Accuracy in Recursive Minimal State Space Methods *

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Abstract

The existence of a recursive minimal state space (MSS) representation is not always guaranteed. This fact has numerical implications: a constructive existence proof generates a convergent algorithm which maybe different from the standard practice. What are the consequences of computing and simulating a model without this type of proofs? To answer this question, we identify a condition which is associated with a convergent and computable MSS representation in a RBC model with taxes. This condition ensures the existence of a benchmark equilibrium that can be used to test frequently used algorithms. To verify the accuracy of simulations even if this condition does not hold, we derive a closed form recursive equilibrium which contains the MSS representation. Both benchmark representations are accurate and ergodic. We show that state of the art algorithms, even if they are numerically convergent, may underestimate the benefits of capital taxes and overestimate their cost by at least 65%, a figure which is in line with recent findings using accurate benchmarks. If the existence of a MSS equilibrium cannot be proved, we found 2 sources of inaccuracy: the lack of a convergent operator and the absence of a well-defined (stochastic) steady state. Moreover, we identify a connection between the lack of convergence in the MSS algorithm and the equilibrium budget constraint which implies that simulated paths may be distorted not only in the long run but also in any time period. When we have a constructive proof, inaccuracy is generated by the lack of qualitative properties in the computed policy functions; a fact which precludes sup-norm convergence.

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1 Introduction

Sometimes macroeconomics is about new answers to old questions. This is the takeaway point of [Nakamura and Steinsson, 2018]. The authors analysis is based on the lack of exogenous variability in the field. This is one of the reasons behind the lack of a definitive answers to old questions, but maybe not the only one. Even if we were able to find an exogenous policy shock, in case we would like to perform a structural analysis, models may not have closed form solutions. Thus, numerical methods are a fundamental ingredient in macroeconomics. Moreover, policy experiments frequently imply that the welfare theorems do not hold. In this framework, even in the simplest possible dynamic stochastic model, a canonical RBC with taxes on capital, the existence of a computable representation is not always guaranteed (see for instance [Santos, 2002]). This fact has not only theoretical but also practical implications: along with an existence proof, if it is *constructive*, it comes a convergent algorithm.

What are the consequences of computing and simulating a model without a constructive existence proof? In order to answer this question, we first identify a condition which is associated with a computable representation in a RBC model with taxes. This condition, when it holds, ensures the existence of a benchmark equilibrium that can be used to test frequently used algorithms. To verify the accuracy of simulations even if this condition does not hold, we derive a closed form recursive equilibrium. Both benchmark representations are compact, continuous and unique. When the benchmark is numerical, the convergence is uniform, and iterations start in a theoretically identified initial condition. Thus, we know that both benchmark representations are accurate and ergodic. This type of benchmarks are rather infrequent when the welfare theorems does not hold, a fact that give us a unique opportunity to test frequently used methods.

We show that state of the art algorithms, even if they are numerically convergent, may underestimate the benefits of capital taxes, as measured by a change in the composition of aggregate demand in favor of consumption, and overestimate their cost, as determined by an increase in the variability of consumption. Our estimation for these biases is above 65%, a figure which is in line with recent finding using accurate benchmarks (see [Pohl et al., 2018]). We find that simulations using standard algorithms underestimate the long run average and overestimate a dispersion measure of consumption. Our goal is not to criticize these methods as it is not always easy to measure the trade-offs (between numerical efficiency and accuracy) faced by researchers, but to suggest that, if the identified condition holds, it is better to reduce the speed of computations in order to increase the precision of the numerical solution.

Since the seminal paper of [Lucas, 1978] macroeconomists have been used a recursive representation of sequential equilibria to solve and simulate models. There are numerical and theoretical reasons behind this choice. As regards the former, it is easier to numerically approximate a first order stationary dynamic process rather than the sequential representation originally defined. In reference to the latter, a markovian structure allows to define a well-behaved long-term equilibrium (i.e. a steady state) using a recursive equilibrium notion (see for instance [Blume, 1982]). Finally, and more importantly, the theoretical and computational arguments are related with each other since accurate numerical simulations requires a Markovian representation and an appropriate steady state (see for instance [Santos and Peralta-Alva, 2005] among others).

This paper has 2 contributions. First, we identify a critical condition, based on some qualitative properties of the expected marginal utility of consumption, that insure the accurate performance of minimal state space methods in recursive macroeconomics. Our results hold in a RBC model with capital taxes, but the identified condition can be found in a large fraction of applied papers. Second, we measure the bias of solutions using an accurate benchmark. When the critical condition holds (i), we compute a convergent operator which has a continuous and unique fixed point; a fact which insures the ergodicity of simulations. When this condition does not hold (ii), to keep the benchmark accurate, we derive a closed form continuous recursive equilibrium with an expanded state space. More precisely, we obtain a recursive representation with an ergodic invariant measure, a finite number of exogenous shocks and a well-behaved state space (i.e. compact). For case (i), the bias is generated by the lack of qualitative properties of the solutions obtained from state of the art algorithms. A recursive equilibrium with minimal state space exists and it can be computed, but the algorithm cannot find it. For case (ii), we found 2 sources of inaccuracy: the absence of a minimal state space recursive representation and the in-existence of a well-defined (stochastic) steady state.

When it comes to compute recursive equilibrium models, the curse of dimensionality calls for minimal state space (MSS) methods. However, [Kubler and Schmedders, 2002] argued that in the presence of multiple equilibria a MSS recursive representation may not exist. As uniqueness has been an elusive quest in this field[1], this fact justifies the necessity of a condition associated with the existence of a MSS recursive equilibrium. It turns out that when the expected marginal return of savings is monotonic in the endogenous state, we can prove the existence of a MSS representation constructively. If this requirement is not satisfied, to obtain an ergodic markov equilibrium, we must increase the number of endogenous variables which are considered states. By enlarging the state space, we show that it is possible to obtain multiple markovian representations in closed form, one of them continuous with a stationary state space. This last “selection” allows us to derive a well-defined steady state by applying standard results.

We test the accuracy of MSS methods using an accurate benchmark for non-optimal economies. If the welfare theorem holds, we know that standard methods deliver accurate approximations (see for instance [Santos and Peralta-Alva, 2005]) that can be used as a benchmark to test faster methods (see [Arellano et al., 2016]). This is not the case in non-optimal economies as the existence of MSS representations can't always be shown

[1] [Dana, 1993] provided conditions to guarantee the uniqueness of equilibria in an infinite horizon economy with complete markets. There is no analogous result for incomplete markets

constructively. This paper test frequently used methods even if a MSS representation doesn't exist. To serve this purpose, we derive a closed form generalized markovian equilibrium (GME) for a standard version of the RBC model with decreasing taxes on capital presented in [Santos, 2002]. As all MSS recursive equilibria form a subset of all GME, if both equilibrium types are well defined in the long run, any simulation from the latter must be matched using the former. It is shown that even a numerically convergent MSS algorithm may not match the ergodic distribution of the model as the MSS equilibrium might not have a well-defined steady state or may not exist. The bias not only affects long run simulations. We identify a connection between the lack of existence of a MSS recursive equilibrium and the budget constraint which implies that simulated paths are distorted in any time period.

As regards the long run of the model, the monotonic properties of the operator plays a critical role as it ensures the continuity of the MSS recursive equilibrium. When this condition does not hold, we show that it is possible that numerical simulations converge to a fixed point simply because the grid is not sufficiently thin. The steady state of the model may not be well defined as the MSS equilibrium has discontinuity points. In this sense, the results in this paper has direct take away point: as the accuracy of simulations depends on the continuity of the solution, even if the numerical procedure has been declared convergent using a demanding criteria, simulations may be far away from the exact steady state. Numerical procedures assume that the model "lives" in a space with no holes. That's why successive approximations may converge despite the fact the object we are trying to compute doesn't exist. In this sense, the results in [Coleman, 1991], which can be used under the monotonicity requirement, insure the convergence of successive approximations to a continuous MSS recursive equilibrium using the SUP norm. Other, faster, methods when applied to the same model may generate significant biases because they do not preserve some qualitative properties of policy functions, associated with sup-norm convergence.

The paper is organized as follows: section 2 presents an overview of the main results using a non-stochastic simple economy. Section 3 deals with the theory necessary to obtain an accurate benchmark when the Euler equation is not monotonic: presents the canonical model together with the closed form recursive equilibrium and discusses its implications. Section 4 presents the numerical test for this class of models. Section 5 presents the theory and perform the test when the critical condition holds. Section 6 concludes.

1.1 Relation with the literature

Most macro model includes an Euler equation. In particular, the expected marginal utility of consumption satisfies:

$$\beta E_z \left[u'(c_+(K)(1-\tau(K))R(K) \right] \tag{1}$$

Where c_+ denotes consumption "tomorrow", z is an exogenous shock, τ is a tax rate on assets K and R is the gross rate of return. In a model of with production (as in [Santos, 2002]), K denotes capital and R its marginal product. In a small open economy model, R is exogenous, τ is a tax on debt (we must reinterpret K in this case) and represent a macro-prudential policy (see [Bianchi, 2011]). In the default literature, $\tau = 0$ and R depends on the probability of default, which in turn is a function of debt (again, reinterpreting K, see [Arellano, 2008]).

Following [Mirman et al., 2008] and [Coleman, 1991], in this paper we show that if equation (1) is *monotonic* in K it is possible to derive an accurate algorithm that converges to a well-defined recursive equilibrium. For instance, in the small open economy literature, as τ is increasing in K but consumption is typically decreasing, equation (1) is not monotonic. In the default literature, R is increasing in K and thus (1) is increasing, so the results in [Coleman, 1991] insure the existence of a computable and numerically efficient algorithm. In the RBC literature, if τ is decreasing in K, (1) is not monotonic. The purpose of this paper is to measure the accuracy of state-of-the-art algorithms when the monotonicity of (1) is and is not satisfied.

The literature (see for instance [Hatchondo, 2010]) focuses on the sensitivity of the numerical results to different methods without an accurate benchmark or test methods in optimal economies (see [Arellano et al., 2016]), but we were unaware of the size and reasons behind the bias in MSS methods in non-optimal economies. Contrarily to what is done in the numerical literature, we can measure this bias using an accurate solution which also has a well-behaved steady state. Thus, we can measure the short and long run implications of the lack of a constructive existence proof. [Santos and Peralta-Alva, 2005] performed a similar exercise for optimal economies. We extend those results for models with distortions.

From a theoretical point of view, we sharpen the characterization of ergodic recursive equilibrium in [Blume, 1982]. We provide a counterexample for the equivalence between a continuous markovian representation and the uniqueness of the sequential equilibrium. In words of [Blume, 1982]:

"the existence of a continuous selection - tantamount to the uniqueness of equilibrium in each state - is not often satisfied".

We found a stationary (i.e. time independent) recursive representation with multiple equilibrium in some nodes which has a continuous selection. This result is relevant to relax recently found conditions to insure the existence of an ergodic steady state. These conditions are at odds with the computation of the model as they involve many continuations for each node (see [Santos et al., 2012]). The existence of a continuous selection in a model with a finite number of shocks is essential to insure the ergodicity of simulations in a computable framework.

From a numerical perspective, this paper is connected with [Pohl et al., 2018]. We also use accurate benchmarks and find a similar bias with respect to state-of-the-art algorithms. While the results in [Pohl et al., 2018] depend on the presence of a persistent stochastic process, ours can be explained for topological reasons: the lack of continuity and the absence of an order structure in the space of policy functions.

2 Preview of the results in a deterministic economy

In this section we present a simple representation of the results found in this paper. For this purpose, we use a non-stochastic RBC model which is canonical except in the tax function. We divide this section in 2, based on equation (1): first we deal with the fact that the Euler equation is not monotonic in the endogenous state. In this case, to test the accuracy of simulations, we derive a closed form recursive equilibrium, which is more general than the MSS representation. In the second case, where the Euler equation is monotonic, we can use a result due to [Coleman, 1991] in order to prove the existence of a MSS equilibrium. We use the former to test a canonical procedure based on iterations that exploit a map between perceived and actual laws of motion for equilibrium states. The latter is used to test the performance of envelope conditions methods (ECM), as we can ensure that the derivative of the value function is continuous provided that the monotonicity requirement holds. In both cases we can prove that the benchmark equilibrium exist, is unique and is ergodic. Thus, they constitute a proper instrument to test the accuracy of simulations.

2.1 Non-Monotonic Euler Equations

Imagine a canonical RBC model distorted by ad valorem taxes. As a distinctive fact, the aliquot can vary along with the business cycle. It will be assumed that it is *decreasing* in the aggregate state of the economy. There is an infinitely lived representative agent endowed with k_0 units of capital. She must choose a sequence of consumption and savings for each unit of time, denoted $t \geqslant 0$, to maximize her lifetime utility. For simplicity, we will assume for now that there is no uncertainty, and that capital depreciates entirely after 1 period. Accumulated saving is rented to a firm, which is assumed to maximize profits using a decreasing returns to scale technology represented by a strongly concave production function. There is a Government that levies an ad-valorem tax on rental income. As mentioned, the aliquot depends on the aggregate state of the economy, denoted K, even though this connection is not perceived by the agent. The Government rebates back the collected taxes making lump-sum transfers to the agent. Finally, as the agent owns the capital stock, she receives the profits form the firm. Formally:

Time is discrete and infinite, $t = 0, 1, 2....$

Let k denote the supply of capital (services) and K its demand. There is a decreasing return to scale firm which only uses capital as input and its technology is characterized by $y_t = f(K_t)$ with $f' > 0$, $f'' < 0$ and $f(0) = 0$ as usual.

As the firm is owned by the consumer as she is endowed with $k_0 > 0$ units of capital and has two sources of current income: benefits, denoted by π_t, and rents from capital, denoted by $r_t k_t$. The flow of taxes paid and transfers received is $\tau(K_t) r_t k_t$ and T_t respectively.

The problem faced by the consumer is to choose consumption c_t and investment x_t that solves the following problem:

$$\max_{\{c_t, x_t\}} \sum_t \beta^t u(c(z^t)) \tag{2}$$

s.t.

$$k_{t+1} = x_t + (1-\delta)k_t \tag{3}$$
$$c_t + x_t = \pi_t - (1 - \tau(K_t) r(z^t) k_t + T(k_t) \tag{4}$$

$c_t \geqslant 0$ and $k_0 > 0$ given, $\delta \in [0, 1]$ is the depreciation rate and $\beta \in (0, 1)$ the discount factor.

The problem of the firm is standard. Taking r_t as given it solves:

$$\max_{K_t} f(K_t) - r_t K_t. \tag{5}$$

The Government simply transfers to the consumer the tax revenues:

$$T = \tau(K_t) r(z^t) k_t. \tag{6}$$

Thus, the flow budget constraint of the agent is:

$$c_t + x_t = \pi_t(K_t) + (1 - \tau(K_t)) r(K_t) k_t + T_t(K_t)$$

Where τ is the aliquot for the ad-valorem tax, r is the rental rate, π denotes profits, x_t represents investment and, due to full depreciation, k_{t+1} and T_t are transfers. The Government runs a balanced budget, $\tau(K_t) r(K_t) k_t = T_t$, and profit maximization implies $F(K_t) = \pi_t(K_t) + r(K_t) K_t$ where $r(K_t) = F'(K_t)$ and $F(K_t)$ denotes aggregate resources [2]. As there is a single firm, the only price in the economy, r, is a function of aggregate capital, K. Moreover, tax collection can explicitly depend on the aggregate state of the economy in order to capture the interaction between the business cycle and fiscal policy.

Replacing the equilibrium conditions in the flow budget constraint, we get:

[2] When $\delta = 1$ aggregate resources, F, and aggregate output, f, are equal.

$$c_t + x_t = F(K_t) + (k_t - K_t)F'(K_t)$$

The above equation is the aggregate budget constraint. Note that gross rental income is proportional to individual capital holdings, k. In equilibrium, it will be required that $k = K$. Thus, if we can ensure that individual and aggregate capital stocks remains closed to each other along the computed equilibrium trajectories, we will say that the decentralized equilibrium is not distorted. Now suppose we want to solve the model and simulate this economy. The canonical approach since [Rios Rull, 2004] is to use the associated dynamic programming program and the policy functions derived from it. In this framework, the agent is supposed to solve:

$$V(k, K) = Max_{c,x}\ u(c) + \beta V(k', K')$$

Subject to

$$x, c \in [0, \pi_t(K_t) + (1 - \tau(K_t))r(K_t)k_t + T_t(K_t)]$$

$$x + c = \pi_t(K_t) + (1 - \tau(K_t))r(K_t)k_t + T_t(K_t)$$

$$K' = G(K)$$

Where G is the *perceived* law of motion for aggregate capital. A *minimal state space recursive equilibrium* (MSSRE) in this economy is a pair of policy functions $c(k, K), x(k, K)$ such that:

$$c(k, K) + x(k, K) = F(K)$$

$$\tau(K)r(K)k = T$$

$$x(k, K) = G(K)$$

$$k = K$$

$$r(K) = F'(K)$$

We call $x(k, K) = G(K)$ the *rational expectations condition.* Now suppose we want to simulate the economy, given $k_0 = K_0$. We can use the set of policy functions iteratively. In order to take care of rational expectations condition, note that the Bellman equation above define a mapping $T(G_n)(k, K) \longmapsto G_{n+1}(k, K)$, where $T(G_n)(k, K) = x(k, K; G_n)$. In a recursive equilibrium the perceived law of motion G_* satisfies $T(G_*)(K, K) = x(K, K; G_*) = G_*(K)$.

Since [Rios Rull, 2004] it is frequent to iterate on T, starting from an arbitrary initial condition, obtaining a sequence $\{G_n\}_n$. Will G_n converge to G_*? Equivalently, is there a numerically implementable operator that converge to a recursive equilibrium? Since [Mirman et al., 2008], we know that if $(1-\tau(K))F'(K)$ is decreasing (in K), we can provide a positive answer to this question. Unfortunately, there are some cases where this condition does not hold for any K. In this section, as in [Santos, 2002], we assume that τ is decreasing in K, which in turn implies that $(1-\tau(K))F'(K)$ is not monotonic as F is strictly concave.

More to the point, if we iterate on T, the limiting function G_∞ satisfies:

$$c^{'} + x^{'} = F(K') + (x(K, K; G_\infty) - K')F'(K')$$

Where $K' = G_\infty(K)$. Now, if the numerical procedure does not converge, we know that the perceived G and the actual x law of motion for capital will not be equal, at least for some K. That is, $T(G_\infty)(K, K) = x(K, K; G_\infty) \neq G_\infty(K)$. Thus, *the lack of convergence implies directly a bias in the computed long term capital stock* as the resources available to the household are permanently distorted by the numerical procedure.

Note that the convergence criteria in any numerical procedure is relative. That is, the algorithm will be "declared convergent" if for $n \geqslant N(\epsilon)$:

$$SUP_K \left| \frac{x(K,K;G_n) - G_n(K)}{G_n(K)} \right| < \epsilon$$

Where ϵ is the tolerance level. Thus, it is possible that $x(K, K; G_n) - G_n(K)$ may be far away from zero even though the numerical procedure has "converged".

So far, we have discussed the implications of the lack of convergence on equilibrium decisions. What can we say about simulations? The first step is to define a proper steady state, as simulated paths must converge to a meaningful object (i.e. an unconditional moment of a stationary distribution). Since [Futia, 1982] we know that compactness and continuity are sufficient to ensure the existence of a well-behaved steady state (see theorem A.1 in the appendix). Assume that K belongs to a compact set. Since [Stokey, 1989], see chapter 5.1, we know that there are curvature conditions associated with F which ensure the desired compactness, so the assumptions seem mild. However, in non-optimal economies, the continuity of the equilibrium equations remains an open question. For instance, [Coleman, 1991] showed that if $(1-\tau(K))F'(K)$ is decreasing (in K), there is a continuous recursive equilibria. However, in this section, as the net rental income is not monotonic, we cannot use this result. We deal with the numerical implications of the results in [Coleman, 1991] in the next subsection.

Let $g_\tau(k, K) \equiv \pi_t(K_t) + (1-\tau(K_t))r(K_t)k_t + T_t(K_t)$. If $u(g_\tau(k, K) - x)$ is strictly concave (in k, x) and the feasibility correspondence for the recursive problem is convex, we know from [Stokey, 1989], see section A.3 in the appendix, that $V(k, K)$ is strictly concave (in k). Unfortunately, in the present framework, we cannot insure the desired properties and thus the value function may not be concave (see section A.3 for a discussion for the stochastic case). Thus, we need to use more general results. From [Rockafellar, 1981] and [Amir et al., 1991] we know that V has a well-defined directional (left) derivative (see section A.2 in the appendix). As, V is not concave, the standard envelope theorem does not hold even if the return function is differentiable (see section A.3 in the appendix). To see why, note that the differentiability of V would have implied that:

$$V'(k, K) = u'(g_\tau(k, K) - x(k, K))(1 - \tau(K))F'(K)$$

At $k = K$, the strict concavity of V implies that $u'(g_\tau(K) - x(K))(1 - \tau(K))F'(K)$ must be decreasing in K, a fact that requires the monotonicity of $(1-\tau(K))F'(K)$, which does not hold by assumption. Thus, any optimal solution must satisfy:

$$u(g_\tau(k, K) - x(k, K)) = \beta V_1^-(k, K)$$

Where V_1^- is the (left) directional derivative with respect to k. As the left hand side is not continuous (as V is not differentiable), the discontinuity is transferred to the left hand side and thus to $x(K, K)$. Since a well-defined steady state requires continuity, the simulated paths may not be convergent.

In order to test the (numerical) implications of the lack of a convergent operator and / or the discontinuity of the equilibrium laws of motion this paper shows the existence of a continuous and closed form recursive equilibrium in an enlarged state space. We call this equilibrium notion *Generalized Markov Equilibrium* (GME). The qualitative properties of this type of equilibrium allow us to test the size of the bias as any MSSRE must satisfy the requirements of our definition. To ensure stationarity and compactness, we build a modified version of canonical result due to [Duffie et al., 1994] (see section A.5 in the appendix).

Let K, K_+, K_{++} be the capital stock today, tomorrow and the day after tomorrow, respectively. Then, the first order condition associated with the sequential equilibrium for this economy, for interior solutions, is:

$$u(g_\tau(K) - K_+) = \beta u'(g_\tau(K_+) - K_{++})(1 - \tau(K_+))F'(K_+)$$

One of the main contributions of this paper is to find a function, $H(K, K_+) = K_{++}$ continuous, unique and with closed form which satisfy the above equation in an equilibrium path (i.e. when the transfers are budget feasible and the goods market clear). In order to test the implications of our findings on the MSSRE, we can use a result in [Amir et al., 1991]. The authors showed that even if the net rental income is not monotonic, any solution to the dynamic programming program associated with a MSSRE must satisfy:

$$u(g_\tau(K) - x(K)) = \beta u'(g_\tau(x(K)) - x(x(K)))(1 - \tau(x(K)))F'(x(K))$$

Where the right-hand side of the above equation maybe discontinuous. Thus, any MSSRE is a GME as it is restricting K_+ to satify $K_+ = x(K)$.

Suppose that we heuristically find a convergent sequence of functions $\{G_n\}_n$ which is also a MSSRE. In the numerical section below, we provide an example of this type of functions. That is, we avoid the problems associated with $x(K, K; G_n) - G_n(K)$. However, we found that the computed MSSRE converges to a steady state quite far away from the "true" equilibria. The pictures below illustrate the situation at hand: as K_+ is not pin down by any stationary function (i.e. x in the MSSRE), the demarcation lines in the plane (K, K_{++}) are pushed towards the boundary of the system during the whole transition. Of course, this is not the case for the MSSRE.

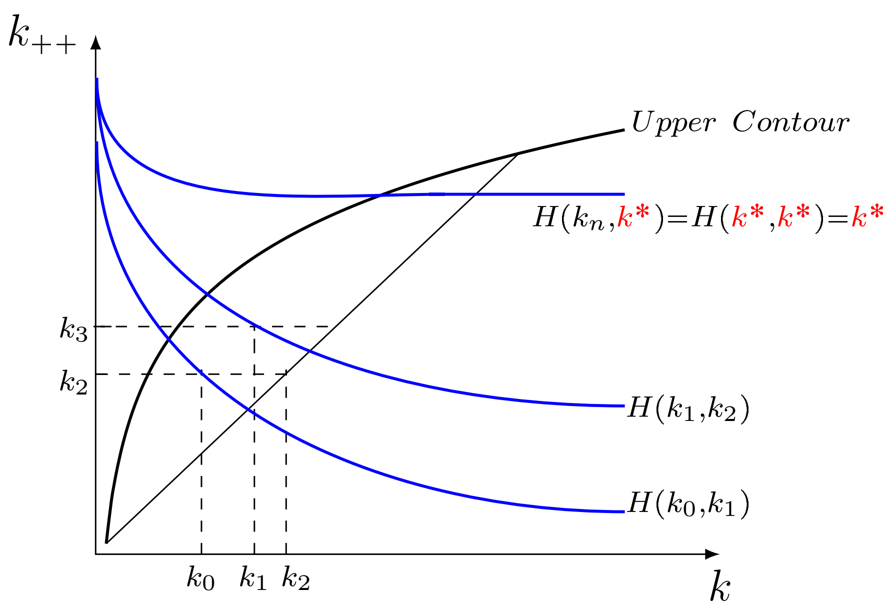

Figure 1: Dynamic Behavior in a Generalized Markov Equilibria (GME)

Figure 1 is borrowed from the numerical section of this papers. It depicts the demarcation lines for K, K_{++} given K_+, which are downward sloping and increasing in K_+. Also, the "upper contour" line reflects the maximal level of K_{++} for a given K, where the boundary reflects the zero consumption pairs. Note that for an arbitrary large n, K_n orbits near the intersection of the 45 ray with upper contour line as the demarcation curves becomes "sufficiently flat" to revert the monotonic dynamic of the capital stock.

Figure 2 illustrates a (numerically) convergent G_*. We know from previous paragraphs that this function will not be continuous, maybe near its intersection with the 45 line.

Moreover, any convergent and continuous MSS with $K_n = K_{MSSRE}, n \geqslant N_\epsilon$, will also satisfy $K_n = H(K_n, K_n)$. This last fact is not depicted in Figure 2 for expositional purposes. We found 2 selections for the closed form GME. However, the state space for one of them is not stationary. Thus, as we only have 1 time independent GME which does not display the same long run behavior of the MSSRE, figure 2 does not represent an equilibrium for the model presented in this paper.

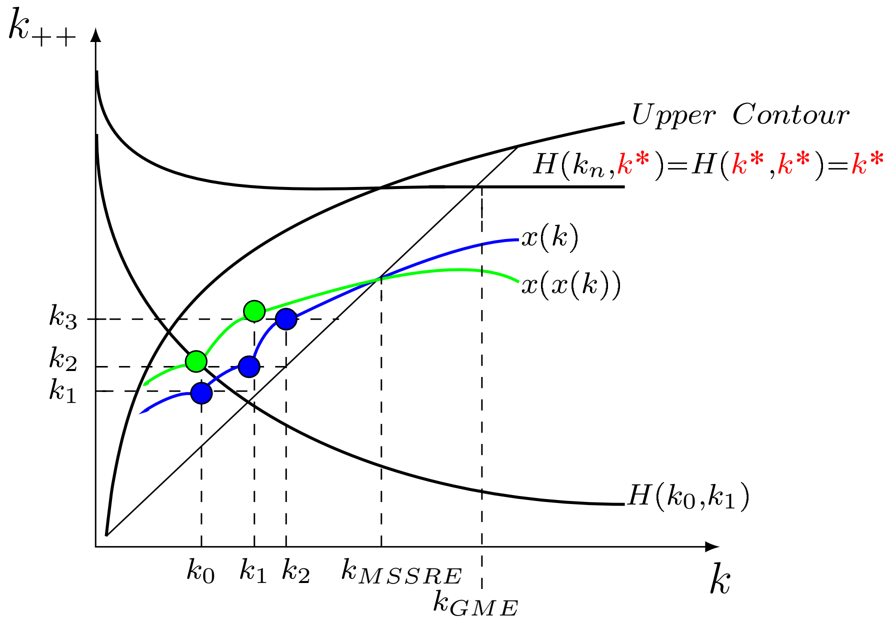

Figure 2: Minimal State Space Markov Equilibria (MSSRE) and GME

Figure 2 shows the pairs $(k, x(k))$ (in blue) and $(k, x(x(k)))$ (in green) which satisfy the equivalence between the 2 equilibrium types. As we are computing the MSSRE in a finite grid, denoted $\{K_j\}$, we choose to plot points, which are interpolated for expositional purposes. Note that eventually, we can find a pair of elements in the grid which satisfy:

$$K_{n+3} = K_{n+2} = Argmax \{W(K_{n+1}, K_{n+1})(K_j)\}_{K_j} = Argmax \{W(K_{n+2}, K_{n+2})(K_j)\}_{K_j}$$

Where W is the objective function of the Bellman equation in the MSS problem. The expression above is the numerical equivalent to $x(x(K_*)) = K_*$. Note that, as we are dealing with a finite set of points, the continuity requirement is trivial as we can endow the function with the discrete topology. Thus, convergence is achieved numerically even if the function is not continuous. Figure 3 depicts a discontinuous mechanism which will be declared convergent by any iterative procedure based on a finite set of (interpolated)

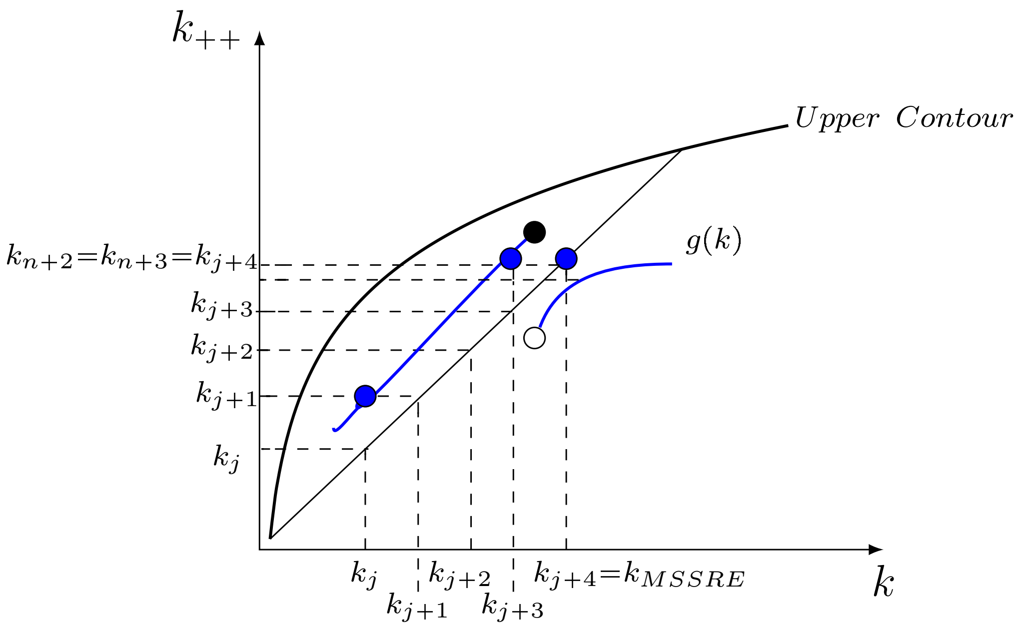

Figure 3: Dynamic Behavior in a Discontinuous MSSRE over an evenly spaced grid $\{K_j\}$

points. This figure illustrates one of the main findings of the paper: a numerically convergent MSSRE which does not have a steady state and a significant bias with respect to the ergodic GME.

The blue dots are the pairs $(K, x(K, K))$. Note that we are plotting an evenly spaced grid $(k_j, ..., k_{j+4})$ and a possible discontinuity point of the MSSRE (in black dots). The actual image of k_{j+4} does not belong to the grid. Moreover, $Argmax\ W(k_{j+4}, k_{j+4})$ is closer to k_{j+4} than any other point in the grid. Thus, as W is typically "bell shaped", the algorithm will pick k_{j+4} as a solution to the maximal problem when the aggregate state is k_{j+4}. Thus, we have $x(k_{j+4}, k_{j+4}) = k_{j+4}$ even though this policy function does not have a steady state. Note that as all MSSRE are a subset of all possible GME, the latter must be discontinuous if we have a bias between the 2 steady states in a convergent solution. If this would have not been the case, the GME would also have a steady state in the same stationary point as the MSSRE. We provide numerical evidence in favor of this hypothesis as we have a GME with 1 ergodic steady state which is not equal to the numerical long run equilibrium in the MSSRE.

2.2 Monotonic Euler Equations

In this case, as $(1 - \tau(K))F'(K)$ is decreasing, we know that the derivative of the value function is continuous and, thus, the classical Euler equation holds. Equipped with this structure, we can define a monotonic operator A. Instead of iterating on G, we will define a procedure based on monotonic consumption functions, c. Along the equilibrium path, i.e. $k = K$, the presence of an optimizing representative agent implies that we can write the Euler equation $u'(c_t) = \beta u'(c_{t+1})(1 - \tau(K_t))F'(K_t)$ as follows:

$$u'((Ac)(K)) = \beta u'(c(F(K) - (Ac)(K)))HR(F(K) - (Ac)(K))$$

Where $HR(K) \equiv (1 - \tau(K))F'(K)$. There are 2 critical facts about operator A: i) it is continuous, ii) it can be defined in the space of equicontinuous monotonic functions once the set containing c is wisely chosen (see the appendix, section A.6). Moreover, the existence of a strictly positive fixed point can be proved using a theorem in the Tarski family, provided the initial condition of the iterative procedure is appropriately chosen. This fact insures, together with some curvature assumptions, that $Ac = c$ is continuous and unique (see the appendix, section A.6). Thus, when we move to the stochastic case, if shocks are drowned from a finite state markov process, we are confident that simulations are ergodic (see the appendix, section A.1). Finally, due to the equicontinuity of the consumption set, the sup norm insures convergence. This is due to the Arzela-Ascoli theorem (see the appendix, section A.6).

The operator A not only insures the presence of a constructive existence proof which can be used to defined an algorithm, it also preserves critical properties when it comes to compute and simulate the model (i.e. uniform convergence and ergodicity). However, it has a cost in terms of CPU time as noted by [Arellano et al., 2016]. To circumvent this problem, the authors defined an algorithm based on the existence of a differentiable value function called Envelope Condition Method (ECM). The operator A allows us to measure the tradeoff inherent in the use of the ECM: numerical efficiency vs. accuracy. Below we briefly discuss the results described in the numerical section for the stochastic case.

The ECM can be briefly described as follows:

1. Pick an initial condition c
2. Compute capital tomorrow $x(K) = F(K) - c(K)$
3. Compute the value function $V(K) = u(c(K)) + \beta V(x(K))$
4. Update consumption $\hat{c}(K) = u'^{-1}\left[\frac{V'(K)}{H(K)}\right]$
5. Continue until convergence under the sup norm $c \to \hat{c} \to \cdots$

Note that we can pick the same initial condition that works for operator A, a monotonic function. However, step 4 does not preserve the monotonicity in $\hat{c}$ as the numerator and the denominator are both decreasing. Moreover, the computed policy function for capital is almost linear with a slope slightly bigger than 1. These facts have at least 2 important implications. Let x_{ECM} be the numerical fixed point found by the ECM. Then: i) as

$x_{ECM}(0) > 0$ and x_{ECM} is linear with a slope bigger than 1, there is no compact ergodic equilibrium. ii) As the ECM does not insure equicontinuity, $c \to \hat{c} \to \cdots$ does not converge under the sup norm. The literature is aware that the ECM is not convergent. However, the arguments typically rely on the fact that the ECM does not insure that the computed policy functions are maximal (i.e. does not satisfy the sufficient conditions associated with the recursive problem). However, in this paper we argue that, even if the ECM does preserve maximality, it does not achieve convergence. The reasons can be found in a combination of facts i) and ii). In particular, the derivative of the value function yields:

$$V'(K) - u'(c(K))HR(K) = [\beta V'(x(K)) - u'(c(K))]\, x'(K) \tag{7}$$

As $x'(K) \approx 1$, we know that imposing the envelope condition (the left hand side in equation (7)) insures that the sufficient condition for the maximality of the policy function holds (the right side in equation (7)) . However, the Euler equation implied by equation (7) and the ECM method has a different structure when compared with operator A. Note that A insures that not only Ac is increasing, but also $F(K) - (Ac)(K)$ has the same property. This last condition is critical to insure that A generates a set of equicontinuous functions $\{c_n\}_n$, with $c_{n+1} = Ac_n$ (see the appendix, section A.6). This is not the case in equation (7). More to the point, fact i) implies that the equilibrium generated by the ECM is not compact and thus the Arzela - Ascoli theorem, which insures the compactness of an equicontinuous set of function under the sup norm, can't hold. Thus, the convergence criteria in step 5 is not appropriate as $c \to \hat{c} \to \cdots$, if it converges, it does so under a weaker topology.

3 Decreasing taxes: non-monotonic Euler equations

3.1 A Stochastic sequential economy with endogenous tax rates

The model is a stochastic version of [Santos, 2002] (section 3.2). Consider a representative agent economy with discrete time, $t = 0, 1, 2....$ Exogenous shocks are markovian and will be denoted z. For the sake of simplicity let us assume that the state space for these shocks is $\{0, 1\}$. An element of the transition matrix will be denoted $p(.,.)$, where the first position denotes rows and the second columns. Let $\{z_t\}$ be a sequence of shocks and Z^t the set of histories up to time t, being a typical element z^t. Using standard results (see [Stokey, 1989], Ch. 8) it is possible to define, for any $z_0 \in \{0, 1\}$, a stochastic process $(\Omega, \sigma_\Omega, \mu_{z_0})$ on Z^∞.

As in this section we are dealing with a sequential economy, k denotes the supply of capital (services) and K its demand. There is a unique decreasing return to scale firm which only uses capital as input and its technology is characterized by $y_t = A(z_t)f(K_t)$ with $f' > 0$, $f'' < 0$ and $f(0) = 0$ as usual. The firm is owned by the consumer as she is endowed with $k_0 > 0$ units of capital. Thus, the agent has two sources of current income derived from her endowment: benefits, denoted by π_t, and rents from capital, denoted by $r_t k_t$. Besides, the flow of taxes paid and transfers received is $\tau(K_t) r_t k_t$ and T_t respectively. Note that the tax rate depends on the stock of capital. It is given by a piecewise linear continuous and decreasing function (see [Santos, 2002], page 87 for details).

The problem faced by the consumer is to choose a pair of functions $c : Z^\infty \longrightarrow \mathbb{R}_+$ and $x : Z^\infty \longrightarrow \mathbb{R}_+$ that solves the following problem:

$$\max_{\{c,x\}} \sum_t \sum_{z^t \in Z^t} \gamma^t u(c(z^t)) \mu_{z_0}(z^t) \tag{8}$$

s.t.

$$k(z^t) = x(z^t) + (1-\delta)k(z^{t-1}) \tag{9}$$

$$c(z^t) + x(z^t) \leqslant \pi(z^{t-1}) - (1 - \tau(z^{t-1}))r(z^t)k(z^{t-1}) + T(z^t) \tag{10}$$

$c(z^t) \geqslant 0, x(z^t) \geqslant 0$ for any $z^t \in Z^t$, z_0 and $k_0 > 0$ given, $\delta \in [0,1]$ is the depreciation rate and $\gamma \in (0,1)$ the discount factor.

Note that we are restricting the maximal random variables (c, x) to take values on $\mathbb{R}_+$. This restriction will be relevant for the recursive representation of the sequential equilibria as boundary conditions will be critical to prove existence of a stationary state space. In what follows $\tau(z^{t-1})$ stands for $\tau(k(z^{t-1}))$ or abusing notation $\tau(k_t(z^{t-1}))$.

That is, the tax rate affects the rents obtained from capital holdings at time t, which is in turn affected by the information contained in z^{t-1} because $k_t(z^{t-1}) = x_{t-1}(z^{t-1}) + (1-\delta)k_{t-1}(z^{t-2})$. A similar argument can be used to understand $r(z^t)$ because the agent knows the clearing condition for the market of factors and the optimality condition for the firm to be described below.

The problem of the firm is standard. Taking r_t as given it solves:

$$\max_{K_t} A(z_t) f(K_t) - r_t K_t, \qquad \text{for any } z_t \in \{0,1\}. \tag{11}$$

Observe that the optimality of the firm implies $r_t = A(z_t) f'(K_t)$. The Government simply transfers to the consumer the tax revenues:

$$T = \tau(z^{t-1}) r(z^t) k(z^{t-1}). \tag{12}$$

Finally, goods and factor markets clear:

$$c(z^t) + x(z^t) = A(z_t) f(K_t) \qquad \text{Goods Market}$$

$$k(z^t) = K_{t+1} \qquad \text{Capital Market}$$

where both equations hold for any $z^t \in Z^t$.

Note that in equilibrium, the optimality condition of the firm and the market clearing equation for capital holdings implies $r_t = A(z_t) f'(k(z^{t-1}))$ which in turn implies

$r_t = r(z^t)$ as claimed. Further, both market clearing conditions imply $c(z^t) + x(z^t) = A(z_t)f(k(z^{t-1})) = y(z^t)$ as expected.

We can now define the sequential equilibrium for this economy:

Definition 1 A Sequential Competitive Equilibrium for this economy is composed by a triad of functions z^t measurable functions (x, c, r) such that:

- Given r, (x, c) solve the Maximization problem of the household.
- For each z^t, given $r(z^t)$, $K(z^t)$ solves the problem of the firm.
- For each z^t, Goods and Capital markets clear.
- For each z^t, the Government runs a balanced budget, equation (12).

3.2 Equilibrium Equation

In this case, the solution to the model can be characterized by the equilibrium Euler equation, which can be obtained by putting the optimality condition for the firm, the budget constraint for the Government and the market clearing conditions into the optimality condition for the consumer.

Assume that $u(c) = \ln(c)$ and $\delta = 1$. Then, the equilibrium equation is given by:

$$\tfrac{1}{C_t} = \gamma \sum_{z_{t+1}=0,1} \frac{A(z_{t+1})p(z_t, z_{t+1})(1-\tau(K_{t+1}))f'(K_{t+1})}{C_{t+1}}, \tag{13}$$

With constrains given by

$$K_{t+1} = A(z_t)f(K_t) - C_t. \tag{14}$$

Note that the market clearing condition for capital implies that *given* z^t the demand for capital K_{t+1} does not depend on the realizations of the exogenous shock at $t+1$. Hence, by replacing C_{t+1} in (13) with its expression obtained from (14) and after some algebra we can rewrite (13) in the following way:

$$\overbrace{\underbrace{\frac{1}{\gamma(A(z_t)f(K_t) - K_{t+1})(1-\tau(K_{t+1}))A(z_t)f'(K_{t+1})}}_{c}}^{c} = \frac{\overbrace{A(0)p(z_t, 0)}^{c_1}}{\underbrace{A(0)f(K_{t+1})}_{d_1} - K_{t+2}} + \frac{\overbrace{A(1)p(z_t, 1)}^{c_2}}{\underbrace{A(1)f(K_{t+1})}_{d_1} - K_{t+2}} \tag{15}$$

One of the purposes of this paper is to find an equation $\Psi : X \longrightarrow X$, where X is an appropriately defined state space and Ψ is a function that maps $x_t \longmapsto x_{t+1}$ with (x_t, x_{t+1}) satisfying equation (15) for any t.

Notice that by standard arguments, by fixing $\delta = 1$ and $f(0) = 0$, K_t stays in $[0, K^{UB}]$ (see [Stokey, 1989], Ch. 5) for any t.

Let $X = [0, K^{UB}] \times [0, K^{UB}] \times \{0, 1\}$. With this state space Ψ becomes a vector valued function of the form $x_t \longmapsto (\Psi_1(x_t), \Psi_2(x_t), \Psi_3(x_t))$ with $x_t = (K_t, U_t, z_t)$.

Let $\{z_n\}$ be a realization of $(\Omega, \sigma_\Omega, \mu_{z_0})$. Then, it is possible to define each coordinate in the image of Ψ as follows:

$$\begin{aligned} K_{t+1} &= \Psi_1(x_t) = U_t \\ z_{t+1} &= \Psi_3(x_t) = \{z_n\}(t+1). \end{aligned}$$

In order define Ψ_2 we could use (15). Notice that (15) takes the form

$$c = \frac{c_1}{d_1 - U_{t+1}} + \frac{c_2}{d_2 - U_{t+1}}, \tag{16}$$

or equivalently,

$$c(d_1 - U_{t+1})(d_2 - U_{t+1}) = c_1(d_2 - U_{t+1}) + c_2(d_1 - U_{t+1}). \tag{17}$$

Due to the fact that this is just a quadratic equation we can get U_{t+1} as a *continuous function* of the parameters, namely:

$$U_{t+1} = \frac{\pm\sqrt{(-d_1 c - d_2 c + c_1 + c_2)^2 - 4c(d_1 d_2 c - c_1 d_2 - c_2 d_1)} + (d_1 + d_2)c - c_1 - c_2}{2c}. \tag{18}$$

Equivalently:

$$U_{t+1} \equiv g(d_1, c, d_2, c_1, c_2)$$

It is important to observe that (18) gives at most *2 different mechanisms* [3] , each of them characterized by a different root of (18). Furthermore, note that $c(K_t, U_t, z_t)$, $d_1(U_t)$, $d_2(U_t)$ and the rest of the parameters in (18) depend on z_t . Thus, Ψ_2 is given by:

[3]Note that (15) implies that this model does not have a trivial solution at $K_t = 0$ as $u = \ln$ and investment is not allowed to be negative. This fact in turn implies that the parameters in (15) are all bounded away from 0. Of course, in order to have two non-trivial solutions it suffices to impose conditions on the discriminant of (18)

$$U_{t+1} = g(d_1, c, d_2, c_1, c_2) \equiv \Psi_2(x_t).$$

Note that once we start iterating the system, it is possible that simulations go outside X. The continuity of Ψ_2 (on K_t and U_t), provided that the state space is well defined across any possible path, seems automatic. It suffices to verify the continuity of C, d_1, d_2 (on K_t and U_t), which is trivially satisfied. However, if we iterate forward equation (18), the restrictions on d_1, c, d_2, c_1, c_2 in order to keep U_{t+1} in $\mathbb{R}$ may affect the empirical performance of the model as the set of parameters (i.e. β, $p(.,.)$, etc) can't be freely choose in the calibration / numerical estimation procedure. Moreover, even if we could find an empirically meaningful parameter set, any solution to (18) may imply a negative consumption level or capital stock. Of course, due to the *log* preferences, these solutions will not be optimal. Thus, we must find a procedure in order to rule out solutions outside $\mathbb{R}_+$ and that imply a non-positive consumption level. In the numerical section, we adapt a canonical result due to [Duffie et al., 1994] to verify that the state space is well defined along equilibrium trajectories. Given the quadratic structure in (18), the stationarity (i.e. time independence) of the state space is sufficient to insure both compactness and continuity of the recursive mechanism.

3.3 Generalized Markov Equilibrium

The previous section describes a recursive mechanism based on an *enlarged* state space X. In particular, we wrote K_{t+2} in terms of (K_t, K_{t+1}, z_t):

$$K_{t+2} = g(K_t, K_{t+1}, z_t).$$

The mechanism, g, is *explicit* and, even more, continuous (of course, this representation has economic content if we can assure that the discriminant in g is positive under reasonable parameterizations for any $x \in X$ and the boundary conditions on endogenous variables are satisfied).

We can now define a Generalized Markov Equilibria.

Definition 2: Generalized Markov Equilibrium (GME)

A GME is a *correspondence* $\Psi : X \to X$ with X compact such that for any $x \in X$, the vector $(x, \Psi(x))$:
i) satisfies the optimality conditions for the household problem, equation (8) s.t. (9) - (3).
ii) The firm solves (11)
iii) Markets clear
iv) The public sector runs a balanced budget. That is, equation (12) holds.

In section 3.2 we show that, if we can ensure the existence of a well-behaved state space, the sequential version of the model presented in this paper has a GME representation. Moreover, Ψ may even have 2 continuous selections. Let Ψ_i be any of the 2 possible selections. Using standard results (see [Stokey, 1989]), we can show that $P_{\Psi_i}(x, A)$ defines a Markov kernel with $P_{\Psi_i}(x, .)$ being a probability measure for any $x \in X$ and $P_{\Psi_i}(., A)$ being a measurable function for any $A \in Borel(X)$. An invariant measure is any fixed point of Ψ_i. Call one of the possible many fixed points μ_i.

Let Ψ_i^j be any numerical approximation to Ψ_i and $P_{\Psi_i^j}(x, A)$, μ_i^j the associated Markov kernel and invariant measure respectively. Since [Santos and Peralta-Alva, 2005], it is known that even if Ψ_i^j converge to Ψ_i, the simulations obtained from Ψ_i^j may differ from the exact ones, generated using Ψ_i. If Ψ_i is equicontinuous and defined over a compact state space, these authors showed that numerical simulations will match the exact long run behavior of the model. However, equicontinuity is associated with very restrictive properties for non-optimal economies as noted in [Coleman, 1991]. If Ψ_i is not continuous / equicontinuous, [Pierri, 2018] provided sufficient conditions which insure that numerical simulations replicate the actual model. Unfortunately, these conditions depend on the cardinality of Z, the set containing exogenous shocks, and will not hold in this framework.

The virtue of this paper is that it allows us to circumvent the mentioned problems. On one hand, we show that *a GME exist* for the problem at hand and thus, it is possible for us to compute it. Moreover, using (13) and (14), we show that Ψ_i has a *continuous closed form representation*, which in turn eliminates the problem associated with the lack of convergence of numerical simulations, provided that we can find a suitable state space.

The (numerical) cost of this representation is the enlargement of the state space with respect to the natural one (i.e. (K_t, z_t)). As we have a closed form solution, these costs are more than compensated by the accuracy of simulations. As discussed in [Kubler and Schmedders, 2002], enlarging the state space might provide a recursive representation. Unfortunately, the results in that paper does not address the continuity of the mechanism; an aspect that has severe consequences for the steady state of the model as discussed in [Duffie et al., 1994].

This paper shows that it is possible to obtain a continuous selection from a correspondence-based recursive representation. After taking care of the boundary conditions, we can insure the compactness of the state space. Coupled with the continuity of the mechanism, Ψ_i, we can show existence of μ_i using canonical results in [Futia, 1982]. See section A.1 in the appendix for a detailed discussion about the existence of invariant measures in compact spaces.

We can use these results to simulate the model. As $U_t := K_{t+1}$, we have now the following iterative system:

Take first an arbitrary initial condition (K_0, U_0, z_0) and a drawn $\{z_n\}$, then

$$K_{t+1} = U_t$$
$$U_{t+1} = g(K_t, U_t, z_t),$$

provides a sequence $\{X_n\}$. Such a sequence defines a Feller mechanism, with compact state space X.

From [Santos and Peralta-Alva, 2005] and [Santos et al., 2012], we know that $P_{\Psi_i}(x,.)$ has an ergodic invariant measure if Ψ_i is equicontinuous. The quadratic structure in (18) ensures that the compactness of the state space and the interiority of solutions are sufficient (if they are satisfied jointly, of course) to guarantee ergodicity. In section 4.2 we will define an operator which allows us to find a state space that it is compact and that ensures that capital and consumption remains positive along equilibrium paths. These facts in turn, guarantee that the derivatives of Ψ_i are finite, which in turn implies equicontinuity. Provided that μ_i is ergodic, the process $\{K_t\}$ has a well-defined invariant measure as well. Moreover, using standard results on laws of large numbers for markov processes (see [Varadhan, 2001]), it can be shown that choosing an appropriate initial condition suffices to guarantee that:

$$\frac{\sum_{t \in 0,\ldots,T} h(X_t)}{T} \quad \text{converges almost surely to} \quad E_\mu(h),$$

Where h is a X-measurable function and μ is one of the possibly many ergodic invariant measures described above.

Finally, note that U_{t+1} is measurable with respect to z^t, which in turn implies that K_{t+2} is measuble with respect to the same filtration. As $Z^t \subset Z^{t+1}$, the measurability requirements in definition 1 are satisfied. This is the cost of working with a Markov structure: we are losing memory inherited from the sequential equilibrium, a fact which may affect the empirical performance of the model as noted by [Pierri and Reffett, 2019].

3.4 Minimal State Space Recursive Equilibrium

This paper deals with global methods, which are widely used in practice. The literature has also made substantial progress in the design of local methods. There is a clear trade-off between these 2 options: while the former can replicate a more flexible dynamic behavior, the latter can deal with large scale models. Any researcher choosing a global method must deal with the limitations implied by the numerical burden associated with the solution of a considerable number of non-linear equations. Thus, it is natural to choose the minimal possible number of states as this option significantly reduces the main disadvantage of global methods.

In this sense, it is critical to understand the limitations of Minimal State Space Recursive Equilibrium (MSSRE) methods. The MSS version of the model described above can be written as follows:

$$V_n\left(k, K, Z; H_j\right) = Max_{y \in \Gamma(k,K,Z)}\, u(g_\tau(k,K,Z)-y)+\beta\sum_{Z'} V_{n-1}(y, H_j(K,Z), Z'; H_j)p(Z,Z') \tag{19}$$

Where the feasibility correspondence is given by:

$$\Gamma(k,K,Z) = [y \in \overline{K}; 0 \leqslant y \leqslant \pi(K,Z) + (1-\tau(K,Z))r(K,Z)k + T(K,Z)]$$

Capital is allowed to fluctuate in a compact set, $[0, K^{UB}] = \overline{K}$. The function g_τ represent disposable income and is defined by:

$$g_\tau(k,K,Z) \equiv \pi(K,Z) + (1-\tau(K,Z))r(K,Z)k + T(K,Z)$$

Where $\pi(K,Z)$ and $\tau(K,Z)$ are defined in (9) and $T(K,Z)$ in (12). The policy function for (19) is given by $h_{n-1,j}(k,K,Z)$, which belongs to the set defined below:

$$argmax\left\{u(g_\tau(k,K,Z)-y) + \beta\sum_{Z'} V_{n-1}(y, H_j(K,Z), Z', H_j)p(Z,Z')\ s.t.\ y \in \Gamma(k,K,Z)\right\}$$

Note, remarkably that: i) the household take a guess at the evolution of the aggregate states using a *perceived law of motion* denoted H_j. ii) The value and the policy function in the dynamic programming problem have to converge in j, which is associated with the rational expectation nature of the problem (i.e. the perceived and the actual law of motion must be equal when $k = K$), and in n, that is guaranteed by the contractive nature of the Bellman operator in (19). iii) The dependence of disposable, $g_\tau(k,.,.)$, on prices, $r(.,.)$, justifies the presence of *equilibrium states* which are represented by capital letters. They affect the household problem through the firm's decisions, given by (11), and market clearing conditions which are contained in the definition of recursive competitive equilibrium, which is given below.

Definition 3 Minimal State Space Recursive Equilibrium (MSSRE)

A MSSRE is a *value function* V_*, a *policy function* $h_{*,*}$ and a *perceived law of motion* H_* such that:
i) the household solves equation (19) obtaining $V_*(k,K,Z;H_*)$ and $h_{*,*}(k,K,Z;H_*)$ for any feasible state k,K,Z.
ii) The firm solves (11)
iii) Markets clear. That is, $k = K$
iv) Expectations are fulfilled. That is, $h_{*,*}(K,K,Z;H_*) = H_*(K,K,Z)$ for any (K,Z)
v) The public sector runs a balanced budget. That is, equation (12) holds.

To understand the connection between the existence of a MSSRE and its computation, we must characterize it. Even under strong curvature and smoothness assumptions on the return function u, which are all satisfied imposing the parametrizations used in sections 3.1 and 3.2, even if we assume the continuity of the feasibility correspondence Γ, for an interior optimal solutions, $h_{*,j}(.,K,.;H_j) \in \Gamma(.,K,.)$, we can't use the envelope theorem in [Stokey, 1989]. The arguments used in section 2 hold *mutatis mutandis*. In particular, Benvenisite and Scheinkman envelope theorem (see [Stokey, 1989] page 266, Th. 9.10) coupled with the strict concavity of V_n (in k) (see [Stokey, 1989] page 265, Th. 9.8) would imply that V_n' is decreasing in k when $k = K$, which will not hold globally as $f'(K)(1-\tau(K))$ is not monotonic. Critically, the feasibility correspondence $\Gamma(k,K,Z)$ is not convex (see section A.3 for a detailed discussion).

Fortunately, using lemmas 3.3. and 3.4 in [Amir et al., 1991] we know that any solution to the dynamic program must satisfy the "classical" Euler equation and, thus, it can be characterized (see section A.4 for a detailed discussion). Formally, a solution to the dynamic programming problem in definition 3 for any pair of individual states (k,Z) and given the aggregate level of capital K must satisfy:

$$u'\left[g_\tau(k,K,Z) - h_{*,j}\right] = \gamma E_Z\left\{u'\left[g_\tau(H_j,h_{*,j}) - h_{*,j}(h_{*,j})\right] Af'(H_j)(1-\tau(H_j))\right\} \quad (20)$$

Where the dependence of $h_{*,j}$ on (k,Z) for each K and of H_j on (K,Z) have been omitted for expositional purposes. Also, equation (20) does not include the equilibrium version of g_τ, the disposable income, as condition iv) in the definition 3 may not hold in this model, even when $k = K$.

Note that (20) defines a mapping T from H_j to $h_{*,j}$. In fact, it is easy to see that any fixed point on this map is a MSSRE. Define the function space B on $K \times Z \equiv S$ as follows:

$$B(S) = \{H(s) \text{ such that } H: S \to K \text{ with } 0 \leqslant H(s) \leqslant A(Z)f(K),\ H \text{ measurable}\}$$

That is, a MSSRE is a fixed point in the functional T as the measurable maximum theorem insures that $h_{*,j} \in B$ when $k = K$. Any attempt to prove the existence of a fixed point in a function space must circumvent the problem associated with the lack of sufficient conditions which insure a convex graph in tractable frameworks. That is, $T(H_j)$ may not be convex for models with a finite number of agents or finite shocks (see [Pierri, 2018] for a detailed discussion). Thus, the literature has turned to the lattice dynamic programming framework because it works in non-convex models. See section A.4 for a review of the results in this literature relevant for the model presented in section 3.1.

Moreover, contrarily to the Fan - Gliksberg theorem, lattice dynamic programming gives us a constructive fixed-point theorem which naturally generates an algorithm. In fact, the numerical procedure in [Rios Rull, 2004] can be proved to be convergent endowing B with an order topololgy if *T is a monotone operator*; which in turn insures the existence of a MSSRE. That is, in order to prove the existence of a MSSRE *and* the convergence of the algorithm in [Rios Rull, 2004] for any $H_j^{'} \geqslant_* H_j$ we must have $T(H_j^{'}) = h_{*,j}^{'} \geqslant_* h_{*,j} = T(H_j)$ where $\geqslant_*$ is the pointwise order in B.

In order to prove the desired properties in T we can borrow from [Mirman et al., 2008] and [Coleman, 1991]. We present the relevant theorems in section A.4. The former proved that it is required to show that $V_*(k, K, Z; H_j)$ has increasing differences (see section A.4 in the appendix) in $(k; K)$ for each (Z, H_j) (lemma 12 and theorems 3 to 6). This condition, in turn, is equivalent to show that $V_{*,1}^{-}(k, K, Z; H_j) = u'(g_\tau(K) - h_{*,j}(K))(1 - \tau(K))r(K)$ is *increasing* in K, where the dependence of $V_{*,1}^{-}$ on $(k, Z; H_j)$ has been omitted in the right hand side of the equation and $V_{*,1}^{-}$ is the left derivative of V_* with respect to k which is finite (see sections A.2 and A.4 in the appendix). Note that $(1 - \tau(K))r(K)$ is decreasing in K if τ is increasing and undefined otherwise.Thus, as τ is decreasing by assumption, the results in [Mirman et al., 2008] does not hold.

[Coleman, 1991] showed that if $u'(g_\tau(K) - h_{*,j}(K))$ is decreasing in K when $k = K$, it is sufficient to assume that τ is increasing to induce an order structure using an operator based on (20). As τ is decreasing by assumption, we have shown that even if $u'(g_\tau(K) - h_{*,j}(K))$ is monotonic in K, as $(1 - \tau(K))r(K)$ is undefined, we cannot have an order structure for this model.

This last fact implies in turn that it is not possible to insure that a sequence of function $\{H_j\}_j$ converging to H_* will "hit" $h_{*.*}$ as required by definition 3. Moreover, *any numerical procedure based on iterations through T* using the uniform metric, as the one described in [Rios Rull, 2004], *cannot be proved to be convergent to a MSSRE* as the induced topology is stronger than the order topology. Thus, $SUP\,|H_{*,j} - h_{*,j}|$ maybe arbitrarily large, a fact which can cause a severe bias in the numerical simulations as discussed in section 2.

4 Numerical Results: non-monotonic Euler equations

The results in section 3 provide a unique opportunity to test the predictive power of MSS methods. As any MSSRE must satisfy equations (13) and (14), the simulations generated by it must converge to one of the possible multiple ergodic distributions obtained using a GME.

To perform this test, we present a standard recursive competitive MSS algorithm with 2 different "updating" rules. The first does not numerically converge to a fixed point between the perceived and actual law of motion and the second does, implying that in the latter case we are dealing only with the effects of a discontinuous equilibrium. Then, the policy functions are simulated, and the results compared with those obtained from equation (18). In order to insure that the exact GME has a state space which generates a pair of equilibrium random variables (c, x) taking values in the in the non-negative real numbers, we adapt a theorem from [Duffie et al., 1994]. We found that only 1 mechanism has a well-defined state space. Thus, we know that any MSSRE is a GME, which is also unique. So, any simulation obtained from the former, must match the latter if this equilibrium exists and it is continuous.

As neither continuity nor existence hold in the MSSRE for the model presented in section 3, we found a significant deviation with respect to the true equilibrium which, in turn, affects the long run distribution of capital. These findings provide evidence in favor of the results in [Hatchondo, 2010] and [Feng et al.,] which suggest the importance of theoretical results in the recursive numerical literature. That is, without sufficient conditions that insure the equivalence between numerical and actual simulations of the model, *a convergent algorithm does not guarantee by itself the absence of biases.*

From a qualitative perspective, the ergodic simulations imply that the true long run capital stock is fluctuating near a zero-tax rate. Thus, according to recent findings (see [Straub and Werning, 2020]), the decentralized equilibrium is not optimal and there is a scope for interventions. *Using the simulations obtained from the MSSRE, contrarily, the economy may be near the optimal tax rate and we would conclude the opposite.*

4.1 MSS Algorithm

We compute a MSSRE as described in definition 3 using the operator T, which in turn follows from equation (20). It is standard in the literature (see for instance, [Rios Rull, 2004]) to pick an arbitrary function H_0 from B and look for uniform convergence. However, as mentioned in section 3, theoretical results do not support such a strong convergence notion. If $(1-\tau)F'$ is increasing in K, it is only possible to show that any iteration starting from a lower or upper bound on T (i.e. $H \in B$ such that $H \leqslant T(H)$ or $T(H) \leqslant H$ respectively) will converge in the order topology. That is, take a sequence of increasing functions generated iteratively from T, $\{H_j\}$ with $H_{j+1} = T(H_j)$. We say that $H_j \rightarrow_{\geqslant_*} H_*$, meaning $\{H_j\}$ converge in the order topology to H_*, if for any j, $H_j \leqslant H_*$ and $H_* \in B$. If $(1-\tau)F'$ is decreasing in K, the convergence will be uniform in the standard sup norm. Unfortunately, as τ is decreasing and F strongly concave, we showed in section 3.4 that T is not a monotonic operator and thus it is not possible to generate a convergent sequence of functions using T.

The discussion in the above paragraph is entirely theoretical. We do not have *known sufficient conditions* which insure the convergence to a MSSRE using T. However, we found *numerically* a fixed point for T in the sup norm. In particular, the procedure described below was found to be convergent using the sup norm for acceptable relative error levels (in the order of 10^{-2})

$$H_0 \rightarrow_{Equation(19)} h_{*,0} \rightarrow_{Definition\ 3}\rightarrow H_1(K,Z) = G^1(H_0, h_{*,0})(K,Z) \rightarrow (...)$$

Where the first $\rightarrow$ means that we are solving equation (19) using H_0 as a guess for the perceived law of motion. The second $\rightarrow$ stands for the fact that we are computing the policy function $h_{*,0}$ along the equilibrium path according to definition 3. The last $\rightarrow$ implies that we are updating the perceived law of motion for aggregates states. The functional G is an updating rule. We use 2 different types of them:

- $G_1^j = \alpha H_j + (1-\alpha)H_{j-1}$, with $\alpha \in (0,1)$
- $G_2^j = \sum_{i=0}^{j} H_i/j$.

The last one was found convergent, that is: $n \geqslant N(\epsilon)$ imply $|G_2^n - h_{*,n}| < \epsilon$. Finally, $\rightarrow$ (...) means that we are starting the loop again if convergence using the sup norm is not achieved.

4.2 Stationary GME

For any arbitrary state space X, with typical element $(K, K_+, Z) \in X$, equation (18) may imply that $K_{++} \notin \mathbb{R}_+$ (i.e. K_{++} may be an imaginary or negative number). Let Z_L be the smallest possible shock. Note that we may have $A(Z_L)f(K) < K_+$ and / or $A(Z_L)f(K_+) < K_{++}$, which in turn imply that consumption is negative.

To solve these problems, we modify theorem 1.2 in [Duffie et al., 1994]. The authors showed that, given the compactness of the sequential equilibria, it is always possible to find a stationary sate space $J \subseteq X$ for the markov equilibrium associated with any root of equation (18) using the following iterative procedure:

$$C_1 = \{x_0 \in X \mid \Psi(x_0) \cap C_0 \neq \emptyset\}$$

Where $C_0 \equiv X \subset \mathbb{R}^3_+$. Moreover, for $n \geqslant N$, $C_n \rightarrow J$. If *the sequential equilibria is compact*, J is non-empty and compact (see section A.5 in the appendix). As some roots of equation (18) may not be a real number, we can use this operator in order to keep those mechanisms, if any, that are self-contained in $\mathbb{R}^3$. Moreover, the authors showed that for any $x \in J$, $\Psi(x) \cap J \neq \emptyset$, which in turn implies that this mechanism can be iterated forward if J is non-empty. As this set is time invariant, it is a state space for Ψ.

However, it is possible that, for some $x \in J$, the vector (c^L, c_+^L, K_{++}) contains a negative number, where $c^L = A(Z_L)f(K) - K_+$ and $c_+^L = A(Z_L)f(K_+) - K_{++}$. In order to circumvent these problems, we use the following modified operator:

$$C_1 = \{x_0 \in X, A(Z_L)f(K) \geqslant K_+ \mid \Psi(x_0) \cap C_0 \neq \emptyset, A(Z_L)f(K_+) \geqslant \Psi_2(x_0)\}$$

Where $\Psi_2(x_0) = K_{++}$ is the second coordinate in the image of the vector valued function which defines the GME. The operator above generates a sequence of sets in $\mathbb{R}_+^3$ with non-negative consumption levels which converge to a possible empty set J. We are interested in finding 1 mechanism Ψ from equation (18) which generates a non-empty state space J.

4.3 Simulations

We now turn to measure the numerical bias. Section 4.1 described the algorithm typically used to compute a MSSRE. The numerical procedure associated with the simulation of a GME is contained in the discussion of sections 3.3 and 4.2.

The task is to compute definitions 2 and 3 using a concrete tax function based on the model in [Santos, 2002], a standard algorithm borrowed from [Rios Rull, 2004] and the refinement for the GME described in section 4.2. In particular, τ is decreasing in K. Thus, the discussion in section 3.4 implies that the operator T is not monotonic and, consequently, it is not possible to prove that a numerical procedure based on iterations using T will converge to a MSSRE. The rest of the parameters are contained in the table below. We are carefully following the preferences and technology structure in [Santos, 2002]. However, as this model is non-stochastic, we are setting the values for the exogenous shocks in set Z and transitions probabilities p_{LH} and p_{HL} in order to insure a well-defined steady state for the GME.

$y = A(Z)f(K) = e^Z K^{1/3}$	$Z_H = 0.2$
$u(c) = ln(c)$	$Z_L = 0.1275$
$\delta = 1$	$p_{LH} = 0.5$
$\beta = 0.99$	$p_{HL} = 0.3$

Table 1: Parameters

The table below contains the results of simulating the procedures described in section 4.1 (MSSRE) and 3.3 (GME). The parameters used are listed in Table 1. We refine the mechanisms for the GME using the operator defined in section 4.2. We found that for the negative root $J = [0.01, 1.50]$ and for the positive root $J = \emptyset$. Thus, we will only report $\Psi_{NR} \equiv \Psi$, where "NR" stands for negative root.

Model	Mean	STD	CV
Ψ	1.1976	0.0079	0.0066
MSSAvg, $K_{UB} = 0.6$	0.4058	0.0117	0.0289
MSSCes, $K_{UB} = 0.6$	0.2662	0.0106	0.0400
MSSCes, $K_{UB} = 1.5$	0.3098	0.0134	0.0431

Table 2: Simulation Results. Statistics for aggregate capital

Where STD stands for standard deviation and CV for the coefficient of variation (standard deviation / mean). The "empirical" distributions are constructed as follows: take an arbitrary initial condition. Simulate a path of 5000 observations for aggregate capital. Store the last 1000 observations. Then, the computed distribution is taken from the relative frequency of 25 grid positions out of these observations. The procedure is repeated for any of the 4 listed distributions.

We report 3 different solutions for the MSSRE, which differs in the updating rule discussed in section 4.1 and in the upper bound (K_{UB}) of the grid. The first one called "MSSAvg", which stands for "average", is not convergent and thus it contains the 2 sources of biases: the lack of a steady state and the lack of convergence. Unfortunately, we can only compute the last one. When we expand the state space, to make it comparable with J, the cesaro updating is not convergent. Below we report the bias associated with the first and the last case.

Model	Min-Max Error	Min-Max Rel. Error	$A(Z_{LB})F'(K^*)$	Bias
MSSAvg, $K_{UB} = 0.6$	$[0.1062, 0.4248]$	$[0.9696, 41.3]$	0.6908	0.1834
MSSCes, $K_{UB} = 0.6$	$[-0.0059, 0.0118]$	$[0.072, 0.1361]$	0.9151	0.0027
MSSCes, $K_{UB} = 1.5$	$[0.0596, 0.3129]$	$[0.3460, 0.8994]$	0.8270	0.1540

Table 3: Lack of Convergence: Implications for the accuracy of simulations

We define an error as the difference between the perceived ($H_{*,j}$) and actual ($h_{*,j}$) law of motion for capital. The columns in the table contains the [minimum - maximum] relative and absolute errors across iterations j. The relative error determines the convergence of the algorithm. Note that only the cesaro updating procedure with a grid of $[0.01, 0.6]$ converged.

The absolute error is used to compute the distortion generated by the algorithm. If we take as a reference value the mean of the capital stock under each procure, denoted K^*, the lack of convergence of the algorithm implies a distortion of $(h_{*,j}(K, K, Z) - H_{*,j}(K, Z))A(Z)F'(K)$ in the equilibrium budget constraint $c + x = F(K)$. That is, on average, the MSSAvg procedure implies that the household receives 0.1834 more units of the consumption good due to the lack of convergence of the algorithm. Thus, as the agent is "wealthier", capital stock is higher when compared with the accurate solution among MSS algorithms (i.e. MSSCes, $K_{UB} = 0.6$).

The numerical solutions in Table 2 has a significant bias, as measured by the difference in mean with respect to the ergodic distribution. The table below presents the relative deviations.

Model	Relative Mean	Relative CV
MSSAvg, $K_{UB} = 0.6$	0.34	4.38
MSSCes, $K_{UB} = 0.6$	0.22	6.06
MSSCes, $K_{UB} = 1.5$	0.25	6.53

Table 4: Relative Bias

Where "Relative" stands for $Mean(MSSAvg, K_{UB} = 0.6)/Mean(\Psi)$, etc. From Table 4 the mean of the ergodic accurate distribution (as measured by the GME) is way above the mean generated by any numerical approximation of the MSSRE. On the contrary, the dispersion is significantly below. Thus, despite the fact that the algorithm for the MSSRE converge for the case of $MSSCes, K_{UB} = 0.6$ using a strong criteria (i.e. the sup norm and a tolerance level of 0.075 for the relative error), the numerical distribution will present a severe bias with respect to the distribution *that we know is well defined*.

The results described above point out to the relevance of a well-defined steady state (i.e. a fixed point of $P_H(K, Z; .)$, where P is the markov kernel defined in section 3.3 but constructed using the perceived law of motion for the MSS, H). From section 3.4 and figure 3 we know that the discontinuity of $V^-_{*,1}$ plays a central role in this fact. Below we show the (numerical) derivative of the value function for the $MSSCes, K_{UB} = 1.5$ when $k = K$. We choose this solution as the state space is comparable with J.

Where the blue line represents the derivative for the low shock. Even though the figure depicts the expected convex shape for a concave function, it has several jumps, suggesting the presence of more than 1 discontinuity. More to the point, the discontinuity set seems large and dependent on the TFP shock, Z. Thus, it would be difficult to know when we have a model with a well or an ill (as depicted in figure 3) behaved steady state. These jumps are especially relevant near the numerical long run distribution. Below we plot figure 4 for the points in the grid that have positive mass in the long run (i.e. $[Mean - 2 * STD, Mean + 2 * STD]$ for the updating rule and grid size given by $MCes, K_{UB} = 1.5$).

Even though near the mean, 0.3098, the derivative seems continuous, the existence of any unconditional moment depends on a well-defined invariant measure (i.e. on a fixed point of the markov kernel P_H). Thus, the jumps to the left and right of the numerical mean are relevant for the existence of a steady state as depicted in figure 3. More

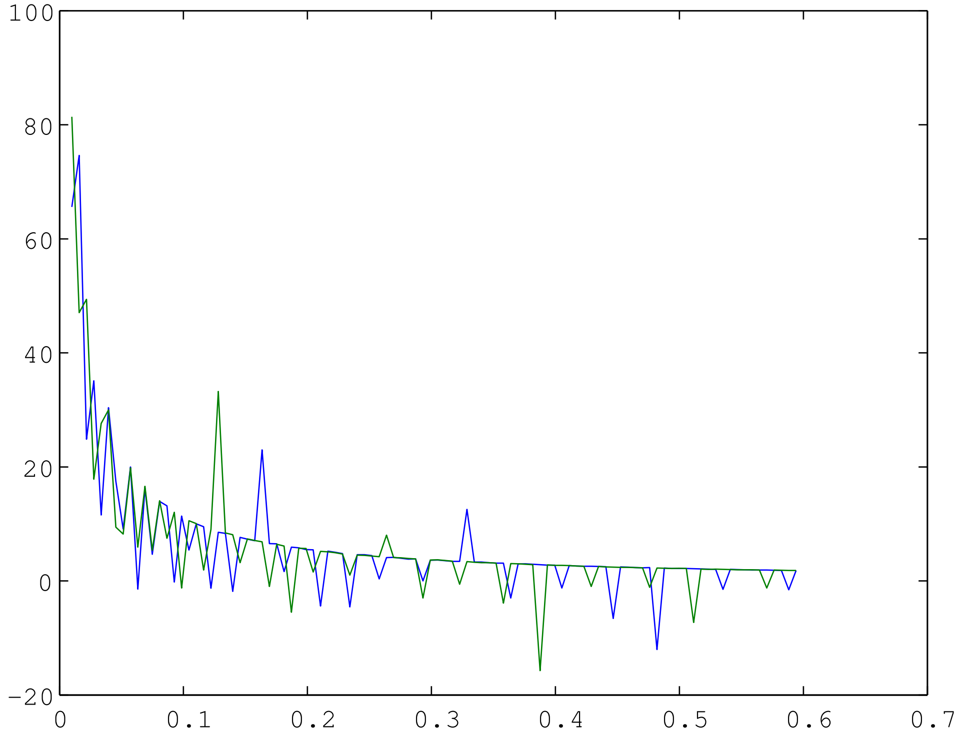

Figure 4: Numerical Derivative of the Value Function when $k = K$

importantly, given the finite cardinality of the grid, the algorithm may not capture the discontinuity and display a well-behaved histogram.

As discussed in previous sections the observed bias could be generated either by the lack of convergence of the perceived to the actual law of motion (i.e. $H_j \rightarrow H_* \nrightarrow h_{*,*}$) or by any difference between the numerical and the actual steady state (i.e. $\mu_i^j \rightarrow \mu_i^* \nrightarrow \mu$ for any computed MSS algorithm $i \in 1, 2, 3$ where μ is the fixex point of P_Ψ). As the regards the former, note that any MSSRE must satisfy equation (20) which in turn insures that any path generated using $h_{*,*}$ along the recursive equilibrium will also be a sequential equilibrium. In other words, any path generated from a MSSRE satisfies equations (13) and (14). The lack of coincidence between the perceived and the actual law of motion will generate a distribution of capital that does not belong to any possible sequential competitive equilibrium, which explains part of the bias as measured in Table 3. Moreover, as a continuous MSSRE may not exist for this model, we cannot ensure the existence of a well behaved steady state for this type of equilibria (i.e. μ_{MSSRE} may not exist). If that is the case, any numerical distribution, namely μ^j_{MSSRE}, could be arbitrarily far away from μ as it is not possible to show that $\mu_{MSSRE} \rightarrow \mu$.

The figure in the appendix show, for the sake of completeness, the phase diagrams

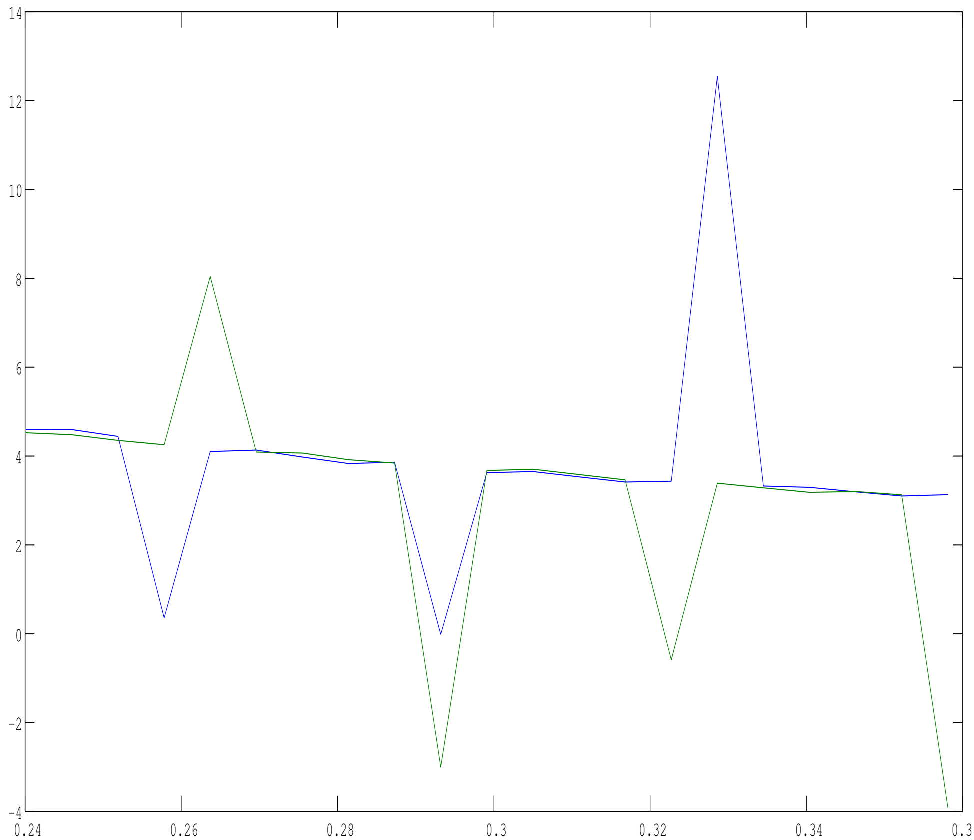

Figure 5: Numerical Derivative of the Value Function in the long run

depicted in figures 1 and 2. The numerical histograms used to compute the results in table 2 are available under request.

We turn to the policy implications of the results in this paper. Note that the mean capital stock for the $MSSCes, K_{UB} = 1.5$ algorithm is well below the ergodic mean. Thus, as τ is decreasing with $\tau \to 0$ for $K \to K_{UB}$, using the MSS to predic the long run behavior of this model we may conclude that the observed tax rate is positive. Recent results, see for instance [Straub and Werning, 2020], have shown that the optimal tax rate is strictly positive in the long run. So, the policy advice would be to slightly change the observed tax rate, if any. However, the true distribution, with a support close to the upper bound of J, calls for a increase in the tax effective tax rate.

We are not considering the effects of data on the parameter set. If we want to test this model empirically, the meaningful parameters will be different depending on the selected recursive equilibrium notion. In this case, to assess the effects of the bias, we would have to perform a comparative statics analysis, which is outside the scope of this paper.

5 Increasing taxes: monotonic Euler equations

In this section we briefly modified the parameter structure in the previous paragraphs in order to match the results in [Coleman, 1991]. We first derive the theoretical structure for the benchmark case. Then we perform the numerical test of the Envelope condition method.

5.1 Preliminary Results

The sequential equilibrium is the same as the one in definition 1. The differences with respect to section 3 and 4 are as follows:

- $\delta = 0.019$
- $u(c) = c^{1-\sigma}/(1-\sigma)$ with $\sigma = 4$
- $f(K) = K^{1/2}$
- $ln(Z_{t+1}) = \rho ln(Z_t) + \epsilon_{t+1}$ with ϵ_{t+1} being a distributed according to a normal distribution with mean 0 and standard deviation 0.007, discretized with a 10 points grid.
- $\tau(K) = 0.1K/K^{UB}$ and $HR(K,Z) \equiv (1-\delta) + Zf'(K)(1-\tau(K))$ is decreasing

Definition 1 implies that any sequential competitive equilibrium must satisfy:

$$u'(c_t) = \beta E_t(u'(c_{t+1})HR(K_{t+1}))$$

Where the dependence on Z_{t+1} has been omitted as the integral in E_t is with respect to this variable. In order to define a recursive equilibrium with minimal state space we can build an Euler equation operator, A. Let C be the space of candidate policy function. In particular:

$$\begin{aligned} C(\mathbb{K},\mathbb{Z}) = \{ & \\ & c : \mathbb{K}\times\mathbb{Z} \to \mathbb{R} \text{ is continuous} \\ & 0 \leqslant c(K,Z) \leqslant F(K,Z) \\ & 0 \leqslant c(K',Z) - c(K,Z) \leqslant F(K',Z) - F(K,Z), K' \geqslant K\} \end{aligned} \tag{21}$$

Where, $F(K,Z) = (1-\delta) + Zf(K)$, $\mathbb{K} \equiv \left[0, K^{UB}\right]$ and $\mathbb{Z}$ is a grid of 10 evenly spaced points. Note that equation (21) implies not only that the policy functions must be continuous but also consumption and investment are increasing. Following [Coleman, 1991] we can defined the following operator:

$$\begin{aligned} u'((Ac)(K,Z)) = \beta E_t\{&u'(c(F(K,Z) - (Ac)(K,Z), Z')) \\ &HR(c(F(K,Z) - (Ac)(K,Z), Z'))\} \end{aligned} \tag{22}$$

Equipped with operator A, we can define and characterize a recursive equilibrium.

Definition 4 Coleman's Minimal State Space Recursive Equilibrium (CMSSRE)

A CMSSRE is a *policy function* c_* such that:
i) is a fixed point of A defined in equation (22).
ii) The firm solves (11)
iii) Markets clear. That is, $x(K,Z) = F(K,Z) - c_*(K,Z)$
iv) The public sector runs a balanced budget. That is, equation (12) holds.

The results in the appendix (see section A.6) allows us to derive the following results, which are all straightforward applications of [Coleman, 1991].

Lemma 1 Properties of a CMSSRE

Let $\{c_n\}$ be a sequence of function generated iteratively using A with $c_0(K,Z) = F(K,Z)$. Then,
i) c_* is the unique strictly positive fixed point of A
ii) c_* is a CMSSRE
iii) $c_*(.,Z)$ is continuous for any $Z \in \mathbb{Z}$ and monotonic in $\mathbb{K} \times \mathbb{Z}$
iv) $\{c_n\}$ converges to c_* in the Sup Norm
v) Let $K' = K(K,Z) = F(K,Z) - c_*(K,Z)$. Then, the markov process generated by $(\mathbb{K} \times \mathbb{Z}, p)$ has an ergodic measure

Proof: See the appendix, section A.6.

Where p is a Markov kernel (see [Stokey, 1989], chapters 7 to 9). Now we turn to the background results for the ECM. From section 3.4 we know that, if $HR(.,Z)$ is decreasing for any Z, the envelope theorem holds. Thus, along the equilibrium path (i.e. $k = K$) we must have:

$$u'(c)HR(K,Z) = V'_*(K,Z) \tag{23}$$

Note that equation (23) implicitly assumes that $h_{*,*} = H_*$. Following [Mirman et al., 2008] we know that, as HR is decreasing, the iterative procedure described in section

4.1 converges under the sup norm. In particular, we know that the mapping from the perceived to the actual law of motion, $H_n \to h_{n,*}$ induced by the recursive problem (19) generates an ordered space $\{c_n, H_n\}$ and the convergence to H_* is uniform. Moreover, we know also from [Mirman et al., 2008] that H_* is differentiable. However, as discuss in section 2, the ECM does not induce a monotonic operator. That is, for the stochastic case, the ECM can be described as follows:

1. Pick an initial condition $c(K, Z)$
2. Compute capital tomorrow $x(K, Z) = F(K, Z) - c(K, Z)$
3. Compute the value function $V(K, Z) = u(c(K, Z)) + \beta E_Z V(x(K, Z), Z')$
4. Update consumption $\hat{c}(K, Z) = u'^{-1}\left[\frac{V'(K,Z)}{H(K,Z)}\right]$
5. Continue until convergence under the sup norm $c \to \hat{c} \to \cdots$

Even if we choose c in order to insure the joint monotonicity of c and x in steps 1 and 2, as $V'(., Z)$ and $H(., Z)$ are both decreasing, $\hat{c}$ may not be monotonic. Thus, $\{c_n\}$ is not an ordered space of equicontinuous functions and, thus, convergence is not insured. However, this is not because $\hat{c}$ is not maximal in (19) as claimed in [Arellano et al., 2016]. That is, as $x(., Z)$ is differentiable for each Z, we have:

$$V'(K, Z) - u'(c(K, Z))HR(K, Z) = \left[\beta E_t \left[V'(x(K, Z), Z')\right] - u'(c(K, Z))\right] x'(K, Z) \quad (24)$$

It is easy to see that, if $x'(K, Z) \approx 1$ for every (K, Z), step 4 in the ECM algorithm implies that $\{c_n\}$ is maximal in (19). However, as claimed above the ECM does not converge under the sup norm even if $x'(K, Z) = 1$.

5.2 Numerical performance: ECM

The table below illustrates the relative performance of the ECM with respect to its benchmark, the CMSSRE. Each result is classified according to the initial condition in the sequence $\{c_n\}$. For instance, the second row shows the results for simulations obtained using a policy function numerically convergent using the ECM and with $c_0 = c_{UB}/2$, where $c_{UB}(K, Z) = F(K, Z)$; the initial condition for operator A. The ECM algorithm was described in the previous section. The algorithm for the benchmark case, the CMSSRE, is described in the appendix (see section A.4 and A.6).

Model	Relative Mean	Relative CV	Mean x'_{ECM}
ECM, $c_0 = c_{UB}$	0.002	558.8	0.9696
ECM, $c_0 = c_{UB}/2$	0.049	45.4	0.9672
ECM, $c_0 = c_{UB}/4$	0.175	47.8	1.045

Table 5: Relative Performance of the ECM

As in the previous case when HR is non-monotonic, frequently used procedures subestimate the mean and over-estimate dispersion measures with respect to the ergodic equilibrium. Note that there is a pattern: the lower the initial condition, the lower the bias with respect to the mean. This is a consequence of equation (24) and the last column of table 5, which insures the maximality of c, x. In particular, note that as $x'_{ECM} \approx 1$, equation (24) and step 4 in the ECM algorithm implies:

$$\hat{c}(K, Z) = u'^{-1}\left[\frac{\beta E_t(V'(x(K, Z), Z'))}{H(K, Z)}\right] \tag{25}$$

As market clearing holds, we know that x is decreasing in c for each (K, Z). Thus, as $V'(., Z)$ is decreasing, $\hat{c}(., Z)$ is decreasing in c as observed in table 5.

6 Conclusions

This paper presents an example of an economy with multiple equilibria and a continuous policy function. This type of equilibrium is useful for accurately assessing the predictions of the model as it allows to generate reliable simulations which can be used to generate counterfactuals that are useful to evaluate alternative economic policies. We also present a condition, the monotonicity of the Euler equation, that is associated with exact simulations and provide a description of the reasons behind the lack of accuracy of them. We use the closed form nature and the existence of a MSS the recursive equilibrium together with the induced Feller mechanism to test the accuracy of MSS methods. The results in this paper does not depend on any numerical procedure, they constitute a unique opportunity to assess the performance of state-of-the-art algorithms.

From a purely economic perspective, as the model is built to study the effect of economic policies, the presence of a bias constitutes a major drawback for the conclusion generated by the model. In particular, the presence of capital taxes generates a reallocation between consumption and savings which are not captured by state-of-the-art algorithms. Ad valomen taxes on capital must shift the composition of aggregate demand towards consumption, away from investment. As standard procedures subestimates the long run average level of consumption, the estimated response to economic policies will be frequently misrepresented.

The paper also connects two branches of the recursive literature: the one concerned with the existence of a steady state (see for instance [Santos and Peralta-Alva, 2005]) and the one concerned with the existence of a recursive representation of the sequential equilibria ([Kubler and Schmedders, 2002]). We show that there is no equivalence between the continuity of the equilibrium and its uniqueness, a fact that is useful for simulating the model reliably.

The results in this paper have to be generalized. It is necessary to understand the connection between the number of possible exogenous states and the number of distinct economically meaningful recursive equilibria. That is, as the degree of the polynomial in the equilibrium equation is increasing in the number of exogenous states and each root of the polynomial defines a different mechanism (provided that the root is real and consumption / capital are positive), there is a tradeoff between a realistic shock process and the predictive performance of the model as more than one possible mechanism generates a less conclusive model. Moreover, the condition required for (1) is sufficiently general to be used in other branches of the literature such as default models.

7 Appendix

7.1 Figures

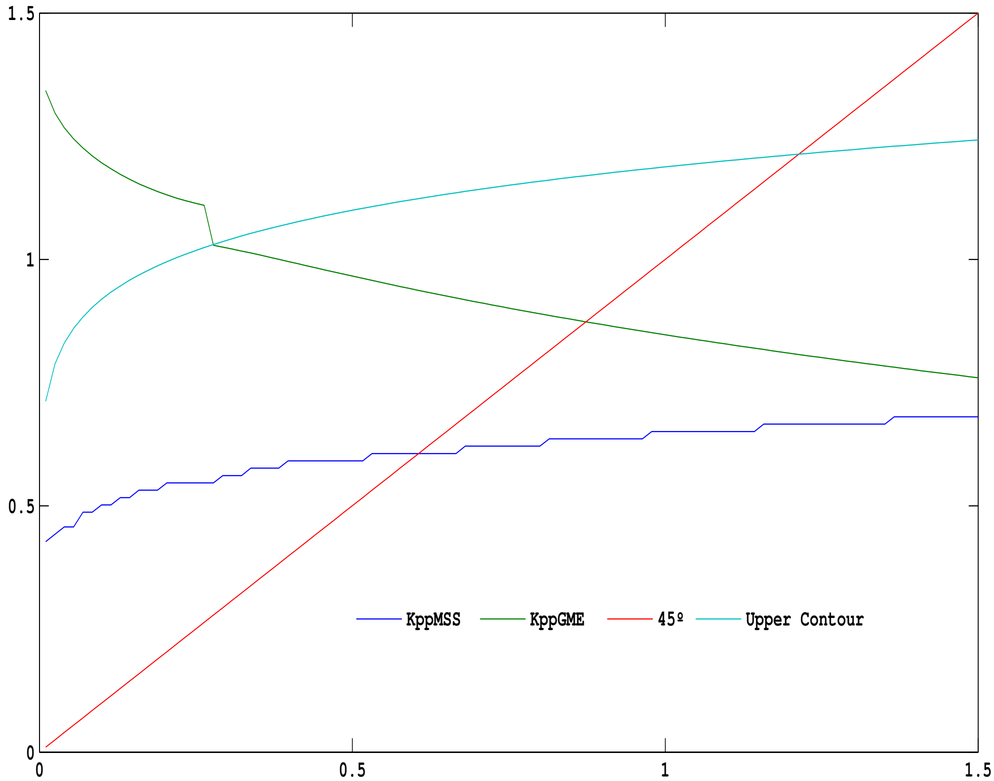

Figure 6: Phase Diagram

The light blue curve is the upper contour for K_{++}, the blue curve is $H(H(K,Z),Z)$ for the MSS and the green line is $H(K,K_{+},Z)$ for the GME, where K_{+} is fixed in the 50th grid point.

7.2 Theorems and useful definitions

A.1 Invariant Measures

Let S be the state space and P a markov operator of the process (A, P). σ_S is the Borel sigma algebra generated by S. P has the Feller property if $P(s, A)$ is continuous (in s) for any $A \subseteq S$. P is tight if S is compact. The operator P maps the space of Borel σ_S -measures $\mathbb{P}$ into itself as follows: $\mu'(A) = \int P(s, A)\mu(ds) \equiv P\mu$.

Theorem A1 (Futia, 1982, page 383, Th. 2.9) If P has the Feller property and is tight, then there is a measure μ such that $\mu = P\mu$

A.2 Supergradients of Concave functions

The following paragraphs are borrowed from [Rockafellar, 1981]. A function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitzian (LL) if : $|f(x'') - f(x')| \leqslant \lambda |x'' - x'|$, where $\lambda \geqslant 0$ and x'', x' belong to a neighborhood of $x \in R^n$. A concave function is LL. Moreover, the generalized directional derivative (GDD) is given by:

$$f^{\circ}(x, v) = limsup_{\ x' \to x,\ t \downarrow 0}\ \frac{f(x' + tv) - f(x')}{t}$$

When f is LL, the GDD is finite. However, the GDD may be "bizarrely disassociated" from f (see [Rockafellar, 1981] page 5). Thus, we need to connect $f^{\circ}(x, v)$ with the "classical" directional derivative (DD):

$$f^{'}(x, v) = lim_{t \downarrow 0}\ \frac{f(x + tv) - f(x)}{t}$$

We know from [Rockafellar, 1981] (see page 6), that when f is concave, $f^{\circ}(x, v) = f^{'}(x, v)$ for all x, v. Of course if f is differentiable $f^{'}(x, v) = f'(x).v$, where f' is the gradient. Moreover, let the superdifferential ∂f be defined as:

$$\partial f = \{p \in \mathbb{R}^n \mid f(x) + p \cdot (y - x) \geqslant f(y),\ x, y \in \mathbb{R}^n\}$$

For concave functions at interior points ∂f is non-empty, finite and $p \in \partial f$ satisfies $p.v \geqslant f^{'}(x, v)$. Thus, we have a connection between the tangent p of a concave function and it's DD, which is finite at interior points. Finally, if $f : \mathbb{R} \to \mathbb{R}$, the left and right derivative ($f'(x^-), f'(x^+)$ respectively) satisfy $f'(x^-) \geqslant f'(x^+)$ and f' (the derivative) has at most a countable discontinuity set. The left derivative is a minor simplification with respect to $f^{'}(x, v)$:

$$f^{'}(x^-) = lim_{t \downarrow 0}\ \frac{f(x - t) - f(x)}{t}$$

A.3 Classical Dynamic Programming

The following paragraphs are borrowed from [Stokey, 1989]. Note that equation (19) and the feasibility correspondence Γ define a standard dynamic programming program, as in [Stokey, 1989], with states (k, Z) for a given K. In order to prove the strict concavity of $V(k, ., .)$ (Theorem 9.8, page 265) we need the following set of assumptions: i) $k \in X \subset \mathbb{R}^n$, ii) $Z \in \mathbb{Z}$ and $\mathbb{Z}$ is countable, iii) $\Gamma(k, K, Z)$ is continuous in k, iv) Let A be the graph of $\Gamma(k, K, Z)$. Then, $u : A \to \mathbb{R}$ is bounded and continuous, v) u is strictly concave for each $Z \in \mathbb{Z}$, vi) $\Gamma(k, K, Z)$ is convex for each $Z \in \mathbb{Z}$. If additionally, we assume that u is differentiable in the interior of A for each $Z \in \mathbb{Z}$, $V(k, ., .)$ is continuously differentiable (Theorem 9.10, page 266).

Unfortunately, when we look at Γ when $k = K$, we loose some properties listed above. As τ is decreasing, $(1 - \tau(K))K$ is increasing and, as F is strictly concave in K, $r(K, Z)$ is decreasing. Moreover, given the functional for for F in Table 1, as τ is piecewise linear continuous, (see [Santos, 2002]), $(1 - \tau(K))Kr(K)$ is convex, which implies that for some $y \in \Gamma(x, Z)$, $y' \in \Gamma(x', Z)$, we have $\theta y + (1 - \theta)y' \notin \Gamma(\theta x + (1 - \theta)x', Z)$. Thus, we may fail to have a concave and differentiable value functions as property vi) is not satisfied. We need thus to use some properties of LL functions, which are described below.

A.4 Lattice Dynamic Programming and Supermodularity

The following paragraphs are borrowed from [Mirman et al., 2008] and [Amir et al., 1991]. If equation (19) has interior solutions, $V_*'(k^-, ., .), V_*'(k^+, ., .)$ exist every where for each K, Z, in particular when $k = K$ (see [Amir et al., 1991], Lemma 3.3). As $u(c) = ln(c)$, we know that solutions will be interior. From Lemma 12 in [Mirman et al., 2008] we know that V_* is LL. The results in A.2 imply in turn that $V_*^{\circ}(k, ., .)$ is finite, thus, $V_*'(k^-, ., .), V_*'(k^+, ., .)$ are finite.

As $(k, K) \in \left[0, \overline{K}\right] \times \left[0, \overline{K}\right]$ and $\mathbb{Z}$ has finite cardinality and it is bounded, we know that the domain of V_* is a complete lattice (a POSET endowed with the pointwise order such that each pair of elements in it, has a least upper bound $\wedge$ and a greatest lower bound $\vee$ that belong to $\left[0, \overline{K}\right] \times \left[0, \overline{K}\right]$). Then, V_* is supermodular if: $V_*(x \vee y, Z) + V_*(x \wedge y, Z) \geqslant V_*(x, Z) + V_*(y, Z)$. This concept is not really useful. Fortunately, we have an alternative characterization which is called increasing difference (ID). V_* has ID if: $V_*(k, K_1, Z) - V_*(k, K_2, Z)$ is non-decreasing in k for $K_1 \geqslant K_2$. [Mirman et al., 2008] propose a set of sufficient conditions (see assumption E in page 78) in order to insure that V_* has ID which in turn imply that the operator T is convergent in a very precise sense. We will only mention the assumptions that are not satisfied in the model presented in this paper.

Assumption A.4.1 $u'(c(k, K, Z))(1-\tau(K))r(K)$ is increasing in K and $0 \leqslant c(k, K', Z) - c(k, K, Z) \leqslant F(K', Z) - F(K, Z)$ with $K' \geqslant K$.

If assumption A.4.1 is satisfied, not only V_* has ID (Lemma 12) but also: $T^{\vee j}(F) \to h^{\vee}_{*,*}$, where $T^{\vee j}(F)$ is the j-th iteration of T starting at F taking the supremum of each maximal element in $ArgmaxV_*$ and $h^{\vee}_{*,*}$ is the supremum in the set of fixed points of T. Moreover, $T^{\wedge j}(0) \to h^{\wedge}_{*,*}$, where the interpretation is analogous. Now we turn to the result in [Coleman, 1991], which are generalized in [Mirman et al., 2008].

Assumption A.4.2 $u'(c(k,K,Z))(1-\tau(K))r(K)$ is decreasing in K and $0 \leqslant c(k,K',Z) - c(k,K,Z) \leqslant F(K',Z) - F(K,Z)$ with $K' \geqslant K$.

If assumption A.4.2 is satisfied we can define an operator, based on (15), which insures that there exist a MSSRE and that it can be computed by successive approximations (see [Mirman et al., 2008], theorem 10, page 86). As $(1-\tau(K))r(K)$ is non-monotonic, we can not use any of these results.

A.5 Stationary Markov Equilibria

The results in this section are borrowed from [Duffie et al., 1994]. Let $\Psi : X \to X$ be the correspondence which defines the GME (see definition 2). Let $C_0 = \left[0, \overline{K}\right] \times \left[0, \overline{K}\right] \times \mathbb{Z}$. Then, we can define a sequence of sets as follows:

$$C_1 = \{x_0 \in X \mid \Psi(x_0) \cap C_0 \neq \emptyset\} \equiv Q(C_0)$$

Let $\{C_i\}_i$ be the sequence of sets generated iteratively using Q. If C_i is non-empty and compact, then $\cap_i C_i = J$ is non-empty and compact and satisfies the self-generation property (i.e. $x \in J$ implies $\Psi(x) \cap J \neq \emptyset$). Intuitively, J is a stationary state space for the markov process generated by P_Ψ. Note that we have modified Q in section 4.2 in order to insure that consumption is positive along the equilibrium path. Thus, we can not use this theorem in order to prove that J is well defined. However, we found a parameter structure and a mechanism which gives a stationary state space numerically.

A.6 Euler Equation Operators

This section contains a summary of the main results in [Coleman, 1991] and the required definitions to make the paper self-contained.

Each element in a set of *equicontinuous functions* $\{c_n\}$ defined on a compact set $\mathbb{K}$ satisfies: for each $k \in \mathbb{K}$ and n in a countable set, if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|c_n(k) - c_n(k')| < \epsilon$ with $|k - k'| < \delta$. The critical fact in an equicontinuous set of functions is the existence of a bound, ϵ, which is uniform across n, k. Note that a collection of function in (21) form a set of equicontinuous functions and this is closely connected with the fact that, both, consumption and investment are increasing in K. The Arzela-Ascoli theorem states that a closed metric space of bounded real valued functions (i.e. $\{c_n\}$ endowed with the sup norm, $|.|$) defined on a compact set (i.e. $\mathbb{K}$) is *compact* if it is equicontinuous. Thus, convergence in $\{c_n\}$ is *uniform* (using the sup norm).

Tarki's fixed point theorem states that a continuous, monotone self map of a nonempty partially order (see section A.4) compact set (the candidate is A defined over C in section

5.1) in which some element c_0 is mapped downwards (i.e. $Ac_0 \leqslant c_0$) has a fixed point (in A) and a sequence of functions generated iteratively from this map (i.e. $\{c_n\}$, with $Ac_n = c_{n+1}$) converges to a maximal fixed point (i.e. in case there are more than one). In order to use this theorem, we need to show that: i) A is continuous and monotone, ii) $\{c_n\}$ is a partially ordered compact set, iii) c_0 maps down the sequence. [Coleman, 1991] shows i) in propositions 4 and 5, ii) proposition 3 and 5, iii) it follows by setting $c_0(K, Z) = F(K, Z)$. Moreover, $\{c_n\}$, with $Ac_n = c_{n+1}$, converges uniformly to the maximal fixed point (proposition 6). It remains to show that the fixed point of A is unique and strictly positive. Under the parameter structure in section 5.1, these facts follows respectively from theorem 11.

The last 2 paragraphs coupled with section A.1 suffice to *show Lemma 1.*

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