x_{uc}^*(d,y,S;c_n,C^T), C^T(S))(d,y) = \inf_n\{x_{uc}^*(d,y,S;c_n,C^T), A_c(C^T)(S)\}$$

so we have the following:

$$\begin{aligned}
A(\vee c_n, C^T(S))(d,y) &= \inf\{x_{uc}^*(d,y,S;\vee c_n,C^T), A_c(C^T)(S)\} \\
&= \inf\{\vee x_{uc}^*(d,y,S;c_n,C^T), A_c(C^T)(S)\} \\
&= \vee \inf_n\{x_{uc}^*(d,y,S;c_n,C^T), A_c(C^T)(S)\} \\
&= \vee A(c_n; C^T(S))(d,y)
\end{aligned}$$

where the second equality follows again from $U_1(c,y^N)$ is continuous and $C(\vee c_n; C^T) = \vee C(c_n; C^T)$ for each C^T,and for the third equality again uses the fact that $\inf_n$ here is an increasing pointwise in n, and the sup is then taken over $n \in \mathbf{N}$. Hence, $A_{uc}(c; C^T(S))(d,y)$ is order continuous in $\mathbf{C}^p$ for each fixed $C^T \in \mathbf{C}^f$, which completes the proof of Step 2.

Remark 10 *Before proceeding to step 3, as equilibrium fixed point comparative statics be an important question in Steps 4 and 5 (and in the proof of the comparative statics theorem in Theorem 6), for the remaining steps of the proof of this lemma, we shall add to the notation for our operator for the parameters of interest, and remark that the operator $A_{uc}(c; C^T(S), \beta, \kappa, R)(d,y)$ is increasing in κ, and decreasing in (β, R) for fixed (c, C^T, d, y, S). To see this, noting $c \in \mathbf{C}^p$ is decreasing in d, U_1 is decreasing in c under assumption 1, Z_{uc}^* in (26) is decreasing in (R,β). So, the root $x_{uc}^*(d,y,S,c,C^T;\beta,\kappa,R)$ is decreasing in (β,R). As Z_{uc}^* is independent of κ, but $A_c(C^T;\kappa,R)$ is increasing in κ, $A_{uc}(c; C^T(S), \beta, \kappa, R)(d,y)$ is increasing in κ.*

Remark 11 *By the previous remark, as $A_{uc}(c; C^T(S), \beta, \kappa, R)(d,y)$ is decreasing in (β, R), and increasing in κ, and $A_c(C^T;\kappa,R)$ is decreasing in R, and increasing in κ, our operator $A(c; C^T(S), \beta, \kappa, R)(d,y)$ is also decreasing in (β, R) and increasing in κ under Assumption 1.*

Step 3. *Existence and computation of the greatest fixed point of* $A_{uc}(c; C^T(S), \beta, \kappa, R)(d, y) \in \mathbf{C}^p$. Fix $C^T \in \mathbf{C}^f$, and denote by $\Psi_A(C^T(S), \beta, \kappa, R)(d, y) \subset \mathbf{C}^p$ the set of fixed points of mapping of $A_{uc}(c; C^T(S), \beta, \kappa, R)(d, y) \in \mathbf{C}^p$. Then, by Tarski's theorem, as $A_{uc}(c, C^T(S)(\beta, \kappa, R)(d, y)$ is monotone on a complete lattice $\mathbf{C}^p$, its set of fixed points $\Psi_A(C^T(S), \beta, \kappa, R)(d, y) \subset \mathbf{C}^p$ is a nonempty complete lattice.[42] By definition, the least fixed point is trivial, and is $c^* = 0$ for all $y \in Y$. By step 2 above, $A_{uc}(c; C^T(S), \beta, \kappa, R)(d, y)$ is order continuous on $\mathbf{C}^p$. Further, $A_{uc}(c^{\max}; C^T(S))(d, y) \leq c_{\max}$ (with strict inequality for some states (d, y)). Hence, by the Tarski-Kantorovich theorem (e.g, Dugundji and Granas ([33], p.15), the greatest fixed point $c^*(C^T(S))(d, y)$ can be computed as:

$$\wedge A^n_{uc}(c^{\max}; C^T(S), \beta, \kappa, R)(d, y) \to c^*(C^T(S), \beta, \kappa, R)(d, y) > 0$$

where the strict positivity of $c^*(C^T(S), \beta, \kappa, R)(d, y) \in \mathbf{C}^p > 0$ follows from the Inada condition on $U_1(c; y^N)$ in its first argument, and we note the dependence of $c^*(C^T(S), \beta, \kappa, R)(d, y)$ on deep parameters for later reference. That proves the existence of a strictly positive greatest fixed point.

Step 4. *Fixed point comparative statics of greatest fixed point.* By standard fixed point statics argument for order continuous operators, the greatest fixed point $c^*(C^T(S), \beta, \kappa, R)(d, y)$ is increasing in $(C^T(S), \kappa)$, and decreasing in (β, R) (e.g., by a parameterized version of the Tarski-Kantorovich Theorem).

Step 5: $c^*(C^T(S), \beta, \kappa, R)(d, y)$ *the unique strictly positive fixed point.* This follows from an application of Corollary 4.1 in Li and Stachurski ([43]) for each $C^T \in \mathbf{C}^f$. To see this, for fixed $C^T \in \mathbf{C}^f$ put

$$\varsigma(d, y) = u'(A_c(C^T)(d, y)) \tag{53}$$

and restrict the first step operator $A_{uc}(c; C^T(S), \beta, \kappa, R)(d, y)$ to the set $\mathbf{C}^{p*}_{++}(\mathbf{S}; \varsigma) \subset \mathbf{C}^p_{++} = \{c \in \mathbf{C}^p | c(d, y) > 0\}$. where in our notation we make explicit the dependence of the the space $\mathbf{C}^{p*}_{++}(\mathbf{S}; \varsigma)$ on the upper bound in (53). Let c_1 and c_2 be elements of $\mathbf{C}^{p*}(\mathbf{S}; \varsigma)$. Equipped the space the $\mathbf{C}^p_{++}(\mathbf{S}; \varsigma(d, y))$ with the norm

$$\rho(c_1, c_2) = \parallel u'_\varsigma \circ c_1 - u' \circ c_2 \parallel$$

where $\parallel u' \circ c_1 - u' \circ c_2 \parallel < \infty$, where $u'(c) = U'(A(c))A_1(c^T, y^{NT})$ is strictly decreasing in c^T under Assumption 1, and give $\mathbf{C}^p_{++}(\mathbf{S})$ its relative distance structure.

Clearly, from the arguments in Step 1 of this proof, $A(c; C^T(S), \beta, \kappa, R)(d, y)$ maps $\mathbf{C}^p_{++}(\mathbf{S}; \varsigma)$ into itself. By Li and Stachurski ([43], Proposition 4.1.a), the pair ($\mathbf{C}^p_{++}(\mathbf{S}; \varsigma)$,$\rho$) is a complete metric space. As $\beta R < 1$, by Li and Stachurski ([43], Proposition 4.1.c), for each $A_{uc}(c; C^T(S), \beta, \kappa, R)(d, y)$ is a contraction of modulus $0 < \beta R < 1$ in ($\mathbf{C}^p_{++}(\mathbf{S}; \varsigma)$,$\rho$). Then, by the contraction mapping theorem, $A(c; C^T(S), \beta, \kappa, R)(d, y)$ has exactly one fixed point in ($\mathbf{C}^p_{++}(\mathbf{S}; \varsigma)$,$\rho$). So, $c^*(C^T(S), \beta, \kappa, R)(d, y)$ is the unique strictly positive fixed point of $A_{uc}(c; C^T(S), \beta, \kappa, R)(d, y)$ for $C^T \in \mathbf{C}^f$.

Finally, as $A_{uc}(c; C^T(S), \beta, \kappa, R)(d, y)$ is easily shown to be continuous in $C^T(S) \in \mathbf{C}^f$ in the topology of pointwise convergence and monotone on $\mathbf{C}^f$, by a standard application of the parameterized version of the Tarski-Kantorovich theorem, as

$$\inf_n A^n_{uc}(c_{\max}, C^T)(s^e) = c^*(C^T(S), \beta, \kappa, R)(d, y)$$

is monotone increasing in C^T on $\mathbf{C}^f$. Further, by the Bonsall-Nadler theorem for parameterized contractions, $c^*(C^T(S), \beta, \kappa, r)(d, y)$ is also continuous in the topology of pointwise convergence on $\mathbf{C}^f$ (e.g., see Nadler ([54], Theorem 1)). Hence, $c^*(C^T(S), \beta, \kappa, R)(d, y)$ is order continuous on $\mathbf{C}^f$. ■

[42] The space $\mathbf{C}^p$ is an closed equicontinuous collection of functions, hence compact set in the space of continuous function in the topology of uniform convergence. To see this, as $c(d, y)$ is decreasing (resp., increasing) in d (resp., y) such that $-d'(d, y) = R(y^T - d) - c(d, y)$ is decreasing (resp., increasing) in d (resp., in y), we have $|\mathrm{c}(\mathrm{d}', y') - c(d, y)| \leq R \; |(y' - d') - (y - d)|$ when $(d', y') \geq (d, y)$, hence $\mathbf{C}^p$ is an (uniformly) equicontinuous collection of continuous functions. Noting $y \in \mathbf{Y}$ is discrete, $\mathbf{C}^p$ is therefore a compact set in the topology of uniform convergence, and hence is chain complete. (See Amann ([3], lemma 3.1)). As $\mathbf{C}^p$ is additionally a lattice, $\mathbf{C}^p$ is a complete lattice. So the set of fixed points of $A_{uc}(c; C^T(S), \beta, R, \kappa)(d, y)$ for each (β, κ, R) given by $\Psi_A(C^T(S), \beta, R, \kappa)(d, y)$ is a nonempty complete lattice by Tarski's theorem (e.g., see Tarski ([76])). So the existence of fixed points in Step 3 is not the question. We simply want to verify in Step 3 that greatest fixed point $\vee \Psi_A(C^T(S), \beta, R, \kappa)(d, y)$ is strictly positive.

Proof of Lemma 3

Proof. For states $s = (d, y, S)$ when $d = D$ and $y = Y$, we first sharpen the characterization of the fixed point mapping $c^*(C^T)(d, y)$ for the operator $A_{uc}(c, C^T)(d, y)$ in Lemma 2, step 5 when the operator $A_{uc}(c; C^T)(d, y)$ is restricted to $C^T \in \mathbf{C}^{f*}(\mathbf{S})$. As $\mathbf{C}^{f*}$ is compact in the topology of uniform convergence, the arguments in step 1, lemma 2 imply $A(c, C^T)$ is continuous jointly in the topology of uniform convergence in $(c, C^T) \in \mathbf{C}^p_{++} \times \mathbf{C}^{f*}$. Therefore, as the parameterized contractions $A(c; C^T)$ are jointly continuous in (c, C^T), by Nadler's theorem (Nadler ([54], Theorem 1), the fixed point mapping $c^*(C^T(S))(s^e)$ is continuous on $\mathbf{C}^{f*}$. Also, note for each $C^T \in \mathbf{C}^{f*}$, $c^*(C^T)(d, y)$ is an element of a a (uniformly) equicontinuous collection of continuous functions over (d, y). [43] Further, for each (d, y),each $C^T(S) \in \mathbf{C}^{f*}$ which is also an element of an (uniformly) equicontinuous collection $\mathbf{C}^{f*}(\mathbf{S})$. In both cases the pair $(c^*(C^T)(d, y), C^T(d, y)$ satisfy the exact same bounds on the variation of (d, y). (i.e., for $c^*(C^T)(d, y)$ (resp, $C^T(D, Y) \in \mathbf{C}^{f*}$, we have $c^*(C^T)(d, y)$ (resp., $C^T(S)$) is decreasing in d (resp., increasing y) such that $|\mathrm{c}(\mathrm{d}', y') - c(d, y)| \leq R\ |(y' - d') - (y - d)|$ when $(d', y') \geq (d, y)$ (resp., $c(d', y') - c(d, y) \leq R\ |(y' - d') - (y - d)|$ when $(d', y') \geq (d, y)$ where $c^*(C^T(d, y))(S)$ must satisfy the budget constraint in equilibrium, as does $C_c(d, Y, C^T(d, y) \in \mathbf{C}^{f*}$ by construction. As $c^*(C^T(d, y))(d, y)$ is also continuous in $s^e = (d, y, d, y)$, by Berge's theorem, $C(s^e) - \inf\{c^*(d, y, C^T(d, y), C_c(d, Y, C^T(d, y))$ is continuous and also satisfies $|\mathrm{C}(\mathrm{d}', y') - C(d, y) \leq R\ |(y' - d') - (y - d)|$. Hence, the space $\mathbf{C}^*$ forms a collection of (uniformly) equicontinuous elements over $\mathbf{S}^e$. As $\mathbf{C}^*(\mathbf{S}^e)$ is also closed, $\mathbf{C}^*(\mathbf{S}^e)$ is compact (and hence, chain complete). As $\mathbf{C}^*(\mathbf{S}^e)$ is also a lattice, $\mathbf{C}^*$ is a complete lattice. ■

Proof of Lemma 4

Proof. By Lemma 2, the mapping $c^*(C^T(S), \beta, \kappa, R)(d, y)$ is order continuous on $\mathbf{C}^f$. It is therefore order continuous on the restriction of $c^*(C^T(S), \beta, \kappa, R)(d, y)$ to $\mathbf{C}^{f*}(\mathbf{S})$. As $\mathbf{C}^{f*}$ is compact in the topology of uniform convergence, $A_{uc}(c; C^T)(d, y)$ is actually continuous in the topology of uniform convergence. Hence, by a stronger version Bonsall-Nadler theorem on parameterized contractions, $c^*(C^T(S), \beta, \kappa, r)(d, y)$. (e.g., see Nadler ([54], Theorem 2 and Lemma, p. 581)), the mapping $c^*(C^T(S), \beta, \kappa, R)(d, y)$ restricted to $\mathbf{C}^{f*}(\mathbf{S})$ is continuous in the topology of uniform convergence. As $A_c(C^T)(s^e)\}$ is also continuous in the topology of uniform convergence restricted to the compact domain $\mathbf{C}^{f*}$, mapping $A^*(C; \beta, \kappa, R)(s^e))$ in (34) where $C = \inf\{c^*(C^T(s^e), A_c(C^T)(s^e) \in \mathbf{C}^*(\mathbf{S}^e)$ for $C^T \in \mathbf{C}^{f*}(\mathbf{S})$ continuous in the topology of uniform convergence on $\mathbf{C}^*(\mathbf{S}^e)$ by Berge's theorem (and hence, continuous in the topology of pointwise convergence), so $A^*(C; \beta, \kappa, R)(s^e))$ is order continuous on $\mathbf{C}^*$. ■

Proof of Theorem 5.

Proof. Let $\Psi \subset \mathbf{C}^*$ be the set of fixed points of the mapping $A^*(C)(s^e))$ defined in (34). That $A^*(C)(s^e)) \in \mathbf{C}^*$ is immediate (as by construction, or fixed S, when $d = D$, $y = Y$, $c^*(C^T(S); \beta, R, \kappa)(d, y)$ $\in \mathbf{C}^p$, and (b) when (d, y) is fixed, $\inf\{c^*(C^T(S); \beta, R, \kappa)(d, y),\ A_c(C^T)(S))\} \in \mathbf{C}^*$ where $C^T \in \mathbf{C}^{f*}$ (as if s^e is a collateral constrained state, then in addtion to $A_c(C^T)(d, y$ being increasing in y, decreasing in d, we also have $d(s^e) = \kappa\{y^T + p(C^T(d, y))y^N)$ is increasing in y, and decreasing in d. Further, as $c^*(C^T)(d, y)$ in lemma 2. step 5 and $A_c(C^T)(s) = (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(d, y))y^{NT}$ in equation (30) are both order continuous (hence monotone increasing) as they are both continuous in the topology of pointwise convergence and monotone, the operator $A^*(C)(s^e)) = \inf\{c^*(C^T)(d, y),\ A_c(C^T)(d, y)\}$ is is also continuous in the topology of pointwise convergence (via Berge's theorem) and hence order continuous on $\mathbf{C}^*$.

Then as $A^*(C)(s^e))$ is monotone, and $\mathbf{C}^*$ is a nonempty complete lattice, by Tarski's theorem (Tarski ([76], theorem 1), $\Psi^*$is a nonempty complete lattice.

Further, as $A^*(C)(s^e))$ is such that when $C^T = 0$, $0 \leq A(0)$, and $0 < A(0)$ for states (d, y) such that $A_c(0) = (1 + \frac{\kappa}{R})y^T - d + \frac{\kappa}{R}p(0)y^N > 0$, we now iterate on $A^n(0)$. As $A_c(A^n(0))$ is an increasing chain the states (d, y) such that $A_c(0) = (1 + \frac{\kappa}{R})y^T - d + \frac{\kappa}{R}p(A^n(0))y^N > 0$ increasing, we have $0 \leq 0 \leq A^n(0)$

[43] See the proof in the footnote of Step 3 of this proof.

, and $0 < A^n(0)$ on increasing subsets of states $\mathbf{S}_+^{ne} \subset \mathbf{D} \times \mathbf{Y}$ of equilibrium states $(d, y) \in \mathbf{D} \times \mathbf{Y}$. Let $C_m = \sup_n A^n(0)$ where $C_m(s^e) > 0$ in the set $\mathbf{S}^{e*} = \sup_n \mathbf{S}_+^{ne}$ (where the sup is taken related to set inclusion). As $A^*(C)$ is order continuous, $\sup_n A^*(0) = C_m = \wedge\Psi$ with minimal state space $\mathbf{S}^{e*}$ by the Tarski-Kantorovich theorem. Further, $\inf_n A^{*n}(c_{\max}) = \vee\Psi$ also by the Tarski-Kantorovich theorem, where we have

$$\sup_n A^*(0) = \wedge\Psi \leq \vee\Psi = \inf_n A^n(c_{\max})$$

for the the minimal (resp., maximal) fixed points $\wedge\Psi$ (resp., $\vee\Psi$). ■

Proof of Theorem 6

Proof. (i) Let $\Psi^*(R, \kappa, \beta) \subset \mathbf{C}^*$ be the set of fixed points of the mapping $A^*(C; \beta, R, \kappa)(s^e)) = \inf\{c^*(C^T(S); \beta, R, \kappa)(d, y), A_c(C^T)(s^e)\}$ defined in (34). as $A^*(C, \beta, R, \kappa)(s^e)) \in \mathbf{C}^*$. As by Remark 11, $A_{uc}(c, C^T)(\beta, \kappa, R)(d, y)$ is decreasing in (β, R) and increasing in κ, $A^*(C, \beta, R, \kappa)(s^e))$ is decreasing in (β, R), and increasing in κ (as $A_c(C^T)(\kappa, R)$ is decreasing in R and increasing in κ the inf operation preserves the relevant comparative statics). So by Veinott's comparative statics version of Tarski's theorem (see Veinott ([78]), also see Topkis ([77], Theorem 2.5.2), the least and greatest selections of $\Psi^*(R, \kappa, \beta)$ exist as fixed points, and are decreasing in (β, R), and increasing in κ.

(ii) As $A^*(C, \beta, R, \kappa)(s^e)) = \inf\{c^*(C^T(S); \beta, R, \kappa)(d, y), A_c(C^T)(s^e)\}$ is order continuous under pointwise partial orders on $\mathbf{C}^*$,

$$\sup_n A^{*n}(0; \beta, \kappa, R)(s^e) \to C^*_\wedge(\beta, \kappa, R)(s^e)$$
$$\inf_n A^{*n}(c_{\max}; \beta, \kappa, R)(s^e) \to C^*_\vee(\beta, \kappa, R)(s^e)$$

the result follows from the Tarski-Kantorovich theorem (e.g., Dugundji and Granas ([33], p.15) with $C^*_\wedge(\beta, \kappa, R)(s^e)$ (resp, $C^*_\vee(\beta, \kappa, R)(s^e)$) decreasing in (β, R) and increasing in κ. ■

Proof of Theorem 8

Proof. Noting the order continuity of the RCE operator, the result follows directly from an application of the main result in Balbus, et. al. ([8], Proposition 3). ■

Proof of Theorem 9

Proof. With the addition of Assumption 2, the we can modify the proof of our main existence result in Theorem 5 into a single-step operator and then construct the actual RCE in a single step via Lemma 2.

In particular, in equation 24, recalling the definition of the space $\mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma) \subset \mathbf{C}^p_{++} = \{c \in \mathbf{C}^p | c(d, y) > 0\}$ in the proof of Lemma 2, for $c \in \mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma)$, define

$$C(c, C^T)(d, y, S) = C(c, c_c^{T*}(d, y)) = \inf\{c(d, y), x^*(s)\}$$

where $x^*(s)$ is the unique solution in equation (44) (i.e., the tradables consumption over collateral constrained states is unique determined by equation 44 under Assumption 2).

Then in the definition of our fixed point operator $A(c)(s^e)$ in equation 31 with $A_c(C^T)(S) = x^*(s)$ so our fixed point operator $A(c)(s^e)$ under Assumption 3 now simplifies to the following

$$\begin{aligned} A(c)(s^e) &= \inf\{A_{uc}(c)(s^e), x^*(s^e)\} \text{ when } c > 0 \\ &= 0 \text{ else} \end{aligned} \tag{54}$$

where in the constrained states, the RCE tradables consumption is $c_c^{T*}(d, y)$ and unique.

The resulting operator $A(c)(s^e)$ is a monotone contraction and has a unique strictly positive fixed point $c^*(s^e)$ via the application of the Li-Stachurski version of the contraction mapping theorem in the complete metric space $c \in \mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma)$, as discussed in step 5 in Lemma 2. Hence, for any initial $c \in \mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma)$,

$$A^{*n}_{uc}(c)(d, y) = c^*(d, y)$$

where the unique RCE for tradables consumption is given by:

$$C^*(d, y) = \inf\{c^*(d, y), x^*(d, y)\}$$

■