Useful Results for the Simulation of Non-Optimal Economies with Heterogeneous agents

Damian Pierri [1]

Universidad Carlos III, Madrid and IIEP-BAIRES (UBA-CONICET)

August 2021

Abstract

This paper deals with infinite horizon non-optimal economies with aggregate uncertainty and a finite number of heterogeneous agents. It derives sufficient conditions for the existence of a recursive structure, an ergodic, a stationary, and a non-stationary equilibria. It also gives an answer to the following question: is it possible to derive a general framework which guarantees that numerical simulations truly reflect the behavior of endogenous variables in the model? We provide sufficient conditions to give an affirmative answer to this question for endowment economies with incomplete markets and uncountable exogenous shocks. These conditions guarantee the ergodicity of the process and hold for a particular selection mechanism. For economies with finitely many shocks or for an arbitrary selection in economies with uncountable shocks, it is only possible to show that a computable, time independent and recursive representation generates a stationary Markov process. The results in this paper suggest that often a well-defined stochastic steady state in heterogenous agent models is sensitive to the initial conditions of the economy; a fact which imply that heterogeneity may have irreversible long-lasting effects.

Keywords: non-optimal economies, Markov equilibrium, heterogeneous agents, simulations.

JEL Classification: C63, C68, D52, D58.

---

[1] www.damianpierri.com / damian.pierri@gmail.com. I would like to thank the comments and suggestions of K. Reffett, J. Stachurski, F. Kubler, H. Moreira, A. Abraham, Y. Balasko, A. Manelli, R. Fraiman, J. Martinez, H. Seoane, D. Heymann, J. Garcia Cicco, F. Ciocchini, R. Raad, C. Ferreira and I. Esponda. Last but not least, I would like to thank especially (in alphabetical order) to P. Gottardi and E. Kawamura. Without their encouragement and support this project would have not been possible. A previous version of this paper circulated as "Computable Stationary Markov Equilibria".

1 Introduction

This paper provides a *general framework for the characterization, computation, and simulation* of non-optimal economies with a finite number of heterogenous agents and aggregate uncertainty. Moreover, it gives conditions which guarantee that the parameters obtained from a calibrated model are consistent with the ergodic behavior of endogenous variables. We show that it is possible to accurately evaluate the quantitative performance of these economies. Further, we characterize ergodic equilibrium selections and differentiate them with respect to time-independent, stationary, and non-stationary ones. We show that the requirements for stationarity and time-independence are milder with respect to the sufficient conditions for ergodicity. Thus, our results go beyond the existence of a stochastic steady state (as in Dufffie, at. al. (1994)) as we can identify qualitative properties of stochastic transition functions (i.e., selections) connected with ergodicity and, in the absence of them, we can still have a sharper characterization of the long run of the model.

Frequently, researchers seek to investigate the long and short run effects of economic policies on the structure of stochastic dynamic equilibria. To achieve this objective, the variables in the model are *computed, simulated,* and *compare with its empirical counterpart*. This is done since general equilibrium models often do not have a closed form solution. Thus, computing and simulating an economy is the most immediate way to explore its quantitative equilibrium properties and, at the same time, provides a basis to characterize the stochastic structure of dynamic behavior. Unfortunately, there is *no general method* to compute and simulate non-optimal economies. The commonly used procedures generate different outcomes (see Hatchondo, et. al., 2010, De Groot, et. al., 2013, among others) and the simulations obtained from them may not provide accurate representations of the economies depicted by the models (see for instance Feng, et.al., 2014). *This is the gap that this paper bridges by answering the following question:*

Is it possible to characterize, approximate, simulate, and empirically evaluate an infinite horizon non-optimal economy with aggregate uncertainty and heterogenous agents using a general framework which guarantees that numerical simulations truly reflect the behavior of endogenous variables in the model?

We give sufficient conditions under which a positive answer to this important question can be provided in the context of a stochastic endowment economies with incomplete markets, finitely many heterogeneous agents and uncountable exogenous states as in Mas-Colell and Zame (1996). We show that accuracy is deeply connected with ergodicy and this property requires to refine the selection process substantially. In the absence of this refinement, we show that selections are generally time-independent, and the associated stochastic process is stationary. Thus, the results in this paper suggest that the stochastic steady state of heterogenous agent models with aggregate uncertainty is frequently sensitive to the initial conditions of the economy, *a fact which suggest that heterogeneity may have long lasting effects*.

1.1 Literature Review

The empirical performance of infinite horizon economies with complete markets or with a representative agent has been repeatedly questioned in the literature (see, for instance, Singleton, 1990, Mehra and Prescott, 1985, Guvenen, 2011, Hayashi et. al. 1996, Cochrane, 1991, Attanasio and Davis 1996, among many others). Because of the inability of these economies to match observed behavior, the literature has moved in different directions. The relevance of financial market incompleteness and market failures for economic analysis were recognized by the empirical evidence long time ago (see Pijoan – Mas (2007), Heathcote (2005), Krueger (1999) and Akyol (2004)). More recently, there is an important debate about the implications of heterogeneity for the design of fiscal and monetary policy (see for instance Jappelli and Pistaferri (2014) and Kaplan, Moll and Violante (2018)). Thus, we choose to apply the results in this paper to the simplest incomplete-markets models with heterogeneity and aggregate uncertainty: a closed endowment economy with numerary 1 period assets offered in zero net supply. This structure allows a sharp and rigorous characterization of selections due the existence of previous results for models with uncountable shocks (i.e., Mas-Colell and Zame, 1996).

The most closely related papers to this one in the literature are Santos and Peralta Alva (2013 and 2015), Brumm, et. al (2017), and Cao (2020) (see also Duggan (2012), and He and Sun (2017)). In these papers there are features of the model that are used to achieve either stochastic stability of the Markovian equilibrium or the existence of a recursive representation. Assumptions are connected with a continuum of households in Cao (2020) or with restrictions on exogenous shocks in Duggan (2012), Santos and Peralta Alva (2013 and 2015), Brumm et. al. (2017) and He and Sun (2017). With respect to Santos and Peralta (2013 and 2015), our paper identifies a particular selection taken from the equilibrium correspondence and characterize it to obtain an ergodic, a stationary, and a time-independent equilibrium. In Santos and Peralta, although it is proved that such equilibrium exists, they do not characterize a particular ergodic selection nor differentiate it with respect to a stationary one. They show that there exist at least one selection and characterize a transition function. We go further in this direction and found that ergodicity relates to the continuity of the selection with respect to exogenous shocks, allowing for a large discontinuity set with respect to endogenous states as it is found in the literature (see for instance Kubler and Schmedders 2002). We provide a selection mechanism and show that a restriction in the number of available assets suffices to achieve ergodicity.

The existence of a recursive representation is necessary but not sufficient to guarantee the accuracy of numerical simulations in the long run. This is the main difference with respect to Brumm et. al. (2017). To prove accuracy, we adapt the results due to Santos and Peralta Alva (2005) to non-optimal economies. We show that the uniform convergence of numerical approximations is essential to prove this result, a fact which requires to adapt the approximation procedure to deal with discontinuous selections using state of the art techniques (see Silanes, et. al. 2001).

Cao (2020) showed the existence of an ergodic recursive equilibrium as a selection from an equilibrium correspondence. However, neither the author can characterize the ergodic selections, nor differentiate them from stationary and non-stationary ones, nor relate such a selection to the accuracy of simulations.

We provide a unified theoretical framework to characterize non-optimal heterogenous agents models with aggregate uncertainty. The necessity of it comes from the failure of methods frequently used (i.e., Kydland and Prescott, 1982, Krusell and Smith, 1998, Cooley and Quadrini, 2001, Chari, Kehoe and McGrattan, 2002, among others) in providing simultaneously: (i) an adequate representation of the stochastic steady state, (ii) a well-defined stationary law of motion for the endogenous variables and (iii) a result which guarantee the accuracy of simulations. Kubler and Schmedders (2002) and Santos (2002) showed that there is a tension between computability and continuity of the recursive stationary equilibrium. This fact is associated with the presence of multiple equilibria in non-optimal economies, which may prevent the convergence of some algorithms. Santos and Peralta Alva (2005) and Feng, et. al. (2014) proved that simulations may not reflect accurately the steady state of the model, even provided a convergent algorithm.

The literature in general equilibrium, with the notable exception of Santos and Peralta Alva (2013 and 2015), Cao (2020) and Brumm, et. al. (2017), has not addressed problems (i) to (iii) at the same time. Duffie, et. al. (1994) and Blume (1982) showed the existence of an ergodic invariant measure for some non-optimal economies, but they did not take care of numerical part of the problem. Feng, et. al. (2014) derived a time invariant recursive representation, but they did not prove ergodicity. This paper fills the gap in the literature by dealing with with facts (i) to (iii) at the same time and provide identifiable conditions related to ergodicity.

The strategies used for the proofs differ from previous results. One of the consequences of allowing for multiple equilibria is that the selected transitions may not be continuous. This fact causes a serious problem for the existence of an invariant measure. The literature has circumvented this problem by using a fixed-point theorem for correspondences. Unfortunately, this approach requires conditions which affect the computability of transitions (like the convexification technique used in Duffie, et. al. or the impossibility to identify an appropriate selection in Blume). The strategy in this paper is to derive verifiable conditions on each computable transition that restore the continuity of the Markov process. Once ergodicty is established, the paper shows the accuracy of simulations by using Birkhoff's ergodic theorem and adapting a result in Santos and Peralta Alva (2005).

1.2 Preview of the results and outline of the paper

For economies with uncountable shocks, a recursive equilibrium notion due to Feng, et. al. (2014) is refined to derive a time-independent Markov process. This fact provides the first step to compute and simulate an infinite horizon non-optimal economy as it endows the sequential equilibrium with a dynamically simple and computable representation. This is done in Lemma 2 in section 3.1. Using our derived representation, it can be shown that: a) under additional restrictions on the stochastic transition functions, the equilibrium process is stationary (theorem 1) or ergodic (theorem 2), where the requirements associated with the latter are stronger. Primitive conditions of the model associated with these facts are described in lemmas 3 and 4, propositions 1 and 2. All these results are contained in section 3.1, 3.3 and 3.4. b) Assuming uniform convergence of numerical approximations and an additional restriction on the computed transition functions, the simulations obtained from them asymptotically replicate the actual ergodic long-run behavior of the model (see theorems 3 and 4 in section 4). c) It is possible to derive an accurate calibration procedure then based on a) and b). For economies with finitely many shocks, it is only possible to show that the recursive equilibrium in Feng, et. al. (2014) generates a time-independent stationary Markov process (section 3.2). Section 5 glues all the pieces together, applying the results to a concrete parametrization of the model. The appendix contains technical details for all sections and proofs.

2. Recursive equilibrium in heterogenous agent models

This section uses an infinite horizon general equilibrium model with incomplete markets to introduce novel recursive equilibrium concepts, discussed its existence and several properties which are useful to simulate the model. A part of this section is devoted to keep the paper self-contained. Thus, some results and definitions will be described in the appendix. The reader who is familiar with the concepts of sequential competitive equilibrium, recursive and wealth recursive equilibrium, Duffie, et. al.'s time homogeneous markov equilibrium and Feng, et. al.'s recursive equilibrium is invited to go directly to sections 2.4, which discusses the existence of "standard" recursive equilibrium concepts, and 2.6 that addresses the applicability of "modern" recursive equilibrium notions for the purposes in this paper.

Sections 2.3 to 2.6 are devoted to justifying the use of a refined version of the recursive equilibrium in Feng, et. al. Further, these sections explain why standard recursive equilibrium notions are unsuitable in the presence of multiple equilibria and how an ergodic Markov process can be used to accurately calibrate the model. The only directly related paper is Santos and Peralta Alva (2013). These authors also study the accuracy of numerical simulations for non-optimal economies. This paper refines and extends some of the results in Santos and Peralta Alva. A detailed discussion of the connection between these 2 papers is postponed to section 3.1 as it requires some investment in notation. There are other connected papers such as Brumm, et. al. (2017) and Cao (2020). However, none of these articles addressed simultaneously the existence of a Markov equilibrium, its ergodicity, and the accuracy of numerical simulations.

2.1 Structure of the Economy

The model is a standard infinite horizon discrete time pure exchange economy. A Markov chain defines the law of motion for the exogenous state variable[2]. For every period t, a shock s_t occurs; $s_t \in S$ and $S = (1,2,\dots,\mathrm{S})$. To model the evolution of uncertainty, an event tree approach is assumed. Each tree $\mathfrak{T}$ has a unique root, $\sigma_0 = s_0$. A typical element will be denoted $\sigma_t = (s_0, s_1, \dots, s_t)$. Each σ_t has a unique predecessor $\sigma_t^* = (s_0, s_1, \dots, s_{t-1})$ and S sucessors, $\sigma_t s$, for each $s \in \mathrm{S}$. Since the exogenous shocks follow a first order Markov process, when S is finite, the evolution of $\{s_t\}_{t=0}^{\infty}$ can be characterized by a transition matrix, $p = \left[p(s_i, s_j)\right]$. When postpone the details for uncountable shocks to section 3. For any given s_0, the probability of σ_t is $\mu_t(\sigma_t) = \prod_{t=1}^{t} p(s_{t-1}, s_t)$ and $\mu_0(\sigma_0) = \delta_{s_0}$, where δ_{s_0} is the Dirac measure at s_0.
The number of agents is assumed to be finite, with a typical element denoted $i \in I$. Each agent is endowed with $e^i(\sigma_t)$ units of the single consumption good. For simplicity, the endowment process is supposed to be iid: $e^i(\sigma_t s) = e^i(s)$, with $e^i\colon S \to \mathbb{R}_{++}$. The vector of endowments at any node is denoted $(\sigma_t) = \left\{e^i(\sigma_t)\right\}_{i=1}^{I}$. Each agent has an additively separable well behaved[3] utility function which is used to evaluate consumption streams, $c = \{c(\sigma_t)\}_{\sigma_t \in \mathfrak{T}}$:

$$U_i(c) = \sum_{t=0}^{\infty} (\beta^i)^t \sum_{\sigma_t^* s} \left[u_s^i(c^i(\sigma_t^* s))\right] \mu_t(\sigma_t^* s)$$

The asset structure is characterized by J one period numeraire real assets, offered in zero net supply, and traded at each node of the tree, $\sigma_t \in \mathfrak{T}$. An asset held by agent

[2] The set of exogenous shocks is assumed to be finite in all the equilibrium concepts mentioned in this section. This is done because the conditions to guarantee the existence of the sequential equilibria are well known. The Time Homogeneous Markov Equilibrium in Duffie, et. al. (1994), Kubler and Schmedders' Markov equilibrium and Feng's Recursive equilibrium can be defined for an arbitrary set of exogenous shocks (see Duffie, et.al. page 749 and Santos and Peralta Alva page 6 respectively). The conditions for the existence of the sequential equilibria with an uncountable, atomless and iid shocks, which is essential for the results in sections 3 and 4, are presented in section 5.

[3] To the conditions stated in Duffie, et. al. (1994) page 765, Kubler and Schmedders (2002) implicitly added the assumption that u_s^i has uniformly bounded gradients. This is done to satisfy a terminal condition on the discounted expected marginal utility (see equation 1 in page 288) which in turn is used to obtain a definition of equilibria based on first order and market clearing conditions. This last definition is essential for the recursive equilibrium literature as can be seen in sections 2.3, 2.5 and the appendix.

i is denoted $\theta_j^i(\sigma_t) \in \mathbb{R}$ and pays dividends $d_j(\sigma_t s) \in \mathbb{R}_+$, only at the S immediate successors of σ_t [4]. The portfolio of agent i at node σ_t will be denoted $\theta^i(\sigma_t) \in \mathbb{R}^J$. It is assumed that the dividend process is also iid: $d_j(\sigma_t^* s) = d_j(s)$, where $d_j: S \to \mathbb{R}_+$ [5]. Further, the $J \times S$ payoff matrix, d, is supposed to have full row rank and a column of d will be denoted $d(\sigma_t)$. Consequently, market incompleteness follows directly from $J < S$. Finally, the price of security θ_j at node σ_t will be denoted $q_j(\sigma_t) \in \mathbb{R}_+$, asset prices will be collected at the vector $q(\sigma_t) \in \mathbb{R}_+^J$ and the net wealth of agent i will be written as $w^i(\sigma_t) = e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t)$.

2.2 Sequential Competitive Equilibrium [6]

An economy $\mathcal{E}$ is characterized by the endowments, payoffs, the structure of preferences and the initial distribution of assets: $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^I, \{\theta_-^i\}_{i=1}^I\right]$. A sequential equilibrium for $\mathcal{E}$ can then be defined as follows,

<u>Definition 1</u>. A <u>sequential competitive equilibrium</u> for $\mathcal{E}$ is a collection of:

i) consumption vectors $\left[\{c^i(\sigma_t)\}_{i=1}^I\right]_{\sigma_t \in \mathfrak{T}}$,

ii) portfolio holdings $\left[\{\theta^i(\sigma_t)\}_{i=1}^I\right]_{\sigma_t \in \mathfrak{T}}$,

iii) asset prices $[\, q(\sigma_t)]_{\sigma_t \in \mathfrak{T}}$

Such that for $s_0 \in S$ *and* $\{\theta_-^i\}_{i=1}^I$ *satisfy:*

a) (Feasibility) For all $\sigma_t \in \mathfrak{T}$, $\sum_{i=1}^I \theta^i(\sigma_t) = \vec{0}$, *where* $\vec{0} \in \mathbb{R}^J$.

b) (Optimality) For each agent $i \in I$ *and prices* $[\, q(\sigma_t)]_{\sigma_t \in \mathfrak{T}}$:

[4] Agents are allowed to short sale every asset θ_j. To define a Time Homogeneous Markov Equilibrium, Duffie, et. al. assumed that there are no short sales and a different asset structure (J Lucas trees). However, Braido (2013) showed that the equilibrium concepts in Duffie, et. al. still holds if short sales are permitted for a general asset structure, which includes one period real assets offered in zero net supply, provided that marginal utilities are uniformly bounded above.

[5] Except in section 2.4, where the equilibrium has closed form, for economies with $\#S < \infty$, it will be assumed that the dividend structure has a riskless bond as in assumption A.6 in Magill and Quinzii (1994) (i.e., $d_j(s) = 1$ for any $s \in S$ and $j \in \{1, \dots, J\}$.

[6] This concept is analogous to the Financial Market Equilibrium in Magill and Quinzii (1996), page 228, extended to an infinite horizon economy.

$$\left[c^i(\sigma_t), \theta^i(\sigma_t)\right]_{\sigma_t \in \mathfrak{T}} \in argmax\{\ U^i(c)\ subject\ to\ \ c(\sigma_t) = w^i(\sigma_t) - \theta^i(\sigma_t).q(\sigma_t)\ for\ all\ \sigma_t \in \mathfrak{T}\ and\ sup_{\sigma_t \in \mathfrak{T}}\left|\theta^i(\sigma_t).q(\sigma_t)\right| < \infty\}.$$

As the payoff matrix does not depend on prices, its (row) rank is constant for any period $0 \leq t \leq \infty$. Consequently, the excess demand function of all agents can be shown to be continuous[7]. To establish the existence of equilibria, an implicit debt constrained is added in condition b). Magill and Quinzii (1994) showed that this condition rules out Ponzi schemes, it is never binding and is sufficient for existence.

2.3 Function Based Recursive Equilibria

In this section we discuss the existence of a time invariant *continuous function* that maps different state spaces (i.e., exogenous shocks, wealth, etc.) into the rest of payoff relevant variables. It will be clear in section 2.4 that the existence of this function depends on the uniqueness of equilibrium; a property that has not been proved in general equilibrium models with infinite horizons and incomplete markets. Most of the relevant definitions will be described in the appendix[8].

Section III of the appendix formally states the definition of weakly recursive equilibrium (WR) and wealth recursive equilibrium (WhR). The WR / WhR requires the existence of 2 continuous functions g^j, f^i / g^j_{WhR}, f^i_{WhR} mapping (s,θ) / $(s,w) \mapsto q^j, \theta^i$ respectively. Unfortunately, as it will be illustrated in section in the next section, the equilibrium concepts described above do not always exist.

[7] See Magill and Quinzii 1996, exercise 3, page 276 for a counterexample for the case of long-lived assets.
[8] See section A.1.1, technical appendix of section 2.3, page 12 of the supplementary appendixes.

2.4 A counter example

We present, for the sake of concretness, an example borrowed from Kubler and Schmedders (2002) which shows that continuous policy functions may not exist in the presence of multiple equilibria. There will be no wealth recursive equilibria if for the same pair of states (s, w) there are at least 2 possible asset prices. This happens because wealth is not a sufficiently state variable: for 2 different portfolio distributions, wealth may the same, but asset prices may differ. In this sense, wealth is insufficient to capture the heterogeneity of agents' decisions and thus constitutes an inappropriate state space for function based recursive equilibrium notions. The authors also presented an economy with no weakly recursive equilibria. For the sake of concreteness, this paper will only discuss the first case, but it should be kept in mind that multiplicity is common in non-optimal economies and affects not only endowment models and the (s, w) state space but also production economies (see Santos 2002) and more "informative" state spaces like (s, θ). The economy is a particular parametrization of the model described in section 2.1. The characterization used for this economy is available in the appendix, section A.1.1.

The section III of the appendix shows a function selected from the equilibrium correspondence which arises after imposing (s, w) as a state space. This type of correspondences may not have a continuous selection[9]. However, as it has closed graph, it is possible to take an appropriate (i.e., measurable[10]) selection to describe the dynamic behavior of the model. Because of the multiplicity of equilibrium, each selection will describe different and equally likely dynamic behavior. This paper, in contrast to Pierri and Reffett (2021), does not propose a selection mechanism.

[9] The correspondence in figure 1 can be shown to be compact valued and upper hemi continuous. For a definition see for instance Stokey, Lucas and Prescott, ch. 3. There are robust examples of this type of correspondences with no continuous selections.

[10] The correspondence observed in figure 1 does not have a continuous selection in the Euclidean metric. However, because it has been established that the steady state in this example is a set of finite cardinality, it is possible to endow the model with the discrete metric and solve all the problems related with the lack of continuity of the transition function. Unfortunately, there are no general conditions on the cardinality of the steady state. For an example of a model with discontinuous transition functions, see Santos (2002).

The discussion above suggests the strength of the continuity assumption with respect to *endogenous states* in definition 2 and 3 presented in the appendix. As this paper derives a general framework, it is necessary to derive the theoretical results without this assumption. The next section addresses this topic.

2.5 Correspondence Based Recursive Equilibria

Contrarily to the equilibrium concepts discussed in section 2.3, the modern recursive literature allows for multiple equilibria and requires a correspondence in order to capture the first order dynamic behavior of the economy. There are 3 seminal papers in this branch of the literature: Duffie, et. al. (1994), Kubler and Schmedders (2003) and Feng, et. al. (2014). All these papers show the existence of a time independent first order recursive structure under mild assumptions.

Section 2.5.1 introduces the results in Duffie, et. al. and discusses its usefulness and limitations for the purposes of this paper. As the recursive structure in Kubler and Schmedders (2003) uses Duffie, et. al.'s results, it shares the same properties and thus will be omitted[11]. Section 2.5.2 discusses the recursive equilibrium in Feng, et. al., which is the starting point of the results in this paper. As before, for the sake of concreteness, details are left to the appendix.

2.5.1 Duffie's et. al. (1994) Time Homogeneous Markov Equilibria

This section illustrates how Duffie, et. al.'s results can be used to: i) show the existence of a sequential equilibrium (fact 2.5.2), a result that will be used in applications in section II of the appendix; ii) derive a time invariant recursive

[11] One of the main contributions of Kubler and Schmedders (2003) is a correspondence based recursive structure, called Markov Equilibria, with minimal state space. As this paper is not concern with the numerical properties of the algorithms involved in the computation of the recursive structure, Kubler and Schmedders' results could be replaced with Feng, et. al.'s which are not affected by the problems in Duffie, et. al. but may have a larger state space. It would be interesting to derive Kubler and Schmedders' Markov equilibria from Feng, et. al.'s structure.

structure and to generate a stationary Markov process[12]; iii) simulate the process (fact 2.5.1), a result that will be used in section 4.2.

This section also discusses the limitations of the results in Duffie, et. al., most of them concern numerical simulations. These facts are essential to understand how the results in Feng, et. al. (2014) fit the purposes of this paper as they solve all the problems in Duffie, et. al. but preserves all its useful properties.

Two facts are worth mentioning from Duffie, et. al.'s equilibrium notion:

Fact 2.5-1): the state space J is the smallest[13] set that can be used to define an equilibrium correspondence as it contains all initial states of any infinite horizon sequential competitive equilibrium and is time independent. It can be used to iterate forward a *first order dynamic stochastic process with a time invariant state space*. This property is frequently called "self-generation" and is weaker than its analogous in games with endogenous states called "self-justification"[14].

Fact 2.5-2): The existence of J requires the existence of a temporary equilibrium for a truncated economy with finite time and all these equilibria must be uniformly compact. Then the former is typically extended to infinity by induction (see lemmas 3.4 and 3.5, page 768), the latter follows from the existence of uniform bounds on endogenous variables. The non-emptiness of the temporary equilibria is directly connected to the "consistency" requirement in the games literature as in Phelan and Stacchetti (2001).

These 2 facts combined with an argument on the optimality of the sequences generated from J using the equilibrium correspondence (see section 3.4 in Duffie, et.

12 See definition 4 in the appendix, page 18.

13 In the temporary equilibrium framework of Hildenbrand and Grandmont (1974) it is possible to set $J = Z$ as overlapping generation agents only live 2 periods. In this type of economies, agents live infinitely many periods and thus it is possible that the backward induction procedure implied by equations 1 and 2 converges to an empty set. Fact 2.5.1) show that this is not the case for economies with compact K and $C_j \neq \emptyset$ for $j \geq 1$.

14 I would like to thank K. Reffett for pointing this out to me. See Phelan and Stacchetti (2001).

al.) can be used to show the existence of a sequential competitive equilibrium. Although this result has already been applied to other incomplete market economies for the case of finite shocks (see Kubler and Schmedders, 2003, Lemma 2), it is not generally used in economies where S is assumed to be uncountable and compact. For the results in this paper, the last structure of exogenous shocks turns out to be important[15]. Thus, this type of existence proof will be discussed in section II of the appendix which involves applications. In the model presented in section 2.1 and 2.2, the Mas-Colell and Zame (1996) framework allows showing the existence of J and the compactness of equilibria. The optimality argument in section 3.4 of Duffie, et. al. can be straightforwardly extended in that model to the case of uncountable shocks. A time invariant Markov process is constructed using 2 building: a state space and a Markov operator. In the context of Duffie, et. al., the state space is J [16]. The Markov operator is denoted π and is a selection of the equilibrium correspondence denoted G (denoted $\pi \sim G$) such that $\pi: J \rightarrow \mathcal{P}(J)$. A pair (J, π) is called Time homogeneous Markov Equilibria, which is formally defined in the appendix. Even though the results in Duffie, et. al. can be used to guarantee the existence of a recursive structure, a THME *is not a computable representation of the* sequential competitive equilibrium as *the time invariant transition functions of the recursive equilibrium depend on unobservable variables, which are basically a selection devise*. This fact is illustrated by the following lemma.

Suppose that the state space, $J \subseteq Z_D$, can be written as $Z_D = S \times \hat{Z}$, where $\hat{Z} = \left\{[\theta_-, c, q, \theta] \in \mathbb{R}^{IJ} \times \mathbb{R}^I \times \mathbb{R}^J \times \mathbb{R}^{IJ} \middle| \sum_{i=1}^I \theta_-^i = \vec{0}, \sum_{i=1}^I \theta^i = \vec{0}\right\}$. Considering the exogeneous nature of S, this assumption can be imposed without loss of generality. The

[15] Fact 2.5.2) implies that any truncated economy ($j \leq T < \infty$) which has uniformly bounded endogenous variables (contained in K) can be used to prove the existence of a sequential infinite horizon equilibria. That is, any recursive equilibrium is a sequential equilibrium. However, there may be some sequential equilibria that are not recursive or that do not have a terminal debt level equal to 0. So, fact 2.5.2) can be used to prove the existence of a subset of all possible sequential equilibria. I would like to thank A. Manelli for pointing this out to me.

[16] It is standard to assume that Z is a Borel Space. As the Cartesian product of a finite set and a finite dimensional Euclidean space is a complete, separable and metric space, the product space is a Polish space. Thus, Z is a measurable subset of a Polish space. If $\mathcal{B}_{[Z]}$ is the Borel sigma-algebra generated from Z, $(Z, \mathcal{B}_{[Z]})$ is a Borel Space. Consequently, measurable will always mean Borel measurable and any measure will be a Borel measure.

equilibrium correspondence G maps $Z_D \mapsto \mu$ with $\mu \in \rho(Z_D)$, where $\rho(Z_D)$ is the space of measures generated out of Z_D. Then,

Lemma 1: *If (J, π) constitute a THME any realization of a process $\{z_t\}$ satisfies as $\hat{z}_{t+1} = f(s_{t+1}, \alpha_{t+1}, z_t)$ where f is a measurable function and $\alpha_{t+1} \in [0,1]$ is uniformly distributed and i.i.d.*

Proof: See Lemma 2.22 page 34 In Kallenberg (2006).

Duffie, et. al. (1994) interprets α_{t+1} as sunspots. Note that lemma 1 implies that for each state z_t, any exogenous shock s_{t+1} could be associated with a continuum of possible continuation states in $\hat{Z}$, each one of them derived from the realization of an *unobservable variable* (α_{t+1}). Consequently, a tree structure with a finite number of branches after each node would not be an appropriate representation of $\{z_t\}_{t=0}^{\infty}$.

A THME has limited predictive power about the evolution of the state process. Thus, a "refinement" is required to obtain a computable object. This is done in definition A.1 in the technical appendix of section 3.1. Duffie, et. al. also provided sufficient conditions for the existence of this refined equilibria. If S is a *finite set*, a subset of G, denoted g, is also an equilibrium correspondence. g induces a compact state space[17] for a Markov process that has realizations without sunspots.

For the purposes of this and Duffie, et. al.'s papers, this refinement is insufficient as the Markov process associated with (J, π) may not be *stationary nor ergodic.*

The concept of conditionally spotless THME, also presented in section II of the appendix, was introduced to address this topic and to derive a notion of steady state, called *ergodic measure*[18]. This equilibrium refines the notion of self-generation in fact 2.5.1 to transitions without sunspots. An invariant measure guarantees that the

[17] The expectation correspondence $g \subset G$ is obtained by restricting μ in equation 2 to the set $\mathcal{P}_F(S \times \hat{Z}) \subset \mathcal{P}(Z)$ defined in setion A.1.1 in the appendix. The set of conditions on K and C_j mentioned above can still be used to guarantee the existence of a self-justified set for g.

[18] See Theorem 1.1 and Proposition 1.3 in Duffie, et. al., page 750 and 757.

Markov process associated with the THME is stationary and the ergodicity of the steady state *implies that the process generates convergent sample paths.*

The authors argued that a conditionally spotless THME implies that any sequence $\{z_t\}_{t=0}^{\infty}$ in J can be described by a random variable representation of the expectations correspondence: a function f that satisfies $\hat{z}_{t+1} = f(s_{t+1}, \alpha_t, z_t)$ for any t, where $\alpha_t \in [0,1]$ is uniformly distributed, i.i.d and represents another type of sunspot which is used to convexify the equilibrium correspondence. As f is not a computable object, because of the presence of sunspots at t, it is inappropriate for this paper.

2.5.2 Feng, et. al.'s Recursive Equilibria

The virtue of Feng, et. al. approach is to derive a recursive structure that exists even in the presence of multiple equilibria as in Duffie, et. al. but at the same time generates computable time invariant transitions. These facts imply that the results in Feng, et. al. can be used to derive laws of motion for the endogenous variables that do not depend neither on unobservable variables nor time. This is a consequence of the definition of the expectation correspondence, which now maps to the space of random variables directly. Besides, this structure sometimes has a lower dimensional state space when is compared to Duffie, et. al.'s, a property that is desirable from a numerical point of view.

In order to obtain these results, It is necessary to restrict the number of possible recursive equilibria (see the technical appendix of this section for additional details) and derived a correspondence $\Phi: \tilde{Z} \times S \rightrightarrows \tilde{Z}$ that maps $(\tilde{z}_t, s_{t+1}) \longmapsto \tilde{z}_{t+1}$, which can be used to construct the analogous of Duffie, et. al.'s equilibrium correspondence, G[19] but with a different image. The state space, denoted Z_F and defined in the appendix, is composed by a subset of the state space in Duffie, et. al, $Z_F \subseteq Z_D$, and an auxiliary variable m which is typically restricted by $z_F \in Z_F$. This additional

[19] The procedure to derive the analogous of G in Duffie, et. al. from Φ will be presented at the beginning of section 3.

variable relates to the derivative of the value function in the sequential problem and under some conditions can be shown to be envelopes of a recursive problem, connecting both types of equilibrium directly through the Euler equation. Thus, $\tilde{z} \in \tilde{Z}$ is of the form $[z_F, m(z_F)]$. As the image of Φ is in the space of realizations, an additional state variable (S) must be added to the domain of the correspondence.

A procedure like the one described in Duffie, et. al., available in the appendix, can be used to refine the state space from $\tilde{Z}$ to $\tilde{J}$, where $\tilde{z} \in \tilde{Z}$ contain all the possible initial variables, $\tilde{z}_0 = [s_0, q_0, \theta_0, m_0]$, each of them associated with a different SCE.

From the discussion in the appendix, 4 facts are relevant for this equilibrium:

Fact 2.5-3): The realizations of the process depend only on observable variables.

Fact 2.5-4): The equilibria in Feng, at. al. is a subset of those in Duffie, et. al.

Fact 2.5-5): For *some* models which fit definition 1, there is a selection $\varphi \sim \Phi$ which can be chosen to be continuous in s_+ for each $\tilde{z} \in \tilde{J}$. This fact does not imply the uniqueness of the equilibrium. For details, see lemma 5 and the technical appendix of this section, in section II.2 and III of the appendix respectively.

Fact 2.5-6): The state space in Feng, et. al. is smaller than the one in Duffie, et. al.

2.6 Computability, Simulations and Empirical Validity

In most applications it may be interesting to test the empirical performance of the model. As general equilibrium economies typically do not have closed form solutions, it is necessary to approximate the endogenous variables model.

The dimension of the sequential competitive equilibrium notion hinders its computability[20]. In particular, any sequential competitive equilibrium can be thought as an infinite sequence of measurable functions $\{\bar{z}_t(\sigma_t)\}_{t=0}^{\infty}$, $\bar{z}_t: \mathfrak{T} \rightarrow \mathbb{R}^I \times \mathbb{R}^J \times \mathbb{R}^{IJ}$, which satisfy conditions a) and b) of definition 1[21] and $\bar{z}_t(\sigma_t) = [\bar{c}_t(\sigma_t), \bar{q}_t(\sigma_t), \bar{\theta}_t(\sigma_t)]$ for some $\sigma_t \in \mathfrak{T}$. Thus, to compute $\{\bar{z}_t(\sigma_t)\}_{t=0}^{\infty}$, it is necessary to solve an infinite number of nonlinear systems of equations. As discussed in Judd, et. al. (2002), this task is rarely achieved in finite CPU time.

Moreover, the stochastic process generated from the sequential equilibrium is generally nonstationary[22]. This property follows from the associated measure μ_t defined in section 2.1, that is allowed to change over time, and turns the empirical assessment of the model a difficult task even if a closed form solution is available (see the technical appendix for a discussion on the relevance of an ergodic Markov process to achieve accurate simulations).

A natural procedure to get empirically relevant predictions is to compute a time invariant recursive structure, composed by a state space and a transition function, with an associated ergodic Markov process. Then, it is possible obtain simulations that approximate the unconditional expected value of the endogenous variables in the (exact) model, which in turn can be used to match the (observed) time series behavior. This fact requires that simulations converge to unconditional moments; a property that is achieved if the invariant measures associated with the true and approximated recursive structures are *ergodic*.

Formally, it is necessary to insure that:

> Fact 2.6-i) $T^{-1}\left[\sum_{t=1}^{T} f\left(\bar{z}_t^{j,\vartheta}\right)\right] \rightarrow_{a.s.} E_{\mu_{j,\vartheta}^*}(f)$ and $T^{-1}\left[\sum_{t=1}^{T} f\left(\bar{z}_t^{\vartheta}\right)\right] \rightarrow_{a.s.} E_{\mu_{\vartheta}^*}(f)$ where j denotes the j^{th} numerical approximation of the model and f is a $\bar{z}$-measurable,

[20] Recently Ferraro and Pierri (2018) presented an algorithm to compute "directly" the SCE in a production non-optimal economy with idle capacity utilization.

[21] Note that $\mathfrak{T} = S^{\infty}$, where $S^{\infty} = S \times S \ldots$ is the infinite Cartesian product of finite sets.

[22] If $\{s_t\}_{t=0}^{\infty}$ is an i.i.d process, a sequential competitive equilibrium $\{\bar{z}_t\}_{t=0}^{\infty}$ is stationary (see Stokey, Lucas and Prescott (1989) page 224, exercise 8.6).

possibly continuous, function. Further, $\mu^*_{j,\vartheta}$, μ^*_{ϑ} is any ergodic measure associated with the j^{th} approximation and with true model, respectively, both taking parameters values $\vartheta \in \Lambda$. Typically, ϑ contains parameters related with preferences and endowments and Λ is compact. The convergence of simulations will be assumed to be almost surely ($a.s.$) in a measure defined in section 4.2.

Fact 2.6-ii) $E_{\mu^*_{j,\vartheta}}(f) \rightarrow_{Weak*} E_{\mu^*_{\vartheta}}(f)$, the convergence is in the $weak *$ topology. This result ensures the accuracy of simulations and is the missing ingredient in papers like Cao (2020), as they only show the existence of an ergodic measure.

Fact 2.6-iii) $Min_{\vartheta\in\Lambda}\left\|E_{\mu^*_{\vartheta}}(f) - \bar{X}_f\right\|$, $\bar{X}_f$ is the mean of the empirical analogous of $f(\bar{z})$ and $\|.\|$ is the Euclidean norm (see Dave, et. al. 2007).

This paper derives conditions which guarantee that facts 2.6-i) and 2.6-ii) hold in endowment general equilibrium models with aggregate uncertainty, incomplete markets and a finite number of heterogeneous agents, allowing for the possibility of multiple equilibria. The technical appendix contains further details about these facts.

3. Stationarity and Ergodicity

To obtain accurate and empirically meaningful numerical simulations[23] some notion of stationarity is required. Time homogeneity is desirably, but it is not enough. A reliable procedure requires an ergodic measure. Heuristically, this fact was stated in section 2.6. This section formally proves first the existence of an invariant measure and later its ergodicity for a computable, correspondence based recursive structure.

[23] The precise meaning of "numerical simulations" will be given in section 4. The results in this paper assume the existence of a uniformly convergent method to compute a recursive equilibrium in the sense of Feng, at. al.

Sections 3.1 and 3.2 together establish the existence of an invariant measure for models with at most a finite number of exogenous shocks that fit into Feng, et. al's framework. Sections 3.1 and 3.3 show the existence of an invariant (theorem 1) and of an ergodic (theorem 2) measure for models with an uncountable number of shocks[24]. Section 3.4 provides sufficient conditions for stationarity (associated with an invariant measure, proposition 1) and ergodicity (proposition 2).
The figure below illustrates the connection between all the theoretical results in this section. Details of the figure are contained in the appendix, section I.

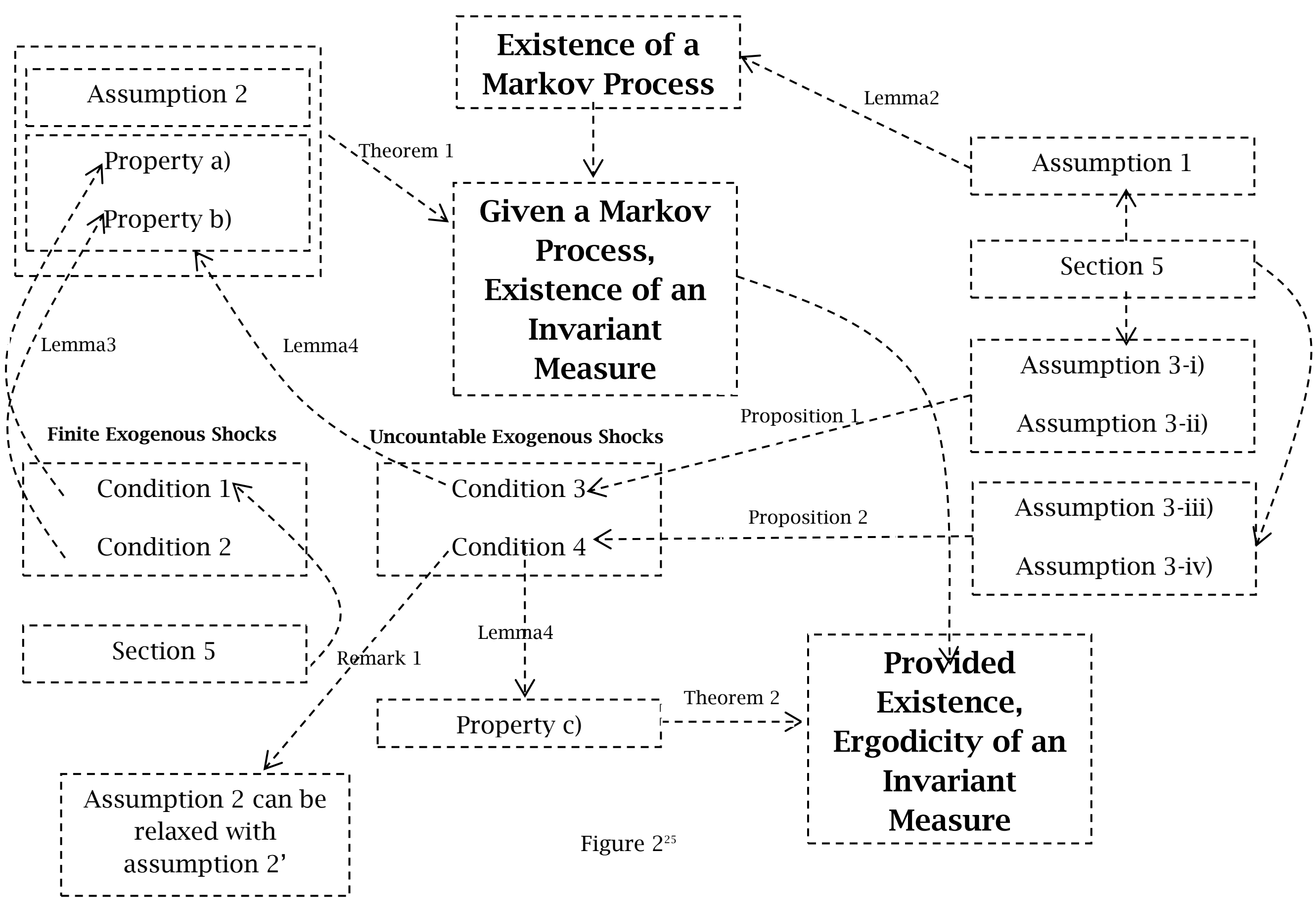

Figure 2[25]

[24] Economies with an infinite but countable number of shocks are intentionally left out as they represent a particularly challenging case for the purpose of this paper: the existence of equilibria requires the same strength of assumptions as in the case of uncountable shocks (see Mas Collel and Zame, 1996) and the existence of an invariant measure is as difficult to show as the case of a finite number of shocks.
[25] Figure 1 is in the supplementary appendix for section 2.4

The figure above implies that theorems are proved given some properties, which are associated with conditions. The relationship between conditions and properties are stated in lemmas. Conditions and properties are stated in terms of endogenous variables and operators, respectively. Then, assumptions are based on primitives and connected with conditions through propositions. Stationarity and ergodicity follow from primitive assumptions 3-i), 3-ii) and 3-iii), 3-iv) respectively.

3.1 Theorems 1 and 2: Existence of an Invariant Measure and Ergodicity

The starting point of this section is a Markov operator for exogenous shocks, $p(s, A) \geq 0$ defined for all $s \in S$ and $A \in \mathcal{B}_S$, where S is compact and $\mathcal{B}_S$ denotes the Borel sets in S, together with the equilibrium correspondence in Feng, et. al., discussed in section 2.5.2. This correspondence is assumed to satisfy:

Assumption 1: *Let $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ be the equilibrium correspondence in definition 5. Then, $\tilde{J}$ is compact and Φ is upper hemi continuous and compact valued.*

These properties can be obtained from mild assumptions on the primitives of the model discussed in section 2, both for finite (Magill and Quinzii, 1994, page 858, assumption 1 to 5) and infinite (Araujo, et. al. 1996, page 122, assumptions 1 and 3) shocks. A detailed discussion is postponed to section II in the appendix.
Assumption 1 together with the following lemma allows defining a Markov operator.

Lemma 2: Let Φ satisfy assumption 1. Then, $\varphi \sim \Phi$ is a $\mathcal{B}_{\tilde{J} \times S}$ -measurable selection of Φ and $P_\varphi(\tilde{z}, A) \geq 0$ is a Markov operator on $(\tilde{J}, \mathcal{B}_{\tilde{J}})$, where P_φ is given by:

5) $P_\varphi(\tilde{z}, A) = p(s, \{s' \in S | \varphi(\tilde{z}, s') \in A\})$, *where* $\tilde{z} = [s, \hat{z}]$

Proof: See Lemma 1 in Hildenbrand and Grandmont (1974), page 260.

Lemma 2 implies the existence of a $\mathcal{B}_{\tilde{J}\times S}$ - measurable function φ, which is the natural candidate to be the time invariant transition function of the process defined by $(\tilde{J}, P_\varphi)$ with typical realization $\{\tilde{z}_t\}_{t=0}^{\infty}$ as it satisfies $\tilde{z}_{t+1} = \varphi(\tilde{z}_t, s_{t+1})$ for any initial condition[26]. Let $B(\tilde{J})$ and $\mathcal{P}(\tilde{J})$ be the space of bounded $\mathcal{B}_{\tilde{J}}$-measurable functions and the space of probability measures on $\tilde{J}$ respectively. Let $\hat{P}_\varphi : B(\tilde{J}) \rightarrow B(\tilde{J})$ and $P_\varphi^* : \mathcal{P}(\tilde{J}) \rightarrow \mathcal{P}(\tilde{J})$ be the semigroup and adjoint operators defined by $\hat{P}_\varphi f(\tilde{z}) = \int f(\tilde{z}')P_\varphi(\tilde{z}, d\tilde{z}')$ and $P_\varphi^*\mu(A) = \int \mu(d\tilde{z})P_\varphi(\tilde{z}, A)$. Standard results[27] imply that $\hat{P}_\varphi f(\tilde{z}) \in B(\tilde{J})$ and $P_\varphi^*\mu(A) \in \mathcal{P}(\tilde{J})$ provided that $f \in B(\tilde{J})$ and $\mu \in \mathcal{P}(\tilde{J})$, respectively.

Theorem 1 establishes *properties* which guarantee that the Markov process $(\tilde{J}, P_\varphi)$ has an invariant measure, $\mu \in \mathcal{P}(\tilde{J})$ with $\mu = P_\varphi^*\mu$, provided that under assumption 1 φ *may not be continuous*. An invariant, not necessarily ergodic, measure is a fixed point of P_φ^* and implies the stationarity of $(\tilde{J}, P_\varphi)$.

From the discussion in the appendix (see the technical appendix of section 3.1) the discontinuity of φ is at the heart of the problem as it breaks the continuity of the adjoint operator. *Theorem 1 restores this property by restricting μ and φ such that the discontinuities in the transition function are negligible in an appropriate sense*. The following assumption formally states the mentioned restriction on φ.

<u>Assumption 2:</u> Let $\varphi \sim \Phi$ be a $\mathcal{B}_{\tilde{J}\times S}$ – measurable selection of the correspondence defined in assumption 1 and $\Delta\varphi$ its discontinuity set*. Then, $\Delta\varphi$ is a collection of at most a countable number of points.*

Under assumption 1, the range of φ is uniformly bounded. Thus, assumption 2 allows φ having at most a countable number of jump discontinuities. See the technical appendix for more details on the implications of assumption 2.

Now it is possible to state one of the main results in this paper:

[26]A careful definition of the stochastic process associated with $(\tilde{J}, P_\varphi)$ will be given in section 4.

[27] Stokey, Lucas and Prescott, 1989, page 213 to 216

<u>*Theorem 1 (Existence of an Invariant Measure-Stationarity of the Process)*</u>*:*
Let $\varphi \sim \Phi$ satisfies assumptions 1 and 2. Suppose additionally that a) $P_\varphi^*: \mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ and b) $\mathcal{P}_0(\tilde{J})$ is weak closed, where $\mathcal{P}_0(\tilde{J})$ is the set of atomless measures in $\mathcal{P}(\tilde{J})$. Then there is a measure $\mu \in \mathcal{P}_0(\tilde{J})$ such that $\mu = P_\varphi^* \mu$.*
Proof: see section IV of the appendix.

Note that a) and b) are “properties” of the process $(\tilde{J}, P_\varphi)$ and together imply that the discontinuity set of φ is negligible. That is, $\mu_n \longrightarrow_{weak*} \mu$ and $\mu(\Delta\varphi) = 0$. Sections 3.2 to 3.4 relate these properties with verifiable “conditions” on P_φ, φ and S. If property a) is satisfied, it suffices to assume that the set $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_0(\tilde{J})\}$ is *weak** closed. This is the strategy taken here and is concerned with the variability of the image of the transition functions, which cannot accumulate mass at any given point. This property requires variability in the image of a possible vector valued function as we move through the coordinates in the domain (see section V.1 in the appendix). Let $IM(\varphi, \mathcal{P}_1) = \{\mu \in \mathcal{P}_1(\tilde{J}) | \mu = P_\varphi^* \mu\}$, where $\mathcal{P}_1(\tilde{J}) \subseteq \mathcal{P}_0(\tilde{J})$. That is, $IM(\varphi, \mathcal{P}_1)$ is a set of invariant measures of $(\tilde{J}, P_\varphi)$ which belong to $\mathcal{P}_1(\tilde{J})$, the set of absolutely continuous measures with respect to the Lebesgue measure on $\tilde{J}$, denoted θ. Under assumptions 1) and 2), if properties a) and b) hold for $\mathcal{P}_1(\tilde{J})$, the non-emptiness of $IM(\varphi, \mathcal{P}_1)$ can be assured using theorem 1 as long as $\mu \in \mathcal{P}_1(\tilde{J})$, $\mu(\Delta\varphi) = 0$ *and μ is the* weak* limit of a sequence of measures generated by P_φ^*. To show that $IM(\varphi, \mathcal{P}_1)$ is compact, which is essential for the existence of an ergodic measure, it is necessary to impose stronger conditions on P_φ, and consequently on φ, than the ones that arise from theorem 1. Once this strengthening has been made, the closedness of $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_1(\tilde{J})\}$ follows from the same result used for $\mathcal{P}_0$ which insures the non-emptyness of $IM(\varphi, \mathcal{P}_1)$. This result cannot be applied to show the compactness of $IM(\varphi, \mathcal{P}_1)$. A suitable proof of this result is available in the appendix. A discussion of these issues can be found in the supplementary appendix to sections 3.1, 3.3 and 3.4.

A set $A \in \mathcal{B}_{\tilde{J}}$ is called invariant under P_φ if $P_\varphi(z, A) = 1$ for any $z \in A$. Let $IM(\varphi)$ be the set of invariant measures associated with selection φ. We say that $\mu \in IM(\varphi)$ is ergodic if either $\mu(A) = 0$ or $\mu(A) = 1$ for any invariant set under P_φ. The next theorem presents properties of $IM(\varphi)$ which guarantee that there exists an ergodic measure.

Theorem 2 (Existence of an Ergodic Measure-Ergodicity of the Process):
Let $\varphi\sim\Phi$ satisfies assumptions 1 and 2. Suppose additionally that $IM(\varphi, \mathcal{P}_1) \neq \emptyset$, where $\mathcal{P}_1$ is the set of absolutely continuous measures with respect to the Lebesgue measure. If c) $IM(\varphi, \mathcal{P}_1)$ is closed, then $IM(\varphi, \mathcal{P}_1)$ contains an ergodic measure.

Proof: The closedness of the set implies its compactness from proposition 2.8 in Futia (1982, page 385). As $IM(\varphi, \mathcal{P}_1)$ is convex, the Krein-Milman theorem (see Simon, 2011, theorem 8.14, page 128) implies that the set of extreme points of $IM(\varphi, \mathcal{P}_1)$, denoted $\mathcal{E}(IM(\varphi, \mathcal{P}_1))$, is non-empty. Remark 6.3 in Varadhan (2001, page 190) implies that if $\mu \in \mathcal{E}(IM(\varphi, \mathcal{P}_1))$, then μ is ergodic.

Theorems 1 and 2 are the first attempt to show separately the existence of an invariant and an ergodic measure for a *computable* correspondence based recursive equilibrium. The difference with respect to previous results can be found in section III of the appendix. For the case of uncountable shocks, we found sufficient conditions for stationarity and ergodicity by characterizing a *particular selection* and connecting it with primitive conditions of the model (see section 3.4 and fact 2.5-5 in section 2.5 with the associated supplementary appendixes). These facts ensure that our results can be used for computation and estimation of heterogenous agent models with aggregate uncertainty and incomplete markets.
Sections 3.2 and 3.3 identify conditions on P_φ which guarantee properties a), b) and c) associated with theorems 1 and 2. These conditions will be traced back to the primitives of certain type of economies in sections 3.4.

3.2 The case of a finite number of shocks

Theorem 1 requires 2 properties. Namely, that the adjoint operator associated with some Markov process $(\tilde{J}, P_\varphi)$ maps the set of atomless measures, $\mathcal{P}_0(\tilde{J})$, into itself (property a) and that $\mathcal{P}_0(\tilde{J})$ is closed (property b). The relationship between these properties and certain conditions of the Markov operator P_φ allows connecting the existence of an invariant measure with primitive assumptions in the model (i.e., restrictions on preferences, shocks, etc) as they affect $\varphi \sim \Phi$ and thus P_φ.

This section takes the first step towards that direction by restricting S, the set which contain the exogenous shocks, to be of finite cardinality. Let $\mu_{n,\theta}$ be a sequence of measures generated by applying P_φ^* iteratively on some $\theta \in \mathcal{P}(\tilde{J})$. Then, the following lemma states conditions on P_φ which guarantee properties a) and b).

Lemma 3 (Conditions for stationarity in models with finite shocks):

Let Φ satisfy assumption 1 and $\#S < \infty$. Then, the measurable space $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ has an atomless measure θ. Let $\{a\}$ be any point in $\tilde{J}$. Suppose that for some $\varphi \sim \Phi$:

1) $\theta(\{a\}) = 0$ implies $P_\varphi(z, \{a\}) = 0$ θ-almost everywhere

2) $Sup_n \mu_{n,\theta}(\{a\}) = 0$.

Then, properties a) and b) in theorem 1 are satisfied.

Proof: see section IV of the appendix.

Note that lemma 3 requires *conditions* 1 and 2 to hold *simultaneously* to guarantee properties a) and b). As can be seen in the appendix, condition 1 is associated with property a) and condition 2 with property b). While sections II and V in the appendix present mild sufficient conditions on the primitives of the economy $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^{I}\right]$, defined in section 2.1, which guarantee condition 1, it is still an open question how to assure that condition 2 holds in a general equilibrium non-optimal economy[28]. Martinez and Pierri (2021) and Pierri and Reffett (2021) try to fill this gap by imposing restrictions on S and φ respectively. Thus, a strong assumption is required to assure the weak*-closedness of $\mathcal{P}_0(\tilde{J})$ when the state space is of the form $\tilde{J} = S \times \hat{Z}$, S is finite and $\hat{Z}$ is uncountable.

3.3 The case of an infinite number of shocks

This section presents conditions on the Markov operator P_φ for economies with an uncountable number of shocks s. Lemma 4 below is analogous to lemma 3 for this type of models. However, there are 3 important differences with respect to the case presented in section 3.2. First, the existence of an invariant measure follows only from 1 condition, a strengthening of condition 1) in lemma 3. Second, it is possible to define conditions on P_φ which guarantee the ergodicity of the invariant measure separately (i.e., condition 4). Third, we can connect properties a), b) and c) in theorems 1 and 2 respectively with conditions on the set of shocks, its distribution and $\varphi \sim \Phi$. This last fact will be proved in section 3.4.

Lemma 4 (Conditions for stationarity and ergodicity with uncountable shocks):
Let Φ satisfy assumption 1 and 2. Further, suppose that S be an uncountable compact set. Then, the measurable space $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ has an atomless measure θ. Let $\{a\}$ and B be, respectively, any point and a Borel measurable set in $\tilde{J}$. Suppose that for some $\varphi \sim \Phi$:

3) *Stationarity:* $\theta(\{a\}) = 0$ *implies* $P_\varphi(z, \{a\}) = 0$ *for any* $z \in \tilde{J}$ *and* $z \notin \Delta\varphi$.

4) *Ergodicity:* $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $\theta(B) < \delta$ *implies* $P_\varphi(z, B) < \varepsilon$ *for any* $z \in \tilde{J}$.

If condition 3) holds, then properties a) and b) in theorem 1 are satisfied.
If condition 4) holds, then property c) in theorem 2 is satisfied.

[28] Ito (1964, page 177) gave an example of a discontinuous function $\varphi \sim \Phi$ satisfying conditions 1)-2). However, it is not clear how to derive sufficient conditions on the primitives or how to characterize selections to ensure that condition 2) holds. However, this can be done if we assume an uncountable number of shocks as we do in section 3.3 and 3.4.

Proof: see section IV of the appendix.

Remark 1: Condition 4) implies condition 3). Further, lemma 4 showed that $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_1(\tilde{J})\}$ is weak closed. Thus, assumption 2) can be replaced with the following, milder version:*

Assumption 2'): Let $\varphi \sim \Phi$ be a $\mathcal{B}_{\tilde{J}\times S}$-measurable selection of the correspondence in assumption 1 and $\Delta\varphi$ its discontinuity set. Then, $\Delta\varphi$ has zero Lebesgue measure[29].

Condition 3) states that $P_\varphi(z,.)$ is an atomless measure for any z in the state space which does not belong to the discontinuity set $\Delta\varphi$[30]. Note that condition 1) in lemma 3 only requires $P_\varphi(z,.)$ to be atomless almost everywhere. Thus, condition 3) is stronger than 1). The technical appendix of this section contains example 1, which illustrates these differences. Moreover, section II.1 in the appendix illustrates how to verify this condition on the economy defined in section 2. Condition 4) states that $P_\varphi(z,)$ is absolutely continuous w.r.t. θ uniformly in $z \in \tilde{J}$.

The discussion in the appendix[31] highlights a tension between the existence of a time invariant recursive equilibrium, and the stationarity or ergodicity of the associated Markov process. With finite shocks, the former can be proved by imposing mild requirements on the primitives of the economy, but the existence of an invariant measure involves a strong restriction on S (see Martinez and Pierri, 2021) or on φ (see Pierri and Reffett, 2021) and the absence of heterogeneous agents. With uncountable shocks, ergodicity follows from the requirements presented below.

[29] As any point in $\tilde{J}$ has zero Lebesgue measure, the result follows from Billingslley (1995, see equation 32.4 in page 422). The possibility to replace assumption 2 by 2' once condition 4 has been imposed follows from the fact that $\mu(\Delta\varphi) = 0$ if μ is the *Weak** limit of $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_1(\tilde{J})\}$ and $\Delta\varphi$ has zero Lebesgue measure.

[30] The equilibrium correspondence in Feng, et. al. has an image in a separable finite dimensional space. Thus, it suffices to consider at most a countable set of selections (see Hildenbrand and Grandmont, 1974). This fact in turn implies that $P_\varphi(z,\{a\}) = 0$ for any $z \in \Delta\varphi$ as $\Delta\varphi$ is a finite set and there is an uncountable number of exogenous shocks. The existence of an atomless measure θ in models where condition 4 is not guaranteed to hold is shown in the proof of lemma 3 in section IV of the appendix.

[31] Section II.2, the technical appendix of this subsection in III and V.2, all in the appendix.

3.4 Sufficient conditions for stationarity and ergodicity

The conditions stated in lemma 4 allow to connect the properties associated with the existence of an invariant and an ergodic measure (properties a) to c) in theorems 1 and 2) with primitive conditions in the model, described in assumption 3 below:

Assumption 3: Let S be the set containing the exogenous shocks, $p(s,.)$ its distribution, $\Phi\colon \tilde{J}\times S \rightrightarrows \tilde{J}$ the equilibrium correspondence presented in definition 5 (see the technical appendix of section 2.5.2) and $\Delta\varphi$ the discontinuity set of $\varphi\sim\Phi$. Assume that:

- *i)* *S is uncountable and compact*
- *ii)* *$p(s,.)$ is atomless $\forall s \in S$*
- *iii)* *Suppose that assumption 2 holds. Let $(\tilde{z},s') \in \Delta\varphi$ and $\{s_n\}$ a sequence with $s_n \to s$. In addition, suppose that $\lim\limits_{(\tilde{z},s_n')\to(\tilde{z},s')} \varphi\,(\tilde{z},s_n') = \varphi(\tilde{z},s')\ \ \forall \tilde{z}\in\tilde{J}$*
- *iv)* *$p(s,.) = U[\underline{s},\overline{s}]\ \forall s \in S$, where $U[\underline{s},\overline{s}]$ is the uniform distribution on $[\underline{s},\overline{s}]$, a closed bounded interval of $\mathbb{R}$.*

Assumption 3-iii) allows for some path $(\tilde{z}_n,s_n')$ to be discontinuous. For any $(\tilde{z},s') \in \Delta\varphi$ there may exist $(\tilde{z}_n,s_n')$ with $lim_{(\tilde{z}_n,s_n')\to(\tilde{z},s')}\,\varphi\,(\tilde{z}_n,s_n') \neq \varphi(\tilde{z},s')$ but continuity is required on S for each $\tilde{z}\in\tilde{J}$. This assumption allows us to connect rectangles in the range of $\varphi(\tilde{z},.)$ with closed sets in S. Then, in proposition 2 below, the countable union of these rectangles will be associated with a small measure set to derive condition 4. Assumption 3-iii), continuity on s_+ for each $\tilde{z}\in\tilde{J}$, follows from mild restrictions[32] on the recursive equilibrium in Feng, et. al. As it was stated in section 2.5.2 (see fact 2.5-5), the procedure in Feng, et. al. can be used to construct a selection φ which satisfies assumption 3-iii) (see the technical appendix for section 2.5.2).
The next 2 propositions connect assumption 3 with conditions 3 and 4.

[32] See fact 2.5-5) in section 2.5.2 and the technical appendix for this section in section III of the appendix.

<u>Proposition 1 (Sufficient Condition for Stationarity)</u>:

Suppose that assumption 1, assumption 3-i) and 3-ii) hold. Then, condition 3) is satisfied: $\theta(\{a\}) = 0$ *implies* $P_\varphi(z, \{a\}) = 0$ *for any* $z \in \tilde{J}$ *for an arbitrary point* $\{a\} \in \tilde{J}$

Proof: see section IV of the appendix.

<u>Proposition 2 (Sufficient Condition for Ergodicity)</u>:

Suppose that assumption 1, assumption 3-iii) and 3-iv) hold. Then, condition 4) is satisfied. That is, $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $\theta(B) < \delta$ *implies* $P_\varphi(z, B) < \varepsilon$ *for any* $z \in \tilde{J}$*.*

Proof: see section IV of the appendix.

Clearly, proposition 2 calls for stronger assumptions than proposition 1. This is because it involves verifying not only the non-atomicity of $P_\varphi(z,.)$, which requires only taking care of points in $\tilde{J} \subset \mathbb{R}^K$, but also its absolute continuity, which demands proving that sets of the form $\{a_1\} \times [a_2, b_2] \times \dots \times [a_K, b_K]$ also have zero Lebesgue measure. Each of these sets can be "matched" with a sequence of rectangles which can be traced back to $p(s,.)$ under assump. 3-iii).

<u>Remark 2</u>: proposition 2 holds under a different version of assumption 3-iv).

Assumption 3-iv'): Let $(s,.) = U[LB(s), UB(s)]$ $\forall s \in S$*, where* $U[LB(s), UB(s)]$ *is the uniform distribution on* $[LB(s), UB(s)]$*.*

Proof: see section IV of the appendix.

Assumption 3-iv') is weaker than 3-iv) as it allows the exogenous states to follow a Markov process instead of being i.i.d. However, the theorem which guarantees the accuracy of simulations and the existence of an equilibrium correspondence for an economy with uncountable shocks, both require assumption 3-iv). The former result will be shown in section 4 and the latter in section 5. Thus, allowing for Markov shock implies that the existence of a sequential equilibrium and the accuracy of simulations, both, must hold by assumption.

4 Implications of Ergodicity

The first step to compute an infinite horizon process is to find a time invariant transition function. This was given by Feng, et. al. Theorems 1 and 2 allow taking matters a step further by proving, respectively, the existence of an invariant measure and its ergodicity. However, unconditional invariant measures are not easily computable. So, to complete the picture, a general result that allows approximating several characteristics of those invariant measures is required.

In this section that task is achieved by assuming the existence of a sequence of functions, $\{\varphi_j\}$ with $\varphi_j \sim \Phi_j$ and each j represent a different step in an iterative numerical procedure, approximating a selection of the equilibrium correspondence $\varphi \sim \Phi$. For a sufficiently large j, it is shown that for *any initial condition that belongs to a positive P_{φ_j}-invariant measure set*[33], almost every average constructed using a series obtained from this approximated function φ_j converges to the mean of some P_φ-invariant measure. These results imply that any model that fits the framework described in this paper can be calibrated accurately using a sufficiently large series. The figure illustrates the structure of this section and highlights the relevance of ergodicity to achieve accurate simulations (see section I of the appendix for details).

[33]The algorithms in Kubler and Schmedders (2003) and Feng, et. al. (2014) generate such a sequence of functions under the uniformity assumption. As those procedures aren't simulation based (like Marcet´s PEA, 1988), it is possible to first compute the equilibrium correspondence and then simulate from it.

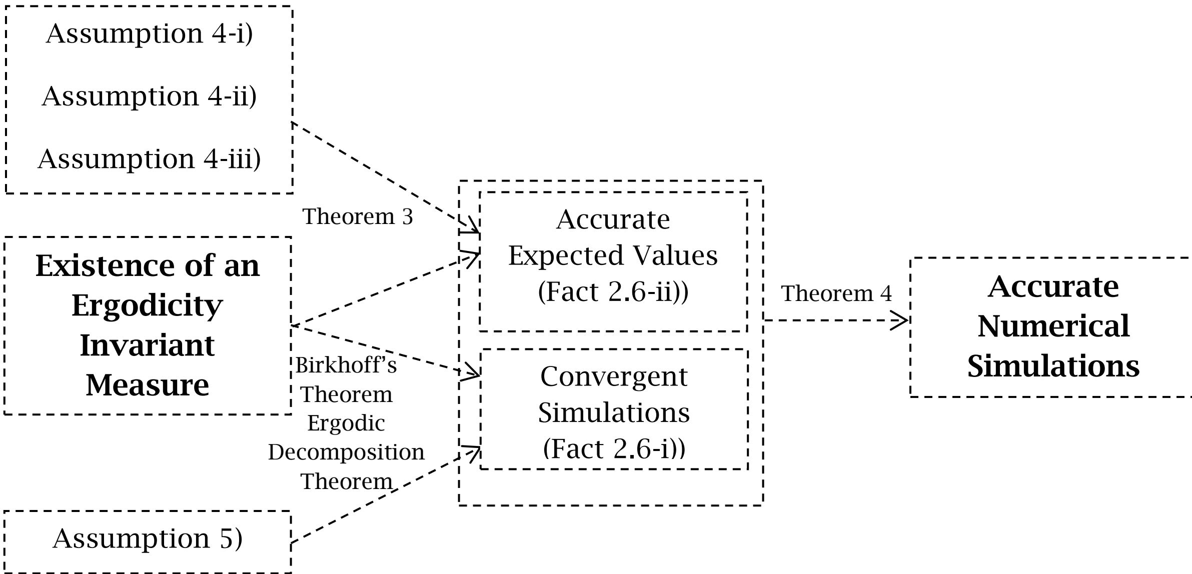

Figure 3

4.1 Accurate Expected Values.

Figure 3 suggests that, to obtain accurate numerical simulations, *it is necessary to guarantee that approximated economies have an ergodic measure and that these measures converge to the true model*. This subsection proves these facts.

Let $\varphi_j \sim \Phi_j$ be an approximation of $\varphi \sim \Phi$ with φ_j converging to φ in a metric to be specified below; theorem 3 state conditions which insure that $E_{\mu_j}(f)$ converges weakly to $E_\mu(f)$, where $\mu_j \in IM(\varphi_j, \mathcal{P}_1)$, $\mu \in IM(\varphi, \mathcal{P}_1)$ and $f \in C(\tilde{J})$.

There is also a similar result in Santos and Peralta Alva (2005). In that paper the authors assumed that the Markov operators, associated with the true and all approximated economies, have the Feller property and that the computed functions converge in a metric induced by a norm weaker than the sup-norm [34].

[34] The authors use the following metric: $d(\varphi_j, \varphi) = Max_{z \in \tilde{J}} \left\| \int \varphi_j(z, s') - \varphi(z, s') dU(s') \right\|$, where $\|.\|$ is the Max norm in $\mathbb{R}^K$.

Unfortunately, the Feller property is not adequate for non-optimal economies. To restore the continuity of the Markov operator, section 3 requires assumption 3-iii), that is, $\varphi(z,.)$ must be continuous on s' for all z[35]. Even if this assumption is imposed on all φ_j, the liming function must preserve this notion of continuity to be an appropriate candidate for φ. As all endogenous variables are assumed to be contained in a compact set, after taking care of the discontinuity set, the sup-norm serves this purpose. Assumption 4-ii) formally states this claim.

Assumption 4-iii) ensures that absolutely continuity is preserved after numerical computations, and it follows from a standard characterization of weak*-limits. See the technical details of this section in the appendix, section III.

Let $P_{\varphi_j}(z,A) = p\big(s,\{s' \in S|\ \varphi_j(z,s') \in A\}\big)$ be the Markov operator of approximated transitions. Then, we can state the main theorem of this section.

Theorem 3 (Accurate expected values):

Let $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^{I}\right]$ be the sequential economy in definition 1 and K be a compact set, with K compact and $f \in C(K)$. Suppose that assumption 1 hold on $\{\Phi_j\}$ and Φ. Further assume that assumptions 2') and 3-iii) hold on $\{\varphi_j\}$ and φ and assumption 3-iv) hold on $p(s,.)$. Finally suppose that:

Assumption 4-i): Payoff relevant variables in $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^{I}\right]$ are contained in K.

Assumption 4-ii): $lim_{j\to\infty}\|\varphi_j - \varphi\| = 0$, $\|\varphi\| \equiv SUP_{(K\times S)\setminus\Delta\varphi}\|\varphi\|_\infty$, $\|.\|_\infty$ is the Max-norm.

Assumption 4-iii): for any open set with $\theta(A) < \delta$, $\varphi_j^{-1}(z,.)(A) \subseteq \varphi^{-1}(z,.)(A)$, $j \to \infty$.

Let a.c. w.r.t. denote "absolutely continuous with respect to". Then,

1) *There is a sequence of measures $\{\mu_j\}$, with $\mu_j = P_{\varphi_j}\mu_j$, and μ_j is a.c. w.r.t. to θ*
2) *$\{\mu_j\}$ has a weakly convergent subsequence, $\mu_j \to \mu$, and μ is a.c. w.r.t. θ.*
3) *μ satisfies $\mu = P_\varphi \mu$.*

[35] For expositional purposes, in this section, z will denote an element of $\tilde{J}$.

Proof: see section IV of the appendix.

There are 2 things to be noted in theorem 3. First, the convergence is uniform on $(K \times S)\backslash\Delta\varphi$. Under assumptions 2') and 3-iii) on $\{\varphi_j\}$, this type of convergence assures that assumption 3-iii) is a valid hypothesis for φ. Further, this convergence notion fits the requirements of spline algorithms for discontinuous functions (see for instance Silanes, et. al. 2001) which makes the theoretical structure in figure 3 suitable for applications. Second, some algorithms (for instance Feng, et. al.) computes outer approximations of the range and domain of φ_j. Thus, assumption 4-iii) suggests that this type of procedures must be modified to fit into this framework. The preliminary remark of theorem 3, in the appendix, provides details on the relationship between theorem 3 and state of the art recursive algorithms.

4.2 Convergent Simulations.

The main result in this section is a direct application of Birkhoff's ergodic theorem and the ergodic decomposition theorem for Markov process. Thus, the results will be stated without proof, and they will be presented just to keep the paper self-contained. This section follows closely chapter 6 of Varadhan (2001) and the reader who is familiar with the literature is invited to go directly to section 4.3. The technical details are contained in the appendix of this section.
As in Santos and Peralta-Alva (2013), Kamihigashi and Stachurski (2015) or chapter 14 of Stokey, Lucas and Prescott, a simulation is *convergent* if it obeys a strong law of large numbers. In contrast to what is stated in those papers, convergence will be achieved only for a subset of all possible initial conditions. This is because the assumptions necessary to guarantee convergence starting from an arbitrary initial condition are too strong for the purpose of this paper (see the appendix for details). From section 2.6 and figure 3, it follows that the accuracy of numerical simulations requires the existence of an ergodic measure. The conditions for stationarity (i.e., the

existence of an invariant not necessarily ergodic measure) are milder, can be traced back to primitives and do not require to construct a tailor-made selection as in the case of an ergodic equilibrium. These facts imply that in practice we may *not* found this last class of equilibrium. However, we cannot ensure accuracy in a stationary non-ergodic equilibrium. This is because the strong law of large numbers for this class of processes (see Meyn and Tweedie, 1993, chapter 17) implies that simulations will not converge to the expected values in theorem 3. Contrarily, ergodic Markov processes will do. Thus, in this section and the next one it will be supposed that assumptions 1), 2'), 3-iii) and 3-iv') hold. Remark 2 allows $\{s_t\}$ to be generated by a Markov process (S, p) if S is an uncountable compact set of $\mathbb{R}$.
Let $\boldsymbol{P}_{\varphi,z_0}$ and $\boldsymbol{P}_{\varphi,\mu}$ the measures defined in the technical appendix of section 4.2. The following facts follow from Varadhan (2001, pages 179 and 187-192):

Fact 4.2-i): $\mu \in IM(\varphi)$ then $\boldsymbol{P}_{\varphi,\mu}$ is stationary and the process $(\tilde{J}, P_\varphi)$ is stationary.

Fact 4.2-ii): μ is ergodic if and only if $\boldsymbol{P}_{\varphi,\mu}$ is ergodic

Fact 4.2-iii): $\boldsymbol{P}_{\varphi,\mu} = \int \boldsymbol{P}_{\varphi,v} Q(dv)$, where v is an ergodic measure in $IM(\varphi)$ and $Q: \mathcal{P}(\tilde{J}) \rightarrow [0,1]$ a measure on $\mathcal{E}(IM(\varphi))$, the set of extreme points of $IM(\varphi)$.

Fact 4.2-iv): $lim_{n\rightarrow\infty}[\sum_{t=1}^{n} f(z_t)]n^{-1} = \int f(z)\mu(dz)$ for almost every $\{z_t\}$ with respect to $\boldsymbol{P}_{\varphi,z_0}$ if z_0 belong to a set of positive μ-measure and $\mu \in IM(\varphi)$.

Fact 4.2-iv) follows from the previous 2 facts: as the ergodicity of μ is equivalent to the ergodicity of $\boldsymbol{P}_{\varphi,\mu}$ (fact 4.2-ii), theorem 2 suffices to show the existence of a Markov ergodic process $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$. Then fact 4.2-iii), the ergodic decomposition theorem for Markov processes, implies that Birkhoff's ergodic theorem can be applied to any initial condition in a positive μ-measure with $\mu \in IM(\varphi)$. That is, μ can be assumed to be ergodic w.l.o.g. $IM(\varphi)$ must have an ergodic measure, v, which is guaranteed by the compactness of the set of invariant measures.

The stochastic process derived directly from a sequential equilibrium may not be stationary. Fact 4.2-i) illustrates the importance of the results in section 3: even if an invariant measure cannot be shown to be ergodic, it suffices to prove the existence of a stationary process associated with the sequential equilibrium, which typically follows from mild requirements. This is because assumptions 3-i) and 3-ii) can be verified from primitive conditions of the model. However, the results in Durret (2019, see section 7.2) imply that the convergence in fact 4.2-iv) cannot be achieved. If the process is stationary but not ergodic, the cesaro average will converge to a random variable which realizations depends on the initial condition of the model.

4.3 Accurate numerical Simulations.

This section connects all the pieces and proves the convergence of simulations. For any selection $\varphi \sim \Phi$, given $z_0 \in \tilde{J}$ and $\{s_t\}$ generated from (S, p). Let $\Omega = \tilde{J} \times \tilde{J} \times \ldots$ be the space of sequences with a typical realization $\omega \in \Omega$. As $z_0 = [s_0, \hat{z}_0]$, it is possible to define a sample path $\{z_t(\omega)\}$ inductively as follows: $z_1(z_0, \omega, \varphi) = \varphi(z_0, s_1)$, for any (t, z_t) satisfies $z_t(z_0, \omega, \varphi) = \varphi(z_{t-1}(z_0, \omega, \varphi), s_t)$. $\{z_t^j(\omega)\}$ can be defined in a similar way by replacing φ with φ_j and $\tilde{J}$ with K [36]. Finally, for any $f \in C(\tilde{J})$, it is possible to define a "time average" as $(N)^{-1} \sum_{i=1}^{N} f\big(z_i(z_0, \omega, \varphi)\big)$.

The following theorem follows from the results in sections 4.1 and 4.2.

<u>Theorem 4 (Accuracy of numerical simulations)</u>:

Suppose all the assumptions in theorem 3 hold. Additionally suppose:

[36] $\{s_t\}$ is a draw from a process defined analogously to $\big(\Omega, \mathcal{F}, \mu^{\infty}(z_0, .)\big)$, the stochastic process for $\{z_t\}$. See the technical appendix of section 4.2, section III of the appendix, and preliminary remark of theorem 4, section IV of the appendix, for details.

Assumption 5: z_0^j *belong to a set of positive* μ_j*-measure and* $\mu_j \in IM(\varphi_j, \mathcal{P}_1)$*, where* j *is sufficiently large such that* $\mu_j \rightarrow_{Weak*} \mu$ *and* $\mu \in IM(\varphi, \mathcal{P}_1)$*. Then*

$$\left|(N)^{-1} \sum_{i=1}^{N} f\left(z_i^j\left(z_0^j, \omega, \varphi_j\right)\right) - \int f(z)\mu(dz)\right| \rightarrow 0 \text{ as } N \rightarrow \infty$$

Convergence is achieved for almost every $\{z_t^j(z_0^j, \omega, \varphi_j)\}$ w.r.t. $\boldsymbol{P}_{\varphi_j, z_0^j}$ and $\mu \in IM(\varphi, \mathcal{P}_1)$.

Proof: see section IV of the appendix.

The results above imply, by the convexity of $IM(\varphi, \mathcal{P}_1)$, that every limit point of $(N)^{-1} \sum_{i=1}^{N} f\left(z_i^j\left(z_0^j, \omega, \varphi_j\right)\right)$ converges to some state average generated by a measure in $IM(\varphi, \mathcal{P}_1)$ provided that z_0^j *have been appropriately chosen*. If μ_j defines a "steady state" of the economy associated with φ_j, assumption 5 implies that z_0^j fluctuates around its steady state values. This result can be used to justify the usual practice in the applied literature where the first "1000" simulations are "thrown away" to assure that the simulated paths are fluctuating around some ergodic set (see Guerron-Quintana, Fernandez-Villaverde, Rubio-Ramirez and Uribe, 2011)[37].

Assumption 5 is the strongest in this paper. It is possible to construct an example of an economy that has a transient and an ergodic set[38], and simulations will fluctuate around the steady state once the transient set has been abandoned.

5 Applications

We apply the theoretical results presented before to a concrete parametrization of the economy in section 2. Following figure 2, the requirements to achieve the existence of an ergodic measure can be categorized in 3: properties (a-c), conditions

[37] I would like to thank H. Seoane for pointing this out to me.

[38] See for example 2 in Stokey, Lucas and Prescott, page 322. The first "1000" may be insufficient to leave the transient set with probability 1 and, although infrequently, it is possibly that z_0^j may not reflect the long run behavior of the model as desired. Thus, the "practitioners' approach" must be repeated several times to avoid this type of problems.

(1-4) and assumptions (1-3). Section 3.2 and 3.3 connected conditions, mostly on the Markov operator P_φ, with properties of the associated process $(\tilde{J}, P_\varphi)$.

Section 3.4 shows that conditions 1-4 can be generated by assumptions for the case of uncountable shocks. While most of these assumptions, 3-i), 3-ii) and 3-iv), are stated in terms of the primitives of the model, there are 2 which are still stated in terms of endogenous variables. This section connects one of these assumptions, 3-iii), with primitives of a version of the model presented in section 2 that is borrowed from Mas – Colell and Zame (1996). The other one, assumption 2 or 2', are left for future research. Further details are contained in section II and V of the appendix.

The first step is to prove the existence of a compact sequential equilibria. This fact leads to the existence of an appropriate recursive structure in the sense of Feng, et. al. The presence of uncountable shocks requires imposing additional assumptions with respect to the canonical model with incomplete markets. This assumption requires total wealth (i.e., $e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t)$, $\sigma_t \in \mathfrak{T}$) to be bounded away from zero. Due to the presence of short sale constraints, this requirement is mild. The detailed list of assumptions to prove existence is contained in the appendix, section II.2.

Once this additional hypothesis holds, there is an important gain in terms of the predictive power of the model as the theory developed in this paper allows showing not only that the model has a well-behaved steady state (theorem 1) but also that it is ergodic (theorem 2) and that simulations are accurate (theorems 3 and 4).

Except for assumption 2'), the remaining hypothesis can be directly traced back to primitive conditions of the model. This last fact can be obtained by dealing with assumption 3-iii) either by, a) for any model satisfying the hypotheses required to prove the existence of a sequential compact equilibrium, building a selection as described in section 2.5.2 (see fact 2.5-5)), or b) for a model with a particular asset structure described in the appendix (section II.2 and V.2), proving an additional lemma which holds for any measurable selection (see lemma 5). Then, propositions 1 and 2 con be used to derive conditions 3 and 4 and thus theorems 1 and 2. Assumptions 4-i) to 4-iii) and 5 ensure the accuracy of simulations.

6 Conclusions and directions for future research

This paper develops the theoretical foundations for an accurate calibration method for incomplete markets general equilibrium models with aggregate uncertainty and heterogeneous agents. Considering the lack of robustness of frequently used procedures, the results in this paper are relevant as they provide a set of assumptions which ensure that empirically relevant models can be taken to data accurately. The parameters obtain are then reliable to perform policy experiments which could be welfare enhancing.

The paper provides a set of results which allow characterizing incomplete markets general equilibrium models beyond existence. Further, it distinguishes between the predictive performances of models with different degree of uncertainty as measured by the cardinality of the set which contains exogenous shocks. Also, this article presents a set of sufficient conditions that guarantee that the parameters obtained by appropriately designed algorithms reflects accurately the long and short run behavior of general equilibrium models.

Although the results are quite general and assumptions rather mild, there is scope for future research both in models with a finite number or with uncountable shocks. For the former, condition 2, which ensures the existence of an invariant measure, must relate to primitive conditions. Further, these conditions must also guarantee the ergodicity of the measure as theorem 2 requires even stronger assumptions than theorem 1 as illustrated by properties b) and c). For the case of uncountable shocks, an extensive numerical test must be performed on the algorithm design in section 4. Moreover, the conditions for the accuracy of simulations, especially assumption 4-iii), must be implemented in an algorithm which combines the Feng, et. al. and the Silanes, et. al. procedures.

Bibliographical References

Akyol, A. (2004): “Optimal monetary policy in an economy with incomplete markets and idiosyncratic risk”, Journal of Monetary Economics, 51(6), 1245-1269, September.

Aliprantis, C. and Border, K. (2006): “Infinite Dimensional Analysis”, Springer, Third Edition.

Attanasio, O. and Davis, S. (1996): “Relative Wage Movements and the Distribution of Consumption,” Journal of Political Economy, 104 (6), 1227–62, December.

Braido, L. (2013): “Ergodic Markov Equilibrium with Incomplete Markets and Short Sales”, Theoretical Economics, vol. 8 (1), 41-57

Brumm, J., Kryczka, D. and Kubler, F. (2017), “Recursive Equilibria in Dynamic Economies With Stochastic Production”. Econometrica, 85: 1467-1499.

Cao, D. (2020). "Recursive equilibrium in Krusell and Smith (1998)," Journal of Economic Theory, Elsevier, vol. 186.

Chari, V. V., Kehoe, P. J. and McGrattan, E. R. (2002): “Can Sticky Price Models Generate Volatile and Persistent Exchange Rates?”, Review of Economic Studies, 69, 533-563.

Cochrane, J. (1991): “A Simple Test of Consumption Insurance,” Journal of Political Economy, 99 (5), 957–76, October.

Cooley, T. F. and Quadrini, V. (2001): “Financial Markets and Firm Dynamics”, The American Economic Review, 91(5) , 1286-1310, December.

De Jong, D. and Dave, C. (2007) “Structural Macroeconometrics”, Princeton University Press.

De Groot, O., Durdu, C. and Mendoza, E. (2013): “Solving DSGE Models with Incomplete Asset Markets: Global vs. Local Methods'', mimeo.

Duffie, D., Geanakopolos, J., Mas-Colell, A., McLennan, A. (1994): “Stationary Markov Equilibria”, Econometrica, 62 (4), July, 745-81.

Duggan, J. (2012), “Noisy Stochastic Games”, Econometrica, 80(5), 2017-2045.
Durret, R. (2019), “Probability: Theory and Examples”, Cambridge University Press.
Feng, Z., Miao, J., Peralta-Alva, A. and Santos, M. (2014), “Numerical simulation of Nonoptimal Dynamic Equilibrium Models”. Int. Economic Review, 55: 83-110.
Fernandez-Villaverde, J., Guerron-Quintana, P., Rubio-Ramirez, J. and Uribe, M. (2011): "Risk Matters: The Real Effects of Volatility Shocks," American Economic Review, 101(6), October, 2530-61.
Ferraro, D. and Pierri, D. (2018): “Tax Policy with idle capacity”, Mimeo.
Grandmont, J-M. and Hildenbrand, W. (1974): “Stochastic Process of Temporary Equilibria”, Journal of Mathematical Economics, 1, 247-277.
Guvenen, F. (2011): “Macroeconomics with Heterogeneity: A Practical Guide”, NBER Working Paper Nº 17622.
He, W and Sun, Y. (2017) “Stationary Markov perfect equilibria in discounted stochastic games”, Journal of Economic Theory 169, 35-61
Hoelle, M. (2014). The relation between sunspot effects and multiplicity in incomplete markets models with numeraire assets. Journal of Mathematical Economics, 50, 119-127.
Ito, Y. (1964): “Invariant Measures for Markov Processes”. Transactions of the American Mathematical Society, (110), 152-184
Jappelli, T., and Pistaferri, L. 2014. "Fiscal Policy and MPC Heterogeneity." American Economic Journal: Macroeconomics, 6 (4): 107-36.
Judd, K., Kubler, F. and Schmedders, K. (2002): “A solution method for incomplete asset markets with heterogeneous agents”, mimeo.
Kallenberg, O. (2006). “Foundations of modern probability”. Springer Science & Business Media.
Kaplan, G., Moll, B. and Violante, G. L. (2018). "Monetary Policy According to HANK." American Economic Review, 108 (3): 697-743.
Kubler, F. and Schmedders K. (2002): “Recursive Equilibria in Economies with Incomplete Markets”, Macroeconomic Dynamics, 6, 284-306.

Kubler, F. and Schmedders K. (2003): “Stationary Equilibria in Asset Pricing Models with Incomplete Markets and Collateral”, Econometrica, 71, 6, November, 1767-93

Kubler, F. and Polemarchakis, H. (2004): “Stationary Markov equilibria for overlapping generations”, Economic Theory, 24(3), 623-643.

Krusell, P. and Smith, A. (1998): “Income and Wealth Heterogeneity in the Macroeconomy”, Journal of Political Economy, 106, 867-896.

Kydland, F. E. and Prescott, E. (1982): “Time to Build Aggregate Fluctuations”, Econometrica, 50, 1345-1370.

Hatchondo, J. C., Martinez, L. and Sapriza, H. (2010): “Quantitative properties of sovereign default models: solution methods matter”, Review of Economic Dynamics, 13(4), 919–933, October.

Heathcote, J. (2005): “Fiscal Policy with Heterogeneous Agents and Incomplete Markets” Review of Economic Studies, 72(1), 161-188.

Hildenbrand, W. (1974): “Core and Equilibria of a Large Economy”, Princeton University Press.

Lucas, R. (1978): “Asset Prices in an Exchange Economy”, Econometrica, 46, 1429-1445.

Magill, M. and Quinzii, M (1994): “Infinite Horizon Incomplete Markets”, Econometrica, 62 (4), July, 853-80.

Magill, M. and Quinzii, M (1996): “Theory of Incomplete Markets, Vol. 1”, MIT Press.

Martinez, J. and Pierri, D. (2021): “Accuracy in Recursive Minimal State Space Methods”, Working Paper, Universidad de San Andrés.

Mehra, R. and Prescott, E. (1985): “The equity premium: A puzzle,” Journal of Monetary Economics, March 1985, 15 (2), 145–161.

Molchanov, I. and Zuyev, S. (2011): “Advanced Course in Probability: Weak Convergence and Asymptotics”, Lecture Notes.

Phelan, C. and Stacchetti, E. (2001): “Sequential Equilibria in a Ramsey Tax Model”. Econometrica, 69: 1491-1518

Pierri, D. and Reffett, K. (2021): “Memory, Collateral and Emerging Market Crises”, UC3M working Paper.

Pijoan-Mas, J. (2007): “Pricing Risk in Economies with Heterogeneous Agents and Incomplete Markets”, Journal of the European Economic Association, 5(5), 987-1015.

Santos, M. (2002): “On Non-existence of Markov Equilibria in Competitive-Market Economies”, Journal of Economic Theory 105, 73–98

Santos, M. and Peralta Alva, A. (2005): “Accuracy of Simulations for Stochastic Dynamic Models”, Econometrica, 73 (6), Nov, 1939-1976.

Santos and Peralta Alva (2013): “Ergodic Invariant Measures for Non-optimal Dynamic Economies”, mimeo.

Santos, M. and Peralta Alva, A. (2015): “Laws of Large Numbers for the Numerical Simulation of Competitive-Market Economies with Frictions”, mimeo.

Singleton, K. (1990): “Specification and Estimation of Intertemporal Asset Pricing Models”, Handbook of Monetary Economics, North Holland.

Raad, R. (2013): “Approximate Recursive Equilibria with Minimal State Space”, working paper Nº 737, Getulio Vargas Foundation.

Radner, R. (1972): “Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets”, Econometrica, 40, 289-303.

Stokey, N., Lucas, R. and Prescott, E. (1989): “Recursive Methods in Economic Dynamics”, Harvard University Press.

Supplement to "Useful Results for the Simulation of Non-Optimal Economies with Heterogeneous agents"

Damian Pierri

Damian.pierri@gmail.com

Universidad Carlos III, Madrid and IIEP-BAIRES (UBA-CONICET)

Index

Section I: Figures 2 and 3

Figure 2

In the context of Duffie, et. al., section 2.5.1 discussed the 2 building blocks of any Markov process, namely: a state space and a Markov operator. Lemma 2 in section 3.1 shows that the same objects can be defined for the framework in Feng, et. al. under standard assumptions. Then, theorems 1 and 2, respectively, state *properties* of the underlying Markov process which guarantee the existence of an invariant measure and its ergodicity.

Section 3.2, for the case of a finite number of exogenous shocks, states *conditions* on the Markov operator defined in lemma 2 which guarantee the properties required in theorem 1. Section 3.3 states *conditions* on the Markov operator which guarantee the properties in theorem 1 and 2 for the case of uncountable exogenous shocks. Section 3.4 states *sufficient conditions* (in terms of the number of possible exogenous shocks, its distribution and the stationary transition $\varphi \sim \Phi$ defined in section 2.5.2) which guarantee that the related Markov operator satisfies the conditions stated in section 3.3.

The results obtained in sections 3 do not depend on the specific structure contained in the non-optimal general equilibrium economy, described in section 2. As in Duffie, et. al., they could also be applied to OLG stochastic economies and repeated games. Thus, in this section and in the next one it will only be assumed the existence of a time invariant correspondence based recursive structure, $\Phi: \tilde{J} \times S \rightarrow \tilde{J}$, where Φ is closed graph and compact valued, $\tilde{J} = S \times \hat{Z}$, S is the set of exogenous shocks and $\hat{Z}$ contain the endogenous variables in some model.

Just as an example of the implications of figure 2, note that assumption 3, by means of propositions 1 and 2, guarantee that conditions 3 and 4 on the Markov operator hold for the case of an uncountable number of exogenous shocks. Then, lemma 4 guarantees that condition 3 (associated with assumptions 3-i) and 3-ii)) imply properties a and b, which, together with assumption 2 prove the existence of an invariant measure using theorem 1. Lemma 4 guarantee that condition 4 (associated with assumptions 3-iii) and 3-iv)) implies property c) that is sufficient for the ergodicity of the invariant measure. Assumption 3 can be traced back to the primitive conditions of the sequential economy. The continuity with respect to exogenous states was discussed in the technical appendix of section 2.5.2, specifically in fact 2.5-5).

Note that only condition 2 and assumption 2 are "disconnected" from the results in this paper. This is because the study of sufficient conditions based on the primitives of the sequential equilibrium that led to these properties are left for future research due to its difficulty.

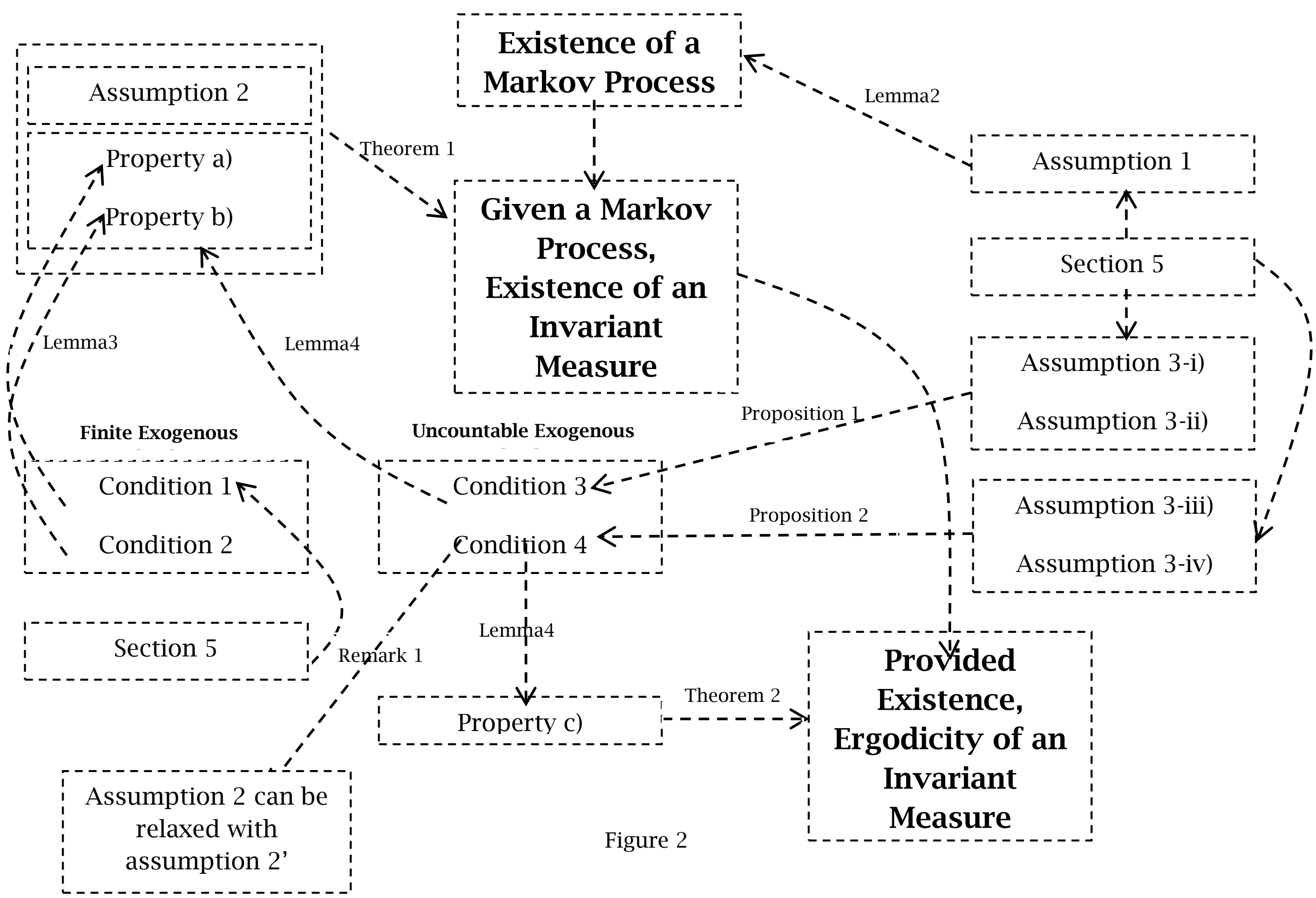

Figure 2

Figure 3

Section 4.1 proves *theorem 3*, which is an extension to non-optimal economies of a similar result in Santos and Peralta Alva (2005). The authors suppose that the transition function has the Feller property, an assumption which does not hold in the present context. To compensate for this fact, *assumption 4-ii) relax* the convergence criteria used for approximating functions with respect to the one assumed in Santos and Peralta Alva. *Assumption 4-iii)* imposes a restriction on the preimage of φ_j to preserve the absolute continuity of the measures generated out of numerical approximations. The connection between assumption 4-iii) and deep parameters in the model is beyond the scope of this paper.

The results presented in section 4.2 will be stated without proof as they are applications of Birkhoff's theorem and the ergodic decomposition theorem.

Section 4.3 proves *theorem 4* which states that, given an appropriately chosen initial condition defined in *assumption 5*, numerical simulations approximate the true steady state of the model.

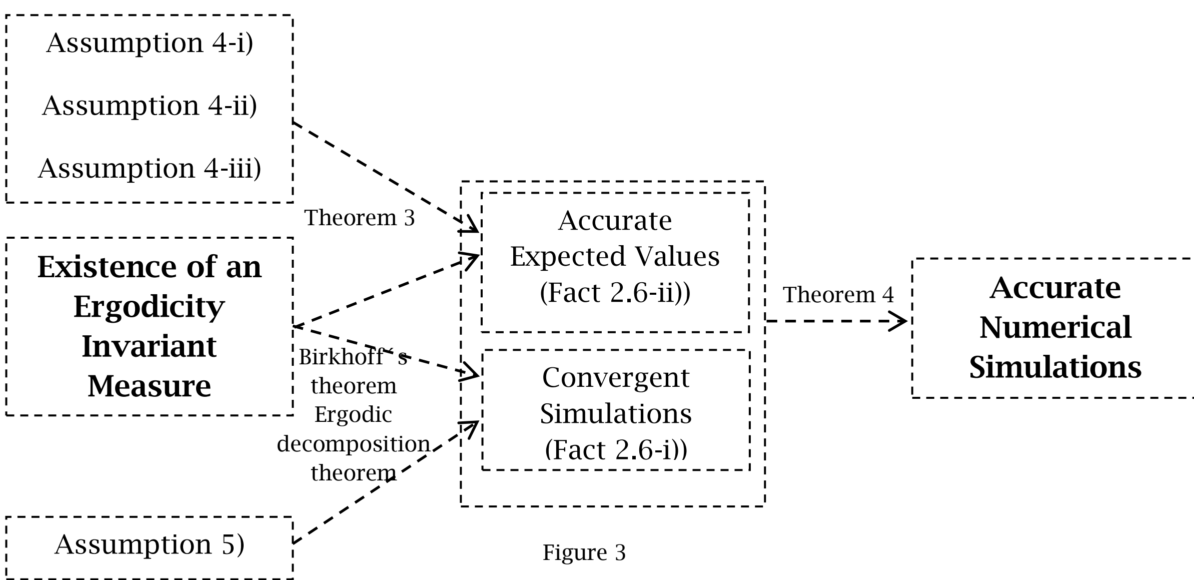

Figure 3

Section II: Applications

For the case of uncountable shocks, typically, see fact 2.5-2), once the existence of equilibria for a truncated (finite time) economy has been shown as in Mas-Colell and Zame (1996), it suffices to prove the uniform compactness of endogenous variables to insure the existence of the recursive equilibrium in Feng, et. al. To guarantee assumption 3-iii), it is possible to refine this recursive equilibrium notion as it was discussed in the appendix of section 2.5.2 or directly restrict the sequential equilibria. Lemma 5 proves this claim. Unfortunately, assumption 2', which restricts the cardinality of the discontinuity set, cannot be associated with primitive conditions. It is a matter of future research to investigate the relationship between the cardinality of the discontinuity set and the equilibrium set.

For the case of finite shocks, the existence of an invariant measure is guaranteed by conditions 1 and 2: the first one connects the Markov process with the set of atomless measures, the second one guarantees the closedness of this last set. Section II.1 shows how to derive the first condition from the curvature of the utility function using the implicit function theorem, which is assumed to hold almost surely and uniformly in the state space except in the discontinuity set. Note that the results in section II.1 can also be used to verify condition 3 in lemma, which ensures stationarity.

Condition 2, however, cannot be derived from primitive conditions of the model and thus deserves to be study in detail. Recently, Martinez and Pierri (2021) provide an example of an economy which illustrates the difficulty of the question at hand. If the economy has finite shocks and discontinuous Markov equilibria in the natural state space (as in Santos 2002), it is possible to prove the existence of an invariant measure by enlarging the state space as sometimes this procedure restores the continuity of the Markov operator.

The requirements that ensure the accuracy of numerical simulations, described in assumptions 4 and 5, are outside the scope of this paper as they require developing an algorithm which can compute the equilibrium correspondence in definition 5 while keeping track of the requirements that preserves the absolute continuity of the measures involved in the successive computations. This type of algorithm has not been developed yet and thus requires a careful separate treatment.

II.1 Finite shocks and implicit function theorem for condition 3

The model is the same as the one described in section 2.1. Following figure 2, the first step to prove the stationarity of the model is to derive a recursive representation for the sequential equilibria. As discussed in section 2.5.2, *the existence of a recursive structure is guaranteed by the existence of the sequential competitive equilibria and the compactness of the equilibrium set*. In the present framework, these properties will be shown to be implied by the assumptions listed in this subsection. Moreover, all the assumptions required for the existence of an invariant measure are presented below.

Assumptions 6.1-i) to 6.1-v) ensure the existence of a non-empty compact equilibrium set which will be shown to be sufficient to derive a Markov representation of equilibria. Provided this representation, to show the existence of an invariant measure, it suffices to impose assumption 2, property a) and property b) (presented in section 3.1, theorem 1). The first and the last are stated as a hypothesis below (assumptions 6.1-vi and 6.1-vii respectively) and the second one will be derived from primitive conditions of the model which are implicit in assumptions 6.1-i) to 6.1-v).

<u>Assumption 6.1).</u> Suppose an incomplete market economy as the one described in section 2.1. To that structure add the following assumptions:

i) *The utility function in the optimality condition of definition 1 is:*

$$U_i(c) = \sum_{t=0}^{\infty}(\beta)^t \sum_{\sigma_t^* s}\left[u_s^i(c^i(\sigma_t^* s))\right]\mu_t(\sigma_t^* s)$$

Where $u_s^i\left[c^i(\sigma_t^* s)\right] = 1 - e^{-\lambda c^i(\sigma_t^* s)}$ *with* $\lambda > 0$.

ii) *The realizations of the exogenous shock* s_t *lie in set* S *of finite cardinality for any time period* $= 0,1, \dots$.

iii) *Endowments satisfy:* $e^i(\sigma_t) > 0$ *and* $\sum_{i=1}^{I} e^i(\sigma_t) < K$ *with* $K > 0$ *for any agent* $i \in \{1, \dots, I\}$ *and node* σ_t *. Idiosyncratic endowments are strictly positive and aggregate endowments are uniformly bounded. There is aggregate and idiosyncratic uncertainty.*

iv) *There is a finite number,* J *, of numerarie short lived assets with (uniformly) bounded dividends and short sale constraints. That is, for each agent* i *and any node* σ_t *the portfolio is given by* $\theta^i(\sigma_t) \geq -B$, $B \in \mathbb{R}_+^J$, *the associated dividends by* $d(\sigma_t s) \in M \subset \mathbb{R}_+^J$, *where* M *is uniformly bounded, and the budget equation by*

$$c^i(\sigma_t) = e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t) - \theta^i(\sigma_t).q(\sigma_t)$$

Where $q(\sigma_t)$ *is the price of the portfolio in terms of the numerarie for every node* σ_t *and* σ_t^* *is the predecessor of* σ_t.

v) *There is a riskless bond. There is an asset* l *which has associated dividends given by* $d_l(\sigma_t s) = 1$ *for any* $s \in S$ *and any node* σ_t.

vi) *Assumption 2 holds (i.e., the discontinuity set of any measurable selection of the equilibrium correspondence has at most finite cardinality).*

vii) *Condition 2 holds (i.e., provided that the adjoint operator maps the set of atomless measures into itself, this set is weakly closed).*

Except the assumption on u_s^i, the short sale constraints, 6.1-vi) and 6.1-vii), the rest are standard in the literature. The results in Magill and Quinzii (1994) imply that under assumptions 6.1-i) to 6.1-v), excluding the restriction on u_s^i, the economy describe in section 2.1 has a non-empty compact equilibrium set (see assumption A.1 to A.6 and the discussion that follows in pages 858-60).

The chosen instantaneous return function u^i on assumption 6.1-i) guarantees that marginal utility is bounded on the entire feasible consumption set which, because of assumption 6-iii), is given by $[0, K]$. Kubler and Schmedders (2002) shows that assumptions 6.1-i) to 6.1-v), including the restriction on the return function but excluding the short sale constraints, imply that any sequence of consumption bundles $\{\{c^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$, portfolios $\{\{\theta^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$ and prices $\{q(\sigma_t)\}_{\sigma_t\in\mathfrak{T}}$ which satisfy the feasibility requirement $\sum_{i=1}^{I}\theta^i(\sigma_t) = \vec{0}$, where $\vec{0} \in \mathbb{R}^J$ for any $\sigma_t \in \mathfrak{T}$, and the Kuhn Tucker conditions listed in equation 3 and 4 (see the technical appendix to section 2.5.2) meet the optimality and feasibility conditions in definition 1 and thus constitutes a sequential equilibrium. The compactness of the equilibrium set follows from Magill and Quinzii (1994).

Short sale constraints are standard in the recursive literature since Duffie, et. al. (1994). Braido (2013) showed that a recursive equilibrium in the sense of Duffie, et. al. exists even if explicit short sale constraints are removed. This is possible as Magill and Quinzii (1994) showed that there is a uniform bound on assets even in the absence of short sale constraints. However, the theoretical results in this paper depend on Feng, et. al. recursive equilibria which, as discussed in section 2.5.2, are a subset of all possible recursive equilibria in Duffie, et. al. It is not clear that Braido's results hold in Feng, et. al.'s framework. Thus, short sale constraints are imposed to guarantee the existence of an appropriate (sunspots free) recursive equilibrium.

As seen in section 2.5.2 (see also Feng, et. al. section 2.2), if the equilibrium set is compact and can be generated by the set of equations implied by the Kuhn Tucker and feasibility conditions, the equilibrium correspondence Φ in definition 5 (see the technical appendix to section 2.5.2 in section III of this appendix) satisfies the assumptions in lemma 2 and thus P_φ, as defined in equation 5, is a well defined Markov operator and $(\tilde{J}, P_\varphi)$ defines a (compact) Markov process with typical state $\tilde{z} = [s, \theta, q, m] \in \tilde{J}$ and $m_j^i = d^j(s)\left(u_s^i(c^i)\right)'$.

Given the existence of a Markovian representation $(\tilde{J}, P_\varphi)$, theorem 1 implies that to prove the existence of an invariant measure, it suffices to impose assumption 2, condition a) and condition b). The first and the last are listed in assumptions 6.1-vi) and 6.1-vii).

Property a), namely that the adjoint operator associated with P_φ maps the space of atomless measures into itself, holds if the implicit function theorem can be applied to the system of equations defined by equations 3, 4 and $\sum_{i=1}^{I} \theta^i = \vec{0}$ in a full lebesgue measure set[1]. Let $z = [s, \theta, q]$ and $F(z, z_+) = \vec{0}$ be the system of $J + J \times I$ equations that can be obtained by replacing equation 3 into 4 and considering only interior solutions[2]. Section V.1 in this appendix will show that, under assumptions 6.1-i) to 6.1-v), $D_{z_+}F(z, z_+)$ has full rank a.e. in z, where $D_{z_+}F$ is the Jacobian matrix of F with respect to z_+.

Once this property has been established, it suffices to apply lemma 3. That is, lemma 3 connects condition 1 (i.e., $\mu(\{a\}) = 0$ *implies* $P_\varphi(z, \{a\}) = 0$ z*-a.e. with respect to an atomless measure* μ) with property a) (i.e., $P_\varphi^*: \mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ where P_φ^* is the adjoint operator, $\tilde{J}$ is the state space of the process and $\mathcal{P}_0(\tilde{J})$ the space of atomless measures in $\mathcal{P}(\tilde{J})$). The arguments in the preliminary remark

[1] See the discussion in the preliminary remark of lemma 3 in section III of this appendix implies for details.

[2] The discussion in section A.2.1 in the appendix connects Φ with F and $\tilde{z}$ with z. Once Φ is defined, it suffices to note that $\tilde{z} = [z, m]$ and m is defined by the additional equation given above.

of lemma 3 and section V.1, in section III and V of this appendix respectively, show that the full rank of $D_{z_+}F(z, z_+)$ is sufficient to guarantee condition 1.

Notice that the implicit function theorem is required to hold a.e. in z in condition 1 and in every z which does not belong to the discontinuity set $\Delta\varphi$. Thus, there is no contradiction between this property and the possible discontinuity of φ as, considering assumption 2, the discontinuity set of φ has finite cardinality. This property implies that: a) $\Delta\varphi$ has zero measure on μ and thus can be excluded from the states z in condition 1, b) we can exclude $\Delta\varphi$ from $\tilde{J}$ in condition 3 without loss of generality as the equilibrium correspondence can be fully characterized by a countable number of selections (constructed for a given point in the domain) as its image is contained in a finite dimensional compact set (see Hildenbrand and Grandmont, 1974). Thus, if $z \in \Delta\varphi$ and $P_\varphi(z, B)$, then B has at most a countable number of elements and thus has zero atomless measure. Of course, $\int_A P_\varphi(z, B)\mu(dz) > 0$ with A possibly containing a discontinuity point.

While assumptions 6.1-i) to 6.1-vi) are relatively mild, assumption 6.1-vii) is strong as it directly implies the weak-closedness of $\mathcal{P}_0(\tilde{J})$ (i.e., property b). Further, this assumption cannot be connected with primitive conditions of the model. Fortunately, it is possible to obtain properties a) and b) jointly by strengthening condition 1. This is done by lemma 4, that requires only condition 3, which strengthens condition 1 by requiring it to hold *uniformly in all continuity points*. Proposition 1 shows that condition 3 holds if the model is allowed to have uncountable exogenous shocks s. Considering the distinctive nature of this type of economies, they must be treated separately. Section 5 in the body of the paper and II.2 below addresses this point.

II.2 Uncountable Shocks

The discussion in the preceding section sets a tradeoff: to get rid of unverifiable assumptions like property b), the structure of exogenous shocks must be modified. Unfortunately, proving the existence of the sequential equilibria (and thus the existence of an appropriate recursive structure in the sense of Feng, et. al.) with uncountable shocks requires imposing an additional assumption on 6.1-i) to 6.1-v). This assumption, labeled 6.2-ii) below, was extensively discussed in the literature (see for instance Mas-Colell and Zame, 1996, or Araujo, et. al. 1996). Assumption 6.2-ii) implies the existence of a positive wealth in each node. Given the presence of short sale constraints, the boundedness of dividends and endowments, in the present context, it is rather mild.

Assumption 6.2 contained all the sufficient conditions to show the existence of an ergodic invariant measure in the model discussed in section 2, except assumption 3-iii) which will be treated separately in a lemma below.

Assumption 6.2). *Suppose an incomplete market economy as the one described in section 2.1. To that structure add the following assumptions:*

- *i)* *Assumptions 6.1-i), 6.1-iii) and 6.1-iv) hold.*
- *ii)* $e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t) > 0$, $\sigma_t \in \mathfrak{T}$
- *iii)* *Assumptions 3-i) and 3-iv) hold (i.e., the set of exogenous shocks is* $S = [\underline{S}, \overline{S}] \subset \mathbb{R}$ *and* $p(s,.) = U[\underline{S}, \overline{S}]$, *where* U *is the uniform distribution).*
- *iv)* *Assumption 2' holds (i.e., the discontinuity set is at most of zero lebesgue measure).*

Assumptions 6.2-i) to 6.2-iii) guarantees the existence of the sequential equilibria. The proof follows immediately by extending the induction argument in Mas-Colell and Zame (1996) for $T = \infty$ as in Duffie, et. al. (1994, see fact 2.5-2 in section 2.5.1). Theorem 4.1 in Mas-Colell and Zame allows proving the non-emptiness C_j for $1 \le j \le T < \infty$, where C_j is the set of initial states of a $j+1$ period economy defined in the technical appendix of section 2.5.2. The compactness of K, the set that includes all payoff relevant states, follows from theorem 4.2 also in Mas-Collel and Zame. The induction argument in section 5 of that paper can be used to set $T = \infty$. The optimality argument in Duffie, et. al. (section 3.4) can be immediately extended to the Mas-Colell and Zame framework as theorem 4.1 and 4.2 hold $\mu_s^\infty(s_0,.)$-a.e. for $s_0 \in S$ and θ_-^i satisfying assumption 6.2-ii), where $(\Omega, \mathcal{F}, \mu_s^\infty(s_0,.))$ is the stochastic process defined in section 4.2 but restricting the state space Ω to contain only an infinite sequences of exogenous shocks $\{s_t\}$.

The compactness of K and the upper hemi continuity (in z_+) of the system of equations defined by 3), 4) and the feasibility of assets guarantees that the equilibrium correspondence, Φ in definition 5 in the technical appendix of section 2.5.2, satisfies the assumptions required by lemma 2. Thus, there is at least 1 measurable selection $\varphi \sim \Phi$ and $(\tilde{J}, P_\varphi)$ defines a Markov process.

Once an appropriate Markov process have been shown to exist, proposition 2 implies that assumptions 6.2-iii), 6.2-iv) and 3.iii) are sufficient to show the ergodicity of the process $(\tilde{J}, P_\varphi)$. The following lemma shows that if there is only 1 asset or the recursive equilibrium notion in Feng, et. al. is appropriately restricted (see fact 2.5-5 in section 2.5.2), assumption 3-iii) can be omitted.

<u>Lemma 5</u>: Suppose that fact 2.5-5 holds or $J = 1$ (i.e., there is 1 asset). Then, under assumptions 6.2-i) to 6.2-iv), $(\tilde{J}, P_\varphi)$ has an ergodic invariant measure.

Proof: see section V of this appendix.

Section III: Supplementary Material

Technical appendix of section 2.3

The simplest notion of function based recursive equilibria, often called *strongly recursive,* can be found in Lucas (1978). In a representative agent economy with complete markets, Lucas can show that the endogenous variables in definition 1[3] can be written solely as a function of the current realization of the exogenous state variable. In this case, S constitutes a sufficient state space to describe the evolution of the economy. Unfortunately, market incompleteness and agent heterogeneity make this equilibrium notion too restrictive. As agents will insure against each other, it is likely that asset positions will differ across agents at any given period even in the same (exogenous) state. Thus, wealth will also differ at the beginning of next period, affecting the consumption possibility set for all $s \in S$ (see Kubler and Schmedders 2002 for a detailed discussion).

As this paper focus on non-optimal general equilibrium economies with heterogeneous agents, broader notions of recursive equilibria are required. We formally define them below.

Definition 2: *A sequential equilibrium is called Weakly Recursive if there exist continuous functions* $f^i : S \times \mathbb{R}^{IJ} \to \mathbb{R}^J$ *for all* $i \in I$ *and* $g^j : S \times \mathbb{R}^{IJ} \to \mathbb{R}_+$ *for all* $j \in J$ *such that for any* $\sigma_t \in \mathfrak{T}$ *and* $s \in S$ *,* $q_j(\sigma_t^* s) = g^j\left(s, \{\theta^i(\sigma_t^*)\}_{i=1}^I\right)$ *and* $\theta^i(\sigma_t^* s) = f^i\left(s, \{\theta^i(\sigma_t^*)\}_{i=1}^I\right)$*, where* $\{\theta^i(\sigma_t^*)\}_{i=1}^I$ *is feasible.*

[3] Lucas (1978) assumed a complete set of trees. Thus, Definition 1 should be modified to account for this fact, as assets are offered in positive net supply and the payoff matrix depends on asset prices.

<u>Definition 3:</u> An equilibrium is called <u>Wealth Recursive</u> if it is weakly recursive and if there are <u>continuous functions</u> $f^i_{WhR}: S \times \mathbb{R}^I \to \mathbb{R}^J$ for all $i \in I$ and $g^j_{WhR}: S \times \mathbb{R}^I \to \mathbb{R}_+$ for all $j \in J$ such that $g^j_{WhR}(s, w(\sigma^*_t s)) = g^j\left(s, \left\{\theta^i(\sigma^*_t)\right\}^I_{i=1}\right)$ and $f^i_{WhR}(s, w(\sigma^*_t s)) = f^i\left(s, \left\{\theta^i(\sigma^*_t)\right\}^I_{i=1}\right)$, where $w(\sigma^*_t) = \left\{w^i(\sigma^*_t)\right\}^I_{i=1}$.

Frequently, the macroeconomic literature (i.e. Arellano, 2008) assumes the existence of a recursive equilibrium based on several standard properties of the Bellman equation (see, for example, Stokey, Lucas and Prescott, 1989). Formally, this equilibrium notion can be thought as an extension of Mehra and Prescott's recursive competitive equilibrium (1980) to an economy with heterogeneous agents and incomplete markets. Typically, the equilibrium is defined as:

<u>Definition 3.1</u>[4]: A <u>recursive equilibrium</u> is composed by a set of I value and price functions, $\left\{V^i(s,w)\right\}^I_{i=1}$ and $\left\{q^i(s,w)\right\}^I_{i=1}$ respectively, which satisfy the following properties:

i) (Optimality) Each household $i \in I$ solves

$$V^i(s,w) = Max_{\theta^i \epsilon \Delta}\, u^i_s\left(w^i - q^i(s,w)\theta^i\right) + E_p\left(\beta^i V^i(s',w')\right)$$

Where the wealth distribution is $w' = \{w'_i\}^I_{i=1} = \left\{e^i(s') + d(s')\theta^i\right\}^I_{i=1}$, E_p is the expected value taken with respect to $p(s,.)$, and the feasible set Δ is compact[5].

ii) (Market Clearing) $\sum_i \theta^i = \vec{0}$

iii) (Expectations) $q^i(s,w) = q^l(s,w) = q(s,w)$ for all $i, j \in I$

---

[4] This definition does not include models of the Hugget (1993) style as this type of models does not assume the existence of aggregate uncertainty (i.e. $\#S = 1$) and the degree of heterogeneity is higher as Hugget suppose the existence of a continuum of distinct agents and idiosyncratic uncertainty.

[5] To achieve this property it is sufficient to impose a short sale constraint on assets.

Provided the existence of continuous price functions $\{q^i(s,w)\}_{i=1}^{I}$ which satisfy iii), the continuity of $\{\theta^i(s,w)\}_{i=1}^{I}$ follows from mild curvature conditions on u_s^i (see Stokey, Lucas and Prescott, Ch. 9 and 10). Thus, definition 3.1 is equivalent to definition 3 in the sense that both imply a recursive structure based on continuous functions that depends on exogenous shocks and wealth distribution.

Technical appendix of section 2.4.

Assume that $I = 2, J = 3, \#S = 5, \beta^i = \beta^{i'} = 5/6$. Preferences, endowments and dividends are given by:

$$u_s^i = a_s^i \frac{[C(\sigma_t^* s)]^{1-5}}{1-5}\ , a_s^1 = [1, 1024, 1],\ a_s^2 = [1, 1, 1024] \text{ for } s = 1,2,3$$

$$u_s^1 = \frac{-[C(\sigma_t^* s)]^{-2}}{2} \text{ for } s = 4,5;\ u_4^2 = \frac{-[C(\sigma_t^* s)]^{-2}}{2},\ u_5^2 = \frac{-6.05[C(\sigma_t^* s)]^{-2}}{2}$$

$$e^1 = [e^1(1), \dots, e^1(5)] = [4,12,1,10,8.69],\ \ e^2 = [e^2(1), \dots, e^2(5)] = [4,1,12,10,11.31]$$

$$d^1 = [d^1(1), \dots, d^1(5)] = [1,0,0,0,0],\ d^2 = [0,1,0,0,0],\ d^3 = [0,0,1,0,0]$$

The transition matrix is given by:

$$[p(s,s')] = \begin{bmatrix} 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.05 & 0.05 & 0.3 \end{bmatrix}$$

As consumption of each agent is bounded above by aggregate consumption, which is in turn uniformly bound by $e^1(5) + e^2(5)$, U_i can be assumed to be bounded above without lost of generality because Bernoulli utility functions are assumed to be strictly increasing. Thus, the arguments in Duffie, et. al. (page 765) imply that consumption is uniformly bounded below by some positive constant. Consequently, marginal utilities are uniformly bounded which in turn imply that any equilibria, if it exists, can be characterized by agent's first order conditions and feasibility constraints (see Kubler and Schmedders, 2002, page 288).

The following tables contain asset prices and portfolios which satisfy the optimality (first order) and feasibility conditions in definition 1. Each table can be seen as a time independent function of the exogenous shocks and the distribution of assets. There are 2 equilibrium portfolios which are computed "by hand". Namely, $\Theta_1 = [\theta_1^1; \theta_1^2] = [0, -1.6, 1.6; 0, 1.6, -1.6]$ and $\Theta_2 = [\theta_2^1; \theta_2^2] = [0, -0.98, 2.28; 0, 0.98, -2.28]$. Thus, these tables define a WRE.

Provided that the initial portfolio distribution $\{\theta_-^i\}$ is either Θ_1 or Θ_2, tables 1 and 2 can be used to generate a unique sequential competitive equilibrium according to definition 1.

Asset Prices (q)					
	S=1	S=2	S=3	S=4	S=5
Θ_1	[0.25, 2.15,2.15]	[0.03, 0.25,0.25]	[0.03, 0.25,0.25]	[0.24, 2.10,2.10]	[0.10, 1.54,0.08]
Θ_2	[0.25, 3.57,1.22]	[0.01, 0.25,0.08]	[0.05, 0.73,0.25]	[0.24, 2.10,2.10]	[0.10, 1.54,0.08]

Table 1

Portfolio $[\theta^1, \theta^2]$					
	S=1	S=2	S=3	S=4	S=5
Θ_1	Θ_1	Θ_1	Θ_1	Θ_1	Θ_2
Θ_2	Θ_2	Θ_2	Θ_2	Θ_1	Θ_2

Table 2

Kubler and Schmedders (2002) showed (numerically) that the endogenous variables in tables 1 and 2 are the only ones that satisfy the optimality and feasibility conditions in definition 1 (see page 301). Then, to show that this economy has no wealth recursive equilibria, it suffices to show that for some pair of states (s, w), there are at least 2 possible asset prices.

Heuristically, it can be argued that the endogenous variables in the tables above define a steady state[6]: once the economy starts either at Θ_1 or Θ_2, it will never leave the state space defined by $S \times \{\Theta_1; \Theta_2\}$.

To verify the existence of a wealth recursive equilibrium, we can describe the dynamic behavior of this economy using $s \in S$ and $w^i(s,\Theta) \equiv e^i(s) + d(s)\theta^i$, where the asset position can take 2 values $\Theta \in \{\Theta_1; \Theta_2\}$, distributed across agents $i = 1,2$ with portfolios $\Theta = [\theta^1, \theta^2]$, as can be seen in table 1 and 2 in section A.1.1 of the appendix. Suppose that $\Theta_{t=0} = \Theta_1$ and take the sequence of exogenous shocks given by $\{s_0, s_1, s_2, \dots\} = \{2,4,1,\dots\}$. This economy has 2 sequential competitive equilibria. There are 2 different sequences of asset prices $q_t(\sigma_t)$, for $\sigma_t \in \mathfrak{T}$, that satisfy the optimality and feasibility conditions if in definition 1 $\{\theta_-^i\}$ is replaced by $\{w_-^i\}$. Note that this last change is necessary to allow the sequential economy to be generated out of a wealth recursive equilibrium. Figure 1 illustrates this result by mapping $w(s_t,\Theta_t) \equiv [w^1(s_t,\Theta_t), w^2(s_t,\Theta_t)]$[7], the wealth distribution, into $q_t^2(s,\Theta)$, the price of asset $j = 2$.

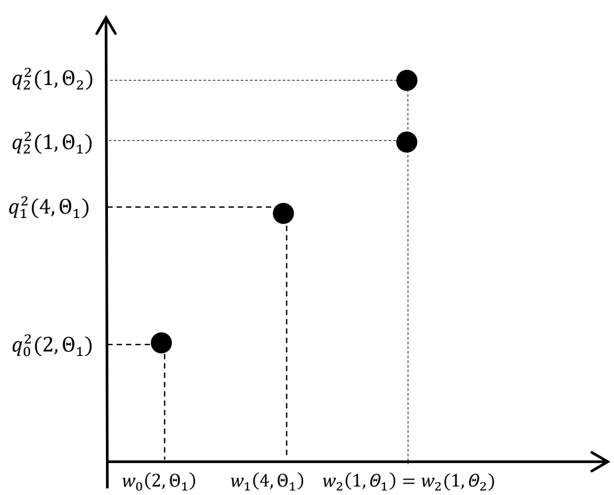

Figure 1: Wealth Equilibrium Correspondence

[6] A steady state for an appropriately defined Markov representation of the sequential competitive equilibria will be formally defined in section 3.

[7] The definition of wealth in section 2.1 would imply $w_t = w(s_t,\Theta_{t-1})$. For expositional purposes, it is convenient to define $w_t = w(s_t,\Theta_t)$. The results in this section will not change using either definition.

As $w_2(1,\Theta_1) = w_2(1,\Theta_2)$ but $q_2^2(1,\Theta_2) \neq q_2^2(1,\Theta_1)$, there are 2 possible images for the same element in the domain of this function. Using table 2 in the appendix it is easy to see that in $s = 1$ the 2 admissible portfolios have $\theta_1^i = 0$ for $i = 1{,}2$. As dividends are 0 for the other 2 assets, *wealth does not vary with the asset distribution.* Thus, the selected state space is insufficient to describe the evolution of endogenous variables using a *continuous function* and there is no wealth recursive equilibrium for this economy. Figure 1' illustrate this fact.

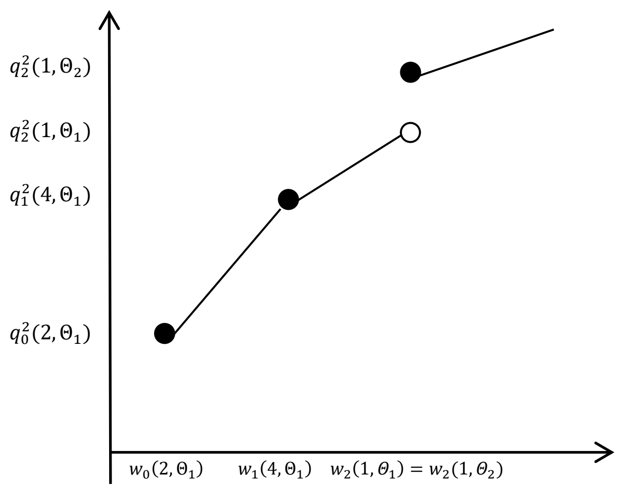

Figure 1': Computed wealth recursive function

As regards the genericity of this example, Hoelle (2014)[8] finds a positive measure set of economies 2 period economies, $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^I, \{\theta_-^i\}_{i=1}^I\right]$, indexed by e, which have multiple equilibria. The discussion above suggests that each of these economies could potentially generate wealth levels with

[8] See Hoelle (2014), page 124. The author finds a strictly positive measure set of 2 period economies with no uncertainty (i.e., $\#S = 1$) and multiple equilibria. This type of economies can be contained in definition 1 by simply letting $p(s,s') = 0$ for $s \neq s'$ and for some $s \in S$ (i.e., s is an isolated state). As assumptions A.1 to A.6 in Magill and Quinzii (1994) does not restrict p, an economy with this "degree" of multiplicity may exist.

$w_0(s,\Theta) = w_0(s,\Theta')$; creating a discontinuity in the wealth map. The question is deep and, thus, a formal result on the robustness of the counter example presented in this section is left for future research.

Technical appendix of section 2.5.1.

Duffie, et. al. showed that a recursive structure, called Time Homegeneous Markov Equilibria (THME), can be derived by imposing only mild assumptions on the primitives of the model if the correspondence based temporary equilibrium framework in Grandmont and Hildenbrand (1974) is applied to an enlarged state space, Z, that includes all equilibrium variables and, thus, circumvent the problems discussed in section 2.4. The virtue of this approach is its generality and its robustness to the presence of multiple equilibria.

Definitions and related concepts are presented below. We formally state the definition of 4 objects that can be used to define a Markov process in the context of Duffie, et. al. Heuristically, J is the state space, called "self-justified", G the equilibrium or "expectation" correspondence that maps states into possible continuations which satisfy the definition of sequential equilibrium, ρ is the space of probability measures over Borel measurable sets, C_j contains the initial payoff relevant variables of a truncated economy with j periods analogous to the one presented in definition 1.

A THME is build using 3 preliminary elements: an expectation correspondence, a self-justified set and a transition function. $Z_D = \left\{[s,\theta_-,c,q,\theta] \in S \times \mathbb{R}^{IJ} \times \mathbb{R}^{I} \times \mathbb{R}^{J} \times \mathbb{R}^{IJ} \middle| \sum_{i=1}^{I} \theta_-^i = \vec{0}, \sum_{i=1}^{I} \theta^i = \vec{0}\right\}$ is the state space.

An *expectation correspondence* is a map, $G\colon Z_D \to \mathcal{P}(Z_D)$, where $\mathcal{P}(Z_D)$ is the set of probability measures generated from Z. It will be said that $\mu \in G(z_0)$, if z_0 and any realization of the random variable z_1, which has conditional distribution given by μ, satisfy the optimality conditions implied by b) in definition 1. Typically, and without loss of generality, G is supposed to have a closed graph.

The purpose of the expectation correspondence is to obtain a sequence $\{z_t\}_{t=0}^{T}$, with $T \in \mathbb{N}$, such that the conditional distribution of z_t is contained in $G(z_{t-1})$. This sequence can be constructed as follows: it will be said that $\mu \in G(z_{t-1})$ and $z_t \sim \mu$, where $z_{t-1} = [s_{t-1}, \theta_{t-2}, c_{t-1}, q_{t-1}, \theta_{t-1}]$ is feasible, if for each $i \in I$

1) $c_{t-1}^{i} = e^{i}(s_{t-1}) + \theta_{t-2}^{i} d(s_{t-1}) - \theta_{t-1}^{i} q_{t-1}$
2) $q_{t-1}\left(u_{s_{t-1}}^{i}\left(c_{t-1}^{i}\right)\right)' = \beta E_{\mu}\left[d(s_t)\left(u_{s_t}^{i}\left(c_t^{i}\right)\right)'\right]$

Where E_{μ} is the expectation with respect to μ[9], which is an arbitrary probability measure on $\mathcal{P}(Z)$, and $\left(u_{s_{t-1}}^{i}\left(c_{t-1}^{i}\right)\right)'$ is the partial derivative of $u_{s_{t-1}}^{i}$.

Let $K \subset Z$ be any measurable set such that $z_t \in K$ for any $\{z_t\}_{t=0}^{T}$ and any $T \in \mathbb{N}$. The existence of this set, typically compact, for economies with short lived assets and finite shocks is guaranteed by the results in Maguill and Quinzii (1994, see page 871). Define $C_0 \equiv K$. Then, the set of all initial states of any 2 period (truncated) economy[10] is contained in the following set: $C_1 = \left\{z \in K \middle| \exists\ v\ \in G(z)\ and\ sup_{D \subset C_0} v(D) = 1\right\}$, where sup denotes the supremum. Inductively, a sequence of nested sets $\{C_j\}$ for $j \geq 1$ can be constructed with C_j containing the initial states of any j-period economy.

It follows from Theorem 1.2 in Duffie, et. al. (page 754) that $J = \bigcap_{j=0}^{\infty} cl\left(\overline{C_J}\right)$ is nonempty and compact, where cl denotes the clousure of a set, if K is compact and $C_j \neq \emptyset$ for $j \geq 1$. In the present context both conditions are

[9] Duffie, et. al. adds 2 technical conditions to 1) and 2). The first one restricts the marginal distribution of s_{t-1} and θ_{t-2} and the second one affects the support of μ. For a detailed discussion see for instance Duffie, et. al. pages 763 and 767.

[10] As long as Z is compact, it is clear that any $\{z_t\}_{t=0}^{T}$ contained in G is a sequential competitive equilibrium. For an arbitrary state space, a set of uniform (stationary) bounds are required. For instance, this is done by Duffie, et. al. (1994) using Radner's (1972) existence result (Lemma 3.4, page 768) and by Kubler and Schmedders (2003) using several elements of the Geanakolos and Zame (2002) existence proof (Lemma 3, page 1777).

guaranteed to hold by corollary 5.3 in Maguill and Quinzii (page 868)[11]. J is *called self-justified set.*

Any selection π of the expectation correspondence must have the following 2 properties, which are standard in the literature: $\pi(.,A)$ must be measurable and $\pi(z,.)$ must be a probability measure for any measurable set A and $z \in J$ respectively. This last condition follows directly from the definition of the expectation correspondence. If J is closed, then the Kuratowski measurable selection theorem (see for instance Hildenbrand 1974, page 55) implies that the restriction of G to J has a measurable selection.

Thus, for the economy described in section 2.1, which satisfy all the relevant assumption in Magill and Quinzii (1994, see page 858), the results in Duffie, et. al. guarantees the existence of correspondence based recursive structure on an enlarged state space which the authors called Time Homogeneous Markov Equilibrium (THME):

Definition 4: *A pair (J,π) is a THME for G if π is a Markov operator and J is a set that satisfies $\pi(z) \in G(z)$ for all $z \in J$.*

Technical appendix of section 2.5.2

Preliminary Remarks

In this framework the state space, $\tilde{Z}$, can be decomposed in 2 parts: payoff relevant variables Z_F and auxiliary variables m. In particular, let $Z_F \equiv \left\{[s,q,\theta] \in S \times \mathbb{R}^J \times \mathbb{R}^{IJ} \middle| \sum_{i=1}^{I} \theta^i = \vec{0}\right\}$, $m^{i,j} \equiv d^j(s)\left(u_s^i(c^i)\right)'$, where m is the vector of shadow values of the marginal return to investment for all assets and all agents. Assume, additionally to the hypothesis stated in section 2.1, that there

[11] Duffie, et. al. (section 3) established the existence of a compact set K (page 767, Lemma 3.1 and 3.2) and $C_j \neq \emptyset$ for any $j \in \mathbb{N}$ (Lemma 3.4 and 3.5, page 768) in a heterogeneous agent economy with a finite number of Lucas trees and short sales constraints. Braido (2013) extended these results for a general asset structure under mild assumption on preferences.

exists a short sale constraint $\bar{B} > 0$ such that $\theta^{i,j} \geq -\bar{B}$. Using the budget constraint, equation 1, it is possible to define a correspondence V that maps $(z) \mapsto m$ as follows: for each $z \in Z_F$, $c^i \in [e^i(s) + \theta^i d(s) - I\bar{B}q, e^i(s) + \theta^i d(s) + I\bar{B}q]$ defines a selection $m \sim V(z)$ which is obtained by taking some $\theta_+^{i,j} \geq -\bar{B}$ for all $i \in I$ and $j \in J$. Provided, as discussed in section 2.5.1, that all endogenous variables in the model are (uniformly) contained in a compact set K, V is compact valued and $Gr(V)$ is compact.

Then, as in the previous subsection, it is possible to derive a time invariant compact state space, which is analogous to Duffie, et. al.'s self-justified set. Let $\widetilde{K} \subset K$ and $\widetilde{K} \equiv Gr(V_0)$. The first order conditions of the model can be written as:

3) $c^i = e^i(s) + \theta^i d(s) - \theta_+^i q$

4) $\left[q\left(u_s^i(c^i)\right)' - \beta E_{p(s,.)}(m_+^i)\right]\left[\theta_+^i - \bar{B}\right] = \vec{0}$

Where $E_{p(s,.)}$ is the expectation with respect to $p(s,.)$, the conditional distribution of s_+ given s, and $m_+^i(s_+) \sim V(\theta_+, q_+, s_+)$. Thus, 4) is defined using the expected value with respect to $p(s,.)$ over $[\theta_+, q_+](s_+)$.

Selection Mechanism (Continuity along $\mathbf{s}_+$)

The functions θ_+ and q_+, mapping $s_+ \mapsto \theta_+$ and $s_+ \mapsto q_+$ respectively, can be chosen to be *continuous* provided that S in an uncountable set. Each of these functions is associated with a predecessor in $\tilde{Z}$.

Following equation 2), in the technical appendix of section 2.5.1, the expected value should have been taken with respect to any possible distribution of z_+, μ. Thus, *equation 4) captures a subset of all possible* z *for any given* z_+[12].

Now it is possible to define the analogous of a "self-justified set" in Feng, et. al. framework. To begin with, the set of all states, $\tilde{z} \in \widetilde{K}$, of any 2-period economy is contained in:

$$Gr(V_1) = \left\{ \tilde{z} \in \widetilde{K} \middle| \, \exists \, \tilde{z}_+ \in Gr(V_0) \; with \; \tilde{z}, \tilde{z}_+ \; satisfying \; eq. 3) \; and \; 4) \right\}$$

That is, $[s, q, \theta, m] \in Gr(V_1)$ if $c^i(\theta_+^i)$ obtained from 3) for all $i \in I$ satisfy equation 4) for some $m_+^i(s_+) \sim V(\theta_+, q_+, s_+)$ with $[\theta_+, q_+, s_+] \in V_0$. For any arbitrary iteration j, notice that for each $s_+ \in S$ there could be more than 1 possible pair (θ_+, q_+). However, as θ_+ is chosen at time "t", to satisfy the restrictions of the SCE, it must be s^t-measurable, where s^t is a branch of the tree $\mathfrak{T}$ defined in section 2.1. Thus, $\theta_+(s_+)$ can be chosen to be constant and thus *continuous for each* $\tilde{z} \in \widetilde{K}$. Moreover, any possible discontinuity in $q(s_+)$ can be ruled out by appropriately changing θ_{++} in $[e^i(s_+) + \theta_+^i d(s_+) - I\bar{B}q_+, e^i(s_+) + \theta_+^i d(s_+) + I\bar{B}q_+]$ with $\theta_{++} \in [-I\bar{B}, I\bar{B}]$. If we set $Gr(V_0) = \widetilde{K}$, we can completely characterize $q(s_+)$ as $Gr(V_j) \subseteq \cdots \subseteq Gr(V_0)$ and the state space of the Markov process is the intersection of this countable collection of compact sets. The upper hemi-continuity and compact valuedness of V implies that "explosions and implosions" are small enough to be ruled out by any perturbation in $[-I\bar{B}, I\bar{B}]$ (see Hildenbrand and Grandmont (1974, page 31 theorem 4). Not surprisingly, it is only possible to ensure the continuity of $[\theta_+, q_+](s_+)$ for an interior path. Note that this procedure can be defined for each $[\theta, q, s] \in V_0$.

[12] Duffie, et. al. also restricts $[\theta_+, q_+]$ to be a function of s_+ and that $s_+ \sim p(s,.)$. However, it is still possible to find a distribution, $z_+ \sim \mu$, which satisfy this restrictions and $E_{p(s,.)} \neq E_\mu$.

Time independent State Space

Let $Gr(V_j) = C_j$. Iterating on $Gr(V_1)$, it is possible to derive a sequence of nested sets $\{C_j\}$ for $j \geq 1$ where C_j contains all $\tilde{z}_0$ of any j-period economy. Note that this procedure defines an operator $G_K: Gr(V) \rightarrow Gr(V)$, where V is some set $\ldots V_j \ldots V_0$. The non-emptiness and compactness of each C_j follows from the arguments in section 2.5.1 as, respectively, equations 3) and 4) are identical to the optimality conditions implied by the definition of "equilibrium with explicit debt constraint" in Magill and Quinzii (page 862) and the recursive equilibria in Feng, at. al. are a subset of those in Duffie, et. al.[13]

As G_K maps compact sets to compact sets, Feng, et. al. showed (theorem 2.1 in page 6) that $V_n \rightarrow V^*$, where V^* is the analogous of Duffie, et. al.'s self justified set. Thus $Gr(V^*) = \tilde{J}$ contains all possible first period payoff relevant variables $\tilde{z}_0(\sigma_0)$ for the sequential competitive equilibrium in definition 1.

Finally, $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ is defined as follows: take any $\tilde{z} = [\tilde{s}, \tilde{\theta}, \tilde{q}, \widetilde{m}] \in \tilde{J}$, $\tilde{z}_+ \in \Phi(\tilde{z}, \tilde{s}_+)$ if $\tilde{z}_+ \in \tilde{J}$ and $(\tilde{z}, \tilde{z}_+)$ satisfy equations 3) and 4) with $m(s_+) \sim V^*(\theta_+, q_+, s_+)(s_+)$. The following definition summarizes this discussion:

<u>Definition 5:</u> Let $\tilde{J} = Gr(V^*)$ and $\tilde{J} \subseteq \widetilde{K}$. $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ is an <u>equilibrium correspondence</u> if $\tilde{z}_{t+1} \in \Phi(\tilde{z}_t, s_{t+1})$ and $\{\tilde{z}_t\}_{t=0}^{\infty}$ satisfy the optimality conditions in equations 3)-4) and the feasibility restrictions in the definition of Z.

The procedure described above can be repeated an infinite number of times as $\tilde{J}$ contain all possible initial conditions $\tilde{z}_0(\sigma_0)$ for any $T \in \mathbb{N}$ period economy. A *time invariant transition function* is obtained by taking a selection of Φ, denoted $\varphi \sim \Phi$. This function is measurable, as Φ has closed graph and is compact valued (see Stokey, Lucas and Prescott, page 60 theorem 3.4 and 184

[13] Section 5.1 will provide some additional details about these facts.

theorem 7.6), and does not depend on unobservable variables, thus *it constitutes the starting point of the theoretical results in section 3.*

Technical appendix of section 2.6

Remarks on ergodicity and accurate simulations

Any calibration procedure may require matching the unconditional expected value of some endogenous variables against its empirical counterpart. As it was discussed in section 2.5, $\bar{z}_t(\sigma_t)$ can be proved to be uniformly bounded, which suggests that time series must be detrended. Further, it is typically assumed that the unconditional expected value of the transformed time series is time invariant (see De Jong and Dave, 2007). The non-stationarity of $\{\bar{z}_t(\sigma_t)\}_{t=0}^{\infty}$ implies that $E_{\mu_t}(\bar{z}_t)$ will change over time, making the calibration of the model not feasible.

An appropriate recursive structure solves one of these problems as it endows the model with a time invariant transition function that depends on a low dimensional observable state space. This function can be computed in finite time using the algorithms discussed in Feng, et. al. or Kubler and Schmedders (2003). However, the stationarity of the stochastic process generated from the recursive structure can be difficult to obtain in a correspondence-based framework.

Equipped with a recursive representation of the sequential equilibrium and a time invariant state space, as it is the case in Feng, et. al., it is possible to define a Markov stochastic process, which may or may not be stationary. Formally, a necessary and sufficient condition to guarantee this last property is the existence of an *ergodic measure* for the process, which can be seen as a notion of steady state in the sense that the unconditional distribution of the payoff relevant variables $\bar{z}_t$ does not change over time. This delicate issue, is essential to obtain empirically meaningful dynamics as it allows to obtain time invariant unconditional moments $E_{\mu^*}(\bar{z}_t) = E_{\mu^*}(\bar{z}_{t'})$ for any $t \neq t'$, where μ^*

is any invariant measure of the Markov process, and this moments can be matched with data.

Remarks on facts 2.6-i) to 2.6-iii) and their connection with the paper

Fact 2.6-i) requires approximating the recursive structure in Feng, et.al., which is the only known recursive structure that generates computable (i.e., that depend on observable variables) time independent transitions and allows for multiple equilibria. This fact also assumes that exact and numerical simulations converge almost surely to unconditional moments, which are obtained by integrating against an ergodic invariant measure. The conditions to prove the existence of an invariant measure and its ergodicity will be presented in section 3.

This paper derives the conditions required to prove the existence of an invariant measure separately from the strengthening necessary to insure its ergodicity. From a pure theoretical point of view, as the stochastic process associated with the sequential equilibria may not be stationary, establishing conditions which guarantee the first property is also desirable.

Section 4 contains the implications of the existence of an ergodic invariant measure for facts 2.6-i) and 2.6-ii). Section 4.2 includes the sufficient conditions which guarantee that simulations converge almost surely, provided the existence of an ergodic invariant measure. Feng, et. al. assumed that the convergence of the approximated transitions is *uniform,* a hypothesis that will be slightly modified to prove fact 2.6-ii) in section 4.1. Thus, the ergodic nature of the invariant measure is only required for the numerical part of the paper as it ensures the accuracy of simulations. Section II in the supplement presents an economy which satisfies the conditions necessary to guarantee facts 2.6-i) and 2.6-ii).

Using facts 2.6-i) to 2.6-iii) together to perform an empirically meaningful exercise is beyond the scope of this paper. The design of an algorithm that satisfies the conditions which guarantee fact 2.6-ii) for an economy like the one presented in section II in the supplement are also left for future research.

Technical appendix of section 3.1

Comment on Assumption 2

Lemma 9.5 in Stokey, Lucas and Prescott (page 261) implies that: $\#\left(\Delta\hat{P}_{\varphi}f\right) \leq \#(\Delta f(\varphi)) \leq \#(\Delta\varphi)$, where # denotes the cardinality of a set.

The discussion in section 2.4 suggests that $\#(\Delta\varphi)$ is typically related with the cardinality of the equilibrium set and depends crucially on the selected state space. Often, an upper bound for the cardinality of the equilibrium set can be obtained for regular economies (see, Geanakolos and Polemarchakis, 1986). In the presence of short sale constraints or an uncountable number of exogenous shocks, standard regularity theorems do not hold. Thus, given the state space, assumption 2 puts an upper bound on the number of economies, each of them associated with a different initial condition, that are allowed to have multiple equilibria. In applications assumption 2 will be taken as given. It should be a topic of future research to relate this assumption with primitive conditions of the sequential economy.

Comment on Theorem 1

Let $C(\tilde{J})$ be the space of continuous functions on $\tilde{J}$. It will be said that P_{φ} has the Feller property if the semigroup operator maps $C(\tilde{J})$ into itself. Lemma 9.5 in Stokey, Lucas and Prescott (page 261) shows that if $f \in C(\tilde{J})$, $\hat{P}_{\varphi}f(\tilde{z}) \in C(\tilde{J})$.

The absence of the Feller property also affects the continuity of the adjoint operator, which is critical to guarantee the existence of a fixed point of it. As P_φ^* is defined over an infinite dimensional space, to discuss its continuity, it is necessary to select an adequate topology. The *weak** topology, the coarsest topology that makes the linear functional $\{\mu \mapsto \int f d\mu,\ f \in C(\tilde{J})\}$ continuous, is frequently chosen. This is because P_φ^* generate sequences of *weak** convergent measures under mild assumptions[14]. Under assumption 1, $\tilde{J}$ is a compact subset of a finite dimensional Euclidean space. Thus, Helly's theorem (Stokey, Lucas and Prescott, page 374) implies the existence of a *weak** - convergent subsequence in $\mathcal{P}(\tilde{J})$, which is the starting point of most existence theorems.

As discussed in Aliprantis and Border (2006, page 47), the choice of a weak topology implies a tradeoff: there are a lot of weakly convergent sequences but there are few weakly continuous functionals. Thus, the Feller property is used to guarantee the *weak** continuity of P_φ^*: $\mu_n \to_{Weak*} \mu$ implies $P_\varphi^* \mu_n \to_{Weak*} P_\varphi^* \mu$ if $\hat{P}_\varphi$ has the Feller property (see Stokey. Lucas and Prescott, page 376).

If φ can be shown to be continuous, under assumption 1, Theorem 2.9 in Futia (1982, page 383) would imply the existence of an invariant measure for P_φ^*. It only suffices to take a sequence of measures generated by applying P_φ^* iteratively on some $\mu_0 \in \mathcal{P}(\tilde{J})$ that is robust to cyclical behavior and fits into the framework of Helly's theorem. Let $\mu_{n_k} \to_{Weak*} \mu$ be the subsequence generated by Helly's theorem. The continuity of P_φ^* implies $P_\varphi^* \mu_{n_k} \to_{Weak*} P_\varphi^* \mu$. Subtracting both subsequences, the desired result follows. *Theorem 1 in this paper shows the existence of an invariant measure for $(\tilde{J}, P_\varphi)$ even if φ is allowed to have (a certain type of) discontinuities.*

[14] This is not the case of the *strong* topology, which is the topology generated by the total variation norm. Stokey, Lucas and Prescott (page 335 to 337) provides an example of a Markov process that generates sequences that converge in the *weak** topology but not in the strong (norm) topology.

The strategy of the proof for Theorem 1 goes along the lines of Hildenbrand and Grandmont (1974). It borrows from theorem 12.10 in Stokey, Lucas and Prescott (1989) (page 376), theorem 3.5 in Molchanov and Zuyev (2011, page 15) and proposition 1 in Ito (1964, see page 155). The following subsection contains a detailed description of the procedures used *up to* now to prove the existence of an invariant measure and the reasons that make them unsuitable for this paper.

Using proposition 1 in Ito and theorem 3.5 in Molchanov and Zuyev it is possible to restore the continuity of P_φ^* without the Feller property. As P_φ^* and $\hat{P}_\varphi$ can be interchanged (see for instance Stokey, Lucas and Prescott page 216), if $\mu_{n_k} \rightarrow_{Weak*} \mu$, for some $f \in C(\tilde{J})$:

$$\int f(\tilde{z}) P_\varphi^* \mu_{n_k}(d\tilde{z}) = \int \hat{P}_\varphi f(\tilde{z}) \mu_{n_k}(d\tilde{z}) \nrightarrow \int f(\tilde{z}) P_\varphi^* \mu(d\tilde{z}) = \int \hat{P}_\varphi f(\tilde{z}) \mu(d\tilde{z})$$

As $\hat{P}_\varphi f(\tilde{z})$ may not be continuous. $\hat{P}_\varphi f(\tilde{z})$ is bounded and $\mathcal{B}_{\tilde{J}}$-measurable. Theorem 3.5 in Molchanov and Zuyev implies that $\int \hat{P}_\varphi f(\tilde{z}) \mu_{n_k}(d\tilde{z}) \rightarrow \int \hat{P}_\varphi f(\tilde{z}) \mu(d\tilde{z})$ if $\mu(\Delta \hat{P}_\varphi f) = 0$, where $\Delta \hat{P}_\varphi f$ is the *set of discontinuities of* $\hat{P}_\varphi f$.

Thus, it only suffices to show that the discontinuity set generated by φ *is sufficiently small under the limiting measure.* To achieve this property, proposition 1 in Ito is used to show that P_φ^* maps the set of *atomless measures, which will be denoted* $\mathcal{P}_0(\tilde{J}) \subset \mathcal{P}(\tilde{J})$*, into itself. The proof will be complete if it can be shown that* $\mu \in \mathcal{P}_0(\tilde{J})$ *and* $\mu(\Delta \hat{P}_\varphi f) = 0$. As a measure is atomless if and only if $\mu(\{a\}) = 0$, $\{a\} \in \tilde{J}$ (i.e. it assigns zero measure to points, see Hildenbrand and Grandmont 1974, page 45), it suffice to restrict the cardinality of $\Delta \hat{P}_\varphi f$ to be at most countable and to show that $\mathcal{P}_0(\tilde{J})$ is closed. The latter property will be insured by imposing conditions on P_φ in section 3.2

and 3.3. The former will be guaranteed by restricting the discontinuity set of φ as in assumption 2 and 2'.

Related results on the existence of Invariant measures.

Because the expectation correspondence in Duffie, et. al. G is defined as map $z \mapsto \mu$, with $\mu \in \mathcal{P}(Z)$, using the arguments presented in section 2.5.1 and 3.1 it is easy to show that a THME (J, π) can be used to define a sequence of measures $\{\lambda_t\}_{t=0}^{\infty}$ in $\mathcal{P}(J)$ such that $\lambda_{t+1} = \pi^*\lambda_t$, where π^* is the adjoint operator associated with (J, π).

Grandmont and Hildenbrand showed that the continuity of π^* is sufficient to show the existence of an invariant measure λ, provided that J is a compact set and G is constructed from an equilibrium correspondence: every $\pi \sim G$ satisfies $\pi = \pi_\varphi$ with $\varphi \sim \Phi$ and $\Phi: J \times S \rightarrow J$. Provided that assumption 1 holds, the existence of π_φ follows from Lemma 2. As discussed in the supplementary appendix of section 3.1, π^* is continuous iff $\hat{\pi}$ has the Feller property, where $\hat{\pi}$ is the semigroup operator associated with (J, π). The authors could not show that $\hat{\pi}$ has this property and had to assume it (see Lemma 2 in Grandmont and Hildenbrand, page 263).

The arguments discussed in section 2.4 imply that φ may not be continuous, thus the result in Hildenbrand and Grandmont was considered unsatisfactory. Blume (1982) dispense with this assumption and took a rather different approach. Given a Markovian structure with time homogeneous transitions, the author used Fan's fixed-point theorem to show the existence of an invariant measure for $G_B: \mathcal{P}(Z) \rightarrow \mathcal{P}(Z)$, where $G_B = \{\pi_\varphi^*: \varphi \sim \Phi\}$. As G_B was assumed to be nonempty, to have closed graph and Z to be compact, the required upper-hemicontinuity followed immediately. However, to apply Fan's theorem, G_B must be convex valued[15]. To guarantee this latter property, Blume assumed that S is characterized by an atomless, not necessarily absolutely continuous, measure. Clearly, if S is a finite set, this last assumption is not realistic. The arguments in section 3.2 try to fill this gap. Even if S is a compact and uncountable set, and $p(s,.)$ is atomless $\forall s \in S$, as discussed at the end of section 3.1, the results in Blume only shows that G_B has a fixed point, which is equivalent to $IM(\Phi) \neq \emptyset$ but weaker than $IM(\varphi) \neq \emptyset$ for any $\varphi \sim \Phi$ satisfying assumptions 1, 2 and the additional hypothesis presented in section 3.3. As discussed in section 4, this last fact has important numerical implications as it allows approximating φ instead of Φ.

The results in Blume highlighted the necessity of a "convexified" correspondence, G_B, to prove the existence of an invariant measure. This was the approach taken by Duffie, et. al. (1994), theorem 1.1, to show the existence of an *ergodic* measure. As discussed in section 2.5.1 and its supplementary appendix, provided the existence of time-invariant state space and that G is convex valued, Duffie, et. al. (1994) showed that a refinement of a THME, called conditionally spotless, has an ergodic measure. The following definition states this notion of equilibrium formally:

Definition A1 (Conditionally Spotless THME):

Let $\mathcal{P}_F(S \times \hat{Z}) = \{\mu \in \mathcal{P}(S \times \hat{Z}) | \exists\, h, h: S \to \hat{Z}, with\ Supp(\mu) = Gr(h)\}$. A THME (J,π) is spotless if $\pi(z) \in \mathcal{P}_F(S \times \hat{Z})$ for all $z \in J$. A THME (J,π) is called conditionally spotless if for all $z \in J$, $\exists\ M \subset \mathcal{P}_F(S \times \hat{Z}) \cap G(z)$, $\eta \in \mathcal{P}(M)$, $\pi(z) = \int v\, d\eta(v)$ and G is convex valued.

Note that a spotless THME removes the possibility of sunspots discussed in Lemma 1: given $z_t \in J$, there is a measure $\mu_{z_t} \in G(z_t) \cap \mathcal{P}_F(S \times \hat{Z})$, which gives

[15] G_B is convex-valued if $\lambda_1', \lambda_2' \in G_B(\lambda)$, with $\lambda_1' = \pi^*_{\varphi_1}\lambda$, $\lambda_2' = \pi^*_{\varphi_2}\lambda$ and $\varphi_1, \varphi_2 \sim \Phi$,then $\lambda' \in G_B(\lambda)$ with $\lambda' = (\alpha)\pi^*_{\varphi_1}\lambda + (1-\alpha)\pi^*_{\varphi_2}\lambda$, $(\alpha)\pi^*_{\varphi_1} + (1-\alpha)\pi^*_{\varphi_2} \in G_B$ and $\alpha \in [0,1]$.

the conditional distribution of z_{t+1}, $\hat{z}_{t+1} = h(s_{t+1})$ and $\mu_{z_t}(Gr(h)) = 1$. Intuitively, each pair (z_t, s_{t+1}) is associated with a unique $\hat{z}_{t+1}$ or equivalently $\hat{z}_{t+1} = h_{\mu_{z_t}}(s_{t+1})$ and $\hat{z}_{t+1}$ satisfy the optimality and feasibility requirements contained in the definition of G. Note that it is possible to refine even more a spotless THME by letting $\hat{z}_{t+1} = h_{z_t}(s_{t+1})$, where the measurability of f must be shown and $z_{t+1} \sim \mu \in \mathcal{P}(S \times \hat{Z})$ must be defined accordingly. *The results in section 3 and 4 hold for this last type of equilibria.*

To show the existence of an ergodic invariant measure for a spotless THME the authors proceeded in 2 steps. First, they applied Fan's fixed point theorem to $T \equiv E \circ m_2 \circ m_1^{-1}: \mathcal{P}(J) \to \mathcal{P}(J)$, where $m_1: \mathcal{P}(Gr(G_J)) \to \mathcal{P}(J)$, $m_2: \mathcal{P}(Gr(G_J)) \to \mathcal{P}(\mathcal{P}(J))$ give the marginals of $\mathcal{P}(Gr(G_J))$ and $E\eta \equiv \int \mu d\eta(\mu), \eta \in \mathcal{P}(\mathcal{P}(J))$ is the mean of η, which is uniquely defined by the Riesz representation theorem for continuous function[16]. T is a continuous linear functional and G_J is upper hemi continuous. This was assumed in Duffie, et. al. In the context of this paper, a similar property follows from theorem 3.1 in Blume under assumption 1 provided that G_J is constructed from Φ using Lemma 2. However, as discussed in section 2.5.2, this last procedure only captures a subset of all possible recursive equilibria. Under these 2 properties, T is also upper hemi-continuous [17]. As J is a self-justified set (see the supplementary appendix for section 2.5.1), G_J is nonempty. T is convex valued as G assumed to be so. As $\mathcal{P}(J)$ is nonempty, (weakly) compact and convex, T has a fixed point. Second, the authors showed that any λ with $\lambda = T(\lambda)$ also satisfies $\lambda = \pi \cdot \lambda$. To derive this result, they defined a transition function $P: J \to \mathcal{P}(\mathcal{P}(J))$ and showed that $E \circ P(z) \in G_J(z)$ λ-a.e. Thus, $\pi(z) = \int v \, d\eta(v)$ almost everywhere for $\eta \in \mathcal{P}(\mathcal{P}(J))$.

[16] See Theorem 14.12 in Aliprantis and Border (2006, page 496).
[17] See Grandmont (1983, page 158).

To obtain an ergodic measure for a conditionally spotless THME, which is defined for economies with a *finite number of shocks*, $\mathcal{P}(J)$ should be replaced with $G_J(z) \cap \mathcal{P}_F(S \times \hat{Z})$. This implies that G_J is convex valued: definition A1 assures that for any $z \in J$, there exist an expectation correspondence $\hat{g}$ which is convex valued as it contains any possible randomization $\mathcal{P}(M)$ over spotless transitions $M \subseteq \mathcal{P}_F(S \times \hat{Z}) \cap G(z)$ for any $z \in J$. A selection $\pi(z) \sim \hat{g}(z)$ is constructed by changing $\eta \in \mathcal{P}(M)$ and computing $\pi(z) = \int v \, d\eta(v)$. The assumption that G is convex valued can be done w.l.o.g. once transitions f are allowed to depend on "contemporaneous" sunspots (α_t), which select among randomized spotless transitions. Section II.2 and V.2 of this appendix shows that, if we restrict the class of models with respect to Duffie, et. al, it is possible to construct an equilibrium correspondence that contains a stationary and an ergodic equilibrium for uncountable shocks.

The discussion above implies that the transition functions generated by a conditionally spotless THME are affected by sunspots; a fact that affects the computability of the recursive structure. The authors did not prove the existence of an ergodic measure for a spotless THME, which generate sunspots free transition function. This paper shows this result for a refinement of all possible spotless THME (i.e., those generated from Feng, et. al.'s recursive structure).

Santos and Peralta Alva (2013) show that $IM(\Phi, \mathcal{P}_1) = \{\varphi \sim \Phi, \mu \in \mathcal{P}_1 | \mu = P_\varphi^* \mu\} \neq \emptyset$. Unfortunately, there are some concerns about the Santos and Peralta Alva (2013) framework. First, it is not clear if S is a finite set. If S can be characterized by a Markov process with an atomless Markov operator (i.e., $p(s,.)$ is atomless for all $s \in S$), the non-emptiness of the set of invariant measures $IM(\Phi)$ follows immediately from theorem 3.1 in Blume (1982). *This paper provides conditions which guarantee the non-emptiness of $IM(\varphi)$ for any*

$\varphi \sim \Phi$ that satisfies assumption 1 and 2 which is slightly stronger than $IM(\Phi) \neq \emptyset$. It is also convenient in applications as frequently it is desirably to compute only an approximation of φ. Second, the conditions which guarantee the existence of an ergodic measure in $IM(\Phi)$ have not been established, at least separately from those that guarantee $IM(\Phi) \neq \emptyset$. *Theorem 1 and 2 establishes, respectively, the properties of $(\tilde{J}, P_{\varphi})$ associated with the existence of an invariant and an ergodic measure. The first set of conditions are milder and thus do not require to construct a "tailor-made selection" as we did in section 2.5.2.* Third, the critical assumptions in Santos and Peralta Alva (2013), assumption 2.3 and remark 6.2, have been stated in terms of P_{φ}, not on primitives, and the procedure to compute an ergodic selection is not available. Thus, it may be difficult to identify these assumptions in certain applications.

Technical appendix of section 3.3

The difference between conditions 1) and 3) has 2 important consequences. First, condition 1) allows S being a finite set. This fact is discussed extensively in the preliminary remark of lemma 3 in section III of this appendix (see equations A.5 and A.6). As was argued in section II.1, the existence of a sequential equilibrium follows from mild assumptions for this type of economies. This is the bright side. On the other hand, however, proving the existence of an invariant measure requires condition 2), which is very challenging to derive from primate conditions. That is, if $\#S < \infty$, the existence of the sequential equilibrium and of the recursive structure in Feng, at. al. can be derived from primitive assumptions of the model. As can be seen in Martinez and Pierri (2021) and Pierri and Reffett (2021), the assumptions needed to prove the existence of an invariant measure when $\#S < \infty$ imply restrictions on S, φ and the existence of a representative agent. As this paper deals with heterogeneity, those results cannot be applied. Second, condition 3) allows proving the existence of an invariant measure imposing only this additional requirement to assumptions 1) and 2). Under this strengthening, condition 2 can be replaced by the closedness of the set of atomless measures under the adjoint operator, which is proved in section III. This condition follows from assuming that S is uncountable and from a mild requirement on its distribution, $p(s,.)$, as it only requires variability along one coordinate of the selection φ (see section V.1). However, showing the existence of a sequential equilibrium and of an appropriate recursive structure requires strong restrictions on endogenous variables. This last fact is discussed in sections 2.5.1 (see fact 2), II.2 and V.2.

In summary, there is a tradeoff between the strength of the conditions which guarantee the existence of a recursive structure and its stationarity or between the mildness of the assumptions required to prove the existence of a sequential equilibrium and to prove the existence of an invariant measure.

From the preceding discussion the crucial step in the existence of an invariant measure and its ergodicity is to ensure that the non-atomicity / absolute continuity of a sequence of measures is preserved under *weak** limits. This can be seen by noting that properties b) and c) in theorems 1 and 2 requires, respectively, the closedness of $\mathcal{P}_0$ and $\mathcal{P}_1$ and that, as was shown in lemmas 3 and 4, these properties impose restrictions on the Markov operator. Section 3.4 discussed how these restrictions reflect on φ and the primitives of the model. The following example illustrate the problem at hand.

<u>*Example 1 (non-uniform boundness of densities)*</u>*:* Let $P: S \times \mathcal{B}_S \rightarrow [0,1]$ be a transition function with $S = [0,1]$, $P(s, \{s/2\}) = 1$ and $\theta = U[0,1]$. Condition 1 is satisfied as $P(s, \{a\}) = 0$ except for $s = 2a$ with $\theta(\{2a\}) = 0$. Thus, under lemma 3, $P_\varphi^*: \mathcal{P}_0([0,1]) \rightarrow \mathcal{P}_0([0,1])$, where $\varphi(s) = s/2$. Note then that property a) in

theorem 1 holds. However, property b) will not be satisfied. Let $\mu_1 = P_\varphi^* \theta$ and $A = [0, a]$ with $0 < a < 1$. Then $\mu_1(A) = 2a$, that is, $\mu_1 = U[0,1/2]$ which has a density of 2. In general, $\mu_n = U[0,1/2^n]$ with $\mu_n = P_\varphi^* \mu_{n-1}$. Thus, $\{\mu_n\}$ has an associated sequence of densities of $\{2^n\}$, which is not a uniformly bounded sequence of functions. Kempton and Persson (2015, page 11) show that absolutely continuity is preserved under *weak** limits if the sequence of densities associated with $\{\mu_n\}$ is uniformly bounded.

This paper proved that that absolutely continuity is preserved under *weak** limits by imposing condition 4), that is slightly weaker than the uniform integrability of densities (see Diestel, 1991 for a detailed discussion), which is in turn weaker than the mentioned uniform boundness.

Example 1 shows that $\mathcal{P}_0([0,1])$, the subset of atomless measures in $\mathcal{P}([0,1])$ generated under the action of P_φ^*, is not closed as it contains a sequence of measures weakly converging to a Dirac measure at 0. Note that condition 3) is not satisfied in example 1 as $P_\varphi(z, \{a\}) = 0$ must hold uniformly in z, not a.e.

Condition 2), by lemma 3, and condition 3), by lemma 4, guarantee the closedness of $\mathcal{P}_0$ for the case of finite and uncountable shocks respectively.

As lemma 4 shows, condition 4) assures the closedness of $\mathcal{P}_1$. This condition implies that the family of measures $\{P_\varphi(z,.) | z \in \tilde{J}\}$ is absolutely continuous w.r.t. θ and that sets with a θ-measure smaller than δ have $P_\varphi(z,.)$-measure uniformly bounded by ε, where uniformity means that ε is independent of δ. This last condition is weaker than the uniformly integrability of densities, denoted by $\bar{p}_\varphi(z,z')$, as the latter requires $\int_B |\bar{p}_\varphi(z,z')| \theta(dz') < \varepsilon$ while the former only implies $\int_B \bar{p}_\varphi(z,z') \theta(dz') < \varepsilon$ (see Diestel, 1991). Although the distinction is subtle, it has important consequences: if $\int_B \bar{p}_\varphi(z,z') \theta(dz') < \varepsilon$ implies $\int_B |\bar{p}_\varphi(z,z')| \theta(dz') < \varepsilon$ for any $z \in \tilde{J}$, then $\bar{p}_\varphi(z,z')$ is bounded away from zero in

$\tilde{J} \times \tilde{J}$. But in this case, exercise 11.4 in Stokey, Lucas and Prescott implies that P_φ satisfies the Doeblin condition (i.e. $\theta(B) < \delta$ implies $\int_B \bar{p}_\varphi(z,z')\theta(dz') < 1-\varepsilon$ for any $z \in \tilde{J}$), which is a sufficient for the existence of an ergodic invariant measure (see page 345-8 for a discussion). A similar result holds if $\bar{p}_\varphi(z,z')$ is uniformly bounded above in $\tilde{J} \times \tilde{J}$.

By the discussion in example 1 and in the preceding paragraph, in this paper it will not be assumed that densities are neither bounded nor uniformly integrable as it suffices to restrict the Markov operator only to condition 4.

Note that assumption 2', like assumption 2, represents an upper bound on the genericity of the multiple equilibria problem discussed in section 2.4. Condition 4 is stronger than condition 3. Thus, as any invariant measure under condition 4 is absolutely continuous with respect to the Lebesgue measure, the constraint imposed by $\mu(\Delta\varphi) = 0$ in theorem 1 is now less restrictive: $\Delta\varphi$ can be an uncountable set if it has zero Lebesgue measure.

The strategy in lemma 4 is different from the one used to show the closedness of $\mathcal{P}_0$ under the adjoint operator. Since Futia (1982), ergodicity requires the compactness of the set of invariant measures. As $IM(\Phi, \mathcal{P}_0)$ may contain unbounded density functions, compactness can't be guaranteed[18]. Lemma 4 shows how condition 4 implies that small θ-measure sets have arbitrary small μ_n-measure, where $\{\mu_n\}$ is any sequence in $IM(\Phi, \mathcal{P}_1)$, and that this latter property guarantees that absolute continuity is preserved under *weak** limits of $\{\mu_n\}$.

[18] Condition 4) implies that the Markov operator is absolutely continuous which, it is well known, implies the existence of densities $\bar{p}_\varphi(z,z')$, which are bounded uniformly in z (as required by condition 4) and a.e. in z'. These facts cannot be assured by imposing condition 3) as the Radon–Nikodym theorem cannot be applied.

Technical appendix of section 4.1

Theorem 3 will show that there is a sequence of fixed-point measures $\{\mu_j\}$, $\mu_j = P_{\varphi_j}\mu_j$, which converges weakly to μ that is also absolutely continuous and is also a fixed point of an exact equilibrium selection, $\mu = P_\varphi\mu$. By choosing a selection which satisfies assumptions 3-iii), see fact 2.5-5) in section 2.5.2, and given that assumption 3-iv) are supposed to hold for $p(s,.)$, proposition 2 implies that P_φ is absolutely continuous w.r.t. θ (i.e., satisfies condition 4). Further, a restriction must be imposed on "the size" of $\varphi_j^{-1}(z,.)(A)$ for any open set with $\theta(A) < \delta$ to preserve the absolute continuity of the limiting measure as $\varphi_j \to \varphi$. This restriction on $\{\varphi_j\}$ guarantee that the sequence of measures $\{\mu_j\}$, if they are absolutely continuous and a fixed point of P_{φ_j}, converges weakly to an absolutely continuous measure μ. This is done by imposing assumption 4-iii) which requires $\varphi_j^{-1}(z,.)(A) \subseteq \varphi^{-1}(z,.)(A)$ for any j sufficiently large. Thus, densities generated from $\varphi_j^{-1}(z,.)(A)$ are bounded by those generated from $\varphi^{-1}(z,.)(A)$, which we know that are well defined as theorem 2 hold for this limiting measure.

To ensure $\mu_j \in IM(\varphi_j, \mathcal{P}_1)$ and μ_j is absolutely continuous, assumptions 1), 2'), 3-iii), 3-iv), 4-i) must be imposed on all approximated economies $\varphi_j \sim \Phi_j$. The purpose of all these assumptions was discussed in section 3, except assumption 4-i), which is standard in the recursive numerical literature (see for instance Feng, et. al.). After showing that $IM(\varphi_j, \mathcal{P}_1) \neq \emptyset$, $\mu_j \to_{weak*} \mu$ and μ is absolutely continuous, theorem 3 shows that $\mu \in IM(\varphi, \mathcal{P}_1)$.

Theorem 3 is in fact equivalent to the upper hemi-continuity of $\Gamma: P_j^* \to \{\mu \in \mathcal{P}(K): \mu = P_j^* \cdot \mu\}$, the correspondence of approximated invariant measures with $\mu \in \mathcal{P}(K)$. Thus, the results in theorem 3 can be applied to any sequence $\{\mu_j\}$ with $\mu_j \in IM(\varphi_j)$. Given the multiplicity of invariant measures for each φ_j, the upper hemi continuity of Γ is relevant for applications as we may assume that the desired properties hold for any μ_j with $j \to \infty$. For a discussion the reader is referred to corollary 2 in Santos and Peralta Alva (2005).

Technical appendix of section 4.2

Remark on the local convergence of the Law of Large Numbers

Santos and Peralta Alva (2013) requires that condition 4) holds for any selection of Φ_j and Φ. Kamihigashi and Stachurski (2015) requires that φ be continuous and Breiman's theorem in Stokey, Lucas and Prescott require a unique ergodic measure. In contrast, *theorem 3 requires only that condition 4 holds for some selection* $\{\varphi_j\}$ *and* φ*. Further, theorems 1 and 2 allow* φ *to be discontinuous and* $(\tilde{J}, P_\varphi)$ *to have multiple ergodic measures.* In this kind of setting, there are no results that guarantee the global almost sure convergence of simulations. Thus, a local theorem, like Birkhoff's, must be used.

Details of the Stochastic Process

To present the results for this section some additional definitions are required. Let $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ be a measurable space and $\left(\tilde{J}^t, \mathcal{B}_{\tilde{J}}^t\right) = (\tilde{J} \times \ldots \times \tilde{J}, \mathcal{B}_{\tilde{J}} \times \ldots \times \mathcal{B}_{\tilde{J}})$ the associated product space. Let $A = A_1 \times \ldots \times A_t$ be a measurable rectangle (see Stokey, Lucas and Prescott page 195 for a definition) in $\mathcal{B}_{\tilde{J}}^t$. Let $\varphi \sim \Phi$ and $z_0, \ldots, z_t \in \tilde{J}$. As long as t is finite, by virtue of the Caratheodony and Hahn theorems and theorem 7.13 in Stokey, Lucas and Prescott (1989), the measure $\mu^t(z_0, A)$, defined by $\mu^t(z_0, A) = \int \cdot_{A_1} \ldots \int \cdot_{A_t} P_\varphi(z_{t-1}, dz_t) \ldots P_\varphi(z_0, dz_1)$, can be uniquely extended to a probability measure in any set of $\mathcal{B}_{\tilde{J}}^t$, where $\int \cdot_{A_i}$ denotes integration w.r.t. $P_\varphi(z_{i-1}, dz_i)$.

Analogously, let $B = A_1 \times \ldots \times A_T \times \tilde{J} \times \ldots$be a finite measurable rectangle (see page 221 of Stokey, Lucas and Prescott for a definition) and $\mathcal{L}$ its power set.

Let $\mathcal{M}$ be the algebra generated by finite unions in $\mathcal{L}$ and $\mathcal{F} = \mathcal{B}_{\mathcal{M}}$ (i.e., $\mathcal{F}$ is the sigma field generated by $\mathcal{M}$). Then $\mu^{\infty}(z_0, B) = \int_{\cdot A_1} \dots \int_{\cdot A_T} P_{\varphi}(z_{T-1}, dz_T) \dots P_{\varphi}(z_0, dz_1)$ can be shown to be extended to $\mathcal{F}$ in 2 steps. First, using the Caratheodony and Hahn theorems it is possible to extend $\mu^{\infty}(z_0, B)$ to $\mathcal{M}$ and then to $\mathcal{F}$. Later, using standard arguments for processes with a finite dimension distribution (see Shiryaev 1996, Ch. 9), $\mu^{\infty}(z_0, B)$ can be shown to be countably additive.

Standard results (see for instance exercise 8.6 in Stokey, Lucas and Prescott) imply that $\left(\Omega, \mathcal{F}, \mu^{\infty}(z_0, .)\right)$ is a Markov process with stationary transitions P_{φ}. Let $\Omega = \tilde{J} \times \tilde{J} \times \dots$ with typical realization $\omega \in \Omega$. As Ω is the space of sequences, it is natural to define a $\mathcal{F}_t$-measurable random variable $z_t : \Omega \to \tilde{J}$, where $\omega(t) = z_t = z_t(\omega)$ denotes a typical realization and $\{\mathcal{F}_t\}$ is a sequence of nested sigma algebras on $\{\times_{i=1}^{t} \tilde{J}(i)\}$, where $\tilde{J}(i) = \tilde{J}$ for $i \geq 1$. The shift operator is denoted by $T : \Omega \to \Omega$. A set $A \in \mathcal{F}$ is called *T-invariant* if $TA = A$[19].

Let $\mu^{\infty}(z_0, B) \equiv \boldsymbol{P}_{\varphi, z_0}(B)$. Under the same assumptions, $\boldsymbol{P}_{\varphi_j, z_0}(B)$ can be analogously defined if $\tilde{J}$ is replaced by K, which was supposed to be compact in assumption 4-i). Further, $\boldsymbol{P}_{\varphi,\mu} \equiv \int_{A_0} \boldsymbol{P}_{\varphi, z_0} \mu(dz_0)$ can be used to define a stochastic process $\left(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu}\right)$ which allows to randomize z_0 as μ is a measure on $(\tilde{J}, \mathcal{B}_{\tilde{J}})$. $\left(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu}\right)$ is said to be *stationary* if $\boldsymbol{P}_{\varphi,\mu}[C(t,n)] = \boldsymbol{P}_{\varphi,\mu}[C(t',n)]$ for all $n \geq 0$ and $t \neq t'$ with $C(t,n) = \{\omega \in \Omega : [z_{t+1}(\omega), \dots, z_{t+n}(\omega)] \in C\}$. $\left(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu}\right)$ is said to be ergodic if $\boldsymbol{P}_{\varphi,\mu}(A) \in \{0,1\}$, where A is a T-invariant set.

[19] Exercise 6.2 in Varadhan shows that this definition can be used w.l.o.g.

Section IV: Proofs

Preliminary Remark on $\tilde{J}$

As theorem 1 will show that there exist $\mu \in \mathcal{P}_0$ with $\mu = P_\varphi^* \mu$ (i.e., an invariant measure exists and it is atomless), it is necessary for the state space of the process defined by $(\tilde{J}, P_\varphi)$ to be uncountable. This is because the candidate measure μ_N, with $\mu_{N_k} \rightarrow_{Weak*} \mu$, satisfies $Supp(\mu_K) \subseteq \tilde{J}$ as it is constructed applying iteratively P_φ^*.

Fortunately, the results used to guarantee the non-emptiness of C_j for $j \geq 1$ (i.e., the set which contains all initial states, $\tilde{z}_0$, of any j-period economy) which were discussed in sections 2.5.1 (fact 2) and 2.5.2 can be used to guarantee the desired result. In particular, Theorems 25.1 in Magill and Quinzii (1996) and theorem 4.1 together with section 5 in Mas-Colell and Zame (1996) for economies with finite and infinite number of shocks respectively can be used to show the existence of a sequential competitive equilibrium (see Definition 1) for a truncated economy $\mathcal{E} = \left[e, d, \left\{U^i\right\}_{i=1}^{I}, T\right]$, with $T < \infty$. The optimality conditions in Definition 1 for this economy are:

A1) $c^i = e^i(s) + \theta^i d(s) - \theta_+^i q$

A2) $\left[q\left(u_s^i(c^i)\right)' - \beta E_{p(s,.)}(m_+^i)\right]\left[\theta_+^i - \bar{B}\right] = \vec{0}$

Where short sale constraints $\bar{B}$ are assumed to hold (see sections 2.5.2 in the body of the paper and II in this appendix) and $\theta_+^i = 0$ if $\theta^i = \theta^i(\sigma_{T-1})$. In the sequential economic literature, if $\theta_+^i = \theta^i(\sigma_0)$, it is customary to assume that $\theta_-^i \equiv \theta^i = 0$ and $\sigma_0 \equiv s_0$ is supposed to be fixed. However, in the recursive literature, both θ_-^i and σ_0 are allowed varying as $\tilde{z}_0 = \left[s_0, \theta_-^i, \hat{z}_0\right]$, where $\hat{z}_0$ contains the rest of the state space.

Moreover, the existence of equilibria for $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^{I}, T\right]$ requires that $e^i(s_0) > 0$ (see assumption A.2 in Magill and Quinzii, page 858). Thus, provided the rest of the assumptions mentioned in sections 2.1, 2.5.1 and II hold, as noted by Duffie, et. al. (Lemma 3.4), θ_-^i and s_0 can be chosen arbitrarily as long as $e^i(s_0) + \theta_-^i d(s_0) > 0$, which can be considered the initial endowment of goods if the exogenous state is s_0. Formally, it suffices to assume that:

Definition A2: *The initial distribution of assets θ_- will be called admissible and denoted $\theta_- \in \Lambda$ if is feasible and satisfies $Min_{i \in I, s \in S} e^i(s) + \theta_-^i d(s) > 0$.*

Remark A1: $\tilde{J} = S \times \Lambda \times \hat{Z}$, where $\Lambda \times \hat{Z}$ is uncountable because and has no isolated points: i) Λ is uncountable and has no isolated points according to definition A2, ii) under the assumptions made in sections 2.1, 2.5.2 and II, $C_j \neq \emptyset$ independently of the cardinality of S (i.e. an equilibrium for $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^{I}, \theta_-\right]$ exists independently of the cardinality of S) for any $\theta_- \in \Lambda$ (i.e. for any admissible θ_-).

Remark 1 is frequent in applications: see for instance Duffie, et. al. (1994) section 3 and Kubler and Schmedders (2003) page 1777. Vector θ_- describes any predetermined level of asset holdings or the capital stock. Consequently, in numerical approximations θ_- is supposed to be contained in an uncountable subset of $\mathbb{R}$ and its properties (i.e., compactness) can be defined independently of those characterizing $\hat{Z}$ as (s, θ_-) are initial conditions of some sequential competitive equilibrium. Thus, Λ is compact if and only if it is closed. This last property is easily verifiable as can be seen in Kubler and Schmedders (2003) (see lemma 1, page 1776). As will be seen in the proof of lemma 3, the crucial property of Λ, besides its cardinality, is the lack of isolated points. This property follows w.l.o.g. from definition A2.

In all the proofs, except that it is mentioned explicitly, it will be assumed that the state space can be written as $\tilde{J} = S \times \Lambda \times \hat{Z}$ and that Λ is admissible.

Theorem 1

Preliminary Remark

As discussed in section 3.1, theorem 1 will fail if any selection $\varphi \sim \Phi$ has an uncountable discontinuity set. Fortunately, there are no examples in economics where such a function characterizes the (recursive) equilibrium set. In fact, the literature (see for instance Santos, 2002) has only found examples with jump discontinuities. As will be claimed in the preliminary remark of theorem 2, there are no available methods to numerically approximate a function with an uncountable discontinuity set. Consequently, if a model does not satisfy assumption 2 it can be said to be *non-computable.*

Theorem 3.5 in Molchanov and Zuyev (2011) only requires the discontinuity set to have zero measure under the limiting measure (i.e., $\mu_n \to_{Weak*} \mu$ and $\mu(\Delta\varphi) = 0$). Thus, it is only necessary, under assumption 2, for μ to be atomless. The arguments in sections 3.2 and 3.3 illustrate the usefulness of properties a) and b) to achieve this purpose. Proposition 1 in Ito (1964) holds under quite mild assumptions on the primitives and assures property a). The critical property is then b), which hold under rather different assumptions depending on the cardinality of S.

Theorem 3.5 in Molchanov and Zuyev restores the continuity of the adjoint operator by extending the set of adequate functions for the *weak** topology from continuous to Borel measurable if the limiting measure is atomless and assumption 2 holds. The example below illustrates the importance of the atomless assumption when dealing with function which is only measurable.

Atomic measures and tight spaces [20]: Let $P: S \times \mathcal{B}_S \to [0,1]$ be a transition function with $S = [0,1]$ and $P(s, \{s/2\}) = 1$. Let $\{\lambda_n\}$ be a sequence of Dirac measures with $\lambda_n = \delta_{(1/2)^n}$. Thus, $\lambda_n \to \delta_0$, where the convergence is in distribution. Define the bounded Borel measurable function $f(s) = \{1\ if\ s = 0\ ; 0\ otherwise\}$and $\delta_0 \equiv \lambda$. Then $\int f(s)\, \lambda_n(ds)$ =0 and $\int f(s)\, \lambda(ds) = 1$ which in turn implies that $\lambda_n \nrightarrow_{weak^*} \lambda$. The reason behind the lack of $weak^*$ convergence is the impossibility to reduce the measure of the discontinuous part of f.

Proof of theorem 1

Let Φ be an equilibrium correspondence according to definition 6 which satisfies assumption 1. By Lemma 2 $P_\Phi = \{P_\varphi: \varphi \sim \Phi\} \neq \emptyset$ and upper hemi continuous (see for instance proposition 2.2. in Blume, 1982). If P_Φ is convex valued, an ergodic invariant measure can be shown to exist using proposition 1.3 in Duffie, et. al. (1994) (see page 757).

If P_Φ is not convex valued, suppose that assumption 2 together with properties a) and b) in theorem 1 hold. Choose any $\lambda_0 \in \mathcal{P}(\tilde{J})$ and construct a non-oscillating sequence of measures $\{\mu_N\}$ with $\mu_N = h(\{\lambda_n\})$, where h averages the first N-1 elements of $\{\lambda_n\}$ and λ_n satisfies $\lambda_n = P_\varphi^* \lambda_{n-1}$. The dependence of $\{\mu_N\}$ on λ_0 can be omitted w.l.o.g. as the initial condition is arbitrary.

As $\mu_N \in \mathcal{P}(\tilde{J})$ for any N, Helly's theorem (see Stokey, Lucas and Prescott (1989) page 372 and 374) implies that $\{\mu_N\}$ has a weakly convergent subsequence. That is, $\{\mu_{N_k}\} \to_{weak*} \mu$.

For notational simplicity $P_\varphi^* \lambda$ and $\hat{P}_\varphi f$ will be replaced by $\pi \cdot \lambda$ and $\pi \cdot f$ as P_φ with $\varphi \sim \Phi$ will be held constant throughout the proof.

[20] This example borrows from Stokey, Lucas and Prescott (1989), page 336. Note that $\{\lambda_n\}$ satisfies $\lambda_n = P \cdot \lambda_{n-1}$. That is, it is possible to generate a sequence of non-atomic measures out of the action induced by P. I would like to thank Prof. R. Fraiman for pointing this out to me.

For any $f \in C(\tilde{J})$ note that:

$$\left|\int f(z)\mu(dz) - \int (\pi \cdot f)(z)\mu(dz)\right|$$
$$\leq \left|\int f(z)\mu(dz) - \int f(z)\mu_{N_k}(dz)\right|$$
$$+ \left|\int f(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu_{N_k}(dz)\right|$$
$$+ \left|\int (\pi \cdot f)(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu(dz)\right| \quad (A.3)$$

From the corollary of theorem 8.1 in Stokey, Lucas and Prescott (1989) (page 215), $(\pi \cdot f): Z \to \mathbb{R}$ is a bounded $\mathcal{B}_{[\tilde{J}]}$-measurable function. Further, from property a) and b), μ is atomless. Under assumption 2, $\mu(\Delta\varphi) = 0$. Then, from theorem 3.5 in Molchanov and Zuyev (2011, fact f), the third term in A3 can be made arbitrarily small. Further, noting that $\{\mu_{N_k}\} \rightarrow_{weak*} \mu$ and $f \in C(\tilde{J})$, the first and the third term in A.3 can be made arbitrarily small.

Following the same reasoning as in Stokey, Lucas and Prescott (1989) page 377, the second term satisfies:

$$\left|\int f(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu_{N_k}(dz)\right| \leq 2\|f\|/N \quad (A.4)$$

Where $\|.\|$ is the sup-norm. Thus, for an N arbitrarily large, $\int f(z)\mu(dz) = \int (\pi \cdot f)(z)\mu(dz) = \int f(z)(\pi \cdot \mu)(dz)$, where the last equality follows from theorem 8.3 in Stokey, Lucas and Prescott (1989) (see page 216). Thus, $\int f(z)\mu(dz) = \int f(z)(\pi \cdot \mu)(dz)$. As f was arbitrary, by virtue of corollary 2 of theorem 12.6 in Stokey, Lucas and Prescott (1989) (page 364) $\mu = \pi \cdot \mu$, QED.

■

Lemma 3

Preliminary remark

The proof of this lemma requires π to be θ-nonsingular. A transition function is said to be θ-nonsingular if for any measurable set B, $\theta(B) = 0$ implies $\pi(z, B) = 0$ θ-a.e. As θ is atomless this is equivalent to say that the set D, defined below, is a finite set.

$$D = \{z \in \tilde{J}: \pi(z, B) > 0 \; if \; \theta(B) = 0\} \quad (A.5)$$

Additionally, B was restricted to be a point. For those transition functions defined by lemma 2, Ito (1964) show that any *non-constant possibly discontinuous many-to-one function* $\varphi \sim \Phi$ will generate a θ-nonsingular transition function. This can be seen by written π_φ in lemma 2 as

$$\pi_\vartheta(z, B) = p\{s|s' \in S: \varphi(z, s') = a\} = p\{s|s' \in S: \{s_i'\} \cap \tilde{\varphi}^{-1}(z,.)(a_{\hat{z}})\} \quad (A.6)$$

Where $z = [s, \hat{z}]$, $p(s|.)$ is the s^{th} row of the transition matrix which defines the evolution of the exogenous process $\{s_t\}$, $B = \{s_i'\} \times B_{\hat{z}}$ was restricted to a point $a = \{s_i'\} \times a_{\hat{z}}$, $\varphi(z, s') = [s', \tilde{\varphi}(z, s')]$ is a vector valued function and $\tilde{\varphi}^{-1}(z,.)(B_{\hat{z}})$ is the z-section of the pre-image of $\tilde{\varphi}$ on $B_{\hat{z}}$.

From A.5 and A.6, under assumption 2, $\#D < \infty$ provided that $\tilde{\varphi}(., s')$ is non-constant in z for all $s' \in S$. In section II.1 and V.1 of this appendix the implicit function theorem is used to show that the model defined in section 2.1 generates θ-nonsingular transition functions.

Proof of lemma 3

Let $(\tilde{J}, \mathcal{B}_{\tilde{J}}, m)$ be a measure space. By assumption 1, $\tilde{J}$ is compact and by remark A1 this set could be written as $\tilde{J} = S \times \Lambda \times \hat{Z}$, where Λ contain all admissible states and $\Lambda \times \hat{Z}$ is uncountable and has no isolated points. Further, note that any measure in $(\Lambda, \mathcal{B}_\Lambda)$, denoted m_Λ, is a Radon measure as Λ is a Hausdorff metric space and m_Λ is: i) defined over a Borel sigma-algebra ($\mathcal{B}_\Lambda$), ii) regular as it is a measure on a Hausdorff (compact) metric space (Λ), iii) $\mathcal{B}_\Lambda$-finite as it is a probability measure. Thus, as Λ has no isolated points, $(\Lambda, \mathcal{B}_\Lambda)$ has an atomless measure m_Λ^A (see Bogachev 2007, page 136) which in turn implies by remark A1 that there is a measure m^A in $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ that is also atomless. The first part of the lemma is completed by setting $m^A \equiv \theta$.

Let $\mathcal{P}_0(\tilde{J}) \subset \mathcal{P}(\tilde{J})$ be the set of atomless measures in $\mathcal{P}(\tilde{J})$ generated by π, starting from θ. It follows from proposition 1 in Ito (1964) that π maps $\mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ as π is θ-nonsingular by condition 1. Finally, condition 2 is just the definition of a *weak*-closed* set applied to $\mathcal{P}_0(\tilde{J})$.

■

Example of an atomless measure $\theta \in \mathcal{P}(\tilde{J})$. The reference measure θ could be a mixed joint density: $\theta(s \times A) = P(s = \{s\}, \hat{z} \in A) = \int_A p_{S,\hat{Z}}(s, \hat{z}) d\hat{z}$ where $p_{S,\hat{Z}}(s, \hat{z}) = \theta(s \times \{\hat{z}\}) = 0$ is a density function on $\hat{Z}$ which may vary with any $s \in S$. From fact 14 page 45 in Hildenbrand and Grandmont (1974), θ is atomless.

Lemma 4

Preliminary Remark

The implication of condition 4) requires showing the *weak** closedness of $IM(\varphi, \mathcal{P}_1)$. The proof below shows that $IM(\varphi, \mathcal{P}_1)$ is *weak** sequentially compact: that every *bounded* sequence in $IM(\varphi, \mathcal{P}_1)$ has a *weak** convergent subsequence. As $\mathcal{P}_1$ can be endowed with the Prohorov metric (see Hildenbrand and Grandmont 1974, page 49), sequential compactness implies that $IM(\varphi, \mathcal{P}_1)$ is not only closed but also compact.

Proof of lemma 4

For the existence of an atomless measure on $\tilde{J} = S \times \Lambda \times \hat{Z}$ with S uncountable and compact, let θ be the uniform measure on $\tilde{J}$.

For property a), note that condition 3) implies that P_φ is θ-nonsingular. Thus, proposition 1 in Ito (1964) applies just as in the proof of lemma 3.

To prove property b), note that any point $\{a\} \in \tilde{J}$ has zero Lebesgue measure. Thus, under condition 3):

$$\mu_n(\{a\}) = \int P_\varphi(z, \{a\})\, \mu_{n-1}(dz) = 0$$

Where the second equality follows from condition 3) and implies that the desired result follows automatically.

Property c) will be proved in 3 parts: i) $IM(\varphi, \mathcal{P}_1) \neq \emptyset$. As $\tilde{J}$ is compact, Helly's theorem implies the existence of a weak* converging subsequence in $IM(\varphi, \mathcal{P}_1)$ denoted w.l.o.g. $\mu_n \rightarrow_{weak*} \mu$. It will be shown that: ii) μ is absolutely continuous w.r.t θ, iii) $\mu \in IM(\varphi, \mathcal{P}_1)$.

In what follows it will be assumed w.l.o.g. that $\theta(dz) = dz$. This is done for expositional purposes only.

i) Standard results (See Billingsley 1968, page 422) imply that condition 4) is equivalent to the following statement: for any measurable set B, $\theta(B) = 0$ implies $SUP_{z \in \tilde{J}}[\pi_\vartheta(z, B)] = 0$. Thus, π_ϑ is θ-nonsingular. By proposition 1 in Ito (1964), $\pi_\vartheta: \mathcal{P}_1 \rightarrow \mathcal{P}_1$. Also, under condition 4), an argument identical to the one used to prove property b) implies that $\mathcal{P}_1$ and the adjoint operator generates a *weak** closed set. Under assumption 2, theorem 1 implies that $IM(\varphi, \mathcal{P}_1) \neq \emptyset$.

ii) By the characterization of absolutely continuity in Billingsley (1968, page 422), it suffices to show that for any $\varepsilon > 0$, $\exists\delta > 0$ such that $\theta(B) < \delta$ implies $\mu(B) < \varepsilon$. Condition 4) implies that $\pi_\vartheta(z,.)$ is absolutely continuous w.r.t. θ for any $z \in \tilde{J}$. That is, $\pi_\vartheta(z,dz')$=$\bar{\pi}_\vartheta(z,z')dz'$ where $\bar{\pi}_\vartheta(z,.)$ is the density associated with $\pi_\vartheta(z,dz')$. Take any sequence $\{\hat{\mu}_n\} \in IM(\varphi,\mathcal{P}_1)$. Note that $\{\pi_\vartheta\hat{\mu}_n\}$ is a family of measures that satisfies the hypothesis of Helly's theorem and $\{\pi_\vartheta\hat{\mu}_n\} \in IM(\varphi,\mathcal{P}_1)$.

Let $\pi_\vartheta\hat{\mu}_n \equiv \mu_n$ and note that $\{\mu_n\} \in IM(\varphi,\mathcal{P}_1)$ and has a *weak** limit denoted (passing to a subsequence if necessary) μ.

Note that $\mu_n(B) = \int_B h_n(z')\theta(dz')$ where $h_n(z') = \int \bar{\pi}_\vartheta(z,z')\,\mu_n(dz)$. But now note that $\mu_n(B)$ could be written as:

$$\mu_n(B) = \int_B h_n(z')\theta(dz') = \int \left[\int_B \bar{\pi}_\vartheta(z,z')dz'\right] \mu_n(dz)$$

Condition 4) implies that $\left[\int_B \bar{\pi}_\vartheta(z,z')dz'\right] < \varepsilon$ uniformly in z. Thus $\mu_n(B) < \varepsilon$. The arguments in the first part of lemma 3 imply that $\{\mu_n\}$ and μ are regular measures. Thus, B can be assumed to be open w.l.o.g. Now, the definition of *weak** convergence implies (see theorem 12.3-c in Stokey, Lucas and Prescott, page 358) $\mu(B) \le liminf_n \mu_n(B)$. To complete the proof, by the preliminary remark of this lemma, it suffices to note that $liminf_n \mu_n(B) < \varepsilon$.

iii) It remains to show that $\mu \in IM(\varphi,\mathcal{P}_1)$.

Take $\mu_n \to_{weak*} \mu$. Note that for any $f \in C(\tilde{J})$:

$$\lim_n \int f(z)\mu_n(dz) = \int f(z)\mu(dz)$$

$$= \lim_n \int f(z)[\pi\mu_n](dz) = \lim_n \int [\pi f](z)\mu_n(dz) = \int [\pi f](z)\mu(dz)$$

$$= \int f(z)[\pi\mu](dz) \quad (A.7)$$

Where the first equality in A.7 follows from the definition of *weak** convergence of $\mu_n \to_{weak*} \mu$, the second from $\{\mu_n\} \in IM(\varphi,\mathcal{P}_1)$, the third from theorem 8.3 in Stokey, Lucas and Prescott, the forth from theorem 3.5 in Molchanov and Zayev as μ is absolutely continuous w.r.t. θ (and thus atomless) and the last equality from theorem 8.3 in Stokey, Lucas and Prescott again. Note that A.7 implies $\int f(z)\mu(dz) = \int f(z)[\pi\mu](dz)$. As $f \in C(\tilde{J})$ is arbitrary, the proof is complete.

■

Proof of Proposition 1

Under assumption 1, lemma 2 implies that π is well defined (i.e. is a Markov operator). Under assumptions 3-i) and 3-ii) the result follows from equation A.6) by noting that $\{s_i'\} \cap \tilde{\varphi}^{-1}(z,.)(a_{\hat{z}})$ is either a point in S or $\emptyset$ for any $z \in \tilde{J}$.

■

Proposition 2

Preliminary remark

Arbitrarily selecting $\in \tilde{J}$, it will be shown that $\forall\varepsilon(z) > 0, \exists\delta(z) > 0$ such that $\theta(B) < \delta(z)$ implies $\pi(z,B) < \varepsilon(z)$. As $\tilde{J}$ is compact and $\varepsilon(z), \delta(z)$ are finite (real) numbers, it suffices to take $max_{z\in\tilde{J}}\, \varepsilon(z) = \varepsilon$ and $max_{z\in\tilde{J}}\, \delta(z) = \delta$.

For the first part of the proof the following fact will be useful: let θ be the Lebesgue measure and $R \subseteq \tilde{J} \subset \mathbb{R}^K$ a rectangle and μ^V its volume. That is, $R = [a_1, b_1] \times \ldots \times [a_K, b_K]$ and $\mu^V(R) = [b_1 - a_1] \ldots [b_K - a_K]$. Then, $\theta(B) = 0$ if $\forall\gamma > 0$, $\exists\{R_i\}_{i=1}^{\infty}$ with $B \subseteq \bigcup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$. The proof of the first part the proposition will be completed if it can be shown that for each $\varepsilon(z) > 0$, there exist an $\gamma > 0$ such that $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$ implies $\sum_{i=1}^{\infty} \pi(z, R_i) \leq \varepsilon(z)$ because $\theta(B) = 0$ as long as $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$.

Proof of proposition 2

Note that any positive $\pi_\varphi(z,.)$-measure rectangle, R_i, could be written as

$$R_i = \left[\varphi_1\left(z, s'_{1,i} - 2^{-1}h_{1,i}\right), \varphi_1\left(z, s'_{1,i} + 2^{-1}h_{1,i}\right)\right] \times \dots$$
$$\times \left[\varphi_K\left(z, s'_{K,i} - 2^{-1}h_{K,i}\right), \varphi_K\left(z, s'_{K,i} + 2^{-1}h_{K,i}\right)\right]$$

where the first coordinate is just $\left[s'_{1,i} - 2^{-1}h_{1,i}, s'_{1,i} + 2^{-1}h_{1,i}\right]$, φ_k and $s'_{k,i}$ denote any coordinate of φ for $1 \leq k \leq K$ and the elements of S that generates coordinate k of rectangle i.

Note assumption 3-iii) implies that $\varphi_k(z,.)$ is allowed to oscillate continuously, not necessarily forming a straight line, between $\varphi_k(z,x)$ and $\varphi_k(z,y)$ where $x = s'_{k,i} - 2^{-1}h_{k,i}$ and $y = s'_{k,i} + 2^{-1}h_{k,i}$. Thus, by theorem 2.27 in Aliprantis and Border (2006), $h_{k,i}$ is the length of the interval in the pre-image of $\varphi_k(z,.)$, where $\varphi_k(z,x)$ and $\varphi_k(z,y)$ are exactly the endpoints of the k^{th} coordinate of rectangle R_i.

Now equation A.6) implies that:

$$\pi(z, R_i) \leq p\left(s, \bigcap_{k=1}^{K} \left[s'_{k,i} - 2^{-1}h_{k,i}, s'_{k,i} + 2^{-1}h_{k,i}\right]\right) = p\left(s, \bigcap_{k=1}^{K} \left[0, h_{k,i}\right]\right)$$

Where the inequality follows from the preceding discussion and the equality from assumption 3-iv) after normalizing $p(s,.)$ to be in the unit interval.

Now note that assumption 1 implies that $\mu^V(R_i)$ is finite as the range of any $\varphi \sim \Phi$ is bounded, and $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$ implies $lim_{i \to \infty} \mu^V(R_i) = 0$. Thus,

$$\pi_\varphi(z, R_i) \leq \varepsilon(z)_i 2^{-i} \text{ A.8)}$$

Where $\varepsilon(z)_i = min_k h_{k,i}$.

Also, from $lim_{i\to\infty}\mu^V(R_i) = 0$, equation A.8) implies that $lim_{i\to\infty}(\varepsilon(z)_i)$ is finite. Thus, $SUP_i\varepsilon(z)_i = max_i\varepsilon(z)_i = \varepsilon(z)$ and $\sum_{i=1}^{\infty}\pi(z,R_i) \leq \varepsilon(z)$, as $\sum_{i=1}^{\infty}2^i = 1$.

Now to prove the dependence of γ on $\varepsilon(z)$, let $R_{i,k}$ be the k^{th} coordinate of rectangle R_i. Note that assumption 3-iii) implies, by theorem 2.34 in Aliprantis and Border, that for all $i, \exists k$ with $R_{i,k} = [\varphi_1(z,x), \varphi_1(z,y)]$ and $[x,y]$ has length smaller or equal to $\varepsilon(z)$. Consequently, $\varepsilon(z)$ could be made arbitrarily small as desired and there will always be an associated γ such that A.8) holds. As z is arbitrary, the proof is complete.

■

Proof of remark 2: the result follows from replacing $p(s, \bigcap_{k=1}^{K}[0, h_{k,i}2^{-i}])$ by $p(s, \bigcap_{k=1}^{K}[LB(s), h_{k,i}2^{-i}])$ in equation A.8) and noting that $\varepsilon(z)_i = min_k \frac{h_{k,i-LB(s)}}{UB(s)-LB(s)}$, where z is a vector of the form $z = [s, \hat{z}]$, is a finite number for all $z \in \tilde{J}$.

Theorem 3

Preliminary Remark

As in the case of theorem 1, theorem 3 must be applied to economies that has at least 1 selection φ~Φ with at most a zero Lebesgue measure discontinuity set. This restriction must be extended to all approximated economies which are characterized by φ_j~Φ_j. This is because, as discussed in section 3.1, $IM\{\varphi_j, \mathcal{P}_1\}$ may be empty if the discontinuity set is allowed to have positive Lebesgue measure.

The discussion in section 2.4 suggests that the cardinality of the discontinuity set is associated with the number of possible equilibria. Thus, even though it is theoretically possible to have a discontinuity set with positive Lebesgue measure, *the endogenous laws of motions in this economy may not be computable even using state of the art procedures*.

The algorithm in Feng, et. al. (2014) computes an outer approximation of the equilibrium correspondence (i.e., $Gr(\Phi_j) \supseteq Gr(\Phi)$). Thus, assumption 4-iii) may not hold for this procedure. Further, in this procedure it is not clear how to impose assumption 3-iii) because the interpolation method used to convexify the computed equilibrium correspondence is not specified in the paper (see page 11 for an outline of the algorithm and pages 39 to 41 for details). The procedure in Kubler and Schmedders (2003) circumvent some of these problems as it provides a convenient spline-based interpolation method. Unfortunately, the sequence of approximating functions is assumed to be continuous and to converge in the sup-norm on K. Both facts taken together imply that the limiting function is continuous on K (see page 1782), which may be inadequate in the context of this paper.

There are spline-based procedures which allows computing functions with an uncountable discontinuity set (see for instance Silanes, et. al. 2001). These procedures converge uniformly on $(K \times S)\backslash\Delta\varphi$. Unfortunately, the arguments in the proof of theorem 3 will show that this type of convergence is inadequate under assumption 4-ii) if $\Delta\varphi$ has positive Lebesgue measure.

It is worth noticing that in an algorithm that approximates $\tilde{J}$ using a sequence of correspondences or sets, theorem 3.5 in Santos and Peralta Alva (2013) can be used to prove the desired upper hemi-continuity and compact valuedness of Φ_j (assumption 1 applied to theorem 3). This is the case of the recursive equilibrium algorithm in Feng, et. al. However, as mentioned before, this procedure generates a sequence of correspondence Φ_j with $Gr(\Phi_j) \supseteq Gr(\Phi)$, which may be inadequate under assumption 4-iii). Finally, it is possible to construct φ_j using a policy function $\varrho_j: S \times Z_1 \to \tilde{Z}$ with $Z = S \times Z_1 \times \tilde{Z}$ as in Kubler and Schmedders (2003). This procedure lowers the dimension of the state space and thus the computational burden, measure in CPU time, of the algorithm. The authors provided a detailed spline procedure, but they did not take care of $\Delta\varphi$. It is a matter of future research to establish if the spline procedure in Silanes, et. al., which addresses $\Delta\varphi$ appropriately, fits into the framework of theorem 3. Assumption 4-iii) should be carefully enforced as it involves all zero Lebesgue measure sets which, because of assumption 2'), do not belong to $(K \times S)\backslash\Delta\varphi$.

The other known recursive algorithms (see for instance Raad, 2013) may not suitable for simulations as it is not clear how to fit those procedures into the theoretical framework outlined in this paper.

Consequently, if all stationary laws of motion (i.e., all $\varphi \sim \Phi$) have a positive Lebesgue measure set of discontinuities, this economy may not be accurately computable and thus is beyond the scope of this paper.

The proof of this theorem will proceed in 3 steps: first, it will be shown that there is a sequence of absolutely continuous measures $\{\mu_j\}$, with $\mu_j = \pi_{\varphi_j}\mu_j$, which has a *weak** limit μ that is also absolutely continuous. Second, using the first result, it will be shown that the evaluation map, $Ev(\varphi, \mu) \equiv \pi_\varphi \mu$, is jointly continuous when φ is endowed with the sup-norm topology and μ with the weak topology. Third, using the second result, we show that $\mu = \pi_\varphi \mu$.

Proof of theorem 3

i) Assumptions 1, 2'), 3-iii) and 3-iv) applied to $\{\varphi_j\}$ implies, by theorem 2, that $IM\{\varphi_j, \mathcal{P}_1\} \neq \emptyset$ for all j. Assumption 4-i) implies that the sequence

$\{\mu_j\}$, with $\mu_j = \pi_{\varphi_j}\mu_j$, satisfies the hypothesis of Helly's theorem as $\mathcal{P}_1 \subset \mathcal{P}(K)$. Thus, $\{\mu_j\}$ has a subsequence weakly converging to μ. As assumption 3-iii) and 3-iv) hold for φ, proposition 2 implies $\pi_\varphi(z, A) < \varepsilon$ for any open set A with $\theta(A) < \delta$. Assumption 4-iii) implies that $lim_{j\to\infty}\pi_{\varphi_j}(z, A) \le \pi_\varphi(z, A)$, which in turn implies that $lim_{j\to\infty}\pi_{\varphi_j}\mu_j(A) < \varepsilon$. The same arguments used in lemma 4-ii) implies that μ is absolutely continuous w.r.t. θ as desired.

ii) Let $\mu_j \to_{Weak*} \mu$. It has to be shown that $Ev(\varphi_j, \mu_j) \to_{Weak*} Ev(\varphi, \mu)$. The arguments in Blume (1982, page 63) implies that it suffice to take an arbitrary test function in the unit ball generated by the sup-norm on $C(\tilde{J})$. Thus, the proof will be completed if it can be shown that:

$$\left|\int f(z)(\pi_{\varphi_j}\mu_j)(dz) - \int f(z)(\pi_\varphi \mu)(dz)\right| \to 0 \text{ as } j \to \infty \quad \text{A.9)}$$

Using theorem 8.3 in Stokey, Lucas and Prescott, A.9) could be written as:

$$\left|\iint (\pi_{\varphi_j} f)(z)\mu_j(dz) - \int (\pi_\varphi f)(z)\mu(dz)\right|$$
$$= \left|\iint \left[\iint f\big(\varphi_j(z, s')U(ds')\right]\mu_j(dz) - \int \left[\iint f(\varphi(z, s')U(ds')\right]\mu(dz)\right|$$

Adding and subtracting $\int[\int f(\varphi(z, s')U(ds')]\mu_j(dz)$ and using the triangle inequality the above expression could be written as

$$\left|\iint \left[\iint f\big(\varphi_j(z, s')U(ds')\right]\mu_j(dz) - \int \left[\iint f(\varphi(z, s')U(ds')\right]\mu(dz)\right|$$
$$\le \left|\iint \left[\iint f\big(\varphi_j(z, s')\big)U(ds') - \int f(\varphi(z, s')U(ds')\right]\mu_j(dz)\right|$$
$$+ \left|\iint \left[\iint f(\varphi(z, s')U(ds')\right]\mu_j(dz) - \int \left[\iint f(\varphi(z, s')U(ds')\right]\mu(dz)\right|$$

Because of assumption 2') is supposed to hold for $\{\varphi_j\}$ and φ, the first term is bounded above by $SUP_{(K\times S)\setminus\Delta\varphi}\left\|\varphi_j(z,s')-\varphi(z,s')\right\|_\infty$, which converges to zero by assumption 4-ii). The arguments in the proof of theorem 1 implies that the second term also converges to zero as μ is absolutely continuous w.r.t. θ and assumption 2') holds on φ. These 2 facts taking together implies $Ev(\varphi_j,\mu_j)\rightarrow_{Weak*} Ev(\varphi,\mu)$ as f is arbitrary.

iii) Equation A.9) and $\mu_j=\pi_{\varphi_j}\mu_j$ for any j implies

$$\left|\int f(z)\mu_j(dz)-\int f(z)(\pi_\varphi\mu)(dz)\right|\rightarrow 0 \quad \text{A.10)}$$

Also $\mu_j\rightarrow\mu$ implies

$$\left|\int f(z)\mu_j(dz)-\int f(z)\mu(dz)\right|\rightarrow 0 \quad \text{A.11)}$$

Now, taking $\left|\int f(z)\mu(dz)-\int f(z)(\pi_\varphi\mu)(dz)\right|$ and adding and subtracting $\int f(z)\mu_j(dz)$, the triangle inequality implies

$$\begin{aligned}\left|\int f(z)\mu(dz)-\int f(z)(\pi_\varphi\mu)(dz)\right| &\\ &\leq\left|\int f(z)\mu_j(dz)-\int f(z)\mu(dz)\right|\\ &+\left|\int f(z)\mu_j(dz)-\int f(z)(\pi_\varphi\mu)(dz)\right|\end{aligned}$$

Finally, equation A.10) and A.11) implies

$\left|\int f(z)\mu(dz)-\int f(z)(\pi_\varphi\mu)(dz)\right|\rightarrow 0$ which proves the last part of the theorem.

■

Proof of Theorem 4 (LLN)

Preliminary Remark on the equilibrium correspondence Φ (definition 5)

In section 2.5.2, and in the technical appendix associated with it in section III of this appendix, $\Phi\colon \tilde{J} \times S \rightrightarrows \tilde{J}$ was defined as containing any $\tilde{z}_+ = [\tilde{s}_+, \tilde{\theta}_+, \tilde{q}_+, \tilde{m}_+]$, $\tilde{z} = [\tilde{s}, \tilde{\theta}, \tilde{q}, \tilde{m}] \in \tilde{J}$ simultaneously satisfying equations A.1) and A.2); implying $\tilde{z}_+ \in \Phi(\tilde{z}, \tilde{s}_+)$ with $m(s_+) \sim V^*(\theta_+, q_+, s_+)(s_+)$ and $\tilde{m}_+ \sim V^*(\theta_+, q_+, s_+)(\tilde{s}_+)$. This remark explores this claim in detail as it is essential to understand the meaning of $\{\tilde{z}_t\}$ as a realization $\omega \in \Omega$ of the process $(\Omega, \mathcal{B}_\Omega, P_\mu)$ defined in section 4.2.

For simplicity take a 5-period economy with only 2 exogenous shocks $S = \{s_1, s_2\}$ as will suffice to illustrate the iterative procedure that generates $\{\tilde{z}_t\}$. The figure below illustrates a sequence of $\{c_j\}$, as defined in the technical appendix of section 2.5.2 in section III, obtained from equations A.1) and A.2):

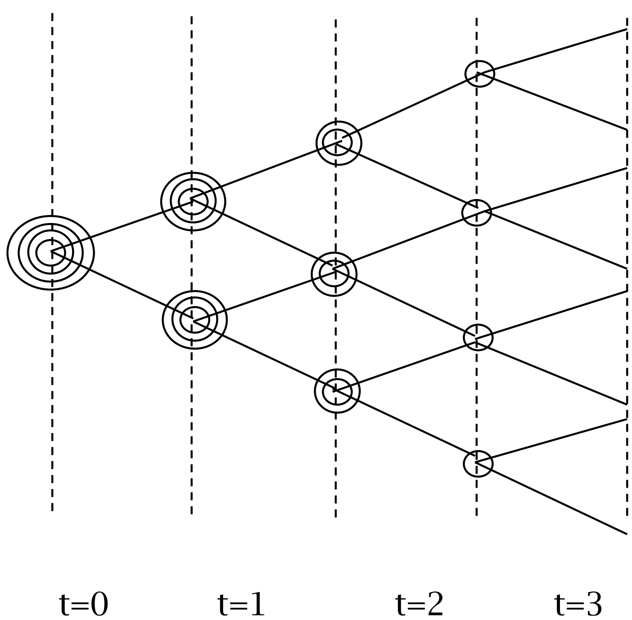

Figure A.1

The nodes with at least 1 circle belong to C_1 (i.e., the set that contains the initial conditions of all 2 period economies in figure A.1), the nodes with at least 2 circles belong to C_2 and so on. Remarkably, note that the only node at t=0 has 4 circles. Thus, *any element* of C_4 has the initial conditions *not only of* *a 5-period economy but also some of all possible initial conditions of any other economy depicted by the figure.* Further, $\tilde{J} = \bigcap_{j=1}^{4} C_j$.

Take any pair of elements in $\tilde{J}$, $[\tilde{z}^i, \tilde{z}^l]$ and let $c_j^i \in C_j$. Note that $[\tilde{z}^i, \tilde{z}^l] = [c_3^i, c_4^l] = [\tilde{z}_+, \tilde{z}]$ where the first equality follows from the definition of $\tilde{J}$ and the second from the definition of C_3 and C_4. Now, w.l.o.g., let $\tilde{z}_+ = [s_1, \tilde{\theta}_+, \tilde{q}_+, \tilde{m}_+]$. From definition 5, $\tilde{z}_+ \in \Phi(\tilde{z}, s_1)$ if $[\tilde{z}_+, \tilde{z}]$ satisfies equations A.1) and A.2). But this fact follows from theorem 1.2 in Duffie, et. al. as, following the discussion in section 2.5.2, the recursive equilibria in Feng, et. al. are a subset of all possible THME (see definition 4) implying $G(\tilde{z}) \cap \mathcal{P}(\tilde{J}) \neq \emptyset$, $G(\tilde{z}) = \{P_\varphi(\tilde{z}, .) : \varphi \sim \Phi\}$ and $\tilde{z} \in \tilde{J}$ as it can be seen from fact 1) in section 2.5.1. Note that theorem 1.2 in Duffie, et. al. requires G to be closed graph. This property follows from standard results in Blume (1982) under assumption 1.

To iterate the process forward using Φ in definition 5, take $\{\tilde{s}, \tilde{q}, \tilde{\theta}, \tilde{m}\} \in \tilde{J}$ and $\tilde{s}_+$. Given $\{\tilde{s}, \tilde{q}, \tilde{\theta}, \tilde{m}\}$ use $m^{i,j} \equiv d^j(s)\left(u_s^i(c^i)\right)'$ and equation A.1 to compute c and $\tilde{\theta}_+$. Take a sequence $\{m_+(s_+)\}_{s_+}$ with $m_+(s_+) \sim V^*(\theta_+, q_+, s_+)(s_+)$ and $\tilde{\theta}_+ = \theta_+(s_+)$. If c and $\{m_+(s_+)\}_{s_+}$ satisfy equation A.2, $\{\tilde{s}_+, q_+(\tilde{s}_+), \tilde{\theta}_+, m_+(\tilde{\theta}_+, \tilde{s}_+)\}$ is the next state.

The existence of $\{m_+(s_+)\}_{s_+}$ satisfying these properties is guaranteed by proposition 1.3 in Duffie, et. al. The fact that m_+ is a function of s_+ follows from the definition of Spotless THME (see definition A.1) applied to this type of economies as the vector $[\theta_+, q_+]$ is allowed to depend measurably on s_+ (see Duffie, et. al. page 767). Note that in this case, $\theta_+(s_+) = \tilde{\theta}_+$ once $\tilde{s}, \tilde{q}, \tilde{\theta}, \tilde{m}$ has been fixed. Thus θ_+ is continuous on s_+, as required by assumption 3-iii).

Proof of theorem 4

The assumptions of theorem 3 implies that $\mathcal{E}\left(IM(\varphi_j, \mathcal{P}_1)\right) \neq \emptyset$ for any j sufficiently large. Let z_0^j be an initial condition which satisfies assumption 5. Then, fact 4.2-iv) implies that $\boldsymbol{P}_{\varphi_j, z_0^j}$ -almost surely:

$$lim_{N\to\infty}[\sum_{t=1}^{N} f(z_i^j(z_0^j, \omega, \varphi_j))]N^{-1} = \int f(z)\mu_j(dz) \text{ with } \mu_j \in IM(\varphi_j, \mathcal{P}_1).$$

By the assumptions in theorem 3, $\mu_j \to_{weak*} \mu$ and $\mu \in IM(\varphi, \mathcal{P}_1)$ or equivalently:

$$\left|\int f(z)\mu_j(dz) - \int f(z)\mu(dz)\right| = 0$$

for j sufficiently large. Then, $\left|[\sum_{t=1}^{N} f(z_i^j(z_0^j, \omega, \varphi_j))]N^{-1} - \int f(z)\mu(dz)\right| = 0$ for N, j sufficiently large as desired.

■

Section V: Proofs for Applications (Section II of the Appendix)

1) Implicit function theorem for finite and uncountable shocks

This section proves that under assumptions 6.1-i) and 6.1-vi) the implicit function theorem can be applied to the system of equations that is equivalent to the sequential competitive equilibrium in definition 1. The result can be applied a.e. and uniformly in any continuity point of the state space.

The results in Magill and Quinzii (1994) and Kubler and Schmedders (2002) imply that under assumptions 6.1-i) to 6.1-v) the following system of equations defines a sequence of consumption bundles $\{\{c^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$, portfolios $\{\{\theta^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$ and prices $\{q(\sigma_t)\}_{\sigma_t\in\mathfrak{T}}$ which satisfy the feasibility and optimality requirements in definition 1:

A.12) $\sum_{i=1}^{I}\theta_+^i = \vec{0}$ with $\vec{0}\in\mathbb{R}^J$

A.13) $q_j u_s^i(e^i(s)+\theta^i d(s)-\theta_+^i q)' - \beta\sum_{s_+\in S} d_j(s_+)p(s,s_+)u_s^i(e^i(s_+)+\theta_+^i d(s_+)-\theta_{++}^i q)' = 0,\ j\in J, i\in I$

Let $z=[s,\theta,q]$ with $\sum_{i=1}^{I}\theta^i=\vec{0}$ and $m^i = d(s)u_s^i(e^i(s)+\theta^i d(s)-\theta_+^i q)'$. Also let $F(z,z_+)=\vec{0}$ be the system of equations defined by A.12) and A.13), where $\vec{0}\in\mathbb{R}^{J+J\times I}$.

The discussion in section 2.5.2 and II.1 of this appendix imply that under assumptions 6.1-i) to 6.1-v) $[z_+,m_+]\in\tilde{J}$ if $[z,m]\in\tilde{J}$, where $\tilde{J}$ is the expanded equilibrium state space in definition 5. Moreover, the same results imply that each θ_{++} implicit in m_+ define a different selection $m_+\sim V^*(z_+)$, where $\tilde{J}=Gr(V^*)$. Thus, as A.12) and A.13) can be used to define a particular selection $\varphi\sim\Phi$, θ_{++} can be assumed to be constant throughout the analysis.

Further, because: a) $s,s_+\in S$ and $\#S<\infty$ and condition 1 is required to hold a.e. in an atomless measure μ and b) for the case of uncountable shocks, condition 3 must hold uniformly in all continuity points, the discussion in the preliminary remark of lemma 3 implies that it suffice to show that $D_{z_+}F(z,z_+)$

has full rank: a) μ -a.e. in z as this implies that $\mu(D) = 0$, where $D = \{[z,m] \in \tilde{J}: P_{\varphi}([z,m],\{a\}) > 0 \; if \; \mu(\{a\}) = 0\}$ was defined in equation A.6), b) all continuity points. Moreover, assumption 6.1-vi) guarantees that $D_{z_+}F(z,z_+)$ is well defined μ -a.e. in z as the discontinuity set of φ is allowed to have up to finite cardinality and F is defined for interior solutions only.

To complete the proof, it suffices to write $D_{z_+}F(z,z_+)$ explicitly to note that this matrix has full rank under assumptions 6.1-i) and 6.1-v) provided that there is more than 1 asset[21].

2) Uncountable Shocks

Preliminary remark of Lemma 5

As discussed in section 3.4), the existence of an ergodic invariant measure can be shown under a slightly weaker assumption than 3-iv). The results hold under assumption 3.iv') which allows $p(s,.)$, the distribution of exogenous shocks, to depend on s. Assume further that,

Assumption A.1): $p(s,.)$ *satisfies assumption 3-iv') and it has the Feller property.*

The proof below assumes that $p(s,.)$ satisfies assumption A.1) provided the existence of a recursive structure Φ. The discussion in section II.2 of this appendix and the results in Mas-Colell and Zame (1996) imply that assumption 3.4) is required to insure the existence Φ in definition 5. Of course, 3.4) implies A.1). However, the proof will be done imposing the less restrictive assumptions in case Φ can be derived under milder restrictions for a different type of economy.

[21] $D_{z_+}F(z,z_+)$ is available under request.

Under assumptions 6.2-i) to 6.2-iv) and 3-iii) the result in lemma 5 follows from proposition 1 and 2 and theorems 1 and 2. Thus the proof of the lemma will only take care of the case of only 1 asset which allows to the continuity imposed by assumption 3-iii). It will be shown that there exist a selection $\varphi \sim \Phi$, with $\varphi(\tilde{z}, s_+) = [s_+, \theta_+(\tilde{z}, s_+), q_+(\tilde{z}, s_+), m_+(\tilde{z}, s_+)]$, that is continuous in each coordinate in s_+. Moreover, considering the incomplete markets nature of the model, $\theta_+(\tilde{z}, s_+)$ will be assumed to be constant. That is, $\theta_+(\tilde{z}, s_+) = \theta_+(\tilde{z})$ for each $s_+ \in [LB(s), UB(s)]$. Once the continuity of $q_+(\tilde{z}, s_+)$ has been shown below, the continuity of $m_+(\tilde{z}, .)$ follows from its definition.

Proof of lemma 5

Assume that $\theta_+(\tilde{z}, s_+)$ is constant in in s_+ for any given $\tilde{z} \in \tilde{J}$. In order to complete the proof, it suffices to show that $q(\tilde{z}, s_+)$ is continuous in s_+ for any given $\tilde{z} \in \tilde{J}$.

Under assumptions 6.2-i) to 6.2-iii) any equilibria in this economy exists satisfies equation A.12, the feasibility requirement, together with

A.14) $q_j u_s^i(e^i(s) + \theta^i d(s) - \theta_+^i q)' - \beta K(s) \int d_j(s_+) u_s^i(e^i(s_+) + \theta_+^i d(s_+) - \theta_{++}^i q)' ds_+ = 0,\ j \in J, i \in I$

Where $K(s)$ is the constant associated with the uniform distribution in assumption 3-iv').

Now suppose that assumption A.1 holds. Then, as mentioned in the preliminary remark, $p(s, .)$ has the Feller property. Then:

A.15) $\lim_{s^n \to s^1} \beta K(s^n) \int m_{++}^{i,j}(x) dx = \beta K(s^1) \int m_{++}^{i,j}(x) dx = q_+^j(s^1) u(e^i(s^1) + \theta_+^i d(s^1) - \theta_{++}^i q_+(s^1))'$

The last equality in A.15) follows because, under assumption 6.2-i) to 6.2-iii), there is a sequential competitive equilibrium for each s^1 which satisfies equation A.14).

After the discussion in section II.2 above, the last equality follows from theorem 4.1, 4.2 and section 5 in Mas-Colell and Zame (1996). Further, under the special form $u_s^i = u$ in assumption 6.2-i), equation A.14 and A.15 implies:

$$\text{A.16)} \lim_{s^n \to s^1} \frac{\beta K(s^n) \int m_{++}^{i,j}(x)dx}{u\left(e^i(s^n)+\theta^i d(s)\right)'} = \lim_{s^n \to s^1} q_+^j(s^n) u(-\theta_{++}^i q_+(s^n))' = q_+^j(s^1) u(-\theta_{++}^i q_+(s^1))'$$

Note that equation A.14 implies the first equality in A16) under u in assumption 6.2-i). Then, as $u\left(e^i(s^n) + \theta^i d(s^n)\right)'$ is bounded above and bounded away from zero for any admissible value of $e^i(s^n) + \theta^i d(s^n)$ under assumptions 6.2-i), equation A.15 implies the last equality.

Now, setting $\lambda = 1$ in u w.l.o.g., the continuity of ln implies

$$\text{A.17)} \underbrace{\lim_{s^n \to s^1} [-\theta_{++}^i q_+(s^n)] + \theta_{++}^i q_+(s^1)}_{A} + \underbrace{ln\,[\lim_{s^n \to s^1} q_+^j(s^n)] - ln\,[q_+^j(s^1)]}_{B} = 0$$

If $B = 0$, then as $\theta_{++}^i \neq 0$ w.l.o.g., A implies $lim\ q_+(s^n) = q_+(s^1)$ as desired.

Suppose that $B \neq 0$. The compactness of the equilibrium set implied by theorem 4.2 in Mas-Colell and Zame (1996) under assumptions 6.2-i) to 6.2-iii) implies that $B \in \mathbb{R}$. Then A.17 under $J = 1$ (i.e., there is only 1 asset) implies:

$$q_+^j(s^1) = \frac{B}{\theta_{++}^i(1 - exp(B))}$$

Note that A.14) implies that $q_+^j(s^1) \geq 0$ and that $\theta_{++}^i > 0$ w.l.o.g. as there are heterogenous agents and the asset is offered in zero net supply. Then, as B is a finite number and it was assumed to be different from zero, then $q_+^j(s^1) < 0$; implying a contradiction with $B \neq 0$.

■

Bibliographical References

Aliprantis, C. and Border, K. (2006): “Infinite Dimensional Analysis”, Springer, Third Edition.

Arellano, C. (2008): “Default Risk and Income Fluctuations in Emerging Economies”, American Economic Review, 98(3), 690-712, June.

Braido, L. (2013): “Ergodic Markov Equilibrium with Incomplete Markets and Short Sales”, Theoretical Economics, vol. 8 (1), 41-57

Bogachev, V. (2007): “Measure Theory, Volume 1”, Springer.

Conesa, J. C. and Krueger, D. (1999): “Social Security Reform with Heterogeneous Agents”, Review of Economic Dynamics, 2(4), 757-795, October.

Duffie, D., Geanakopolos, J., Mas-Colell, A., McLennan, A. (1994): “Stationary Markov Equilibria”, Econometrica, 62 (4), July, 745-81.

Grandmont, J-M. and Hildenbrand, W. (1974): “Stochastic Process of Temporary Equilibria”, Journal of Mathematical Economics, 1, 247-277.

Grandmont, J-M (1983): “Money and Value”, Cambridge University Press.

Hildenbrand, W. (1974): “Core and Equilibria of a Large Economy”, Princeton University Press.

Ito, Y. (1964): “Invariant Measures for Markov Processes”. Transactions of the American Mathematical Society, (110), 152-184

Kempton, T., and Persson, T. (2015). Bernoulli convolutions and 1D dynamics. Nonlinearity, 28(11), 3921.

Kubler, F. and Schmedders K. (2002): “Recursive Equilibria in Economies with Incomplete Markets”, Macroeconomic Dynamics, 6, 284-306.

Magill, M. and Quinzii, M (1994): “Infinite Horizon Incomplete Markets”, Econometrica, 62 (4), July, 853-80.

Magill, M. and Quinzii, M (1996): “Theory of Incomplete Markets, Volume 1”, MIT Press.

Martinez, J. and Pierri, D. (2018): “A bunch of tricks. Continuous Markovian Representations in an enlarged state space”, Mimeo.

Mehra., R. and Prescott, E. (1980) “Recursive Competitive Equilibria: The Case of Homogeneous Households.” Econometrica 48:1365-79

Molchanov, I and Zuyev, S. (2011): Advanced Course in Probability: Weak Convergence and Asymptotics, Mimeo

Pierri, D. and Reffett, K. (2018): “Price Dependent occasionally binding constraints: Towards a unified approach”, mimeo.

Raad, R. (2013): “Approximate Recursive Equilibria with Minimal State Space”, working paper Nº 737, Getulio Vargas Foundation.

Radner, R. (1972): “Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets”, Econometrica, 40, 289-303.

Santos, M. (2002): “On Non-existence of Markov Equilibria in Competitive-Market Economies”, Journal of Economic Theory 105, 73–98

Santos and Peralta Alva (2005): “Accuracy of Simulations for Stochastic Dynamic Models”, Econometrica, 73 (6), Nov, 1939-1976.

Santos and Peralta Alva (2013): “Ergodic Invariant Measures for Non-optimal Dynamic Economies”, mimeo.

Shiryaev, A. (1996): “Probability”, Springer.

Stokey, N., Lucas, R. and Prescott, E. (1989): “Recursive Methods in Economic Dynamics”, Harvard University Press.