Recursive Equilibria, are you there?. Measuring the accuracy of minimal state space methods

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Abstract

The seminal paper of Duffie, et. al. (1994) shows that in non-optimal economies with a finite number of exogenous shocks there is a trade off between the generality of a recursive representation and a well behaved steady state, which is defined by an ergodic invariant measure of an stationary Markov Process. The authors "convexified" the state space using "sunspots" in order to prove the ergodicity of the measure. The purpose of this note is to show that, in certain environments, it is possible to obtain a recursive representation of a non-optimal general equilibrium model with a finite number of exogenous shocks that has an ergodic invariant measure, a compact and stationary state space and no "sunspots". By enlarging the number of variables in the state space, this paper proves the existence of multiple continuous markovian representations; which allows deriving an ergodic invariant measure for each of them using standard results.These facts show, contrarily to what is claimed in Blume (1982), that it is possible to obtain an economy with multiple equilibria and a continuous markovian representation. Moreover, for a stochastic RBC model with taxes, this paper derives a closed form recursive representation. As it is not necessary to use "sunspots", it is possible to simulate the model accurately. These results are then used to test the performance of minimal state space recursive equilibrium methods. Even if the algorithm converges, the numerically simulated distribution does not match any of the possible ergodic measures. We found that minimal state space methods may over/sub-estimate concentration and dispersion measures of the true ergodic distribution. Moreover, when it comes to asses the effect of economic policies, the numerical solution may even predict a different sign with respect to any of the actual distributions.

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1 Introduction

Since the seminal paper of [10] macroeconomists have been concerned with the recursive representation of sequential equilibria in general equilibrium models. This representation is relevant not only for computational purposes but also for theoretical ones. As regards the former, it is easier to numerically approximate a first order stationary dynamic process rather than the sequential representation originally defined. In reference to the latter, a markovian structure allows to define a well behaved long term equilibria (i.e. a steady state) using a recursive equilibrium notion (see for instance [3]). Finally, and more importantly, the theoretical and computational arguments are related with each other since accurate numerical simulations requeres a Markovian representation and an appropriate steady state (see for intance [14] among others).

Unfortunately, there is no free lunch. In order to improve the emprical performance of general equilibrium models, macroeconomists have turned to the incomplete markets framework. Thus, the results in [10] are insufficient. [6] showed that in incomplete market models there is a trade off between the generality of a recursive representation and a well behaved steady state. The authors showed that in the presence of multiple sequential equilibria, Markov equilibria may not be continuous. Thus, in order to obtain an ergodic invariant measure, which is the natural representation of a steady state under the most general markovian environment, it is necessary to artificially convexify an appropriately enlarged state space. This fact affects the predictive performance of the model as each time period has an arbitrarily large number of possible continuations which are called "sunspots". Moreover, with the notable exception of [7], there is no numerical procedure that uses sunspots to convexify multiple equilibria. The standard practice is to proceed as if equilibrium were unique (see for instance [12]). From another perspective, [3] showed that it is possible to prove the existence of an invariant measure using a state space with no sunspots as long as there is an uncountable number of exogenous shocks. This last fact is inconvinient from a numerical point of view as the computed policy functions must be evaluated in an arbitrarily large number of different shocks in order to satisfy the required assumption.

The purpose of this note is to show that, in certain environments, it is possible to obtain a recursive representation with an ergodic invariant measure, a finite number of exogenous shocks and a well behaved state space (i.e. a state space with no sunspots). Moreover, the recursive equilibrium is not *unique* and each of them induces *a Feller mechanism*. One of the main contributions of the paper is to present a closed form continuous Markov equilibrium that satisfies all the requirements of the sequential version of a canonical RBC model with taxes. Equipped with that equilibrium it is possible to test the accuracy of simulations of a standard, numerically convergent, minimal state space algorithm. *We found that a cononical procedure may sub-estimate both concentration and dispersion measures of any of the possible multiple ergodic distributions of the model.* For instance, the mean (the coefficient of variation) of the "true" distribution could be almost 5 (12) times bigger than its numerical counterpart. We argue that this bias could be generated either for the lack of existence of a minimal state space equilibrium, which precludes the theoretical convergence of the algorithm, or by the lack of existence of a well behaved steady state, which affects the performance of the simulations.

When it comes to compute recursive equilibrium models, the curse of dimentionality calls for minimal state space (MSS) methods. However, [9] argued that in the presence of the presence of multiple equilibria a MSS recursive representation may not exist. This fact justifies the necessity of an enlarged state space in an incomplete markets general equilibrium framework as uniqueness has been an elusive quest in this field[1]. By enlarging the number of variables in the state space, it is possible to obtain *multiple* markovian representations, each of them *continuous*, which allows to derive an ergodic invariant measure by applying standard results. This note shows that, in certain cases, an appropriate enlargement is enough to: i) derive at least 1, possible 2, continuous markovian process which represents a subset of all (possibly multiple) sequential equilibria, ii) obtain a sunspots free state space with a finite number of exogenous shocks and, iii) determine a suitable steady state.

Fact i) above is relevant from a theoretical point of view as it provides a counterexample for the equivalence between a continuous markovian representation and the uniqueness of the sequential equilibrium. In words of [3]:

"the existence of a continuos selection - tantamount to the uniqueness of equilibrium in each state - is not often satisfied".

The results in this paper are based on a stochastic version of a RBC model due to [13]. The non-stochastic version of the model illustrates the implications of looking for a computationally efficient recursive representation. In particular, [13] presents an example of an economy with a discontinuous markovian representation and minimal state space. In this note, it is shown that an enlargement similar to the one used in [6] is enough to derive a continuous markovian representation with well behaved steady state and state space for the model in [13] with aggregate productivity shocks.

From a methodological point of view, the results in this paper are used to test the accuracy of simulations in MSS recursive equilibrium methods. In order to prove ergodicity, this paper derives a *closed form* generalized recursive equilibrium (Fact i) for a standard version of the RBC model with decreacing taxes on capital presented in [13]. The state space for exogeneous shocks is finite (Fact ii), which allows for computations using standard procedures. As minimal state space are subset of generalized recursive equilibria, any simulation from the latter, must be matched using the former. It is shown that a numerically convergent MSS algorithm *does not match any of the 2 ergodic distributions of the model* (Fact iii). This bias not only affects the long run simulations

[1] [4] provided conditions to guarantee the uniqueness of equilibria in an infinite horizon economy with complete markets. There is no analogous result for incomplete markets

obtained from MSS methods but also the trajectories obtained from them, which implies in turn that the computed effects of economic policies are also innacurate. In particular, we simulate shocks to preferences and technology. We found that *numerical simulations may over/sub-estimate* the response of endogeneous variables, both taking into account concentration and dispersion measures. Moreover, for some cases, *the numerical comparative statics gives a wrong sign* (i.e. a decrease in the coefficient of variation of the long run distribution of capital, when 1 of the 2 "true" distributions increase their dispersion and the other does not change significantly after the shock).

The paper is organized as follows: section 2 presents the canonical model and the closed form recursive equilibrium and discusses its implications. Section 3 presents the numerical test. Section 4 concludes.

2 A Continuous Recursive Representation

2.1 Setting of the Model

The model is a stochastic version of [13] (section 3.2). Consider a representative agent economy with discrete time, $t = 0, 1, 2....$ Exogenous shocks are markovian and will be denoted z. For the sake of simplicity let us assume that the state space for these shocks is $\{0, 1\}$. An element of the transition matrix will be denoted $p(.,.)$. Let $\{z_t\}$ be a sequence of shocks and Z^t the set of histories up to time t, being a typical element z^t. Using standard results (see [16], Ch. 8) it is possible to define, for any $z_0 \in \{0, 1\}$, a stochastic process $(\Omega, \sigma_\Omega, \mu_{z_0})$ on Z^∞.

There is a unique decreasing return to scale firm which only uses capital as input and its technology is characterized by $y_t = A(z_t)f(k_t)$ with $f' > 0$, $f'' < 0$ and $f(0) = 0$ as usual. The firm is owned by the consumer as she is endowed with $k_0 > 0$ units of capital. Thus, the agent has two sources of current income derived from her endowment: benefits, denoted by π_t, and rents from capital, denoted by $r_t k_t$. Besides, the flow of taxes paid and transfers received is $\tau(k_t)r_t k_t$ and T_t respectively. Note that the tax rate depends on the stock of capital. In particular, it is given by a piecewise linear continuous function (see [13], page 87 for details).

The problem faced by the consumer is to choose a pair of functions $c : Z^\infty \longrightarrow \mathbb{R}$ and $x : Z^\infty \longrightarrow \mathbb{R}$ that solves the following problem:

$$\max_{\{c,x\}} \sum_t \sum_{z^t \in Z^t} \gamma^t u(c(z^t))\mu_{z_0}(z^t) \tag{1}$$

s.t.

$$k(z^t) = x(z^t) + (1 - \delta)k(z^{t-1}) \tag{2}$$
$$c(z^t) + x(z^t) \leqslant \pi(z^{t-1}) - (1 - \tau(z^{t-1}))r(z^t)k(z^{t-1}) + T(z^t) \tag{3}$$

$c(z^t) \geqslant 0, x(z^t) \geqslant 0$ for any $z^t \in Z^t$, z_0 and $k_0 > 0$ given, $\delta \in [0,1]$ is the depreciation rate and $\gamma \in (0,1)$ the discount factor.

In what follows $\tau(z^{t-1})$ stands for $\tau(k(z^{t-1}))$ or abusing notation $\tau(k_t(z^{t-1}))$.

That is, the tax rate affects the rents obtained from capital holdings at time t, which is in turn affected by the information contained in z^{t-1} because $k_t(z^{t-1}) = x_{t-1}(z^{t-1}) + (1-\delta)k_{t-1}(z^{t-2})$. A similar argument can be used to understand $r(z^t)$ because the agent knows the clearing condition for the market of factors and the optimality condition for the firm to be described below.

The problem of the firm is standard. Taking r_t as given it solves:

$$\max_{K_t} A(z_t)f(K_t) - r_t K_t, \qquad \text{for any } z_t \in \{0,1\}. \tag{4}$$

Observe that the optimality of the firm implies $r_t = A(z_t)f'(K_t)$. The Government simply transfers to the consumer the tax revenues:

$$T = \tau(z^{t-1})r(z^t)k(z^{t-1}). \tag{5}$$

Finally, goods and factor markets clear:

$$\begin{aligned} c(z^t) + x(z^t) &= A(z_t)f(K_t) && \text{Goods Market} \\ k(z^t) &= K_{t+1} && \text{Capital Market} \end{aligned}$$

where both equations hold for any $z^t \in Z^t$.

Note that in equilibrium, the optimality condition of the firm and the market clearing equation for capital holdings implies $r_t = A(z_t)f'(k(z^{t-1}))$ which in turn implies $r_t = r(z^t)$ as claimed. Further, both market clearing conditions imply $c(z^t) + x(z^t) = A(z_t)f(k(z^{t-1})) = y(z^t)$ as expected.

2.2 Equilibrium Equation

In this case, the solution to the model can be characterized by the equilibrium Euler equation, which can be obtained by putting the optimality condition for the firm, the budget constraint for the Government and the market clearing conditions into the optimality condition for the consumer.

Assume that $u(c) = \ln(c)$ and $\delta = 1$. Then, the equilibrium equation is given by:

$$\tfrac{1}{C_t} = \gamma \sum_{z_{t+1}=0,1} \frac{A(z_{t+1})p(z_t, z_{t+1})(1-\tau(K_{t+1}))f'(K_{t+1})}{C_{t+1}}, \tag{6}$$

with constrains given by

$$K_{t+1} = A(z_t)f(K_t) - C_t. \tag{7}$$

Note that the market clearing condition for capital implies that *given* z^t the demand for capital K_{t+1} does not depend on the realizations of the exogenous shock at $t+1$. Hence, by replacing C_{t+1} in (6) with its expression obtained from (7) and after some algebra we can rewrite (6) in the following way:

$$\underbrace{\frac{1}{\gamma(A(z_t)f(K_t) - K_{t+1})(1-\tau(K_{t+1}))f'(K_{t+1})}}_{c} = \frac{\overbrace{A(0)p(z_t,0)}^{c_1}}{\underbrace{A(0)f(K_{t+1})}_{d_1} - K_{t+2}} + \frac{\overbrace{A(1)p(z_t,1)}^{c_2}}{\underbrace{A(1)f(K_{t+1})}_{d_1} - K_{t+2}}. \tag{8}$$

The purpose of this note is to find an equation $\Psi : X \longrightarrow X$, where X is an appropriately defined state space and Ψ is a function that maps $x_t \longmapsto x_{t+1}$ with (x_t, x_{t+1}) satisfying equation (8) for any t.

Notice that by standard arguments, by fixing $\delta = 1$ and $f(0) = 0$, K_t stays in $[0, K^{UB}]$ (see [16], Ch. 5) for any t.

Let $X = [0, K^{UB}] \times [0, K^{UB}] \times \{0, 1\}$. With this state space Ψ becomes a vector valued function of the form $x_t \longmapsto (\Psi_1(x_t), \Psi_2(x_t), \Psi_3(x_t))$ with $x_t = (K_t, U_t, z_t)$.

Let $\{z_n\}$ be a realization of $(\Omega, \sigma_\Omega, \mu_{z_0})$. Then, it is possible to define each coordinate in the image of Ψ as follows:

$$\begin{aligned} K_{t+1} &= \Psi_1(x_t) = U_t \\ z_{t+1} &= \Psi_3(x_t) = \{z_n\}(t+1). \end{aligned}$$

In order define Ψ_2 we could use (8). Notice that (8) takes the form

$$c = \frac{c_1}{d_1 - U_{t+1}} + \frac{c_2}{d_2 - U_{t+1}}, \tag{9}$$

or equivalently,

$$c(d_1 - U_{t+1})(d_2 - U_{t+1}) = c_1(d_2 - U_{t+1}) + c_2(d_1 - U_{t+1}). \tag{10}$$

Due to the fact that this is just a quadratic equation we can get U_{t+1} as a *continuous function* of the parameters, namely:

$$U_{t+1} = \frac{\pm\sqrt{(-d_1c - d_2c + c_1 + c_2)^2 - 4c(d_1d_2c - c_1d_2 - c_2d_1)} + (d_1+d_2)c - c_1 - c_2}{c}. \tag{11}$$

Equivalently:

$$U_{t+1} \equiv g(d_1, c, d_2, c_1, c_2)$$

It is important to observe that (11) gives at most *2 different mechanisms* [2] ,each of them characterized by a different root of (11). Furthermore, note that $c(K_t, U_t, z_t)$, $d_1(U_t)$, $d_2(U_t)$ and the rest of the parameters in (11) depend on z_t . Thus, Ψ_2 is given by:

$$U_{t+1} = g(d_1, c, d_2, c_1, c_2) =: \Psi_2(x_t).$$

In order to show the continuity of Ψ (on K_t and U_t), provided that the discriminant in g is positive, it suffices to verify the continuity of C, d_1, d_2 (on K_t and U_t).

2.3 Discussion

Translating this, we just wrote K_{t+2} in terms of (K_t, K_{t+1}, z_t); ie,

$$K_{t+2} = g(K_t, K_{t+1}, z_t).$$

Though g might be awful, it is still *explicit* and, even more, continuous (of course, this representation has economic content if we can assure that the discriminant in g is positive under reasonable parameterizations for any $x \in X$).

The (numerical) cost of this representation is the enlargement of the state space with respect to the natural one (i.e. (K_t, z_t)). As discussed in [9], enlarging the state space might provide a recursive representation. Unfortunately, the results in that paper does not address the continuity of the mechanism; an aspect that has severe consequences for the steady state of the model as discussed in [6].

This paper shows that it is possible to obtain a continuous selection from a correspondence based on a recursive representation. Moreover, this example shows that the restrictions implied by [15] or [3] may not always be necessary as it is possible to have multiple equilibria and continuous mechanisms. In particular, as $U_t := K_{t+1}$, we have now the following iterative system:

Take first an arbitrary initial condition (K_0, U_0, z_0) and a drawn $\{z_n\}$, then

$$\begin{aligned} K_{t+1} &= U_t \\ U_{t+1} &= g(K_t, U_t, z_t), \end{aligned}$$

[2]Note that (8) implies that this model does not have a trivial solution at $K_t = 0$ as $u = \ln$ and investment is not allowed to be negative. This fact in turn implies that the parameters in (8) are all bounded away from 0. Of course, in order to have two non-trivial solutions it suffice to impose conditions on the discriminant of (11)

provides a sequence $\{X_n\}$. Such a sequence defines a Feller mechanism, with compact state space X.

Thus, it has an ergodic invariant measure (see [8]), which guarantees that the process K_t has an invariant measure as well. Moreover, using standard results on laws of large numbers for markov processes (see [17]), it can be shown that choosing an appropriate initial condition it suffices to guarantee that:

$$\frac{\sum_{t \in 0,\dots,T} h(X_t)}{T} \quad \text{converges almost surely to} \quad E_\mu(h),$$

where h is a σ_X-measurable function and μ is one of the possibly many ergodic invariant measures described above.

It is worth mentioning that this "trick" can also be done in the case of discrete shocks if the number of total states is 3 or 4. This relies on the fact that there are explicit expressions for the roots of a polynomial in terms of its coefficients whenever the degree of the equation is smaller or equal than 4. Thus, it is possible to obtain an expression similar to (11) even if we allow for a more realistic exogenous state space.

3 A numerical exploration

The results in section 2.2 provide a unique oportunity to test the predictive power of minimal state space (MSS) methods. As any solution obtained from a well defined MSS recursive equilibrium must satisfy equations (6) and (7), the simulations generated by a MSS method must converge to one of the possible multiple ergodic distributions obtained from these equations. This section presents a standard recursive competitive MSS algorithm that numerically converges to a fixed point between the perceived and actual law of motion. Then, the equilibium policy function is simulated and the results compare with those obtained from equation (11). We find a severe bias in the simulations obtained from the MSS method which, in turn, affects the long run distribution of capital. These findings provides evidence in favor of the results in [2] and [7] which suggests the importance of theoretical results in the recursive numerical literature. That is, without sufficient conditions that insure the equivalence between numerical and actual simulations of the model, a convergent algorithm does not guarantee by itself the absence of biases.

3.1 The Reasons Behind the Bias

In order to rationalize these results, it is possible to borrow from 2 branches of the theoretical literature. First, [5] suggests that a MSS recursive equilibrium for the model described in this paper can't be computed using a standard state of the art algorithm. On the contrary, the existence of a well defined recursive equilibrium calls for a "two step method", which has never been implemented in pactice. Second, the results in [11] and [15], both based on an extension of [14], indicate that even if the model has an invariant ergodic measure and thus a law of large numbers, there maybe a bias between actual and numerical simulations. The results in section 2 insures that neither of these facts are a reason of concern, provided an appropriate enlargement of the state space, as equation (11) provides a closed form solution.

The rest of this section will discuss in detail these 2 claims. Let us begin with the connection between the lack of existence of a MSS recursive equilibrium and state of the art algorithms. The minimal state space version of the model described in section 2 can be written as follows:

$$V_n\left(k,K,Z;H_j\right) = Max_{y\in\Gamma(k,K,Z)}\, u(g_\tau(k,K,Z)-y)+\beta\sum_{Z'}V_{n-1}(y,H_j(K,Z),Z';H_j)p(Z,Z') \tag{12}$$

Where the feasibility correspondence is given by:

$$\Gamma(k,K,Z) = \left[y \in \overline{K}; 0 \leqslant y \leqslant \pi(K,Z) + (1-\tau(K,Z))r(K,Z)k + T(K,Z)\right]$$

Capital is allowed to fluctuate in a compact set, $\left[0, K^{UB}\right] = \overline{K}$. The function g_τ represent disposible income and is defined by:

$$g_\tau(k,K,Z) \equiv \pi(K,Z) + (1-\tau(K,Z))r(K,Z)k + T(K,Z)$$

Where $\pi(K,Z)$ and $\tau(K,Z)$ are defined in (2) and $T(K,Z)$ in (5). The policy function for (12) is given by $h_{n-1,j}(k,K,Z)$, which belongs to the set defined below:

$$argmax\left\{u(g_\tau(k,K,Z)-y)+\beta\sum_{Z'}V_{n-1}(y,H_j(K,Z),Z',H_j)p(Z,Z')\ s.t.\ y\in\Gamma(k,K,Z)\right\}$$

Note, remarkably that: i) the household take a guess at the evolution of the aggregate states using a *perceived law of motion* denoted H_j. ii) The value and the policy function in the dynamic programming problem have to converge in j, which is associated with the rational expectation nature of the problem (i.e. the perceived and the actual law of motion must be equal when $k = K$), and in n, that is guarenteed by the contractive nature of the Bellman operator in (12). iii) The dependence of disposible, $g_\tau(k,.,.)$, on prices, $r(.,.)$, justifies the presence of *equilibrium states* which are represented by capital letters. In particular, they affect the household problem through the firm's decisions, given by

(4), and market clearing conditions which are contained in the definition of recursive competitive equilibrium, which is given below.

Definition 1: Minimal State Space Recursive Equilibrium (MSSRE)

A MSSRE is a *value function* V_*, a *policy function* $h_{*,*}$ and a *perceived law of motion* H_* such that:
i) the household solves equation (12) obtaining $V_*(k, K, Z; H_*)$ and $h_{*,*}(k, K, Z; H_*)$ for any feasible state k, K, Z.
ii) The firm solves (4)
iii) Markets clear. That is, $k = K$
iv) Expectations are fulfilled. That is, $h_{*,*}(K, K, Z; H_*) = H_*(K, K, Z)$ for any (K, Z)
v) The public sector runs a balanced budget. That is, equation (5) holds.

Before computing a MSSRE we must characterize it in order to undertand the difference with respect to the generalized Markov equilibrium described in section 2. Under standard curvature and smoothness assumptions on the return function u, which are all satisfied imposing the parametrizations used in section 2, together with the convexity in the graph of the feasibility restriction Γ, for an interior optimal solutions, $h_{*,j}(., K, .; H_j) \in \Gamma(., K, .)$, we can use the envelope theorem in [16]. Then, a solution to the dynamic programming problem in Definition 1 for any pair of individual states (k, Z) and given the aggregate level of capital K must satisfy:

$$u'\left[A(Z)f(k, K) - h_{*,j}\right] = \gamma E_Z \left\{u'\left[Af(H_j, h_{*,j}) - h_{*,j}(h_{*,j})\right] Af'(H_j)(1 - \tau(H_j))\right\} \quad (13)$$

Where the dependence of $h_{*,j}$ on (k, Z) for each K and of H_j on (K, Z) have been omitted for expositional purposes. Also, equation (13) includes the equilibrium version of g_τ, the disposible income, which explains the dependence of f jointly on (k, K).

Note that (13) defines a mapping T from H_j to $h_{*,j}$. In fact, it is easy to see that any fixed point on this map is a MSSRE. Define the function space B on $K \times Z \equiv S$ as follows:

$$B(S) = \{H(s) \text{ } such \text{ } that \text{ } H : S \to K \text{ } with \text{ } 0 \leqslant H(s) \leqslant A(Z)f(K), \text{ } H \text{ } measurable\}$$

That is, a MSSRE is a fixed point in the functional T as the measurable maximum theorem insures that $h_{*,j} \in B$ when $k = K$. As mentioned before, any attempt to prove the existence of a fixed point in a function space has to circunvent the problem associated with the lack of sufficient conditions which insure a convex graph in tractable frameworks.

That is, $T(H_j)$ may not be convex for models with a finite number of agents or finite shocks (see [11] for a detailed discussion). Thus, the literature has turned to the lattice dynamic programming framework because it works in non-convex models. Moreover, contrarily to the Fan - Gliksberg theorem, it gives us a constructive fixed point theorem which generates an algorithmic proceadure naturaly. In fact, the numerical procedure in [12] can be proved to be convergent endowing B with an order topololgy if T *is a monotone operator*; which in turn insures the existence of a MSSRE. That is, in order to prove the existence of a MSSRE *and* the convergence of the algorithm in [12] for any $H_j^{'} \geqslant_* H_j$ we must have $T(H_j^{'}) = h_{*,j}^{'} \geqslant_* h_{*,j} = T(H_j)$ where $\geqslant_*$ is the pointwise order in B.

In order to prove the desired properties in T we can borrow from [1]. These authors proved that it is sufficient to show that $V_*(k, K, Z; H_j)$ has increasing differences in $(k; K)$ for each (Z, H_j) (see lemma 12 and theorems 3 to 6). Using a standard envelope theorem, this condition is equivalent to show that $V_{*,1}(k, K, Z; H_j) = u'(g_\tau(K) - h_{*,j}(K))(1 - \tau(K))r(K)$ is increasing in K, where the dependence of $V_{*,1}$ on $(k, Z; H_j)$ has been omitted in the right hand side of the equation and $V_{*,1}$ is the derivative of V_* with respect to k. Note that $(1 - \tau(K))r(K)$ is decreasing in K if τ is increasing and undefined otherwise. Moreover, $g_\tau(K) - h_{*,j}(K) = C(K)$ is increasing in K. In order to show this last claim let us define the objective function in (12):

$$u(c) + \beta \sum_{Z'} V_*(y, H_j(K, Z), Z'; H_j)p(Z, Z') \equiv q(c, a, K)$$

Where $y + c = f(K)$. Using standard results, it is possible to show that q is concave in (c, a). From Lemma 2 in [1], it is also increasing in K. Let $\hat{y}(K) \in h_{*,j}(K)$, $\hat{c}(K) = f(K) - \hat{y}(K)$ and $K' > K$. Then, $q(\hat{c}(K), \hat{y}(K), K')$ belongs to the upper contour set of $q(c, a, K)$. Call this set A. By the concavity of $q(.,., K)$ for any K, A is convex. Moreover, as f is increasing in K, it is posible to pick $c' > \hat{c}(K)$ and $y' > \hat{y}(K)$. Thus, $\alpha[\hat{c}(K), y'] + (1-\alpha)[c', \hat{y}(K)] \geqslant_* [\hat{c}(K), \hat{y}(K)]$ and $q(\alpha[\hat{c}(K), y'] + (1-\alpha)[c', \hat{y}(K)], K') \geqslant_* q([\hat{c}(K), y'], K') \geqslant_* g([\hat{c}(K), \hat{y}(K)], K)$ where the first inequality follows from the concavity of $q(.,., K)$ and the second from the monotonicity of $q(c,.,.)$. As a similar inequality holds for $q([c', \hat{y}(K)], K')$, we know that any optimal policy must satisfy $[\hat{c}(K'), \hat{y}(K')] \geqslant_* [\hat{c}(K), \hat{y}(K)]$ as desired.

Thus, we have shown that, when $k = K$, $u'(g_\tau(K) - h_{*,j}(K))$ is decreasing in K. As $(1 - \tau(K))r(K)$ is either undefined or decresing, we cannot have increasing differences. This last fact implies in turn that it is not possible to insure that a sequence of function $\{H_j\}_j$ converging to H_* will "hit" $h_{*.*}$ as required by definition 1. Moreover, *any numerical procedure based on iterations through* T using the uniform metric, as the one described in [12], *cannot be proved to be convergent to a MSSRE* as the induced topology is stronger than the order topology.

We now turn to the second source of "problems", the one related with the convergence of numerical simulations. Taking into account the difficulties mentioned above with the MSSRE, we must define a robust recursive equilibrium notion in order to generate well defined simulations. We call this equilibrium *Generalized Markov.*

Definition 2: Generalized Markov Equilibrium (GME)

A GME is a *correspondence* $\Psi : X \to X$ with X compact such that for any $x \in X$, the vector $(x, \Psi(x))$:
i) satisfies the optimality conditions for the household problem, equation (1) s.t. (2) - (3).
ii) The firm solves (4)
iii) Markets clear
iv) The public sector runs a balanced budget. That is, equation (5) holds.

In section 2.2 we show that the sequential version of the model presented in this paper has a GME representation. Moreover, Ψ has 2 continuous selections. Let Ψ_i be any of the 2 possible selections. Using standard results (see [16]), we can show that $P_{\Psi_i}(x, A)$ defines a Markov kernel with $P_{\Psi_i}(x, .)$ being a probability measure for any $x \in X$ and $P_{\Psi_i}(., A)$ being a measurable function for any $A \in Borel(X)$. An invariant measure is any fixed point of Ψ_i. Call one of the possible many fixed point μ_i.

Let Ψ_i^j be any numerical approximation to Ψ_i and $P_{\Psi_i^j}(x, A)$, μ_i^j the associated Markov kernel and invariant measure respectively. Since [14], it is known that even if Ψ_i^j converge to Ψ_i, the simulations obtained from Ψ_i^j may differ from the exact ones, generated using Ψ_i. If Ψ_i is continuous and defined over a compact state space, these authors showed that numerical simulations will match the exact long run behabior of the model. If Ψ_i is not continuous, which is tipically but not always the case if there are multiple equilibria, [11] provided sufficient conditions which insure that numerical simulations replicate the actual model. Unfortunately, these conditions depend on the cardinality of Z, the set containing exogenous shocks, and will not hold in this framework. In other words, using the Law of Large numbers defined in section 2.3, we know that any cummulative average constructed iteratively using Ψ_i and Ψ_i^j will converge to μ_i and μ_i^j, respectively. However, without requiring additional conditions, it is not possible to show that $\mu_i^j \to \mu_i$.

The virtue of the results in section 2 is that it allows us to circunvent the 2 mentioned problems. On one hand, we show that *a GME exist* for the problem at hand and thus, it is possible for us to compute it. This is not the case for the MSSRE. Moreover, using (6) and (7), we show that Ψ_i has a *continuous closed form representation*, which in turn eliminates the problem associated with the lack of convergence of numerical simulations. The next sub-section will use $\Psi_i, i \in \{1, 2\}$ to measure the size of the possible bias in state of the art MSSRE methods.

3.2 Numerical Simlations

We now turn to measure the numerical bias. We will briefly describe the algorithm tipycally used to compute a MSSRE, which was presented in definition 1. The numerical procedure associted with the computation of a GME is contained in the discussion in sections 2.2 and 2.3 and thus will be omitted.

The task is to compute definition 1 using a concrete tax function based on the model in [13] and a standard algorithm borrowed from [12]. Let τ in equation (5) be given by:

$$\tau(K) = \begin{matrix} 0.1 \quad if \;\; K \leqslant 0.160002 \\ 0.05 - 10(K - 0.1652) \quad if \;\; 0.160002 \leqslant K \leqslant 0.170002 \\ 0 \quad if \;\; K > 0.170002 \end{matrix} \tag{14}$$

Note that τ is decreasing in K. Thus, the discussion in section 3.1 implies that the operator T is not monotonic and, consequently, it is not possible to prove that a numerical procedure based on iterations using T will converge to a MSSRE. The rest of the parameters are contained in the table below. We are carefully following the preferences and technology structure in [13]. Thus, we are able to center the state space in the stable steady state found in that paper (i.e. 0.165002). As the model in [13] is non-stochastic, we set the values for the exogeneous shock state space Z and transitions probabilities p_{LH} and p_{HL} arbitrarily.

$y = A(Z)f(K) = e^Z K^{1/3}$	$Z_H = 0.35$
$u(c) = ln(c)$	$Z_L = 0.25$
$\delta = 1$	$p_{LH} = 0.5$
$\beta = 0.99$	$p_{HL} = 0.2$

Table 1: Parameters

We now turn to the numerical results. We computed a MSSRE as described in definition 1 using the operator T, which follows from equation (13). It is standard in the literature (see for instance, [12]) to pick an arbitrary function H_0 from B and look for uniform convergence. However, as mentioned in section 3.1, theoretical results do not support such a strong convergence notion. Instead, it is only possible to show that any iteration starting from a lower or upper bound on T (i.e. $H \in B$ such that $H \leqslant T(H)$ or $T(H) \leqslant H$ respectively) will converge in the order topology. That is, take a sequence of increasing functions generated iteratively from T, $\{H_j\}$ with $H_{j+1} = T(H_j)$. We say that $H_j \rightarrow_{\geqslant *} H_*$, meaning $\{H_j\}$ converge in the order topology to H_*, if for any j, $H_j \leqslant H_*$ and $H_* \in B$. Unfortunately, under this parametrization, we show in section 3.1 that T is not a monotonic operator and thus it is not possible to generate an increasing sequence of functions using T.

The discussion in the above paragraph is enterily theoretical. We do not have *known sufficient conditions* which insure the convergence to a MSSRE. However, starting from an upper bound on T, we found *numerically* a fixed point for T in the sup norm. *If this heuristic result "hits" any MSSRE, the simulations obtain from it must match any of the GME* as this last type of recursive notion *computes all possible sequential equilibria* and gives us a *closed form, ergodic and Feller recursive representation*, which in turn insures that the 2 sources of bias are absent. In particular, the proceadure described below was found to be convergent using the sup norm for acceptable relative error levels (in the order of 10^{-2})

$$H_0 \rightarrow_{Equation(12)} h_{*,0} \rightarrow_{Definition1} h_{*,0}(K, K, Z) = H_1(K, Z) \rightarrow (...)$$

Where the first $\rightarrow$ means that we are solving equation (12) using H_0 as a guess for the perceived law of motion. The second $\rightarrow$ stands for the fact that we are computing the policy function $h_{*,0}$ along the equilibrium path according to definition 1 and using that object as an update for a perceived law of motion for aggregates states. Finally, $\rightarrow$ (...) means that we are starting the loop again if convergence using the sup norm is not achieved.

The table below contains the result of the simulations using the above proceadure and the one described in section 2, which does not depend on a numerical solution. The parameters used are listed in Table 1 and equation (14).

Model	Mean	STD	CV
NR	1.5169	0.0232	0.0153
PR	0.5451	0.5452	1.0002
Num	0.3087	0.0267	0.0865

Table 2: Simulation Results. Statistics for aggregate capital

The "empirical" distributions are constructed as follows: take as an initial condition the non-stochastic steady state of the model in [13]. Simulate a path of 5000 observations for aggregate capital. Store the last observation. Replicate the proceadure 1000 times. Then, the computed distribution is taken from the relative frequency of 25 grid positions out of 1000 observations [3]. The proceadure is repeated for any of the 3 listed distributions.

The numerical solution, "Num" in Table 2, has a significant bias as measured by the difference in mean with respect to one of the possible "true" distributions. In particular,

[3]For any of the GME we simulate the model for various possible K_1 with K_0 being the Santos non-stochastic steady state and the empirical distributions do not change significantly

"NR" stands for negative root in Table 2 and represent one of the possible equilibrium selections from a GME as presented in definition 2. Note that "STD" stands for standard deviation and "CV" for coefficient of variation. Thus, it can be seen from Table 2 that there is also a significant bias as measured by the difference in dispersion with respect to the other "true" distribution, labaled "PR" which means positive root. The table below presents these deviations with respect to any of the "true" distributions.

Model	Mean	CV
NR	4.9	0.18
PR	1.8	11.6

Table 3: Relative Bias

From Table 3 it is clear that the mean of one of the true distributions is almost 5 times bigger than its numerical approximation ($Mean(NR)/Mean(Num) = 4.9$). Moreover, the dispersion of the other true distribution is 12 times bigger than its numerical approximation. Thus, despite the fact that the algorithm for the MSSRE converge using a strong criteria (i.e. the sup norm an a tolerance level of 0.02 for the relative error), the numerical distribution will present a severe bias with respect to the exact model distributions.

As discussed in section 3.1, this bias could be generated either by the lack of convergence of the percieved to the actual law of motion (i.e. $H_j \to H_* \to h_{*,*}$) or by any difference between the numerical and the actual steady state (i.e. $\mu_i^j \to \mu_i$ for any selection $i \in 1,2$). As the regards the former, note that any MSSRE must satisfy equation (13) for $H_j = h_{*,j} = H_*$ which in turn insures that any path generated using $h_{*,*}$ along the recursive equilibrium will also be a sequential equilibrium. In other words, any path generated from a MSSRE satisfies equations (6) and (7). The lack of coincidence between the percieved and the actual law of motion will generate a distribution of capital that does not belong to any possible sequential competitive equilibrium, which explains part of the bias. Moreover, as a MSSRE may not exist for this model, we cannot insure the existence of a well behaved steady state for this type of equilibria (i.e. μ_{MSSRE} may not exist). If that is the case, any numerical distribution, namely μ_{MSSRE}^j, could be arbitrarily far away from μ_i as it is not possible to show that $\mu_{MSSRE} \to \mu_i$.

We now test the accuracy of the numerical approximation when it is used to asses the effects of a TFP or a preference shock.

	β		TFP	
Model	Mean	CV	Mean	CV
NR	0.0	0.7	4.2	9.1
PR	4.1	0.0	0.0	0.0
Num	9.4	-5.9	4.3	3.0

Table 4: Comparative statics. Difference w.r.t. the benchmark in % points

We simulate a 5% decrease in β and a 10% increase in each of the 2 possible values of z, the TFP shock. We then compute for each of these changes the long run distribution for the approximated MSSRE, "Num" in Table 4, and for every possible root in the GME. Finally, we compare the mean and coefficient of variarion of the obtained distributions with respect to the benchmark case presented in Table 2.

The lack of sensitivity of the positive root can be explained by a "peak of mass" in a corner solution. As investment is irreversible by assumption and $\delta = 1$, capital must be non-negative. For some parameters, the ergodic distribution associated with the positive root accumulate mass in this value. Thus, the change in the mean and in any dispersion measure will be affected by this fact.

Turning to the results in Table 4, the "numerical comparative statics" also suggest a striking difference in the sensibility of the distribution with respect to any of the true possible outcomes. Moreover, there is a difference in the sign of the variation. While in any of the 2 shocks, both distributions increase their dispersion after a preference shock, in the numerical case it decreases.

Finally, since [6] the notion of "sunspots" was used to convexify the space of multiple GME, generating a unique outcome. In this case, the following question arises then: is it possible to interpret the unique numerical MSSRE as a convexification of any of the 2 true equilibria?. Figure 1, in the appendix provides a negative answer to this question. The numerical distribution is not "centered" between the 2 roots, it lies to the left of the 2 of them. Thus, the simulations obtained from a numerical MSSRE cannot be a convex combination of the 2 possible GME for each node.

4 Conclusions

This note presents an example of an economy with multiple equilibria and continuous policy functions (i.e. Ψ is not unique). This type of equilibrium is useful for accurately assessing the predictions of the model as it allows to generate reliable simulations. These simulations can be used to generate counterfactuals which are useful to evaluate alternative economic policies.

The paper also connects two branches of the recursive literature: the one concerned with the existence of a steady state (see for instance [14]) and the one concerned with the existence of a recursive representation of the sequential equilibria ([9]). We show that there is no equivalence between the continuity of the equilibrium and its uniqueness, a fact that is useful for simulating the model reliably.

We use the closed form nature of the recursive equilibrium and the induced Feller mechanism to test the accuracy of MSS methods. As the results in this paper does not depend on any numerical procedure, they constitute a unique opportunity to asses the performence of state of the art algoritms.

It is clear that the results in this note have to be generalized. In particular, it is necessary to understand the connection between the number of possible exogenous states and the number of distinct economically meaningful recursive equilibria. That is, as the degree of the polynomial in g is increasing in the number of exogenous states and each root of the polynomial defines a different mechanism (provided that the root is real and positive), there is a trade off between a realistic shock process and the predictive performance of the model as more than one possible mechanism generates a less conclusive model.

5 Appendix

5.1 Figures

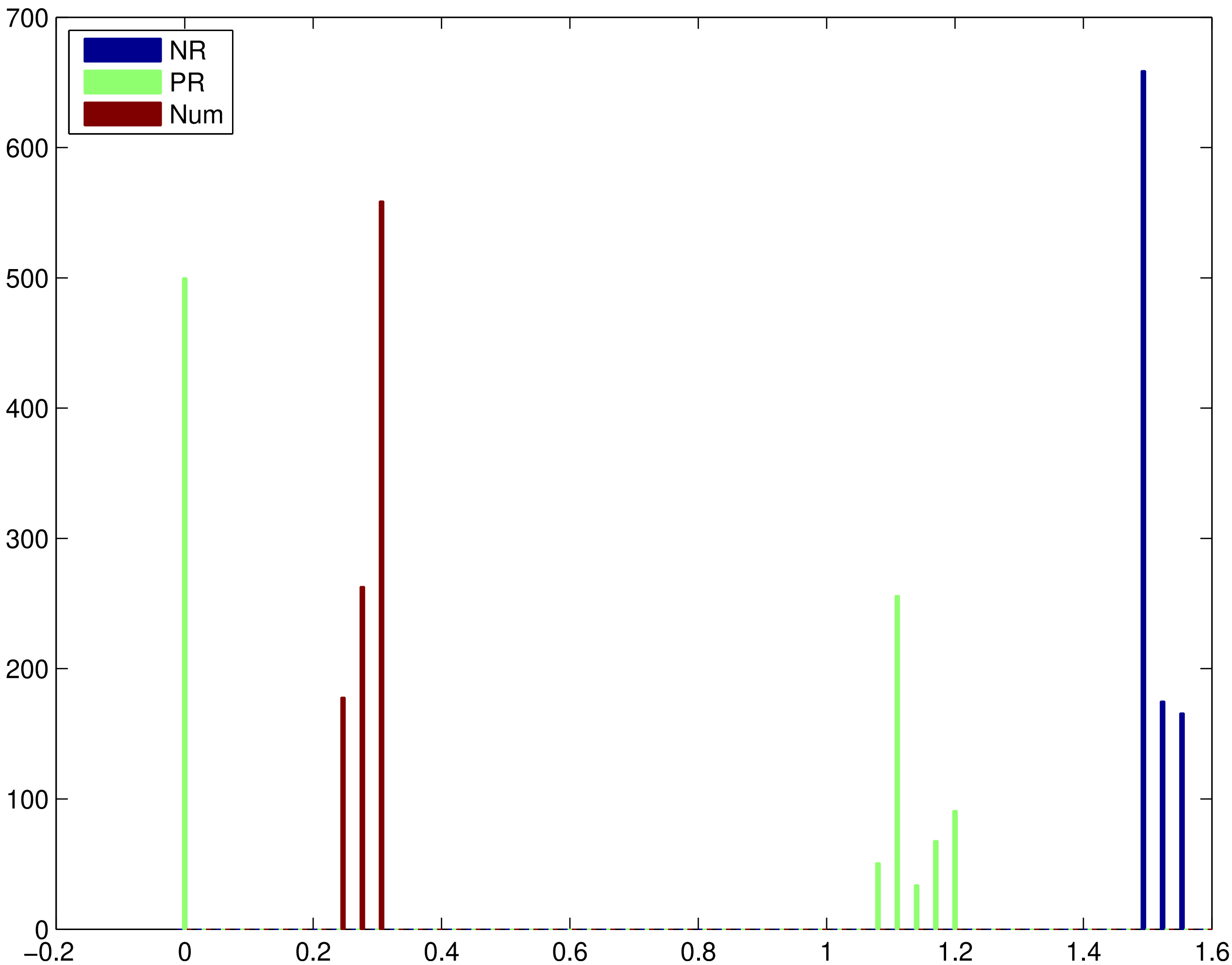

Figure 1: Ergodic Distributions

References

[1] A qualitative approach to markovian equilibrium in infinite horizon economies with capital. *Journal of Economic Theory*, 139(1):75 – 98, 2008.

[2] Quantitative properties of sovereign default models: Solution methods matter. *Review of Economic Dynamics*, 13(4):919 – 933, 2010.

[3] Lawrence E Blume. New techniques for the study of stochastic equilibrium processes. *Journal of Mathematical Economics*, 9(1-2):61–70, 1982.

[4] Rose Anne Dana. Existence and uniqueness of equilibria when preferences are additively separable. *Econometrica*, 61(4):953–957, 1993.

[5] M. Datta, K. Reffett, and L. Woźny. Comparing recursive equilibrium in economies with dynamic complementarities and indeterminacy. *Economic Theory*, 66(3):593–626, Oct 2018.

[6] Darrell Duffie, John Geanakoplos, Andreu Mas-Colell, and Andrew McLennan. Stationary markov equilibria. *Econometrica: Journal of the Econometric Society*, pages 745–781, 1994.

[7] Z. Feng, J. MIAO, A. PERALTA-ALVA, and M. SANTOS. Numerical simulation of nonoptimal dynamic equilibrium models. *International Economic Review*, 55(1):83–110.

[8] Carl A Futia. Invariant distributions and the limiting behavior of markovian economic models. *Econometrica: Journal of the Econometric Society*, pages 377–408, 1982.

[9] Felix Kubler and Karl Schmedders. Recursive equilibria in economies with incomplete markets. *Macroeconomic dynamics*, 6(02):284–306, 2002.

[10] Jr Lucas and E Robert. Asset prices in an exchange economy. *Econometrica: Journal of the Econometric Society*, pages 1429–1445, 1978.

[11] Damian Pierri. Useful results for the simulation of non-optimal economies. *mimeo*, 2018.

[12] Victor Rios Rull. Lecture notes, econ 702. *mimeo*, 2004.

[13] Manuel S Santos. On non-existence of markov equilibria in competitive-market economies. *Journal of Economic Theory*, 105(1):73–98, 2002.

[14] Manuel S Santos and Adrian Peralta-Alva. Accuracy of simulations for stochastic dynamic models. *Econometrica*, 73(6):1939–1976, 2005.

[15] Manuel S Santos, Adrian Peralta-Alva, et al. Ergodic invariant distributions for non-optimal dynamic economics. Technical report, 2012.

[16] Nancy L Stokey. *Recursive methods in economic dynamics.* Harvard University Press, 1989.

[17] Srinivasa RS Varadhan. Probability theory, volume 7 of courant lecture notes in mathematics. *New York University Courant Institute of Mathematical Sciences, New York*, 1:100, 2001.