Accuracy in Recursive Minimal State Space Methods *

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Abstract

The existence of a recursive minimal state space (MSS) representation is not always guaranteed. However, because of its numerical efficiency, this type of equilibrium is frequently used in practice. What are the consequences of computing and simulating a model without a constructive proof? To answer this question, we identify a condition which is associated with a convergent and computable MSS representation in a RBC model with state contingent taxes. This condition ensures the existence of a benchmark equilibrium that can be used to test frequently used algorithms. To verify the accuracy of simulations even if this condition does not hold, we derive a closed form recursive equilibrium which contains the MSS representation. Both benchmark representations are accurate and ergodic. We show that state of the art algorithms, even if they are numerically convergent, may underestimate capital (and thus overestimate the benefits of capital taxes) by at least 65%, a figure which is in line with recent findings using accurate benchmarks. When an existence proof is not available, we found 2 sources of inaccuracy: the lack of a convergent operator and the absence of a well-defined (stochastic) steady state. Moreover, we identify a connection between lack of convergence and the equilibrium budget constraint which implies that simulated paths may be distorted not only in the long run but also in any period. When we have a constructive proof, inaccuracy is generated by the lack of qualitative properties in the computed policy functions.

Keywords: accuracy, recursive equilibrium, state contingent fiscal policy.

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1 Introduction

Sometimes macroeconomics is about new answers to old questions. This is the takeaway point of [Nakamura and Steinsson, 2018]. The authors analysis is based on the lack of exogenous variability in the field. This is one of the reasons behind the lack of a definitive answers to old questions, but maybe not the only one. In case we would like to perform a structural analysis, models may not have closed form solutions. Thus, numerical methods are a fundamental ingredient in macroeconomics. Moreover, policy experiments frequently imply that the welfare theorems do not hold. In this framework, even in the simplest possible dynamic stochastic model, a canonical RBC with state dependent taxes on capital, the existence of a computable representation is not always guaranteed (see for instance [Santos, 2002]). This fact has not only theoretical but also practical implications: along with an existence proof, if it is *constructive*, it comes a convergent algorithm.

What are the consequences of computing and simulating a model without a constructive existence proof? In order to answer this question, we first identify a condition which is associated with a computable representation in a RBC model with taxes. This condition ensures the existence of a benchmark equilibrium that can be used to test frequently used algorithms. These type of benchmarks are available in optimal economies due to the presence of welfare theorems as it is well known since, for instance, [Stokey, 1989]. However, in non-optimal economies it is rather unusual to have the opportunity to accurately test frequently used methods. To verify the accuracy of simulations even if this condition does not hold, we derive a closed form recursive equilibrium. Both benchmark representations are compact, continuous and unique. When the benchmark is numerical, the convergence is uniform, and iterations start in a theoretically identified initial condition. Thus, we know that both benchmark representations are accurate and ergodic. These results taking together give us a unique opportunity to test commonly used methods.

We show that state of the art algorithms, even if they are numerically convergent, may overestimate the benefits of capital taxes, as measured by a change in the composition of aggregate demand in favor of consumption and against investment. Our estimation for these biases is above 65%, a figure which is in line with recent finding using accurate benchmarks (see [Pohl et al., 2018]). We find that simulations using standard algorithms underestimate the long run average of capital (i.e. the computed average is at least 65% below its accurate benchmark). Our goal is not to criticize these methods as it is not always easy to measure the trade-offs (between numerical efficiency and accuracy) faced by researchers, but to suggest that, if the identified condition holds, it is better to reduce the speed of computations in order to increase the precision of the numerical solution.

Since the seminal paper of [Lucas, 1978] macroeconomists have been using a recursive representation of sequential equilibria to solve and simulate models. There are numerical and theoretical reasons behind this choice. As regards the former, it is easier to numerically approximate a first order stationary dynamic process rather than the sequential representation originally defined. In reference to the latter, a markovian structure allows to define a well-behaved long-term equilibrium (i.e., a steady state) using a recursive equilibrium notion (see for instance [Blume, 1982]). Finally, and more importantly, the theoretical and computational arguments are related with each other since accurate numerical simulations requires a Markovian representation and an appropriate steady state (see for instance [Santos and Peralta-Alva, 2005] among others).

Numerical approximations can be done globally or locally. Depending on the question, typically involving large fluctuations, the former one is preferred. However, global methods are more costly numerically, especially in recursive macroeconomics. The number of states are deeply connected with this burden due to the curse of dimensionality. Thus, it is critical to investigate the properties of minimal state space (MSS) methods, which include only 2 states; 1 endogenous and 1 exogenous.

This paper has 2 contributions. First, we identify a verifiable condition, based on some qualitative properties of the expected marginal utility of consumption, that ensure the accurate performance of MSS methods in recursive macroeconomics. Our results hold in a RBC model with state dependent capital taxes, but the identified condition can be found in a large fraction of applied papers. Second, we measure the bias of solutions using an *accurate benchmark*. When the critical condition holds (i), we compute a convergent operator which has a continuous and unique fixed point; a fact which ensures the ergodicity of simulations. When this condition does not hold (ii), to keep the benchmark accurate, we derive a closed form continuous recursive equilibrium with an expanded state space. In both cases, we obtain a recursive representation with an ergodic invariant measure, a finite number of exogenous shocks and a well-behaved state space (i.e., compact). For case (i), the bias is generated by the lack of qualitative properties of the solutions obtained from state of the art algorithms. A recursive equilibrium with minimal state space exists and it can be computed, but the algorithm cannot find it. For case (ii), we found 2 sources of inaccuracy: the absence of a minimal state space recursive representation and the non-existence of a well-defined (stochastic) steady state.

This paper proposes a verifiable condition which is associated with the constructive existence of a MSS recursive equilibrium. [Kubler and Schmedders, 2002] argued that in the presence of multiple equilibria a MSS recursive representation may not exist. As uniqueness has been an elusive quest in this field[1], this fact justifies the necessity of a condition associated with the existence of a MSS recursive equilibrium. We show that when the expected marginal return of assets is monotonic in the endogenous state, we can prove the existence of a MSS representation constructively. In the RBC model, this happens when the tax rate is either constant or increasing. If this requirement is not satisfied, to obtain an ergodic markov equilibrium, we must increase the number of endogenous variables which are considered states. By enlarging the state space, we show that it is possible to obtain multiple markovian representations in closed form, one of

[1] [Dana, 1993] provided conditions to guarantee the uniqueness of equilibria in an infinite horizon economy with complete markets. There is no analogous result for incomplete markets

them continuous with a stationary state space. This last "selection" allows us to derive a well-defined steady state by applying standard results. Formally, we derive a closed form generalized markovian equilibrium (GME) for a standard version of the RBC model with decreasing taxes on capital presented in [Santos, 2002]. As all MSS recursive equilibria form a subset of all GME, if both equilibrium types are well defined in the long run, any simulation from the latter must be matched using the former. It is shown that even a numerically convergent MSS algorithm may not match the ergodic distribution of the model as the MSS equilibrium might not have a well-defined steady state. The bias not only affects long run simulations. We identify a connection between the lack of existence of a MSS recursive equilibrium and the budget constraint which implies that simulated paths are distorted in any time period.

The results in this paper has direct take away point: even if the numerical procedure has been declared convergent using a demanding criteria, simulations may be far away from the exact steady state. If the identified condition holds, the results in [Coleman, 1991] ensure the monotone convergence of successive approximations to an equicontinuous MSS recursive equilibrium using the SUP norm. Then the results in [Santos and Peralta-Alva, 2005] guarantee the accuracy of simulations. When this condition does not hold, we use the closed form GME to compute the biases. As we still need to numerically simulate the exact Markov process, we need to show that simulations are accurate. We prove that the GME is also equicontinous, which ensures the accuracy of simulations. In this case, the MSS steady state of the model may not be well-defined as the equilibrium has discontinuity points. Faster methods when applied to the same model may generate significant biases because they do not preserve some of these qualitative properties, which are deeply connected with sup-norm convergence and accuracy.

The paper is organized as follows: section 2 presents an overview of the main results using a non-stochastic simple economy. Section 3 deals with the theory necessary to obtain an accurate benchmark. Section 4 and 5 presents the numerical test when the critical condition does not hold and holds respectively. Section 6 concludes.

1.1 Relation with the literature

Most macro model includes an Euler equation. In particular, the expected marginal utility of consumption satisfies:

$$\beta E\left[u'(c_+(K)(1-\tau(K))R(K)\right] \tag{1}$$

Where c_+ denotes consumption "tomorrow", exogenous shocks are assumed to be i.i.d., τ is a state dependent tax rate on assets K and R is the gross rate of return. In a model of with production (as in [Santos, 2002]), K denotes capital and R its marginal product. In a small open economy model, R is exogenous, τ is a tax on debt (we must reinterpret

K in this case) and represent a macro-prudential policy (see [Bianchi, 2011]).

Following [Mirman et al., 2008] and [Coleman, 1991], in this paper we show that if equation (1) is *monotonic* in K it is possible to derive an accurate algorithm that converges to a well-defined recursive equilibrium [2]. For instance, in the small open economy literature, as τ is increasing in external debt K but consumption is typically decreasing in it, equation (1) is not monotonic. In the RBC literature, if τ is decreasing in K, (1) is not monotonic. The purpose of this paper is to measure the accuracy of state-of-the-art algorithms when the monotonicity of (1) is and is not satisfied.

The literature (see for instance [Hatchondo, 2010]) focuses on the sensitivity of the numerical results to different methods without an accurate benchmark or test methods in optimal economies (see [Arellano et al., 2016]), but we were unaware of the size and reasons behind the bias in MSS methods in non-optimal economies. Contrarily to what is done in the numerical literature, we can measure this bias using an accurate solution which also has a well-behaved steady state. Thus, we can measure the short and long run implications of the lack of a constructive existence proof. [Santos and Peralta-Alva, 2005] performed a similar exercise for optimal economies. We extend those results for models with distortions.

From a theoretical point of view, we sharpen the characterization of ergodic recursive equilibrium in [Blume, 1982]. We provide evidence against the equivalence between a continuous markovian representation and the uniqueness of the sequential equilibrium. In words of [Blume, 1982]:

"the existence of a continuous selection - tantamount to the uniqueness of equilibrium in each state - is not often satisfied".

We found a stationary (i.e., time independent) recursive representation with multiple sequential equilibrium in some nodes which has a continuous selection. This result is relevant to relax recently found conditions to ensure the existence of an ergodic steady state. These conditions are at odds with the computation of the model as they involve many continuations for each node (see [Santos et al., 2012]). The existence of a continuous selection in a model with a finite number of shocks is essential to ensure the ergodicity of simulations in a computable framework.

From a numerical perspective, this paper is connected with [Pohl et al., 2018]. We also use accurate benchmarks and find a similar bias with respect to state-of-the-art algorithms. While the results in [Pohl et al., 2018] depend on the presence of a persistent stochastic process, ours can be explained for topological reasons: the lack of continuity and the absence of an order structure in the space of policy functions.

[2] As shocks are i.i.d, the state space is composed only by K. If exogenous shocks were Markov, we need to include them as an additional state variable.

2 Preview of the results in a deterministic economy

In this section we present a simple preview of the results that will be found in this paper. For this purpose, we use a non-stochastic RBC model which is canonical except in the tax function. Based on equation (1), we divide this section in 2: first we deal with the fact that the Euler equation is not monotonic in the endogenous state. In this case, to test the accuracy of simulations, we derive a closed form recursive equilibrium, which is more general than the MSS representation. In the second case, where the Euler equation is monotonic, we can use a result due to [Coleman, 1991] in order to prove the existence of a MSS equilibrium. We use the former to test a canonical procedure based on iterations due to [Rios Rull, 2004] that exploit a map between perceived and actual laws of motion for equilibrium states. The latter is used to test the performance of envelope conditions methods (ECM) due to [Arellano et al., 2016], as we can ensure that the derivative of the value function (i.e., an envelope) is continuous provided that the monotonicity requirement holds. In both cases we can prove that the benchmark equilibrium exist, is unique and is ergodic. Thus, they constitute a proper instrument to test the accuracy of simulations.

2.1 Non-Monotonic Euler Equations

Imagine a canonical RBC model distorted with ad valorem taxes. As a distinctive fact, the aliquot can vary along with the business cycle. That is, it is state dependent. It will be assumed that it is *decreasing* in the aggregate state of the economy. There is an infinitely lived representative agent endowed with k_0 units of capital. She must choose a sequence of consumption and savings for each unit of time, denoted $t \geqslant 0$, to maximize her lifetime utility. For simplicity, we will assume for now that there is no uncertainty, and that capital depreciates entirely after 1 period. Accumulated saving is rented to a firm, which is assumed to maximize profits using a decreasing returns to scale technology represented by a strongly concave production function. There is a Government that levies an ad-valorem tax on rental income. As mentioned, the aliquot depends on the aggregate state of the economy, denoted K, even though this connection is not perceived by the agent. The Government rebates back the collected taxes making lump-sum transfers to the agent. Finally, as the agent owns the capital stock, she receives the profits form the firm. Formally, time is discrete and infinite, $t = 0, 1, 2....$ Let k denote the supply of capital (services) and K its demand. There is a decreasing return to scale firm which only uses capital as input and its technology is characterized by $y_t = f(K_t)$ with $f' > 0$, $f'' < 0$ and $f(0) = 0$ as usual.

As the firm is owned by the consumer, as she is endowed with $k_0 > 0$ units of capital, she has two sources of current income: benefits, denoted by π_t, and rents from capital, denoted by $r_t k_t$. The flow of taxes paid and transfers received is $\tau(K_t) r_t k_t$ and T_t respectively.

The problem faced by the consumer is to choose consumption c_t and investment x_t that solves the following problem:

$$\max_{\{c_t, x_t\}} \sum_t \beta^t u(c(z^t)) \tag{2}$$

s.t.

$$k_{t+1} = x_t + (1-\delta)k_t \tag{3}$$
$$c_t + x_t = \pi_t - (1-\tau(K_t))r(z^t)k_t + T(k_t, K_t) \tag{4}$$

$c_t \geqslant 0$ and $k_0 > 0$ given, $\delta \in [0,1]$ is the depreciation rate and $\beta \in (0,1)$ the discount factor. Moreover, τ is the aliquot for the ad-valorem tax, r is the rental rate, π denotes profits, x_t represents investment and, due to full depreciation, k_{t+1}. Finally, T_t are transfers.

The problem of the firm is standard. Taking r_t as given it solves:

$$\max_{K_t} f(K_t) - r_t K_t. \tag{5}$$

The Government simply transfers to the consumer the tax revenues:

$$T = \tau(K_t)r(z^t)k_t. \tag{6}$$

The Government runs a balanced budget, $\tau(K_t)r(K_t)k_t = T_t$, and profit maximization implies $F(K_t) = \pi_t(K_t) + r(K_t)K_t$ where $r(K_t) = F'(K_t)$ and $F(K_t)$ denotes aggregate resources [3]. As there is a single firm, the only price in the economy, r, is a function of aggregate capital, K. Moreover, tax collection can explicitly depend on the aggregate state of the economy in order to capture the interaction between the business cycle and fiscal policy.

Replacing the equilibrium conditions into the flow budget constraint, we get:

$$c_t + x_t = F(K_t) + (k_t - K_t)F'(K_t)$$

The above equation is the aggregate budget constraint. Note that gross rental income is proportional to individual capital holdings, k, and that the problem of the firm imply that profits depend on aggregate states. In equilibrium, it will be required that $k = K$. Thus, if we can ensure that individual and aggregate capital stocks remains closed to each other along the computed equilibrium trajectories, we will say that the decentralized

[3] When $\delta = 1$ aggregate resources, F, and aggregate output, f, are equal.

equilibrium is *not distorted*.

Now suppose we want to solve the model and simulate this economy. The canonical approach since [Rios Rull, 2004] is to use the associated dynamic programming program and the policy functions derived from it. In this framework, the agent is supposed to solve:

$$V(k, K) = Max_{c,x}\ u(c) + \beta V(k', K')$$

Subject to

$$x, c \in [0, \pi(K) + (1 - \tau(K))r(K)k + T(k, K)]$$

$$x + c = \pi(K) + (1 - \tau(K))r(K)k + T(k, K)$$

$$K' = G(K)$$

Where G is the *perceived* law of motion for aggregate capital.

Definition. Minimal State Space Recursive Equilibrium, (MSSRE): A MSSRE in this economy is a pair of policy functions $c(k, K), x(k, K)$ such that:

$$c(k, K) + x(k, K) = F(K)$$

$$\tau(K)r(K)k = T$$

$$x(k, K) = G(K)$$

$$k = K$$

$$r(K) = F'(K)$$

We call $x(k, K) = G(K)$ the *rational expectations condition.*

Now suppose we want to simulate the economy, given $k_0 = K_0$. We can use the set of policy functions iteratively. In order to take care of rational expectations condition, note that the Bellman equation above define a mapping $T(G_n)(k, K) \longmapsto G_{n+1}(k, K)$, where $T(G_n)(k, K) = x(k, K; G_n)$. In a recursive equilibrium the perceived law of motion G_* satisfies $T(G_*)(K, K) = x(K, K; G_*) = G_*(K)$.

Since [Rios Rull, 2004] it is frequent to iterate on T, starting from an arbitrary initial condition, obtaining a sequence $\{G_n\}_n$. It is not clear if G_n will converge to G_* starting from an arbitrary initial condition G_0. Since [Mirman et al., 2008], we know that if $u'(c(k,K))(1-\tau(K))F'(K)$ is decreasing or increasing (in K, i.e., is monotonic), we can provide a positive answer to this question. Unfortunately, there are some cases where this condition does not hold for any K. In this section, as in [Santos, 2002], we assume that τ is decreasing in K, which in turn implies that $u'(c(k,K))(1-\tau(K))F'(K)$ when $k = K$ is not monotonic as F is strictly concave.

More to the point, if we iterate on T, the limiting function G_∞ satisfies:

$$c^{'} + x^{'} = F(K') + (x(K,K;G_\infty) - K')F'(K')$$

Where $K' = G_\infty(K)$. Now, if the numerical procedure does not converge, we know that the perceived G and the actual x law of motion for capital will not be equal, at least for some K. That is, $T(G_\infty)(K,K) = x(K,K;G_\infty) \neq G_\infty(K)$. Thus, as the above equation suggests *the lack of convergence implies directly a bias in the computed long term capital stock* as the resources available to the household are permanently distorted by the numerical procedure.

Note that the convergence criteria in any numerical procedure is relative. That is, the algorithm will be "declared convergent" if for $n \geqslant N(\epsilon)$:

$$SUP_K \left| \frac{x(K,K;G_n)-G_n(K)}{G_n(K)} \right| < \epsilon$$

Where ϵ is the tolerance level. Thus, it is possible that $x(K,K;G_n) - G_n(K)$ may be far away from zero even though the numerical procedure has "converged".

So far, we have discussed the implications of the lack of convergence on equilibrium decisions. However, we have been silent about simulations. The first step is to define a proper steady state, as simulated paths must converge to a meaningful object (i.e., an unconditional moment of a stationary distribution). Since [Futia, 1982] we know that compactness and continuity are sufficient to ensure the existence of a well-behaved steady state (see theorem A.1 in the appendix). Assume that K belongs to a compact set. Since [Stokey, 1989] [4], we know that there are curvature conditions associated with F which ensure the desired compactness, so this assumption seems mild. However, in non-optimal economies, the continuity of the equilibrium equations remains an open question. For instance, [Coleman, 1991] showed that if $(1-\tau(K))F'(K)$ is decreasing (in K), there is a continuous recursive equilibria. However, in this section, as the net rental income is not monotonic, we cannot use this result. We deal with the numerical implications of the results in [Coleman, 1991] in the next subsection.

[4]see chapter 5.1.

Let $g_\tau(k,K) \equiv \pi(K)+(1-\tau(K))r(K)k+T(k,K)$. If $u(g_\tau(k,K)-x)$ is strictly concave (in k,x) and the feasibility correspondence for the recursive problem is convex, we know from [Stokey, 1989], see section A.3 in the appendix, that $V(k,K)$ is strictly concave (in k). Unfortunately, in the present framework, we cannot ensure the desired properties when $k=K$ and thus the value function may not be concave (see section A.3 for a discussion for the stochastic case). Thus, we need to use more general results. From [Rockafellar, 1981] and [Amir et al., 1991] we know that V has a well-defined directional (left) derivative (see section A.2 in the appendix). As, V is not concave, the standard envelope theorem does not hold even if the return function is differentiable (see section A.3 in the appendix). To see why, note that the differentiability of V would have implied that:

$$V'(k',K') = u'(g_\tau(k',K') - x(k',K'))(1-\tau(K'))F'(K')$$

At $k=K$, the strict concavity of V would imply that $u'(g_\tau(K)-x(K))(1-\tau(K))F'(K)$ is decreasing in K, a fact that requires the monotonicity of $(1-\tau(K))F'(K)$ and ensures the sufficiency of first order conditions, which does not hold in this subsection by assumption. Thus, any optimal solution must satisfy:

$$u'(g_\tau(k,K) - x(k,K)) = \beta u'(g_\tau(k',K') - x(k',K'))(1-\tau(K'))F'(K')$$

Where the above equation follows from the necessity of the Euler equation (see [Amir et al., 1991]). As the tax function may generate multiple equilibrium values for $x(K,K)$ that solves this equation (see [Mirman et al., 2008]), the MSSRE *may not be continuous.* Since a well-defined steady state requires continuity, the simulated paths may not be convergent.

In order to test the (numerical) implications of the lack of a convergent operator and / or the discontinuity of the equilibrium laws of motion this paper shows the existence of a continuous and closed form recursive equilibrium in an enlarged state space. We call this equilibrium notion *Generalized Markov Equilibrium* (GME). The qualitative properties of this type of equilibrium allow us to test the size of the bias as any MSSRE must satisfy the requirements of our definition. To ensure stationarity and compactness, we build a modified version of canonical result due to [Duffie et al., 1994] (see section A.5 in the appendix).

Definition. Generalized Markov Equilibrium, (GME): Let K, K_+, K_{++} be the capital stock today, tomorrow and the day after tomorrow, respectively. Then, the first order condition associated with the sequential equilibrium for this economy, for interior solutions, is:

$$u(g_\tau(K) - K_+) = \beta u'(g_\tau(K_+) - K_{++})(1-\tau(K_+))F'(K_+)$$

A GME is a function H mapping (K, K_+) into K_{++} that solves the above equation for any (K, K_+) in the expanded state space $\left[0, \bar{K}\right] \times \left[0, \bar{K}\right]$, where $\bar{K}$ is an upper bound for capital [5].

One of the main contributions of this paper is to find a function, $H(K, K_+) = K_{++}$ continuous, unique and with closed form which satisfy the above equation in an equilibrium path (i.e., when the transfers are budget feasible and the goods market clear). As we do not need to solve for $x(K, K)$, which is the source of the discontinuity in the MSSRE, it is natural to expect a well-behaved solution.

In order to test the implications of our findings on the MSSRE, we can use a result in [Amir et al., 1991]. The authors showed that even if the net rental income is not monotonic, any solution to the dynamic programming program associated with a MSSRE must satisfy:

$$u(g_\tau(K) - x(K)) = \beta u'(g_\tau(x(K)) - x(x(K)))(1 - \tau(x(K)))F'(x(K))$$

Where $x(K, K) = x(K)$ maybe discontinuous. Thus, any MSSRE is a GME as it is restricting K_+ to satify $K_+ = x(K)$.

Suppose that we heuristically find a convergent sequence of functions $\{G_n\}_n$ which is also a MSSRE. In the numerical section below, we provide an example of this type of functions. That is, we avoid the problems associated with $x(K, K; G_n) - G_n(K)$ in the equilibrium budget constraint of the household which may distort every period and we focus on the long run. Despite the fact that the solution is convergent, we found that the computed MSSRE converges to a steady state quite far away from the "true" equilibria. The pictures below illustrate the situation at hand: as K_+ is not pin down by any stationary function (i.e., x in the MSSRE), the demarcation lines in the plane (K, K_{++}) are pushed towards the boundary of the system during the whole transition. Of course, this is not the case for the MSSRE.

Figure 1 is borrowed from the numerical section of this paper (see section 7.1). It depicts the demarcation lines for K, K_{++} given K_+, which are downward sloping and increasing in K_+. Also, the "upper contour" line reflects the maximal level of K_{++} for a given K, where the boundary reflects the zero consumption pairs. Note that for an arbitrary large n, K_n orbits near the intersection of the 45º ray with upper contour line as the demarcation curves becomes "sufficiently flat". That is, if $K_{n-1} < K_n$, with $H(K_{n-1}, K_n) = K_{n+1}$ and K_{n-1}, K_n sufficiently close to each other, because K_n intersects the 45º degree line and the demarcation curve is flat in the neighborhood of K_n we know that $K_n = K_{n+1}$. Then, in the next iteration, we have $H(K_n, K_{n+1}) = K_{n+2} = H(K_n, K_n) = K_n$, where the last equality follows from the fact that $H(K_{n-1}, K_n)$ and

[5] The existence of this bound follows from standard arguments in [Stokey, 1989]. See chapter 5.1.

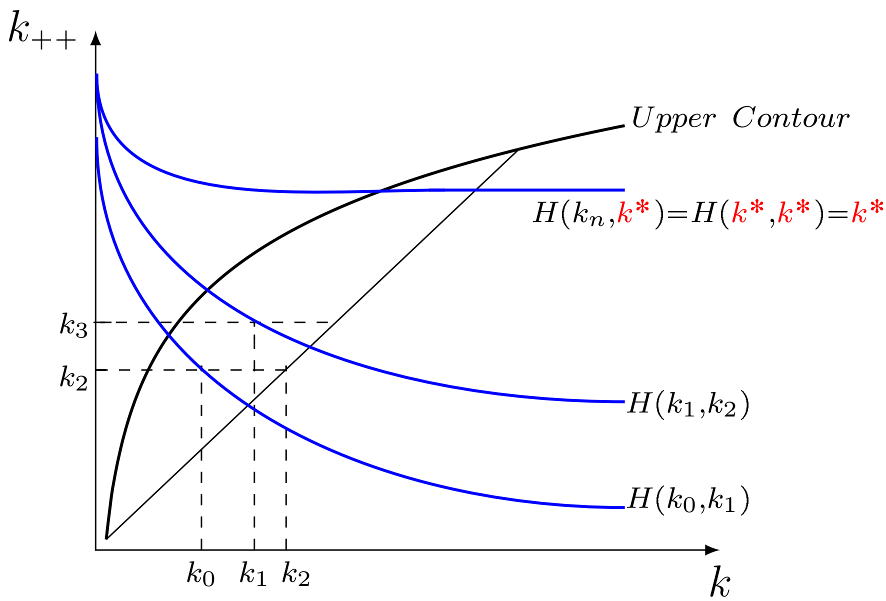

Figure 1: Dynamic Behavior in a Generalized Markov Equilibria (GME)

$H(K_n, K_{n+1})$ are in the same contour. Then, we simply call $K_n = K^*$ by the definition of a (non-stochastic) steady state.

Figures 2 and 3 illustrate 2 different (numerically) convergent G_*. We know from previous paragraphs that the equilibrium policy function in the MSSRE may not be continuous, maybe near its intersection with the $45^{\underline{o}}$ line. Moreover, any convergent and continuous MSS with $K_n = K_{MSSRE}, n \geqslant N_\epsilon$, will also satisfy $K_n = H(K_n, K_n)$. We found 2 selections for the closed form GME. However, as a stationary Markov process is constructed using a time independent transition function and state space, only 1 them is suitable for the purpose of this paper. This is because one of the 2 selections generate a the state space that is not stationary. Thus, as we only have 1 time independent GME, the Markov equilibrium in the expanded state space is unique and, *if the MSSRE is continuous*, the 2 equilibria must display the same long run behavior. Figure 2 depicts this fact.

Figure 2 shows the pairs $(k, x(k))$ (in blue) and $(k, x(x(k)))$ (in green) which satisfy the equivalence between the 2 equilibrium types.

We now turn to the numerical solution of the problem. As we are computing the MSSRE in a finite grid, denoted $\{K_j\}$, we choose to plot points, which are interpolated for expositional purposes. Note that eventually, we can find a pair of elements in the grid which satisfy:

$$K_{n+3} = K_{n+2} = Argmax\,\{W(K_{n+1})(K_j)\}_{K_j} = Argmax\,\{W(K_{n+2})(K_j)\}_{K_j}$$

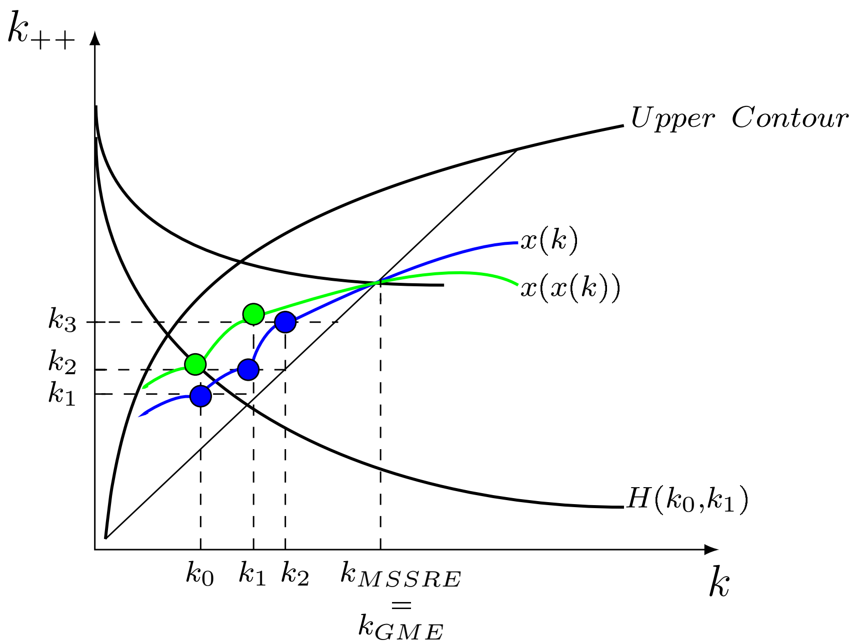

Figure 2: Minimal State Space Markov Equilibria (MSSRE) and GME

Where W is the objective function of the Bellman equation in the MSS problem with $k = K$. The expression above is the numerical equivalent to $x(x(K_*)) = K_*$. Note that, as we are dealing with a finite set of points, the continuity requirement is trivial as we can endow the function with the discrete topology. Thus, convergence is achieved numerically even if the function is not continuous. Figure 3 depicts a discontinuous mechanism which will be declared convergent by any iterative procedure based on a finite set of points. *This figure illustrates one of the main findings of the paper: it is possible that a numerically convergent MSSRE has a significant bias with respect to the exact unique ergodic equilibrium (i.e., the GME) because it does not have a steady state.*

The blue dots are the pairs $(K, x(K))$. Note that we are plotting an evenly spaced grid $(k_j, ..., k_{j+4})$ and a discontinuity point of the MSSRE (in black dots). The actual image of k_{j+4} does not belong to the grid. Moreover, *Argmax* $W(k_{j+4})$ is closer to k_{j+4} than any other point in the grid. Thus, as W is typically "bell shaped", the algorithm will pick k_{j+4} as a solution to the maximal problem when the aggregate state is k_{j+4}. Thus, we have $x(k_{j+4}) = k_{j+4}$ even though this policy function does not have a steady state. Note that because all MSSRE are a subset of all possible GME, as we show that the latter is unique, the former must be discontinuous in the presence of a bias between the 2 steady states. If the MSSRE is continuous, the uniqueness of the GME implies that they must have the same steady state as depicted in figure 2. When we solve the stochastic model, we found several discontinuities in the MSSRE and an important bias with respect to the ergodic GME.

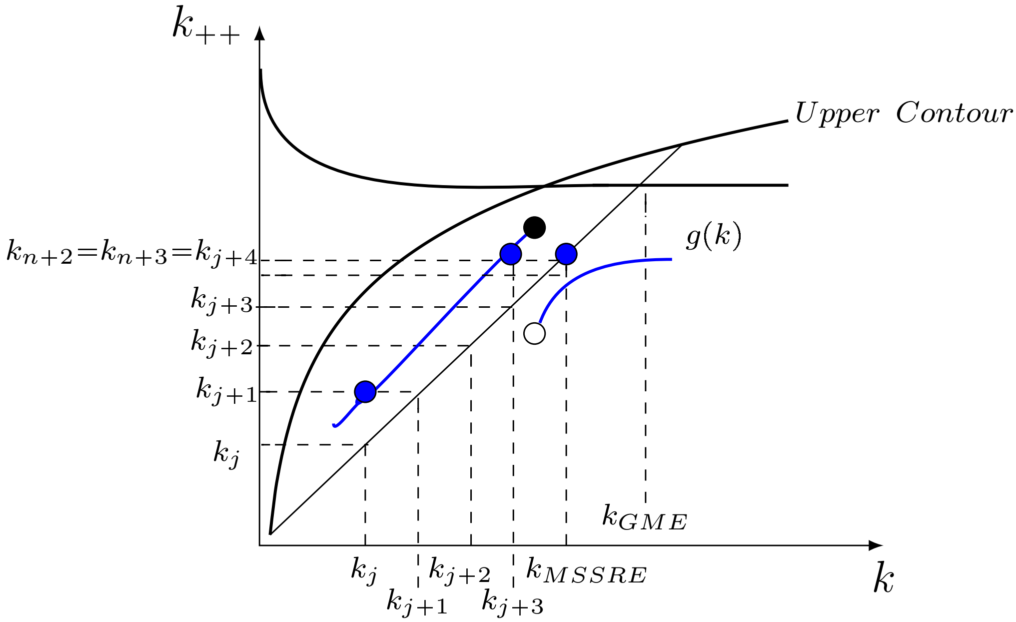

Figure 3: Dynamic Behavior in a Discontinuous MSSRE over an evenly spaced grid $\{K_j\}$

2.2 Monotonic Euler Equations

In this case, as $(1-\tau(K))F'(K)$ is decreasing, we know that the derivative of the value function is continuous and, thus, the classical Euler equation holds. Equipped with this structure, we can define a monotonic operator A. Instead of iterating on G, we will define a procedure based on monotonic consumption functions, c. Along the equilibrium path, i.e., $k = K$, the presence of an optimizing representative agent implies that we can write the Euler equation $u'(c_t) = \beta u'(c_{t+1})(1-\tau(K_t))F'(K_t)$ as follows:

$$u'((Ac)(K)) = \beta u'(c(F(K) - (Ac)(K)))HR(F(K) - (Ac)(K))$$

Where $HR(K) \equiv (1-\tau(K))F'(K)$. There are 2 critical facts about operator A: i) it is continuous, ii) it can be defined in the space of equicontinuous monotonic functions once the set containing c is wisely chosen (see the appendix, section A.6). Moreover, the existence of a strictly positive fixed point can be proved using a theorem in the Tarski family, provided the initial condition of the iterative procedure is appropriately chosen. This fact ensures, together with some curvature assumptions, that $Ac = c$ is continuous and unique (see the appendix, section A.6). Thus, when we move to the stochastic case, if shocks are drowned from a finite state markov process, we are confident that simulations are ergodic (see the appendix, section A.1). Finally, due to the equicontinuity of the consumption set, the sup norm ensures convergence. This is due to the Arzela-Ascoli theorem (see the appendix, section A.6).

The operator A not only ensures the presence of a constructive existence proof which can be used to defined an algorithm, it also preserves critical properties when it comes to compute and simulate the model (i.e., uniform convergence and ergodicity). However, it has a cost in terms of CPU time as noted by [Arellano et al., 2016]. To circumvent this problem, the authors defined an algorithm based on the existence of a differentiable value function called Envelope Condition Method (ECM).

If $u'(c(K))(1-\tau(K))F'(K)$ is decreasing we know that the Euler operator defined above is continuous (see appendix A.4 and A.6 which is based on [Coleman, 1991]), the standard envelope theorem works (see appendix A.3 which is based on [Stokey, 1989]), and the first order conditions are sufficient. Thus, we can use standard results to compute the equilibrium. The operator A allows us to measure the trade-off inherent in the use of the ECM: numerical efficiency vs. accuracy. Below we briefly discuss the results described in the numerical section for the stochastic case.

The ECM can be briefly described as follows:

1. Pick an initial condition c for each K
2. Compute capital tomorrow $x(K) = F(K) - c(K)$
3. Compute the value function $V(K) = u(c(K)) + \beta V(x(K))$
4. Update consumption $\hat{c}(K) = u'^{-1}\left[\frac{V'(K)}{H(K)}\right]$
5. Continue until convergence under the sup norm $c \to \hat{c} \to \cdots$

Note that we can pick the same initial condition that works for operator A, a monotonic function. However, step 4 does not preserve the monotonicy in $\hat{c}$ as the numerator and the denominator are both decreasing. Moreover, the computed policy function for capital is almost linear with a slope slightly bigger than 1. These facts have at least 2 important implications. Let x_{ECM} be the numerical fixed point found by the ECM. Then: i) as $x_{ECM}(0) > 0$ and x_{ECM} is linear with a slope bigger than 1, there is no compact ergodic equilibrium. ii) As the ECM does not ensure equicontinuity (see appendix A.6), $c \to \hat{c} \to \cdots$ does not converge under the sup norm. The literature is aware that the ECM is not convergent. However, the arguments typically rely on the fact that the ECM does not ensure that the computed policy functions are maximal (i.e., does not satisfy the sufficient conditions associated with the recursive problem). However, in this paper we argue that, even if the ECM does preserve maximality, it does not achieve convergence. The reasons can be found in a combination of facts i) and ii). In particular, following [Arellano, 2008], after totally differentiating $V(K) = u(c(K)) + \beta V(x(K))$ with respect to K the derivative of the value function yields:

$$V'(K) - u'(c(K))HR(K) = \left[\beta V'(x(K)) - u'(c(K))\right] x'(K) \tag{7}$$

As $x'(K) \approx 1$, we know that imposing the envelope condition (the left hand side in equation (7)) ensures that the sufficient condition for the maximality of the policy function holds (the right side in equation (7)) . However, the Euler equation implied by (7) and the ECM method has a different structure when compared with operator A. Note that A ensures that not only Ac is increasing, but also $F(K) - (Ac)(K)$ has the same property (see [Coleman, 1991]). This last condition is critical to ensure that A generates a set of equicontinuous functions $\{c_n\}_n$, with $c_{n+1} = Ac_n$ (see the appendix, section A.6). This is not the case in equation (7). More to the point, fact i) implies that the equilibrium generated by the ECM is not compact and thus the Arzela - Ascoli theorem, which ensures the compactness of an equicontinuous set of function under the sup norm, can't hold. Thus, the convergence criteria in step 5 is not appropriate as $c \to \hat{c} \to \cdots$, if it converges, it does so under a weaker topology.

3 Equilibrium Definitions

From the previous section it was clear that when the Euler equation is not monotonic, a fact associated with a decreasing tax function, a Minimal State Space Recursive Equilibrium (MSSRE) is too restrictive to get a continuous representation, which in turn is critical for simulations. In this section we show that by expanding the state space we gain enough degrees of freedom to find a continuous recursive representation, even when the tax function is decreasing. This representation is called Generalized Markov Equilibrium (GME) and it is derived directly from the Sequential Competitive Equilibrium (SCE). To understand the connection between these 3 equilibria the subsections below describe them and relate them with each other.

3.1 Sequential Competitive Equilibrium

The model is a stochastic version of [Santos, 2002] (section 3.2). Consider a representative agent economy with discrete time, $t = 0, 1, 2....$ Exogenous shocks are markovian and will be denoted z. For the sake of simplicity let us assume that the state space for these shocks is $\{0, 1\}$. An element of the transition matrix will be denoted $p(.,.)$, where the first position denotes rows and the second columns. Let $\{z_t\}$ be a sequence of shocks and Z^t the set of histories up to time t, being a typical element z^t. Using standard results (see [Stokey, 1989], Ch. 8) it is possible to define, for any $z_0 \in \{0, 1\}$, a stochastic process $(\Omega, \sigma_\Omega, \mu_{z_0})$ on Z^∞.

Preferences are represented by a utility function U and a instantaneous return function u, where u is continuous differentiable, strictly concave and strictly increasing.

As in this section we are dealing with a sequential economy, k denotes the supply of capital (services) and K its demand. There is a unique decreasing return to scale firm which only uses capital as input and its technology is characterized by $y_t = A(z_t)f(K_t)$

with $f' > 0$, $f'' < 0$ and $f(0) = 0$ as usual. The firm is owned by the consumer as she is endowed with $k_0 > 0$ units of capital. Thus, the agent has two sources of current income derived from her endowment: benefits, denoted by π_t, and rents from capital, denoted by $r_t k_t$. The flow of taxes paid and transfers received are $\tau(K_t) r_t k_t$ and T_t respectively. Note that the tax rate depends on the stock of capital. It is given by a piecewise linear continuous function which can be either decreasing, increasing or constant with respect to K. This is the distinctive feature of the model as it will allow us to classify the recursive equilibrium depending on whether it is monotonic, if the tax function is constant or increasing, or it is not, when taxes are decreasing, (see [Santos, 2002], page 87 for details).

The problem faced by the consumer is to choose a pair of functions $c : Z^\infty \longrightarrow \mathbb{R}_+$ and $x : Z^\infty \longrightarrow \mathbb{R}_+$ that solves the following problem:

$$\max_{\{c,x\}} \sum_t \sum_{z^t \in Z^t} \gamma^t u(c(z^t)) \mu_{z_0}(z^t) \tag{8}$$

s.t.

$$k(z^t) = x(z^t) + (1-\delta) k(z^{t-1}) \tag{9}$$

$$c(z^t) + x(z^t) \leqslant \pi(z^{t-1}) - (1 - \tau(z^{t-1})) r(z^t) k(z^{t-1}) + T(z^t) \tag{10}$$

$c(z^t) \geqslant 0, k(z^t) \geqslant 0$ for any $z^t \in Z^t$, z_0 and $k_0 > 0$ given, $\delta \in [0,1]$ is the depreciation rate and $\gamma \in (0,1)$ the discount factor.

Note that we are restricting the maximal random variables (c, x) to take values on $\mathbb{R}_+$. This restriction will be relevant for the recursive representation of the sequential equilibria as boundary conditions will be critical to prove existence of a stationary state space. In what follows $\tau(z^{t-1})$ stands for $\tau(k(z^{t-1}))$ or abusing notation $\tau(k_t(z^{t-1}))$. That is, the tax rate affects the rents obtained from capital holdings at time t, which is in turn affected by the information contained in z^{t-1} because $k_t(z^{t-1}) = x_{t-1}(z^{t-1}) + (1-\delta) k_{t-1}(z^{t-2})$. A similar argument can be used to understand $r(z^t)$ because the agent knows the clearing condition for the market of factors and the optimality condition for the firm to be described below.

The problem of the firm is standard. Taking r_t as given it solves:

$$\max_{K_t} A(z_t) f(K_t) - r_t K_t, \qquad \text{for any } z_t \in \{0,1\}. \tag{11}$$

Observe that the optimality of the firm implies $r_t = A(z_t) f'(K_t)$. The Government simply transfers to the consumer the tax revenues:

$$T = \tau(z^{t-1}) r(z^t) k(z^{t-1}). \tag{12}$$

Finally, goods and factor markets clear:

$$c(z^t) + x(z^t) = A(z_t) f(K_t) \qquad \text{Goods Market}$$

$$k(z^t) = K_{t+1} \qquad \text{Capital Market}$$

where both equations hold for any $z^t \in Z^t$.

Note that in equilibrium, the optimality condition of the firm and the market clearing equation for capital holdings implies $r_t = A(z_t)f'(k(z^{t-1}))$ which in turn implies $r_t = r(z^t)$ as claimed. Further, both market clearing conditions imply $c(z^t) + x(z^t) = A(z_t)f(k(z^{t-1})) = y(z^t)$ as expected.

We can now define the sequential equilibrium for this economy:

Definition 1 A Sequential Competitive Equilibrium (SCE) for this economy is composed by a triad of functions z^t measurable functions (x, c, r) such that:

- Given r, (x, c) solve the Maximization problem of the household.
- For each z^t, given $r(z^t)$, $K(z^t)$ solves the problem of the firm.
- For each z^t, Goods and Capital markets clear.
- For each z^t, the Government runs a balanced budget, equation (12).

Assuming $\delta = 1$, the SCE can be characterized by:

$$u'(C_t) = \gamma \sum_{z_{t+1}=0,1} A(z_{t+1})p(z_t, z_{t+1})(1 - \tau(K_{t+1}))f'(K_{t+1})u'(C_{t+1}), \tag{13}$$

With constrains given by

$$K_{t+1} = A(z_t)f(K_t) - C_t. \tag{14}$$

3.2 Generalized Markov Equilibrium

The discussion in the appendix (see section A.2) describes a recursive mechanism based on an *enlarged* state space X. In particular, we wrote K_{t+2} in terms of (K_t, K_{t+1}, z_t):

$$K_{t+2} = g(K_t, K_{t+1}, z_t).$$

The mechanism, g, is *closed form* and, even more, continuous (of course, this representation has economic content if we can assure that the boundary conditions on endogenous variables generated by g are satisfied).

Equipped with g we can define a stochastic version of the GME described in section 2.

Definition 2: Generalized Markov Equilibrium (GME)

A GME is a *correspondence* $\Psi : X \to X$ with X compact such that for any $x \in X$, the vector $(x, \Psi(x))$:
i) satisfies the optimality conditions for the household problem, equation (8) s.t. (9) - (3).
ii) The firm solves (11)
iii) Markets clear
iv) The public sector runs a balanced budget. That is, equation (12) holds.

In section A.2 of the appendix we show that, if we can ensure the existence of a well-behaved state space, the sequential version of the model presented in this paper has a GME representation. Let Ψ_i be any selection of Ψ. Using standard results (see [Stokey, 1989]), we can show that $P_{\Psi_i}(x, A)$ defines a Markov kernel with $P_{\Psi_i}(x, .)$ being a probability measure for any $x \in X$ and $P_{\Psi_i}(., A)$ being a measurable function for any $A \in Borel(X)$. An invariant measure is any fixed point of P_{Ψ_i}. Call one of the possible many fixed points μ_i.

Let Ψ_i^j be any numerical approximation of Ψ_i and $P_{\Psi_i^j}(x, A)$, μ_i^j the associated Markov kernel and invariant measure respectively. Since [Santos and Peralta-Alva, 2005], it is known that even if Ψ_i^j converge to Ψ_i, the simulations obtained from Ψ_i^j may differ from the exact ones, generated using Ψ_i. If Ψ_i is equicontinuous and defined over a compact state space, these authors showed that numerical simulations will match the exact long run behavior of the model. However, equicontinuity is associated with very restrictive properties for non-optimal economies as noted in [Coleman, 1991].

The virtue of this paper is that it allows us to circumvent the mentioned problems. On one hand, we show that *a GME exist* for the problem at hand and thus, it is possible for us to compute it. Moreover, using (21) and (22), we show that Ψ_i has a *continuous closed form representation*, which in turn eliminates the problem associated with the lack of convergence of numerical simulations, provided that we can find a suitable state space.

The (numerical) cost of this representation is the enlargement of the state space with respect to the natural one (i.e., (K_t, z_t)). As we have a closed form solution, these costs are more than compensated by the accuracy of simulations. As discussed in [Kubler and Schmedders, 2002], enlarging the state space might provide a recursive representation. Unfortunately, the results in that paper does not address the continuity of the mechanism; an aspect that has severe consequences for the steady state of the model as discussed in [Duffie et al., 1994].

After taking care of the boundary conditions, we can ensure the compactness of the state space. Coupled with the continuity of the mechanism, Ψ_i, we can show existence of μ_i using canonical results in [Futia, 1982]. See section A.1 in the appendix for a detailed discussion about the existence of invariant measures in compact spaces.

We can use these results to simulate the model. Let $U_t := K_{t+1}$, we have now the following iterative system:

Take first an arbitrary initial condition (K_0, U_0, z_0) and a drawn $\{z_n\}$, then

$$\begin{aligned} K_{t+1} &= U_t \\ U_{t+1} &= g(K_t, U_t, z_t), \end{aligned}$$

provides a sequence $\{X_n\}$. Such a sequence defines a Feller mechanism, with compact state space X.

The quadratic structure in (26) ensures that the compactness of the state space and the interiority of solutions are sufficient (if they are satisfied jointly, of course) to guarantee ergodicity (see section A.1 and A.7 of the appendix). Below we will define an operator which allows us to find a state space that it is compact and that ensures that capital and consumption remains positive along equilibrium paths. Provided that μ_i is ergodic, the process $\{K_t\}$ has a well-defined invariant measure as well [6]. Moreover, using standard results on laws of large numbers for markov processes (see [Varadhan, 2001]), it can be shown that choosing an appropriate initial condition suffices to guarantee that:

$$\frac{\sum_{t \in 0,\ldots,T} h(X_t)}{T} \quad \text{converges almost surely to} \quad E_\mu(h),$$

Where h is a X-measurable function and μ is one of the possibly many ergodic invariant measures described above.

Finally, note that U_{t+1} is measurable with respect to z^t, which in turn implies that K_{t+2} is measuble with respect to the same filtration. As $Z^t \subset Z^{t+1}$, the measurability requirements in definition 1 are satisfied. This is the cost of working with a Markov structure: we are losing memory inherited from the sequential equilibrium, a fact which may affect the empirical performance of the model as noted by [Pierri and Reffett, 2019].

3.3 Minimal State Space Recursive Equilibrium

This paper deals with global methods. Any researcher choosing them must deal with the limitations implied by the numerical burden associated with the solution of a considerable number of non-linear equations. Thus, it is natural to choose the minimal possible number of states as this option significantly reduces the main disadvantage of this type of methods.Thus, it is critical to understand the limitations of MSSRE methods. The MSS version of the model described above can be written as follows:

[6]Let μ_i be and invariant measure for $x \equiv (K, U, z)$. Then by integrating out K, z we can find the marginal distribution for U, which is an invariant measure for K'.

$$V_n\left(k,K,Z;H_j\right) = Max_{y\in\Gamma(k,K,Z)}\; u(g_\tau(k,K,Z)-y)+\gamma\sum_{Z'} V_{n-1}(y,H_j(K,Z),Z';H_j)p(Z,Z') \tag{15}$$

Where the feasibility correspondence is given by:

$$\Gamma(k,K,Z) = [y \in \overline{K}; 0 \leqslant y \leqslant g_\tau(k,K,Z)]$$

Capital is allowed to fluctuate in a compact set, $[0, K^{UB}] = \overline{K}$ and g_τ represent disposable income and is defined by:

$$g_\tau(k,K,Z) \equiv \pi(K,Z) + (1-\tau(K,Z))r(K,Z)k + T(K,Z)$$

Where $\pi(K,Z)$ and $\tau(K,Z)$ are defined in (9) and $T(K,Z)$ in (12). The policy function for (15) is given by $h_{n-1,j}(k,K,Z)$, which belongs to the set defined below:

$$argmax\left\{u(g_\tau(k,K,Z)-y)+\gamma\sum_{Z'} V_{n-1}(y,H_j(K,Z),Z',H_j)p(Z,Z')\; s.t.\; y\in\Gamma(k,K,Z)\right\}$$

Note, remarkably that: i) the household take a guess at the evolution of the aggregate states using a *perceived law of motion* denoted H_j. ii) The value and the policy function in the dynamic programming problem have to converge in j, which is associated with the rational expectation nature of the problem (i.e., the perceived and the actual law of motion must be equal when $k = K$), and in n, that is guaranteed by the contractive nature of the Bellman operator in (15). iii) The dependence of disposable, $g_\tau(k,.,.)$, on prices, $r(.,.)$, justifies the presence of *equilibrium states* which are represented by capital letters. They affect the household problem through the firm's decisions, given by (11), and market clearing conditions which are contained in the definition of recursive competitive equilibrium, which is given below.

Definition 3 Minimal State Space Recursive Equilibrium (MSSRE)

A MSSRE is a *value function* V_*, a *policy function* $h_{*,*}$ and a *perceived law of motion* H_* such that:
i) the household solves equation (15) obtaining $V_*(k,K,Z;H_*)$ and $h_{*,*}(k,K,Z;H_*)$ for any feasible state k,K,Z.
ii) The firm solves (11)
iii) Markets clear. That is, $k = K$
iv) Expectations are fulfilled. That is, $h_{*,*}(K,K,Z;H_*) = H_*(K,K,Z)$ for any (K,Z)
v) The public sector runs a balanced budget. That is, equation (12) holds.

To understand the connection between the existence of a MSSRE and its computation, we must characterize it. Even under strong curvature and smoothness assumptions on the return function u, which are all satisfied imposing the parametrizations used in section 3.1, even if we assume the continuity of the feasibility correspondence Γ, for an interior optimal solutions, $h_{*,j}(., K, .; H_j) \in \Gamma(., K, .)$, we can't use the implications of the envelope theorem in [Stokey, 1989] when $k = K$ [7]. In particular, as $V'(K, K, z)$ is not decreasing, the first order conditions are not sufficient. Moreover, as $V'(k, K, z)$ is continuous in k but $V'(K, K, z)$ may not be as $f'(K)(1 - \tau(K))$ is not monotonic, the operator defined by the Euler equation, presented below, may have multiple roots.

Fortunately, using lemmas 3.3 and 3.4 in [Amir et al., 1991] we know that any solution to the dynamic program must satisfy the classical Euler equation and, thus, it can be characterized (see section A.4 for a detailed discussion) using first order conditions. Formally, a solution to the dynamic programming problem in definition 3 for any pair of individual states (k, Z) and given the aggregate level of capital K must satisfy:

$$u'\left[g_\tau(k, K, Z) - h_{*,j}\right] = \gamma E_Z\left\{u'\left[g_\tau(H_j, h_{*,j}) - h_{*,j}(h_{*,j})\right] A f'(H_j)(1 - \tau(H_j))\right\} \quad (16)$$

Where the dependence of $h_{*,j}$ on (k, Z) for each K and of H_j on (K, Z) have been omitted for expositional purposes. 2 things must be noted from equation (16): 1) condition iv) in the definition of MSSRE may not hold in this model because we can not prove that this type of equilibrium exists. 2) As $f'(K)(1-\tau(K))$ is not monotonic, there maybe multiple values of H_j for some (K, z).

Equation (16) defines a mapping T from H_j to $h_{*,j}$. The discussion in section A.2 of the appendix (see the supplementary material for section 3.2) shows that any fixed point of this map is a MSSRE. This discussion also suggests that it is not possible to ensure that a sequence of function $\{H_j\}_j$ converging to H_* will "hit" $h_{*,*}$ as required by the definition of MSSRE. Existence proofs require either a convex policy correspondence or an order structure, which can not be proved in the non-monotonic case covered in this subsection. That is, *any numerical procedure based on iterations through* T using the uniform metric, as the one described in [Rios Rull, 2004], *cannot be proved to be convergent to a MSSRE.* Thus, $SUP\,|H_{*,j} - h_{*,j}|$ maybe arbitrarily large, a fact which can cause a severe bias in the numerical simulations as discussed in section 2.

[7] The arguments used in section 2 hold *mutatis mutandis*. In particular, Benvenisite and Scheinkman envelope theorem (see [Stokey, 1989] page 266, Th. 9.10) coupled with the strict concavity of V_n (in k) (see [Stokey, 1989] page 265, Th. 9.8) would imply that V_n' is decreasing in k when $k = K$, which will not hold globally as $f'(K)(1 - \tau(K))$ is not monotonic.

4 Decreasing taxes: non-monotonic Euler equations

The results in section 3 provide a unique opportunity to test the predictive power of MSS methods. As any MSSRE must satisfy equations (13) and (14), the simulations generated by it must converge to one of the possible multiple ergodic distributions obtained using a GME.

To perform this test, we present a standard recursive competitive MSS algorithm with 2 different "updating" rules. The first does not numerically converge to a fixed point between the perceived and actual law of motion and the second does, implying that in the latter case we are dealing only with the effects of a discontinuous equilibrium as explained in section 2. Then, the policy functions are simulated, and the results compared with those obtained from the exact solution of a GME, as described in the appendix (see section A.2, supplementary material for section 3.2). In order to ensure that the exact GME has a state space which generates a pair of equilibrium random variables (c, x) taking values in the in the non-negative real numbers, we adapt a theorem from [Duffie et al., 1994]. *We found that only 1 mechanism has a well-defined state space. Thus, we know that any MSSRE is a GME, which is also unique. So, any simulation obtained from the former, must match the latter if this equilibrium exists and it is continuous.*

As neither continuity nor existence hold in the MSSRE if $u'(c(K))(1 - \tau(K))f'(K)$ is not monotonic, we found a significant deviation with respect to the true equilibrium which, in turn, affects the long run distribution of capital. These findings provide evidence in favor of the results in [Hatchondo, 2010] and [Feng et al., 2015] which suggest the importance of theoretical results in the recursive numerical literature. That is, without sufficient conditions that ensure the equivalence between numerical and actual simulations of the model, *a convergent algorithm does not guarantee by itself the absence of biases.*

4.1 MSS Algorithm

We compute a MSSRE using the operator T, which in turn follows from equation (16). It is standard in the literature (see for instance, [Rios Rull, 2004]) to pick an arbitrary function H_0 as a candidate for the equilibrium aggregate perceived law of motion and look for uniform convergence. However, theoretical results do not support such a strong convergence notion [8].

[8] If $u'(c)(1 - \tau)F'$ is increasing in K, it is possible to show that any iteration starting from a lower or upper bound on T (i.e., $H \in B$ such that $H \leqslant T(H)$ or $T(H) \leqslant H$ respectively) will converge in the order topology. That is, take a sequence of increasing functions generated iteratively from T, $\{H_j\}$ with $H_{j+1} = T(H_j)$. We say that $H_j \rightarrow_{\geqslant *} H_*$, meaning $\{H_j\}$ converge in the order topology to H_*, if for any j, $H_j \leqslant H_*$ and $H_* \in B$, given the compactness of the state space, this will suffice (see section A.2, the supplementary material for section 3.2, A.4 and A.6 of the appendix for a detailed discussion). If $u'(c)(1 - \tau)F'$ is decreasing in K, the convergence will be uniform in the standard sup norm. Unfortunately, as τ is decreasing and F strongly concave, we showed in section 3.4 that T is not a monotonic operator and thus it is not possible to generate a convergent sequence of functions using T.

We do not have *known sufficient conditions* which ensure the convergence to a MSSRE using T. However, we found *numerically* a fixed point for T in the sup norm. In particular, the procedure described below was found to be convergent using the sup norm for acceptable relative error levels (in the order of 10^{-2})

$$H_0 \rightarrow_{Equation(15)} h_{*,0} \rightarrow_{Definition\ 3} \rightarrow H_1(K,Z) = G^1(H_0, h_{*,0})(K,Z) \rightarrow (...)$$

Where the first $\rightarrow$ means that we are solving equation (15) using H_0 as a guess for the perceived law of motion. The second $\rightarrow$ stands for the fact that we are computing the policy function $h_{*,0}$ along the equilibrium path according to definition 3. The last $\rightarrow$ implies that we are updating the perceived law of motion for aggregates states. The functional G is an updating rule. We use 2 different types of them:

- $G_1^j = \alpha H_j + (1-\alpha)H_{j-1}$, with $\alpha \in (0,1)$
- $G_2^j = \sum_{i=0}^{j} H_i/j$.

The last one was found convergent, that is: $n \geqslant N(\epsilon)$ imply $|G_2^n - h_{*,n}| < \epsilon$. Finally note that "$\rightarrow$ (...)" means that we are starting the loop again if convergence using the sup norm is not achieved.

4.2 Stationary GME

For any arbitrary state space X, with typical element $(K, K_+, Z) \in X$, equation $U_+ = K_{++} = g(K, U, z)$ may imply that $K_{++} \notin \mathbb{R}_+$ (i.e., K_{++} may be an imaginary or negative number). Let Z_L be the smallest possible shock. Note that we may have $A(Z_L)f(K) < K_+$ and / or $A(Z_L)f(K_+) < K_{++}$, which in turn imply that consumption is negative.

To solve these problems, we use theorem 1.2 in [Duffie et al., 1994]. The authors showed that, given the compactness of the sequential equilibria, it is always possible to find a stationary sate space $J \subseteq X$ for the markov equilibrium associated with any root of the system characterizing the closed form GME, see equation (26) in the appendix (section A.2, supplementary material for section 3.2), using an iterative procedure. For simplicity, the details of this procedure are contained in the appendix (see section A.2).

4.3 Simulations

We now turn to measure the numerical bias. in the previous section we described the algorithm typically used to compute a MSSRE and a GME.

The task is to compute definitions 2 and 3 using a concrete tax function based on the model in [Santos, 2002], a standard algorithm borrowed from [Rios Rull, 2004] and the refinement for the GME described in section 4.2. In particular, τ is decreasing in K. Thus, the operator T is not monotonic and, consequently, it is not possible to prove that a numerical procedure based on iterations using T will converge to a MSSRE. The parameters used to compute the model are contained in the table below. We are carefully following the preferences and technology structure in [Santos, 2002]. However, as this model is non-stochastic, we are setting the values for the exogenous shocks in set Z and transitions probabilities p_{LH} and p_{HL} in order to ensure a well-defined steady state for the GME.

$y = A(Z)f(K) = e^Z K^{1/3}$	$Z_H = 0.2$
$u(c) = ln(c)$	$Z_L = 0.1275$
$\delta = 1$	$p_{LH} = 0.5$
$\gamma = 0.99$	$p_{HL} = 0.3$

Table 1: Parameters

The table below contains the results of simulating a MSSRE and a GME. The parameters used are listed in Table 1. We refine the mechanisms for the GME using the operator defined in section A.2 of the appendix (see the supplementary material for section 4.2). We found 2 selections of the GME. For one of them $J = [0.01, 1.50]$ and for the other $J = \emptyset$. Thus, we will only report $\Psi_{NR} \equiv \Psi$, where "NR" stands for negative root.

Model	Mean	STD	CV
Ψ	1.1976	0.0079	0.0066
MSSAvg, $K_{UB} = 0.6$	0.4058	0.0117	0.0289
MSSCes, $\boldsymbol{K_{UB} = 0.6}$	0.2662	0.0106	0.0400
MSSCes, $K_{UB} = 1.5$	0.3098	0.0134	0.0431

Table 2: Simulation Results. Statistics for aggregate capital

Where STD stands for standard deviation and CV for the coefficient of variation (standard deviation / mean). The "empirical" distributions are constructed as follows: take an arbitrary initial condition. Simulate a path of 5000 observations for aggregate capital. Store the last 1000 observations. Then, the computed distribution is taken from the relative frequency of 25 grid positions out of these observations. The procedure is repeated for any of the 4 listed distributions.

We report 3 different solutions for the MSSRE, which differs in the updating rules G_1, G_2 discussed in section 4.1 and in the upper bound (K_{UB}) of the grid. The first one called "MSSAvg", which stands for "average", is not convergent and thus it contains the 2 sources of biases: the lack of a steady state and the lack of convergence. When we expand the state space, to make it comparable with J, the cesaro updating is not convergent. We highlight in bold font the convergent simulation which contains only 1 source of error. Below we report the associated biases.

Model	Min-Max Error	Min-Max Rel. Error	$A(Z_{LB})F'(K^*)$	Bias
MSSAvg, $K_{UB} = 0.6$	$[0.1062, 0.4248]$	$[0.9696, 41.3]$	0.6908	0.1834
MSSCes, $\boldsymbol{K_{UB}} = \mathbf{0.6}$	$[-0.0059, 0.0118]$	$[0.072, 0.1361]$	0.9151	0.0027
MSSCes, $K_{UB} = 1.5$	$[0.0596, 0.3129]$	$[0.3460, 0.8994]$	0.8270	0.1540

Table 3: Lack of Convergence: Implications for the accuracy of simulations

We define an error as the difference between the perceived ($H_{*,j}$) and actual ($h_{*,j}$) law of motion for capital. The columns in the table contains the [minimum - maximum] relative and absolute errors across iterations j using the sup-norm. The relative error determines the convergence of the algorithm. For instance, we declare **MSSCes, $\boldsymbol{K_{UB}} = \mathbf{0.6}$** convergent because for some j the relative error was 0.072. Note that only the cesaro updating procedure with a grid of $[0.01, 0.6]$ converged.

The absolute error is used to compute the distortion generated by the algorithm. If we take as a reference value the mean of the capital stock under each procure, denoted K^*, the lack of convergence of the algorithm implies a distortion of $(h_{*,j}(K, K, Z) - H_{*,j}(K, Z))A(Z)F'(K)$ in the equilibrium budget constraint $c + x = F(K)$. That is, on average, the MSSAvg procedure implies that the household receives 0.1834 more units of the consumption good due to the lack of convergence of the algorithm. Thus, as the agent is "wealthier", capital stock is higher when compared with the accurate solution among MSS algorithms (i.e., MSSCes, $K_{UB} = 0.6$).

The numerical solutions in Table 2 has a significant bias, as measured by the difference in mean with respect to the ergodic distribution. The table below presents the relative deviations.

Model	Relative Mean	Relative CV
MSSAvg, $K_{UB} = 0.6$	0.34	4.38
MSSCes, $\boldsymbol{K_{UB}} = \mathbf{0.6}$	0.22	6.06
MSSCes, $K_{UB} = 1.5$	0.25	6.53

Table 4: Relative Bias

Where "Relative" stands for $Mean(MSSAvg, K_{UB} = 0.6)/Mean(\Psi)$, etc. From Table 4 the mean of the ergodic accurate distribution (as measured by the GME) is way above the mean generated by any numerical approximation of the MSSRE. *We observe that mean capital is at least* 66% *below the accurate mean.* On the contrary, the dispersion is significantly below. Thus, despite the fact that the algorithm for the MSSRE converge for the case of $MSSCes, K_{UB} = 0.6$ using a strong criteria (i.e., the sup norm and a tolerance level of 0.075 for the relative error), the numerical distribution will present a severe bias with respect to the distribution *is well defined*.

The results described above point out to the relevance of a well-defined steady state (i.e., a fixed point of $P_H(K, Z; .)$, where P is the markov kernel defined in section 3.2 but constructed using the perceived law of motion for the MSS, H). From section 3.3 we know that the discontinuity of $V'_{*,1}$ plays a central role in this fact. Below we show the (numerical) derivative of the value function for the $MSSCes, K_{UB} = 1.5$ when $k = K$. We choose this solution as the state space is comparable with J.

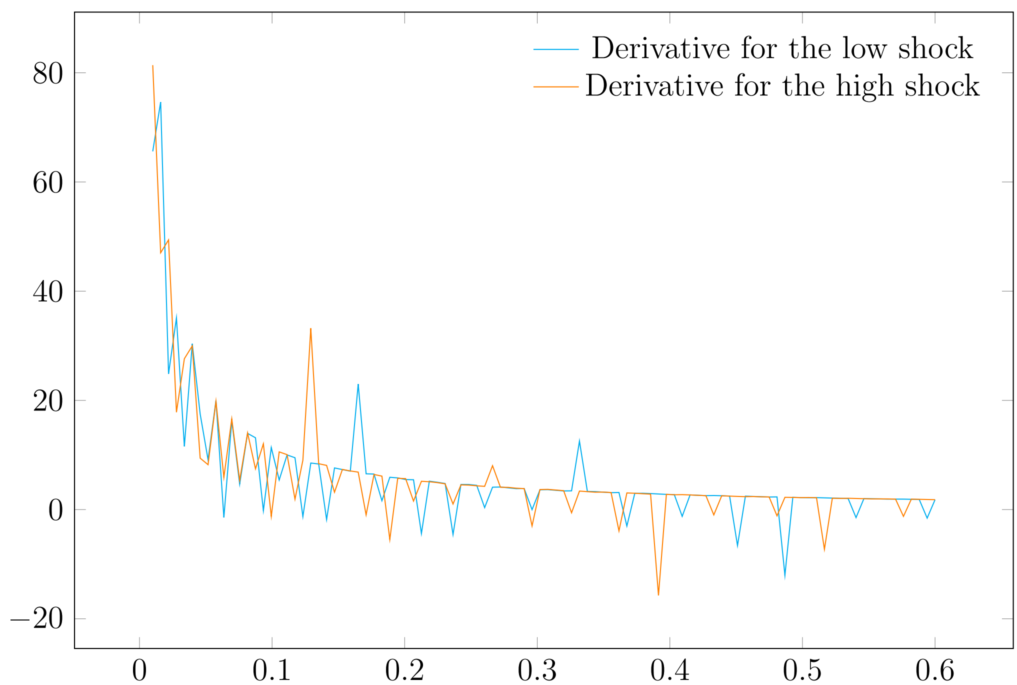

Figure 4: Numerical Derivative of the Value Function when $k = K$

The blue line represents the derivative for the low shock. Even though the figure depicts the expected convex shape for a concave function, it has several jumps, suggesting the presence of more than 1 discontinuity. More to the point, the discontinuity set seems large and dependent on the TFP shock, Z. Thus, it would be difficult to know when we have a model with a well or an ill behaved steady state as depicted in figure 3. These jumps are especially relevant near the numerical long run distribution. Below we plot figure 4 for the points in the grid that have positive mass in the long run (i.e., $[Mean - 2STD, Mean + 2STD]$ for the updating rule and grid size given by $MCes, K_{UB} = 1.5$).

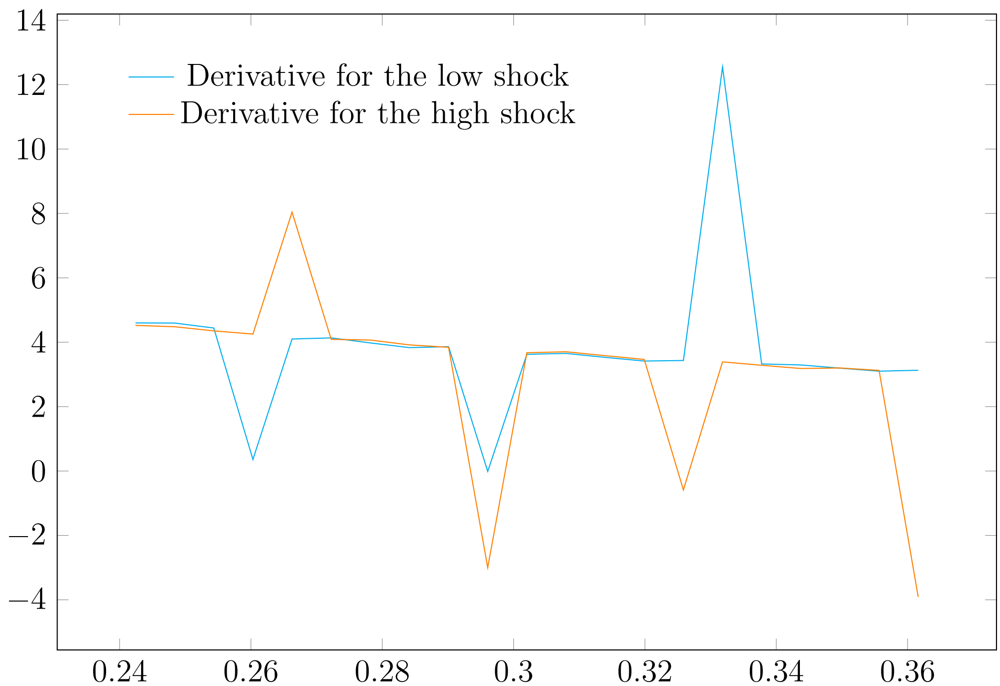

Figure 5: Numerical Derivative of the Value Function in the long run

Even though near the mean, 0.3098, the derivative seems continuous, the existence of any unconditional moment depends on a well-defined invariant measure (i.e., on a fixed point of the markov kernel P_H). Thus, the jumps to the left and right of the numerical mean are relevant for the existence of a steady state as depicted in figure 3. More importantly, given the finite cardinality of the grid, the algorithm may not capture the discontinuity and display a well-behaved histogram.

As discussed in previous sections the observed bias could be generated either by the lack of convergence of the perceived to the actual law of motion (i.e., $H_j \to H_* \nrightarrow h_{*,*}$) or by any difference between the numerical and the actual steady state (i.e., $\mu_i^j \to \mu_i^* \nrightarrow \mu$ for any computed MSS algorithm $i \in 1, 2, 3$ where μ is the fixex point of P_Ψ). As the regards the former, note that any MSSRE must satisfy equation (16) which in turn ensures that any path generated using $h_{*,*}$ along the recursive equilibrium will also be a sequential equilibrium. In other words, any path generated from a MSSRE satisfies equations (13) and (14). The lack of coincidence between the perceived and the actual law of motion will generate a distribution of capital that does not belong to any possible sequential competitive equilibrium, which explains part of the bias as measured in Table 3. Moreover, as a continuous MSSRE may not exist for this model, we cannot ensure the existence of a well behaved steady state for this type of equilibria (i.e., μ_{MSSRE} may not exist). If that is the case, any numerical distribution, namely μ^j_{MSSRE}, could be arbitrarily far away from μ as it is not possible to show that $\mu_{MSSRE} \to \mu$.

The figure in the appendix show, for the sake of completeness, the phase diagrams depicted in figures 1 and 2. The numerical histograms used to compute the results in table 2 are available under request.

We turn to the policy implications of the results in this paper. Note that the mean capital stock for the $MSSCes, K_{UB} = 1.5$ algorithm is well below the ergodic mean. Thus, as τ is decreasing with $\tau \to 0$ for $K \to K_{UB}$, using the MSS to predic the long run behavior of this model we may conclude that the observed tax rate is positive. Recent results, see for instance [Straub and Werning, 2020], have shown that the optimal tax rate is strictly positive in the long run. So, the policy advice would be to slightly change the observed tax rate, if any. However, the true distribution, with a support close to the upper bound of J, calls for a increase in the tax effective tax rate.

5 Increasing taxes: monotonic Euler equations

In this section we briefly modified the parameter structure in the previous paragraphs in order to match the results in [Coleman, 1991]. We first derive the theoretical structure for the benchmark case. Then we perform the numerical test of the Envelope condition method.

5.1 Preliminary Results

The sequential equilibrium is the same as the one in definition 1. The differences with respect to section 3 and 4 are as follows:

- $\delta = 0.019$
- $u(c) = c^{1-\sigma}/(1-\sigma)$ with $\sigma = 4$
- $f(K) = K^{1/2}$
- $ln(Z_{t+1}) = \rho ln(Z_t) + \epsilon_{t+1}$ with ϵ_{t+1} being a distributed according to a normal distribution with mean 0 and standard deviation 0.007, discretized with a 10 points grid.
- $\tau(K) = 0.1K/K^{UB}$ and $HR(K, Z) \equiv (1-\delta) + Zf'(K)(1-\tau(K))$ is decreasing

Definition 1 implies that any sequential competitive equilibrium must satisfy:

$$u'(c_t) = \beta E_t(u'(c_{t+1})HR(K_{t+1}))$$

Where the dependence on Z_{t+1} has been omitted as the integral in E_t is with respect to this variable. In order to define a recursive equilibrium with minimal state space we can build an Euler equation operator, A. Let C be the space of candidate policy function. In particular:

$$\begin{aligned} C(\mathbb{K},\mathbb{Z}) = \{ & \\ & c : \mathbb{K} \times \mathbb{Z} \to \mathbb{R} \text{ is continuous} \\ & 0 \leqslant c(K,Z) \leqslant F(K,Z) \\ & 0 \leqslant c(K',Z) - c(K,Z) \leqslant F(K',Z) - F(K,Z), K' \geqslant K\} \end{aligned} \tag{17}$$

Where, $F(K,Z) = (1-\delta) + Zf(K)$, $\mathbb{K} \equiv \left[0, K^{UB}\right]$ and $\mathbb{Z}$ is a grid of 10 evenly spaced points. Note that equation (17) implies not only that the policy functions must be continuous but also consumption and investment are increasing. Following [Coleman, 1991] we can defined the following operator:

$$\begin{aligned} u'((Ac)(K,Z)) = \beta E_t\{ & u'(c(F(K,Z) - (Ac)(K,Z), Z')) \\ & HR(c(F(K,Z) - (Ac)(K,Z), Z'))\} \end{aligned} \tag{18}$$

Equipped with operator A, we can define and characterize a recursive equilibrium.

Definition 4 Coleman's Minimal State Space Recursive Equilibrium (CMSSRE)

A CMSSRE is a *policy function* c_* such that:
i) is a fixed point of A defined in equation (18).
ii) The firm solves (11)
iii) Markets clear. That is, $x(K,Z) = F(K,Z) - c_*(K,Z)$
iv) The public sector runs a balanced budget. That is, equation (12) holds.

The results in the appendix (see section A.6) allows us to derive the following results, which are all straightforward applications of [Coleman, 1991].

Lemma 1 Properties of a CMSSRE

Let $\{c_n\}$ be a sequence of function generated iteratively using A with $c_0(K,Z) = F(K,Z)$. Then,
i) c_* is the unique strictly positive fixed point of A
ii) c_* is a CMSSRE

iii) $c_*(., Z)$ is continuous for any $Z \in \mathbb{Z}$ and monotonic in $\mathbb{K} \times \mathbb{Z}$
iv) $\{c_n\}$ converges to c_* in the Sup Norm
v) Let $K' = K(K, Z) = F(K, Z) - c_*(K, Z)$. Then, the markov process generated by $(\mathbb{K} \times \mathbb{Z}, p)$ has an ergodic measure

Proof: See the appendix, section A.6.

Where p is a Markov kernel [9]. Now we turn to the background results for the ECM. From sections 3.2 and 3.3 we know that, if $HR(., Z)$ is decreasing for any Z, the envelope theorem holds. Thus, along the equilibrium path (i.e., $k = K$) we must have:

$$u'(c)HR(K, Z) = V_*'(K, Z) \tag{19}$$

Note that equation (19) implicitly assumes that $h_{*,*} = H_*$. Following [Mirman et al., 2008] we know that, as HR is decreasing, the iterative procedure described in section 4.1 converges under the sup norm. In particular, we know that the mapping from the perceived to the actual law of motion, $H_n \to h_{n,*}$ induced by the recursive problem (15) generates an ordered space $\{c_n, H_n\}$ and the convergence to H_* is uniform [10]. Moreover, we know also from [Mirman et al., 2008] that H_* is differentiable. However, as discuss in section 2, the ECM does not induce a monotonic operator. That is, for the stochastic case, the ECM can be described as follows:

1. Pick an initial condition $c(K, Z)$
2. Compute capital tomorrow $x(K, Z) = F(K, Z) - c(K, Z)$
3. Compute the value function $V(K, Z) = u(c(K, Z)) + \beta E_Z V(x(K, Z), Z')$
4. Update consumption $\hat{c}(K, Z) = u'^{-1}\left[\frac{V'(K,Z)}{H(K,Z)}\right]$
5. Continue until convergence under the sup norm $c \to \hat{c} \to \cdots$

Even if we choose c in order to ensure the joint monotonicity of c and x in steps 1 and 2, as $V'(., Z)$ and $H(., Z)$ are both decreasing, $\hat{c}$ may not be monotonic. Thus, $\{c_n\}$ is not an ordered space of equicontinuous functions and, thus, convergence is not ensured. However, this is not because $\hat{c}$ is not maximal in (15) as claimed in [Arellano et al., 2016]. That is, as $x(., Z)$ is differentiable for each Z, we have:

$$V'(K, Z) - u'(c(K, Z))HR(K, Z) = \left[\beta E_t\left[V'(x(K, Z), Z')\right] - u'(c(K, Z))\right]x'(K, Z) \tag{20}$$

[9] p can be defined analogously to P_{Ψ_i} in section 3.2. For further reference, see [Stokey, 1989], chapters 7 to 9.

[10] If we set $H_0 = C_0 = F$ and $h_0 = F - Ac$, as the envelope theorem holds, the convergence is assured.

Where (20) follows from the total differentiation of the value function at the optimum. It is easy to see that, if $x'(K,Z) \approx 1$ for every (K,Z), step 4 in the ECM algorithm implies that $\{c_n\}$ is maximal in (15). However, as claimed above the ECM does not converge under the sup norm even if $x'(K,Z) = 1$.

5.2 Numerical performance: ECM

The table below illustrates the relative performance of the ECM with respect to its benchmark, the CMSSRE. Each result is classified according to the initial condition in the sequence $\{c_n\}$. For instance, the second row shows the results for simulations obtained using a policy function numerically convergent using the ECM and with $c_0 = c_{UB}/2$, where $c_{UB}(K,Z) = F(K,Z)$; the initial condition for operator A. The ECM algorithm was described in the previous section. The algorithm for the benchmark case, the CMSSRE, is described in the appendix (see section A.4 and A.6).

Model	Relative Mean	Relative CV	Mean x'_{ECM}
ECM, $c_0 = c_{UB}$	0.002	558.8	0.9696
ECM, $c_0 = c_{UB}/2$	0.049	45.4	0.9672
ECM, $c_0 = c_{UB}/4$	0.175	47.8	1.045

Table 5: Relative Performance of the ECM

As in the previous case when HR is non-monotonic, frequently used procedures subestimate the mean and over-estimate dispersion measures with respect to the ergodic equilibrium. Note that there is a pattern: the lower the initial condition, the higher the bias with respect to the mean. *We observe that the mean of K is at least 82% below its accurate benchmark.*

6 Conclusions

This paper presents an example of a non-optimal economy with accurate benchmarks. This type of equilibrium is useful to assess the predictions of the model as it allows to generate reliable simulations. We also present a condition, the monotonicity of the Euler equation, that is associated with exact simulations and provide a description of the reasons behind the lack of accuracy of them. We use the closed form nature and the existence of a MSS the recursive equilibrium together with the induced Feller mechanism to test the accuracy of MSS methods. The results in this paper does not depend on any numerical procedure, they constitute a unique opportunity to assess the performance of state-of-the-art algorithms.

From a purely economic perspective, as the model is built to study the effect of economic policies, the presence of a bias constitutes a major drawback for the conclusion generated by the model. In particular, the presence of capital taxes generates a reallocation between consumption and savings which are not captured by state-of-the-art algorithms. Ad valomen taxes on capital must shift the composition of aggregate demand towards consumption, away from investment. As standard procedures subestimates the long run average level of consumption, the estimated response to economic policies will be frequently misrepresented.

The paper also connects two branches of the recursive literature: the one concerned with the existence of a steady state (see for instance [Santos and Peralta-Alva, 2005]) and the one concerned with the existence of a recursive representation of the sequential equilibria ([Kubler and Schmedders, 2002]). We show that when the existence of equilibrium can't be shown constructively biases may arise.

The results in this paper have to be generalized. It is necessary to understand the connection between the number of possible exogenous states and the number of distinct economically meaningful recursive equilibria. That is, as the degree of the polynomial in the equilibrium equation in the closed form of the GME is increasing in the number of exogenous states and each root of the polynomial defines a different mechanism (provided that the root is real and consumption / capital are positive), there is a tradeoff between a realistic shock process and the predictive performance of the model as more than one possible mechanism generates a less conclusive model. Moreover, the monotonicity condition presented in this paper is sufficiently general to be used in other branches of the literature such as default models.

7 Appendix

7.1 Figures

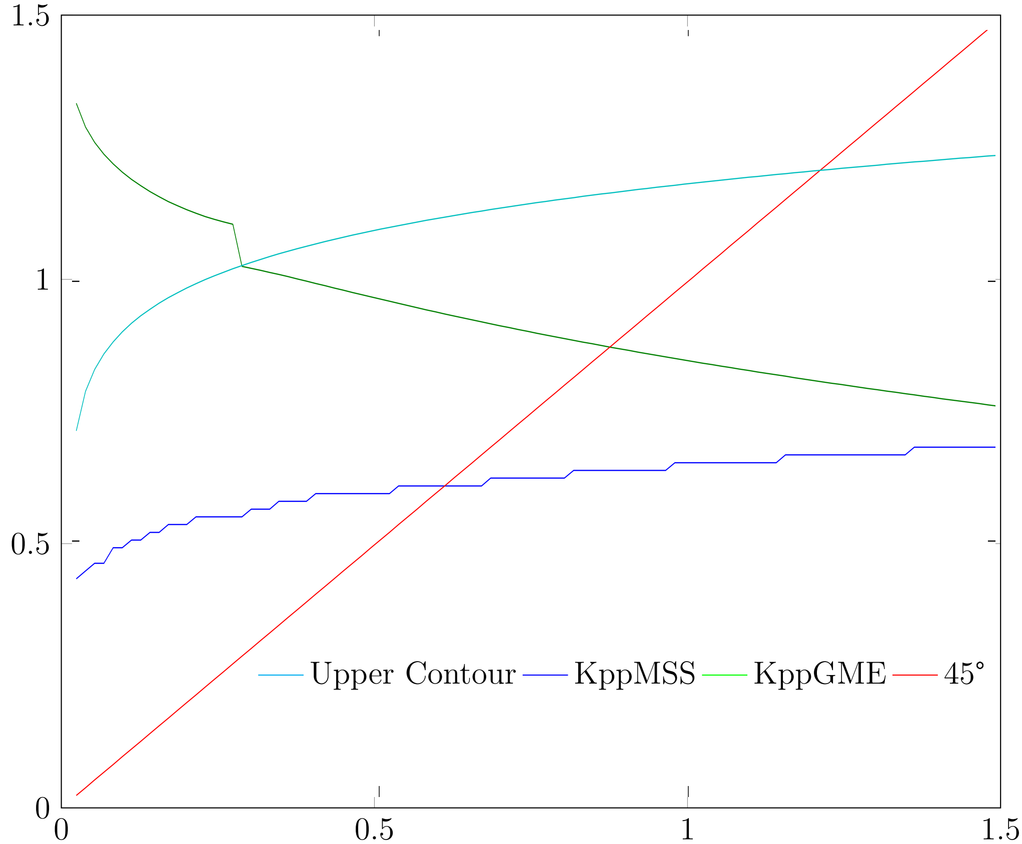

Figure 6: Phase Diagram

The light blue curve is the upper contour for K_{++}, the blue curve is $H(H(K,Z),Z)$ for the MSS and the green line is $H(K,K_+,Z)$ for the GME, where K_+ is fixed in the 50th grid point.

7.2 Theorems and useful definitions

A.1 Invariant Measures

Let S be the state space and P a markov operator of the process (A, P). σ_S is the Borel sigma algebra generated by S. P has the Feller property if $P(s, A)$ is continuous (in s) for any $A \subseteq S$. P is tight if S is compact. The operator P maps the space of Borel σ_S -measures $\mathbb{P}$ into itself as follows: $\mu'(A) = \int P(s, A)\mu(ds) \equiv P\mu$.

Theorem A1 (Futia, 1982, page 383, Th. 2.9) If P has the Feller property and is tight, then there is a measure μ such that $\mu = P\mu$

A.2 Supplementary material for sections 3 and 4

Section 3.2: A Closed form GME

In this case, the solution to the model can be characterized by the equilibrium Euler equation, which can be obtained by putting the optimality condition for the firm, the budget constraint for the Government and the market clearing conditions into the optimality condition for the consumer.

Assume that $u(c) = \ln(c)$ and $\delta = 1$. Then, the equilibrium equation is given by:

$$\tfrac{1}{C_t} = \gamma \sum_{z_{t+1}=0,1} \frac{A(z_{t+1})p(z_t, z_{t+1})(1 - \tau(K_{t+1}))f'(K_{t+1})}{C_{t+1}}, \tag{21}$$

With constrains given by

$$K_{t+1} = A(z_t)f(K_t) - C_t. \tag{22}$$

Note that the market clearing condition for capital implies that *given* z^t the demand for capital K_{t+1} does not depend on the realizations of the exogenous shock at $t + 1$. Hence, by replacing C_{t+1} in (21) with its expression obtained from (22) and after some algebra we can rewrite (21) in the following way:

$$\begin{array}{c} \overbrace{\underbrace{\frac{1}{\gamma(A(z_t)f(K_t) - K_{t+1})(1 - \tau(K_{t+1}))A(z_t)f'(K_{t+1})}}_{c}}^{c} = \\ \frac{\overbrace{A(0)p(z_t, 0)}^{c_1}}{\underbrace{A(0)f(K_{t+1})}_{d_1} - K_{t+2}} + \frac{\overbrace{A(1)p(z_t, 1)}^{c_2}}{\underbrace{A(1)f(K_{t+1})}_{d_1} - K_{t+2}} \end{array} \tag{23}$$

One of the purposes of this paper is to find an equation $\Psi : X \longrightarrow X$, where X is an appropriately defined state space and Ψ is a function that maps $x_t \longmapsto x_{t+1}$ with (x_t, x_{t+1}) satisfying equation (23) for any t.

Notice that by standard arguments, by fixing $\delta = 1$ and $f(0) = 0$, K_t stays in $[0, K^{UB}]$ (see [Stokey, 1989], Ch. 5) for any t.

Let $X = [0, K^{UB}] \times [0, K^{UB}] \times \{0, 1\}$. With this state space Ψ becomes a vector valued function of the form $x_t \longmapsto (\Psi_1(x_t), \Psi_2(x_t), \Psi_3(x_t))$ with $x_t = (K_t, U_t, z_t)$.

Let $\{z_n\}$ be a realization of $(\Omega, \sigma_\Omega, \mu_{z_0})$. Then, it is possible to define each coordinate in the image of Ψ as follows:

$$\begin{aligned} K_{t+1} &= \Psi_1(x_t) = U_t \\ z_{t+1} &= \Psi_3(x_t) = \{z_n\}(t+1). \end{aligned}$$

In order define Ψ_2 we could use (23). Notice that (23) takes the form

$$c = \frac{c_1}{d_1 - U_{t+1}} + \frac{c_2}{d_2 - U_{t+1}}, \tag{24}$$

or equivalently,

$$c(d_1 - U_{t+1})(d_2 - U_{t+1}) = c_1(d_2 - U_{t+1}) + c_2(d_1 - U_{t+1}). \tag{25}$$

Due to the fact that this is just a quadratic equation we can get U_{t+1} as a *continuous function* of the parameters, namely:

$$U_{t+1} = \frac{\pm\sqrt{(-d_1 c - d_2 c + c_1 + c_2)^2 - 4c(d_1 d_2 c - c_1 d_2 - c_2 d_1)} + (d_1 + d_2)c - c_1 - c_2}{2c}. \tag{26}$$

Equivalently:

$$U_{t+1} \equiv g(d_1, c, d_2, c_1, c_2)$$

It is important to observe that (26) gives at most *2 different mechanisms* [11] , each of them characterized by a different root of (26). Furthermore, note that $c(K_t, U_t, z_t)$, $d_1(U_t)$, $d_2(U_t)$ and the rest of the parameters in (26) depend on z_t . Thus, Ψ_2 is given by:

[11] Note that (23) implies that this model does not have a trivial solution at $K_t = 0$ as $u = \ln$ and investment is not allowed to be negative. This fact in turn implies that the parameters in (23) are all bounded away from 0. Of course, in order to have two non-trivial solutions it suffices to impose conditions on the discriminant of (26)

$$U_{t+1} = g(d_1, c, d_2, c_1, c_2) \equiv \Psi_2(x_t).$$

Note that once we start iterating the system, it is possible that simulations go outside X. That is, some simulations could be imaginary, negative or not finite. The continuity of Ψ_2 (on K_t and U_t), provided that the state space is well defined across any possible path, seems automatic. It suffices to verify the continuity of C, d_1, d_2 (on K_t and U_t), which is trivially satisfied. However, if we iterate forward equation (26), the restrictions on d_1, c, d_2, c_1, c_2 in order to keep U_{t+1} in $\mathbb{R}$ may affect the empirical performance of the model as the set of parameters (i.e., β, $p(.,.)$, etc) can't be freely choose in the calibration / numerical estimation procedure. Moreover, even if we could find an empirically meaningful parameter set, any solution to (26) may imply a negative consumption level or capital stock. Of course, due to the *log* preferences, these solutions will not be optimal. Thus, we must find a procedure in order to rule out solutions outside $\mathbb{R}_+$ and that imply a negative consumption level. In the numerical section, we adapted a canonical result due to [Duffie et al., 1994] to verify that the state space is well defined along equilibrium trajectories. Given the quadratic structure in (26), the stationarity (i.e., time independence) of the state space is sufficient to ensure both compactness and continuity of the recursive mechanism.

Section 3.3: An Operator for the MSSRE

Define the function space B on $K \times Z \equiv S$ as follows:

$$B(S) = \{H(s) \text{ such that } H : S \to K \text{ with } 0 \leqslant H(s) \leqslant A(Z)f(K),\ H \text{ measurable}\}$$

That is, a MSSRE is a fixed point in the functional T as the measurable maximum theorem ensures that $h_{*,j} \in B$ when $k = K$. Any attempt to prove the existence of a fixed point in a function space must circumvent the problem associated with the lack of sufficiency of the first order conditions which typically ensure a convex graph in the policy correspondence. That is, $T(H_j)$ may not be convex for models with a finite number of agents or finite shocks (see [Pierri, 2021] for a detailed discussion). Thus, the literature has turned to the lattice dynamic programming framework because it works in non-convex models. See section A.4 for a review of the results in this literature relevant for the model presented in section 3.

Moreover, contrarily to the Fan - Gliksberg theorem, lattice dynamic programming gives us a constructive fixed-point theorem which naturally generates an algorithm. In fact, the numerical procedure in [Rios Rull, 2004] can be proved to be convergent endowing B with an order topololgy if T *is a monotone operator*; which in turn ensures the existence of a MSSRE. That is, in order to prove the existence of a MSSRE *and* the convergence of the algorithm in [Rios Rull, 2004] for any $H_j^{'} \geqslant_* H_j$ we must have $T(H_j^{'}) = h_{*,j}^{'} \geqslant_* h_{*,j} = T(H_j)$ where $\geqslant_*$ is the pointwise order in B.

In order to prove the desired properties in T we can borrow from [Mirman et al., 2008] and [Coleman, 1991]. We present the relevant theorems in section A.4. The former proved that it is required to show that $V_*(k, K, Z; H_j)$ has increasing differences (see section A.4 in the appendix) in $(k; K)$ for each (Z, H_j) (lemma 12 and theorems 3 to 6). This condition, in turn, is equivalent to show that $V'_{*,1}(k, K, Z; H_j) = u'(g_\tau(K) - h_{*,j}(K))(1 - \tau(K))r(K)$ is *increasing* in K, where the dependence of $V'_{*,1}$ on $(k, Z; H_j)$ has been omitted in the right hand side of the equation. Note that $(1 - \tau(K))r(K)$ is decreasing in K if τ is increasing and undefined otherwise.Thus, as τ is decreasing by assumption, the results in [Mirman et al., 2008] does not hold.

[Coleman, 1991] showed that, if $u'(g_\tau(K) - h_{*,j}(K))(1 - \tau(K))f'(K)$ is decreasing in K when $k = K$, the operator based on (16) induces an order structure. As τ is decreasing by assumption, we cannot have an order structure for this model.

Section 4.2: An operator to stationarize the GME

Provided the compactness of the equilibrium, to have a well defined state space in the GME we need 2 things: i) to keep the roots of the equation which defines the GME, (26), into the positive real numbers, 2) to keep consumption positive. The following operator, borrowed from [Duffie et al., 1994], serves this purpose. We start by defining the sets which contain the elements in X that solves the system of equations that form a 2 period economy. We allow the vectors that characterize the second period of this economy to lie in $C_0 \equiv X$:

$$C_1 = \{x_0 \in X \mid \Psi(x_0) \cap C_0 \neq \varnothing\}$$

Where $C_0 \equiv X \subset \mathbb{R}^3_+$. Moreover, for $n \geqslant N$, $C_n \to J$. If *the sequential equilibria is compact*, J is non-empty and compact. As some roots of equation (26) may not be a real number, we can use this operator in order to keep C_n contained in $\mathbb{R}^3$. Moreover, the authors showed that for any $x \in J$, $\Psi(x) \cap J \neq \varnothing$, which in turn implies that the GME can be iterated forward. As this set is time invariant, it is a state space for Ψ.

However, as we are solving for an internal solution, we need to get rid of any vector (c^L, c^L_+, K_{++}) that contains a negative number, where $c^L = A(Z_L)f(K) - K_+$ and $c^L_+ = A(Z_L)f(K_+) - K_{++}$. In order to circumvent these problems, we use the following modified operator:

$$C_1 = \{x_0 \in X, A(Z_L)f(K) \geqslant K_+ \mid \Psi(x_0) \cap C_0 \neq \varnothing, A(Z_L)f(K_+) \geqslant \Psi_2(x_0)\}$$

Where $\Psi_2(x_0) = K_{++}$ is the second coordinate in the image of the vector valued function which defines the closed form solution of the GME (see section A.2 of this appendix, supplementary material for section 3.2). The operator above generates a sequence of sets in $\mathbb{R}^3_+$ with non-negative consumption levels which converge to a possible empty set J. We are interested in finding 1 mechanism Ψ from equation (26) which generates a non-empty state space J.

A.3 Classical Dynamic Programming

The following paragraphs are borrowed from [Stokey, 1989]. Note that equation (15) and the feasibility correspondence Γ define a standard dynamic programming program, as in [Stokey, 1989], with states (k, Z) for a given K. In order to prove the strict concavity of $V(k, ., .)$ (Theorem 9.8, page 265) we need the following set of assumptions: i) $k \in X \subset \mathbb{R}^n$, ii) $Z \in \mathbb{Z}$ and $\mathbb{Z}$ is countable, iii) $\Gamma(k, K, Z)$ is continuous in k, iv) Let A be the graph of $\Gamma(k, K, Z)$. Then, $u : A \to \mathbb{R}$ is bounded and continuous, v) u is strictly concave for each $Z \in \mathbb{Z}$, vi) $\Gamma(k, K, Z)$ is convex for each $Z \in \mathbb{Z}$. If additionally, we assume that u is differentiable in the interior of A for each $Z \in \mathbb{Z}$, $V(k, ., .)$ is continuously differentiable (Theorem 9.10, page 266).

Unfortunately, when we look at Γ when $k = K$, we loose some properties listed above. As τ is decreasing, $(1 - \tau(K))K$ is increasing and, as F is strictly concave in K, $r(K, Z)$ is decreasing. Moreover, given the functional for for F in Table 1, as τ is piecewise linear continuous, (see [Santos, 2002]), $(1 - \tau(K))Kr(K)$ is convex, which implies that for some $y \in \Gamma(x, Z)$, $y' \in \Gamma(x', Z)$, we have $\theta y + (1 - \theta)y' \notin \Gamma(\theta x + (1 - \theta)x', Z)$. Thus, we may fail to have a concave and differentiable value functions as property vi) is not satisfied. We need thus to use some properties of LL functions, which are described below.

A.4 Lattice Dynamic Programming and Supermodularity

The following paragraphs are borrowed from [Rockafellar, 1981]. A function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitzian (LL) if : $|f(x'') - f(x')| \leqslant \lambda |x'' - x'|$, where $\lambda \geqslant 0$ and x'', x' belong to a neighborhood of $x \in R^n$. A concave function is LL. Moreover, the generalized directional derivative (GDD) is given by:

$$f^{\circ}(x, v) = limsup_{\ x' \to x,\ t \downarrow 0} \ \frac{f(x' + tv) - f(x')}{t}$$

When f is LL, the GDD is finite. However, the GDD may be "bizarrely disassociated" from f (see [Rockafellar, 1981] page 5). Thus, we need to connect $f^{\circ}(x, v)$ with the "classical" directional derivative (DD):

$$f^{'}(x, v) = lim_{t \downarrow 0} \ \frac{f(x + tv) - f(x)}{t}$$

We know from [Rockafellar, 1981] (see page 6), that when f is concave, $f^{\circ}(x, v) = f^{'}(x, v)$ for all x, v. Of course if f is differentiable $f^{'}(x, v) = f'(x).v$, where f' is the gradient. Moreover, let the superdifferential ∂f be defined as:

$$\partial f = \{p \in \mathbb{R}^n \mid f(x) + p \cdot (y - x) \geqslant f(y),\ x, y \in \mathbb{R}^n\}$$

For concave functions at interior points ∂f is non-empty, finite and $p \in \partial f$ satisfies $p.v \geqslant f^{'}(x, v)$. Thus, we have a connection between the tangent p of a concave function and it's DD, which is finite at interior points. Finally, if $f : \mathbb{R} \to \mathbb{R}$, the left and right derivative ($f'(x^-), f'(x^+)$ respectively) satisfy $f'(x^-) \geqslant f'(x^+)$ and f' (the derivative) has at most a countable discontinuity set. The left derivative is a minor simplification with respect to $f^{'}(x, v)$:

$$f^{'}(x^-) = lim_{t \downarrow 0}\ \frac{f(x-t) - f(x)}{t}$$

The following paragraphs are borrowed from [Mirman et al., 2008] and [Amir et al., 1991]. If equation (15) has interior solutions, $V_*'(k^-, ., .), V_*'(k^+, ., .)$ exist every where for each K, Z, in particular when $k = K$ (see [Amir et al., 1991], Lemma 3.3). As $u(c) = ln(c)$, we know that solutions will be interior. From Lemma 12 in [Mirman et al., 2008] we know that V_* is LL. The results above imply in turn that $V_*^{\circ}(k, ., .)$ is finite, thus, $V_*'(k^-, ., .), V_*'(k^+, ., .)$ are finite.

As $(k, K) \in \left[0, \overline{K}\right] \times \left[0, \overline{K}\right]$ and $\mathbb{Z}$ has finite cardinality and it is bounded, we know that the domain of V_* is a complete lattice (a POSET endowed with the pointwise order such that each pair of elements in it, has a least upper bound $\wedge$ and a greatest lower bound $\vee$ that belong to $\left[0, \overline{K}\right] \times \left[0, \overline{K}\right]$). Then, V_* is supermodular if: $V_*(x \vee y, Z) + V_*(x \wedge y, Z) \geqslant V_*(x, Z) + V_*(y, Z)$. This concept is not really useful. Fortunately, we have an alternative characterization which is called increasing difference (ID). V_* has ID if: $V_*(k, K_1, Z) - V_*(k, K_2, Z)$ is non-decreasing in k for $K_1 \geqslant K_2$. [Mirman et al., 2008] propose a set of sufficient conditions (see assumption E in page 78) in order to ensure that V_* has ID which in turn imply that the operator T is convergent in a very precise sense. We will only mention the assumptions that are not satisfied in the model presented in this paper.

Assumption A.4.1 $u'(c(k, K, Z))(1 - \tau(K))r(K)$ is increasing in K and $0 \leqslant c(k, K', Z) - c(k, K, Z) \leqslant F(K', Z) - F(K, Z)$ with $K' \geqslant K$.

If assumption A.4.1 is satisfied, not only V_* has ID (Lemma 12) but also: $T^{\vee j}(F) \to h^{\vee}_{*,*}$, where $T^{\vee j}(F)$ is the j-th iteration of T starting at F taking the supremum of each maximal element in $ArgmaxV_*$ and $h^{\vee}_{*,*}$ is the supremum in the set of fixed points of T. Moreover, $T^{\wedge j}(0) \to h^{\wedge}_{*,*}$, where the interpretation is analogous. Now we turn to the result in [Coleman, 1991], which are generalized in [Mirman et al., 2008].

Assumption A.4.2 $u'(c(k, K, Z))(1 - \tau(K))r(K)$ is decreasing in K and $0 \leqslant c(k, K', Z) - c(k, K, Z) \leqslant F(K', Z) - F(K, Z)$ with $K' \geqslant K$.

If assumption A.4.2 is satisfied we can define an operator, based on (23), which ensures that there exist a MSSRE and that it can be computed by successive approximations (see [Mirman et al., 2008], theorem 10, page 86). When $(1-\tau(K))r(K)$ is non-monotonic, we can not use any of these results.

A.5 Stationary Markov Equilibria

The results in this section are borrowed from [Duffie et al., 1994]. Let $\Psi : X \to X$ be the correspondence which defines the GME (see definition 2). Let $C_0 = \left[0, \overline{K}\right] \times \left[0, \overline{K}\right] \times \mathbb{Z}$. Then, we can define a sequence of sets as follows:

$$C_1 = \{x_0 \in X \mid \Psi(x_0) \cap C_0 \neq \emptyset\} \equiv Q(C_0)$$

Let $\{C_i\}_i$ be the sequence of sets generated iteratively using Q. If C_i is non-empty and compact, then $\cap_i C_i = J$ is non-empty and compact and satisfies the self-generation property (i.e., $x \in J$ implies $\Psi(x) \cap J \neq \emptyset$). Intuitively, J is a stationary state space for the markov process generated by P_Ψ. Note that we have modified Q in section 4.2 in order to ensure that consumption is positive along the equilibrium path. Thus, we can not use this theorem in order to prove that J is well defined. However, we found a parameter structure and a mechanism which gives a stationary state space numerically.

A.6 Euler Equation Operators

This section contains a summary of the main results in [Coleman, 1991] and the required definitions to make the paper self-contained.

Each element in a set of *equicontinuous functions* $\{c_n\}$ defined on a compact set $\mathbb{K}$ satisfies: for each $k \in \mathbb{K}$ and n in a countable set, if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|c_n(k) - c_n(k')| < \epsilon$ with $|k - k'| < \delta$. The critical fact in an equicontinuous set of functions is the existence of a bound, ϵ, which is uniform across n, k. Note that a collection of function in (17) form a set of equicontinuous functions and this is closely connected with the fact that, both, consumption and investment are increasing in K. The Arzela-Ascoli theorem states that a closed metric space of bounded real valued functions (i.e., $\{c_n\}$ endowed with the sup norm, $|.|$) defined on a compact set (i.e., $\mathbb{K}$) is *compact* if it is equicontinuous. Thus, convergence in $\{c_n\}$ is *uniform* (using the sup norm).

Tarki's fixed point theorem states that an order continuous, monotone self map of a nonempty partially order (see section A.4) compact set (the candidate is A defined over C in section 5.1) in which some element c_0 is mapped downwards (i.e., $Ac_0 \leqslant c_0$) has a fixed point (in A) and a sequence of functions generated iteratively from this map (i.e., $\{c_n\}$, with $Ac_n = c_{n+1}$) converges to a maximal fixed point (i.e., in case there are more than one). In order to use this theorem, we need to show that: i) A is continuous and monotone, ii) $\{c_n\}$ is a partially ordered compact set, iii) c_0 maps down the sequence. [Coleman, 1991] shows i) in propositions 4 and 5, ii) proposition 3 and 5, iii) it follows by setting $c_0(K, Z) = F(K, Z)$. Moreover, $\{c_n\}$, with $Ac_n = c_{n+1}$, converges uniformly to the maximal fixed point (proposition 6). It remains to show that the fixed point of A is unique and strictly positive. Under the parameter structure in section 5.1, these facts follows respectively from theorem 11.

The last 2 paragraphs coupled with section A.1 suffice to *show Lemma 1.*

A.7 Ergodicity of the invariant measure

Let $\Phi = \{\mu \mid \mu = P\mu\}$ be the correspondence of fixed points defined in section A.1 of this appendix. [Santos and Peralta-Alva, 2005] shows that if the mechanism defining the Markov process, Ψ_i, is equicontinuous, then Φ is weakly compact. Then, [Futia, 1982] shows that μ is ergodic. As Ψ_i is one of the roots of a polinomial of degree 2, equation (23), the fact that the state space contains positive and bounded capital stocks, which at the same time generates positive consumption levels (see sections A.2, supplementary material for section 4.2, and A.4), implies that it has bounded derivatives if X is compact. Thus, Ψ_i is equicontinuous and μ is ergodic.

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