Supplement to "Useful Results for the Simulation of Non Optimal General Equilibrium Economies"

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Section I: Figures

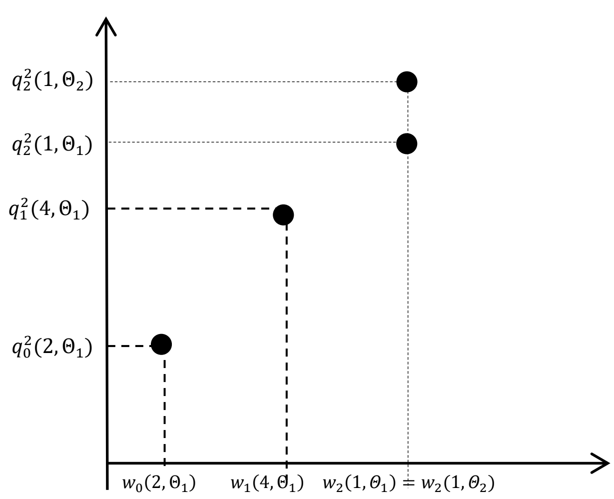

Figure 1: Wealth Equilibrium Correspondence

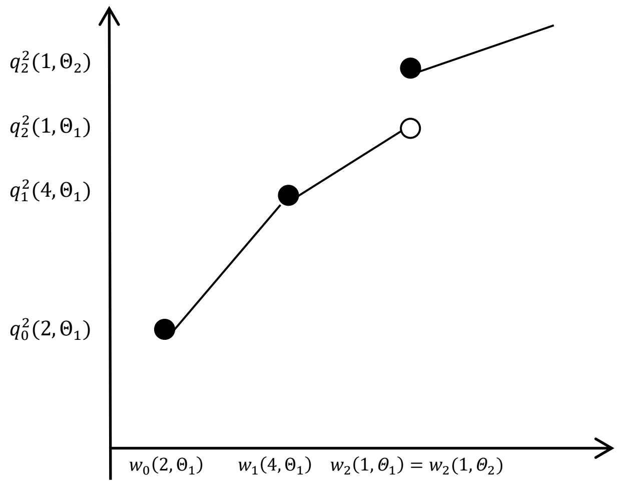

Figure 1': Computed wealth recursive function

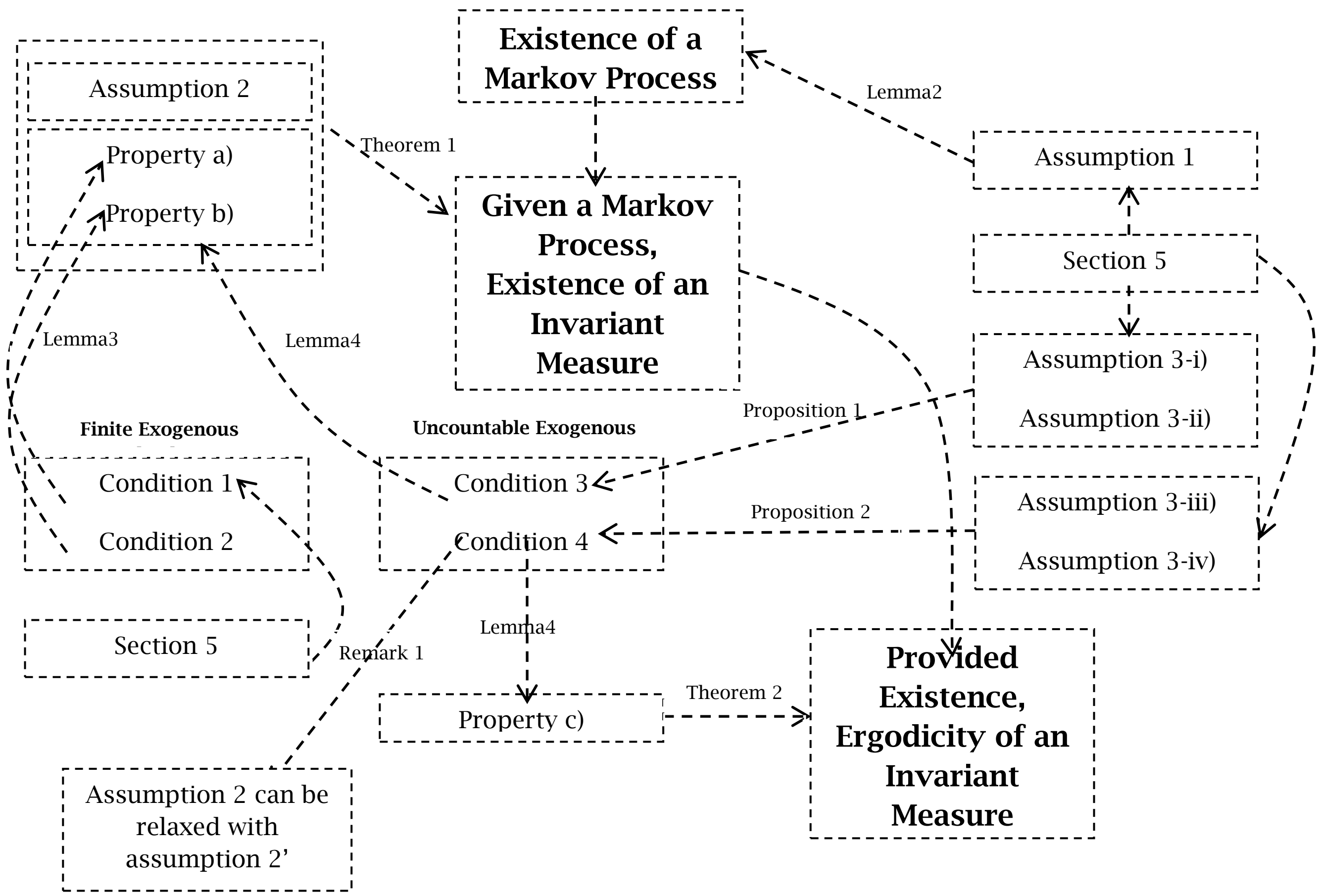

Figure 2

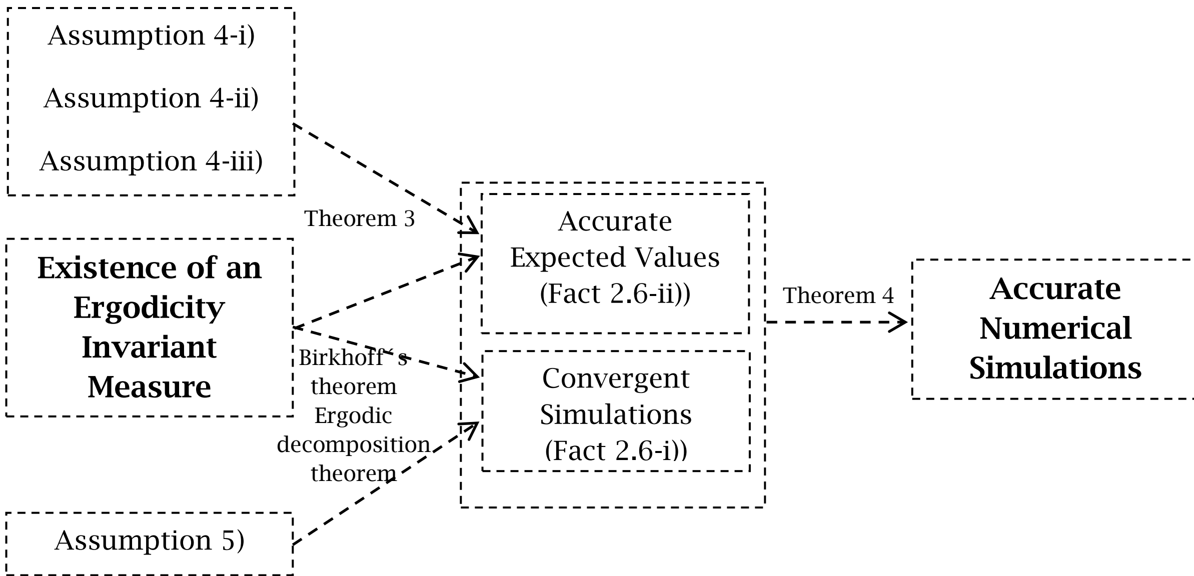

Figure 3

Section II: Applications

This section applies the results in section 3 to a concrete parametrization of the economy described in section 2. Following figure 2 the requirements to achieve the existence of an invariant measure can be categorized in 3: properties (a-c), conditions (1-4) and assumptions (1-3). Section 3.2 and 3.3 connected conditions (mostly on the Markov operator P_φ) with properties of the associated Markov process $(\tilde{J}, P_\varphi)$.

Section 3.4 shows that conditions 1-4 can be generated by assumptions (mostly on the structure of exogenous variables) for the case of uncountable shocks. While the majority of these assumptions -1, 3-i), 3-ii) and 3-iv) - are stated in terms of the primitives of the model, there are 2 which are still stated in terms of endogenous variables of the model. This section connects one of these assumptions, 3-iii), with primitives of a version of the model presented in section 2 that is borrowed from Mas – Colell and Zame (1996). Typically, see fact 2.5-2), once the existence of equilibria for a truncated (finite time) economy has been shown as in Mas-Colell and Zame (1996), it suffice to prove the uniform compactness of endogenous variables to insure the existence of the recursive equilibrium in Feng, et. al. In order to guarantee assumption 3-iii), it is possible to refine this recursive equilibrium notion as it was discussed in the appendix of section 2.5.2 or directly restrict the sequential equilibria. Lemma 5 in section II.2 proves this claim. Unfortunately, assumption 2', which restricts the cardinality of the discontinuity set, cannot be associated with primitive conditions. It is a matter of future research to investigate the relationship between the cardinality of the discontinuity set and the equilibrium set as suggested in section 3.

For the case of finite shocks the existence of an invariant measure is guaranteed by 2 conditions: the first one connects the Markov process with the set of atomless measures, the second one guarantees the closedness of this last set. Section II.1 shows that it is possible to derive the first condition from the curvature of the utility function using the implicit function theorem, which are assumed to hold almost surely. The second condition however, cannot be derived from primitive conditions of the model and thus deserves to be study in detail. Recently, Martinez and Pierri (2017) provide an example of an economy which illustrates the difficulty of the question at hand. If the economy has finite shocks and discontinuous Markov equilibria in the natural state space (as for instance in Santos 2002), it is possible to prove the existence of an invariant measure by enlarging the state space as in Duffie, et. al. (1994) as sometimes this procedure restores the continuity of the Markov operator.

The requirements that insure the accuracy of numerical simulations, described in assumptions 4 and 5, are outside the scope of this paper as they require developing an algorithm which is capable of computing the equilibrium correspondence in definition 5 while keeping track of the requirements that preserves the absolute continuity of the measures involved in the successive computations. This type of algorithm has not been developed yet and thus requires a careful separate treatment.

II.1 Finite Shocks

The model is the same as the one described in section 2.1. Following figure 2, the first step to prove the stationarity of the model is to derive a recursive representation for the sequential equilibria. As discussed in section 2.5.2, *the existence of a recursive structure is guaranteed by the existence of the sequential competitive equilibria and the compactness of the equilibrium set*. In the present framework, these properties will be shown to be implied by the assumptions listed in this subsection. Moreover, all the assumptions required for the existence of an invariant measure are presented below.

Assumptions 6.1-i) to 6.1-v) insure the existence of a non-empty compact equilibrium set which will be shown to be sufficient to derive a Markov representation of equilibria. Provided this representation, in order to show the existence of an invariant measure, it suffices to impose assumption 2, property a) and property b) (presented in section 3.1, theorem 1). The first and the last are stated as a hypothesis below (assumptions 6.1-vi and 6.1-vii respectively) and the second one will be derived from primitive conditions of the model which are implicit in assumptions 6.1-i) to 6.1-v).

<u>*Assumption 6.1).*</u> *Suppose an incomplete market economy as the one described in section 2.1. To that structure add the following assumptions:*

i) *The utility function in the optimality condition of definition 1 is:*

$$U_i(c) = \sum_{t=0}^{\infty} (\beta^i)^t \sum_{\sigma_t^* s} \left[u_s^i(c^i(\sigma_t^* s))\right] \mu_t(\sigma_t^* s)$$

Where $u_s^i\left[c^i(\sigma_t^* s)\right] = 1 - e^{-\lambda c^i(\sigma_t^* s)}$ *with* $\lambda > 0$.

ii) *The realizations of the exogenous shock* s_t *lie in set* S *of finite cardinality for any time period* $= 0,1,\dots$.

iii) *Endowments satisfy:* $e^i(\sigma_t) > 0$ *and* $\sum_{i=1}^{I} e^i(\sigma_t) < K$ *with* $K > 0$ *for any agent* $i \in \{1, \dots, I\}$ *and node* σ_t . *That is, idiosyncratic endowments are strictly positive and aggregate endowments are uniformly bounded.*

iv) *There is a finite number,* J*, of numerarie short lived assets with (uniformly) bounded dividends and short sale constraints. That is, for each agent* i *and any node* σ_t *the portfolio is given by* $\theta^i(\sigma_t) \geq -B,\ B \in \mathbb{R}_+^J$*, the associated dividends by* $d(\sigma_t s) \in M \subset \mathbb{R}_+^J$*, where* M *is uniformly bounded, and the budget equation by*

$$c^i(\sigma_t) = e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t) - \theta^i(\sigma_t).q(\sigma_t)$$

Where $q(\sigma_t)$ *is the price of the portfolio in terms of the numerarie for every node* σ_t *and* σ_t^* *is the predecessor of* σ_t.

v) *There is a riskless bond. That is, there is an asset* l *which has associated dividends given by* $d_l(\sigma_t s) = 1$ *for any* $s \in S$ *and any node* σ_t.

vi) *Assumption 2 holds (i.e. the discontinuity set of any measurable selection of the equilibrium correspondence has at most finite cardinality).*

vii) *Condition 2 holds (i.e. provided that the adjoint operator maps the set of atomless measures into itself, this set is weakly closed).*

Except the assumption on u_s^i, the short sale constraints, 6.1-vi) and 6.1-vii), the rest are standard in the literature. The results in Magill and Quinzii (1994) imply that under assumptions 6.1-i) to 6.1-v), excluding the restriction on u_s^i, the economy describe in section 2.1 has a non-empty compact equilibrium set (see assumption A.1 to A.6 and the discussion that follows them in pages 858-60).

The chosen return function on assumption 6.1-i) guarantees that marginal utility is bounded on the entire feasible consumption set which, because of assumption 6-iii), is given by $[0, K]$. Kubler and Schmedders (2002) shows that assumptions 6.1-i) to 6.1-v), including the restriction on the return function but excluding the short sale constraints, imply that any sequence of consumption bundles $\left\{\{c^i(\sigma_t)\}_{i\in I}\right\}_{\sigma_t\in\mathfrak{T}}$, portfolios $\left\{\{\theta^i(\sigma_t)\}_{i\in I}\right\}_{\sigma_t\in\mathfrak{T}}$ and prices $\{q(\sigma_t)\}_{\sigma_t\in\mathfrak{T}}$ which satisfy the feasibility requirement $\sum_{i=1}^{I}\theta^i(\sigma_t) = \vec{0}$, where $\vec{0} \in \mathbb{R}^J$ for any $\sigma_t \in \mathfrak{T}$, and the Kuhn Tucker conditions listed in equation 3 and 4 (see section A.1.1, technical appendix to section 2.5.2) meet the optimality and feasibility conditions in definition 1 and thus constitutes a sequential competitive equilibrium. The compactness of the equilibrium set follows from Magill and Quinzii (1994).

Short sale constraints are standard in the recursive literature since Duffie, et. al. (1994). Braido (2013) showed that a recursive equilibrium in the sense of Duffie, et. al. exists even if explicit short sale constraints are removed. This is possible as Magill and Quinzii (1994) showed that there is a uniform bound on assets even in the absence of short sale constraints. However, the theoretical results in this paper depend on Feng, et. al. (2013) recursive equilibria which, as discussed in section 2.5.2, are a subset of all possible recursive equilibria in Duffie, et. al. It is not clear that Braido's results hold in

Feng, et. al.'s framework. Thus, short sale constraints are imposed in order to guarantee the existence of an appropriate (sunspots free) recursive equilibria.

As discussed in section 2.5.2 (see also Feng, et. al. 2013 section 2.2), if the equilibrium set is compact and can be generated by the set of equations implied by the Kuhn Tucker and feasibility conditions, the equilibrium correspondence Φ in definition 5 (see section A.1.1, technical appendix to section 2.5.2) satisfy the assumptions in lemma 2 and thus P_φ, as defined in equation 5, is a well defined Markov operator and $(\tilde{J}, P_\varphi)$ defines a (compact) Markov process with typical state $\tilde{z} = [s, \theta, q, m] \in \tilde{J}$ and $m_j^i = d^j(s)\big(u_s^i(c^i)\big)'$.

Given the existence of a Markovian representation $(\tilde{J}, P_\varphi)$, theorem 1 implies that to prove the existence of an invariant measure, it suffices to impose assumption 2, condition a) and condition b). The first and the last are listed in assumptions 6.1-vi) and 6.1-vii).

The discussion in the preliminary remark of lemma 3 in the appendix implies that property a), namely that the adjoint operator associated with P_φ maps the space of atomless measures into itself, is guaranteed to hold if the implicit function theorem can be applied to the system of equations defined by equations 3, 4 and $\sum_{i=1}^{I} \theta^i = \vec{0}$ in a full lebesgue measure set. More precisely, let $z = [s, \theta, q]$ and $F(z, z_+) = \vec{0}$ be the system of $J + J \times I$ equations that can be obtained by replacing equation 3 into 4 and considering only interior solutions[1]. Section A.2.1) in the appendix will show that, under assumptions 6.1-i) to 6.1-v), $D_{z_+}F(z, z_+)$ has full rank a.e. in z, where $D_{z_+}F$ is the Jacobian matrix of F with respect to z_+.

[1] The discussion in section A.2.1 in the appendix connects Φ with F and $\tilde{z}$ with z. Once Φ is defined, it suffice to note that $\tilde{z} = [z, m]$ and m is defined by the additional equation given above.

Once this property has been established, it suffices to apply lemma 3. That is, lemma 3 connects condition 1 (i.e. $\mu(\{a\}) = 0$ *implies* $P_\varphi(z, \{a\}) = 0$ z*-a.e. with respect to an atomless measure* μ) with property a) (i.e. $P_\varphi^*: \mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ where P_φ^* is the adjoint operator associated and $\tilde{J}$ the state space of the process with P_φ and $\mathcal{P}_0(\tilde{J})$ the (sub)space of atomless measures in $\mathcal{P}(\tilde{J})$). The arguments in the preliminary remark of lemma 3 and section A.2.1 in the appendix show that the full rank of $D_{z_+}F(z, z_+)$ is sufficient to guarantee condition 1.

Notice that the implicit function theorem is required to hold a.e. in z. Thus there is no contradiction between this property and the possible discontinuity of φ as, taking into account assumption 2, the discontinuity set of φ has finite cardinality and thus zero measure on μ.

While assumptions 6.1-i) to 6.1-vi) are relatively mild, assumption 6.1-vii) is quite strong as it directly implies the weak-closedness of $\mathcal{P}_0(\tilde{J})$ (i.e. property b). Further, this assumption cannot be connected with primitive conditions of the model. Fortunately, it is possible to obtain properties a) and b) jointly by strengthening condition 1. This is done by lemma 4, that requires only condition 3, which strengthens condition 1 by requiring it to hold *uniformly in* z. Proposition 1 shows that condition 3 holds if the model is allowed to have an uncountable number of exogenous shocks s. Taking into account the distinctive nature of this type of economies, they must be treated separately. Section 6.2 below addresses this point.

II.2 Uncountable Shocks

The discussion in the preceding section sets a trade off: in order to get rid of unverifiable assumptions like property b), the structure of exogenous shocks must be modified. Unfortunately, proving the existence of the sequential equilibria (and thus the existence of an appropriate recursive structure in the sense of Feng, et. al.) with uncountable shocks requires imposing an additional assumption on 6.1-i) to 6.1-v). This assumption, labeled 6.2-ii) below, was extensively discussed in the literature (see for instance Mas-Colell and Zame, 1996, or Araujo, et. al. 1996). While 6.2-ii) was considered unsatisfactory by the sequential equilibrium literature, it have been implicitly assumed in recursive models as can be seen in the preliminary remark of section A.1.2) in the appendix. Thus, in the present context, assumption 6.2-ii) is rather mild.

Once this additional hypothesis has been imposed there is an important gain in terms of the predictive power of the model developed in section 2.1 and 2.2 as the theory developed in section 3 allows showing not only that the model has a well behaved steady state (i.e. an invariant measure, see theorem 1) but also that it is ergodic (see theorem 2). Further, with the notable exception of assumption 2') and 6.2-ii), the remaining hypothesis can be directly traced back to primitive conditions of the model. This last fact can be obtained by proving an additional lemma which allows getting rid of assumption 3-iii) in section 3.4. Once this lemma has been shown, propositions 1 and 2 con be used to derive conditions 3 and 4 and thus theorems 1 and 2 by means of conditions 3 and 4.

As in section 5.1, assumption 6.2 below contained all the sufficient conditions to show the existence of an ergodic invariant measure in the model discussed in sections 2.1 and 2.2 except assumption 3-iii) which will be treated separately in a lemma below.

<u>Assumption 6.2).</u> *Suppose an incomplete market economy as the one described in section 2.1. To that structure add the following assumptions:*

i) *Assumptions 6.1-i), 6.1-iii) and 6.1-iv) hold.*

ii) $e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t) > 0 \ , \sigma_t \in \mathfrak{T}$

iii) *Assumptions 3-i) and 3-iv) hold (i.e. the set of exogenous shocks is* $S = [\underline{S}, \overline{S}] \subset \mathbb{R}$ *and* $p(s,.) = U[\underline{S}, \overline{S}]$, *where* U *is the uniform distribution).*

iv) *Assumption 2' holds (i.e. the discontinuity set is at most of zero lebesgue measure).*

Assumptions 6.2-i) to 6.2-iii) guarantees the existence of the sequential equilibria. The proof follows immediately by extending the induction argument in Mas-Colell and Zame (1996) for $T = \infty$ as in Duffie, et. al. (1994, see fact 2.5-2 in section 2.5.1). In particular, theorem 4.1 in Mas-Colell and Zame allows proving the non-emptiness C_j for $1 \leq j \leq T < \infty$, where C_j is the set of initial states of a $j + 1$ period economy defined in the technical appendix of section 2.5.2. The compactness of K, the set that includes all payoff relevant states, follows from theorem 4.2 also in Mas-Collel and Zame. The induction argument in section 5 of that paper can be used to set $T = \infty$. The optimality argument in Duffie, et. al. (section 3.4) can be immediately extended to the Mas-Colell and Zame framework as theorem 4.1 and 4.2 hold $\mu_s^\infty(s_0,.)$ -a.e. for $s_0 \in S$ and θ_-^i satisfying assumption 6.2-ii), where $(\Omega, \mathcal{F}, \mu_s^\infty(s_0,.))$ is the stochastic process defined in section 4.2 above but restricting the state space Ω to contain only an infinite sequences of exogenous shocks $\{s_t\}$.

The compactness of K and the continuity (in z_+) of the system of equations defined by 3), 4) and the feasibility of assets guarantees that the equilibrium correspondence, Φ in definition 5, satisfies the assumptions required by lemma 2. Thus, there is at least 1 measurable selection $\varphi \sim \Phi$ and $(\tilde{J}, P_\varphi)$ defines a Markov process.

Once an appropriate Markov process have been shown to exist, proposition 2 implies that assumptions 6.2-iii), 6.2-iv) and 3.iii) are sufficient to show the ergodicity of the process $(\tilde{J}, P_{\varphi})$. The following lemma shows that if there is only 1 asset or the recursive equilibrium notion in Feng, et. al. is appropriately restricted (see fact 2.5-5 in section 2.5.2), assumption 3-iii) can be omitted.

Lemma 5: Suppose that fact 2.5-5 holds or $J = 1$ (i.e. there is just 1 asset). Then, under assumptions 6.2-i) to 6.2-iv), $(\tilde{J}, P_{\varphi})$ has an ergodic invariant measure.

Proof: see section A.2.2 of the appendix.

Section III: Appendix

A.1) Sections 2, 3 and 4.

A.1.1) Technical appendix

Technical appendix of section 2.3.

Definition 2: A sequential equilibrium is called Weakly Recursive if there exist continuous functions $f^i: S \times \mathbb{R}^{IJ} \to \mathbb{R}^J$ for all $i \in I$ and $g^j: S \times \mathbb{R}^{IJ} \to \mathbb{R}_+$ for all $j \in J$ such that for any $\sigma_t \in \mathfrak{T}$ and $s \in S$, $q_j(\sigma_t^* s) = g^j\left(s, \{\theta^i(\sigma_t^*)\}_{i=1}^I\right)$ and $\theta^i(\sigma_t^* s) = f^i\left(s, \{\theta^i(\sigma_t^*)\}_{i=1}^I\right)$, where $\{\theta^i(\sigma_t^*)\}_{i=1}^I$ is feasible.

Definition 3: An equilibrium is called Wealth Recursive if it is weakly recursive and if there are continuous functions $f_{WhR}^i: S \times \mathbb{R}^I \to \mathbb{R}^J$ for all $i \in I$ and $g_{WhR}^j: S \times \mathbb{R}^I \to \mathbb{R}_+$ for all $j \in J$ such that $g_{WhR}^j(s, w(\sigma_t^* s)) = g^j\left(s, \{\theta^i(\sigma_t^*)\}_{i=1}^I\right)$ and $f_{WhR}^i(s, w(\sigma_t^* s)) = f^i\left(s, \{\theta^i(\sigma_t^*)\}_{i=1}^I\right)$, where $w(\sigma_t^*) = \{w^i(\sigma_t^*)\}_{i=1}^I$.

Frequently, the macroeconomic literature (i.e. Arellano, 2008) assumes the existence of a recursive equilibrium based on several standard properties of the Bellman equation (see, for example, Stokey, Lucas and Prescott, 1989). Formally, this equilibrium notion can be thought as an extension of Mehra and Prescott's recursive competitive equilibrium (1980) to an economy with heterogeneous agents and incomplete markets. Typically, the equilibrium is defined as:

Definition 3.1[2]*: A recursive equilibrium is composed by a set of* I *value and price functions,* $\{V^i(s,w)\}_{i=1}^{I}$ *and* $\{q^i(s,w)\}_{i=1}^{I}$ *respectively, which satisfy the following properties:*

i) (Optimality) Each household $i \in I$ solves

$$V^i(s,w) = Max_{\theta^i \epsilon \Delta}\, u_s^i\left(w^i - q^i(s,w)\theta^i\right) + E_p\left(\beta^i V^i(s',w')\right)$$

Where the wealth distribution is $w' = \{w_i'\}_{i=1}^{I} = \left\{e^i(s') + d(s')\theta^i\right\}_{i=1}^{I}$, E_p is the expected value taken with respect to $p(s,.)$, and the feasible set Δ is compact[3].

ii) (Market Clearing) $\sum_i \theta^i = \vec{0}$

iii) (Expectations) $q^i(s,w) = q^l(s,w) = q(s,w)$ for all $i,j \in I$

Provided the existence of continuous price functions $\{q^i(s,w)\}_{i=1}^{I}$ which satisfy iii), the continuity of $\{\theta^i(s,w)\}_{i=1}^{I}$ follows from mild curvature conditions on u_s^i (see Stokey, Lucas and Prescott, Ch. 9 and 10). Thus, definition 3.1 is equivalent to definition 3 in the sense that both imply a recursive structure based on continuous functions that depends on exogenous shocks and wealth distribution.

Technical appendix of section 2.4.

[2] This definition does not include models of the Hugget (1993) style as this type of models does not assume the existence of aggregate uncertainty (i.e. $\#S = 1$) and the degree of heterogeneity is higher as Hugget suppose the existence of a continuum of distinct agents and idiosyncratic uncertainty.

[3] To achieve this property it is sufficient to impose a short sale constraint on assets.

Assume that $I = 2, J = 3, \#S = 5, \beta^i = \beta^{i'} = 5/6$. Preferences, endowments and dividends are given by:

$$u_s^i = a_s^i \frac{[C(\sigma_t^* s)]^{1-5}}{1-5}\ ,\ a_s^1 = [1, 1024, 1],\ a_s^2 = [1, 1, 1024] \text{ for } s = 1,2,3$$

$$u_s^1 = \frac{-[C(\sigma_t^* s)]^{-2}}{2} \text{ for } s = 4,5;\ u_4^2 = \frac{-[C(\sigma_t^* s)]^{-2}}{2},\ u_5^2 = \frac{-6.05[C(\sigma_t^* s)]^{-2}}{2}$$

$$e^1 = [e^1(1), \dots, e^1(5)] = [4,12,1,10,8.69],\ \ e^2 = [e^2(1), \dots, e^2(5)] = [4,1,12,10,11.31]$$

$$d^1 = [d^1(1), \dots, d^1(5)] = [1,0,0,0,0],\ d^2 = [0,1,0,0,0],\ d^3 = [0,0,1,0,0]$$

The transition matrix is given by:

$$[p(s,s')] = \begin{bmatrix} 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.3 & 0.05 & 0.05 \\ 0.3 & 0.3 & 0.05 & 0.05 & 0.3 \end{bmatrix}$$

As consumption of each agent is bounded above by aggregate consumption, which is in turn uniformly bound by $e^1(5) + e^2(5)$, U_i can be assumed to be bounded above without lost of generality because Bernoulli utility functions are assumed to be strictly increasing. Thus, the arguments in Duffie, et. al. (page 765) imply that consumption is uniformly bounded below by some positive constant. Consequently, marginal utilities are uniformly bounded which in turn imply that any equilibria, if it exist, can be characterized by agent's first order conditions and feasibility constraints (see Kubler and Schmedders, 2002, page 288).

The following tables contain asset prices and portfolios which satisfy the optimality (first order) and feasibility conditions in definition 1. Each table can be seen as a time independent function of the exogenous shocks and the distribution of assets. There are 2 equilibrium portfolios which are computed "by hand". Namely, $\Theta_1 = [\theta_1^1; \theta_1^2] = [0, -1.6, 1.6; 0, 1.6, -1.6]$ and $\Theta_2 = [\theta_2^1; \theta_2^2] = [0, -0.98, 2.28; 0, 0.98, -2.28]$. Thus, these tables define a WRE.

Provided that the initial portfolio distribution $\{\theta_-^i\}$ is either Θ_1 or Θ_2, tables 1 and 2 can be used to generate a unique sequential competitive equilibrium according to definition 1.

Asset Prices (q)					
	S=1	S=2	S=3	S=4	S=5
Θ_1	[0.25, 2.15,2.15]	[0.03, 0.25,0.25]	[0.03, 0.25,0.25]	[0.24, 2.10,2.10]	[0.10, 1.54,0.08]
Θ_2	[0.25, 3.57,1.22]	[0.01, 0.25,0.08]	[0.05, 0.73,0.25]	[0.24, 2.10,2.10]	[0.10, 1.54,0.08]

Table 1

Portfolio $[\theta^1, \theta^2]$					
	S=1	S=2	S=3	S=4	S=5
Θ_1	Θ_1	Θ_1	Θ_1	Θ_1	Θ_2
Θ_2	Θ_2	Θ_2	Θ_2	Θ_1	Θ_2

Table 2

Kubler and Schmedders (2002) showed (numerically) that the endogenous variables in tables 1 and 2 are the only ones that satisfy the optimality and feasibility conditions in definition 1 (see page 301). Then, in order to show that this economy has no wealth recursive equilibria, it suffice to show that for some pair of states (s, w), there are at least 2 possible asset prices.

Heuristically, it can be argued that the endogenous variables in the tables above define a steady state[4]: once the economy starts either at Θ_1 or Θ_2, it will never leave the state space defined by $S \times \{\Theta_1; \Theta_2\}$.

Technical appendix of section 2.5.1.

[4] A steady state for an appropriately defined Markov representation of the sequential competitive equilibria will be formally defined in section 3.

A THME is build using 3 preliminary elements: an expectation correspondence, a self-justified set and a transition function. $Z_D = \{[s, \theta_-, c, q, \theta] \in S \times \mathbb{R}^{IJ} \times \mathbb{R}^I \times \mathbb{R}^J \times \mathbb{R}^{IJ} \,|\, \sum_{i=1}^{I} \theta_-^i = \vec{0}, \sum_{i=1}^{I} \theta^i = \vec{0}\}$ is the state space.

An *expectation correspondence* is a map, $G: Z_D \to \mathcal{P}(Z_D)$, where $\mathcal{P}(Z_D)$ is the set of probability measures generated from Z. It will be said that $\mu \in G(z_0)$, if z_0 and any realization of the random variable z_1, which has conditional distribution given by μ, satisfy the optimality conditions implied by b) in definition 1. Typically and without loss of generality, G is supposed to have a closed graph.

The purpose of the expectation correspondence is to obtain a sequence $\{z_t\}_{t=0}^{T}$, with $T \in \mathbb{N}$, such that the conditional distribution of z_t is contained in $G(z_{t-1})$. This sequence can be constructed as follows: it will be said that $\mu \in G(z_{t-1})$ and $z_t \sim \mu$, where $z_{t-1} = [s_{t-1}, \theta_{t-2}, c_{t-1}, q_{t-1}, \theta_{t-1}]$ is feasible, if for each $i \in I$

1) $c_{t-1}^i = e^i(s_{t-1}) + \theta_{t-2}^i d(s_{t-1}) - \theta_{t-1}^i q_{t-1}$
2) $q_{t-1}\left(u_{s_{t-1}}^i(c_{t-1}^i)\right)' = \beta E_\mu\left[d(s_t)\left(u_{s_t}^i(c_t^i)\right)'\right]$

Where E_μ is the expectation with respect to μ[5], which is an arbitrary probability measure on $\mathcal{P}(Z)$, and $\left(u_{s_{t-1}}^i(c_{t-1}^i)\right)'$ is the partial derivative of $u_{s_{t-1}}^i$.

Let $K \subset Z$ be any measurable set such that $z_t \in K$ for any $\{z_t\}_{t=0}^{T}$ and any $T \in \mathbb{N}$. The existence of this set, typically compact, for economies with short lived assets and finite shocks is guaranteed by the results in Maguill and Quinzii (1994, see page 871). Define $C_0 \equiv K$. Then, the set of all initial states of any 2

[5] Duffie, et. al. add 2 technical conditions to 1) and 2). The first one restricts the marginal distribution of s_{t-1} and θ_{t-2} and the second one affects the support of μ. For a detailed discussion see for instance Duffie, et. al. pages 763 and 767.

period (truncated) economy[6] is contained in the following set: $C_1 = \{z \in K | \exists\, v \in G(z)\ and\ sup_{D \subset C_0} v(D) = 1\}$, where sup denotes the supremum. Inductively, a sequence of nested sets $\{C_j\}$ for $j \geq 1$ can be constructed with C_j containing the initial states of any j-period economy.

It follows from Theorem 1.2 in Duffie, et. al. (page 754) that $J = \cap_{j=0}^{\infty} cl(\overline{C}_j)$ is non empty and compact, where cl denotes the clousure of a set, if K is compact and $C_j \neq \emptyset$ for $j \geq 1$. In the present context both conditions are guaranteed to hold by corollary 5.3 in Maguill and Quinzii (page 868)[7]. J is *called self-justified set.*

Any selection π of the expectation correspondence must have the following 2 properties, which are standard in the literature: $\pi(.,A)$ must be measurable and $\pi(z,.)$ must be a probability measure for any measurable set A and $z \in J$ respectively. This last condition follows directly from the definition of the expectation correspondence. If J is closed, then the Kuratowski measurable selection theorem (see for instance Hildenbrand 1974, page 55) implies that the restriction of G to J has a measurable selection.

Thus, for the economy described in section 2.1, which satisfy all the relevant assumption in Magill and Quinzii (1994, see page 858), the results in Duffie, et. al. guarantee the existence of correspondence based recursive structure on an enlarged state space which the authors called Time Homogeneous Markov Equilibrium (THME):

[6] As long as Z is compact, it is clear that any $\{z_t\}_{t=0}^{T}$ contained in G is a sequential competitive equilibrium. For an arbitrary state space, a set of uniform (stationary) bounds are required. For instance, this is done by Duffie, et. al. (1994) using Radner's (1972) existence result (Lemma 3.4, page 768) and by Kubler and Schmedders (2003) using several elements of the Geanakolos and Zame (2002) existence proof (Lemma 3, page 1777).

[7] Duffie, et. al. (section 3) established the existence of a compact set K (page 767, Lemma 3.1 and 3.2) and $C_j \neq \emptyset$ for any $j \in \mathbb{N}$ (Lemma 3.4 and 3.5, page 768) in a heterogeneous agent economy with a finite number of Lucas trees and short sales constraints. Braido (2013) extended these results for a general asset structure under mild assumption on preferences.

Definition 4: *A pair (J,π) is a* THME *for G if π is a Markov operator and J is a set that satisfies $\pi(z) \in G(z)$ for all $z \in J$.*

Technical appendix of section 2.5.2

In this framework the state space, $\widetilde{Z}$, can be decomposed in 2 parts: payoff relevant variables Z_F and auxiliary variables m. In particular, let $Z_F \equiv \{[s,q,\theta] \in S \times \mathbb{R}^J \times \mathbb{R}^{IJ} | \sum_{i=1}^{I} \theta^i = \vec{0}\}$, $m^{i,j} \equiv d^j(s)\left(u_s^i(c^i)\right)'$, where m is the vector of shadow values of the marginal return to investment for all assets and all agents. Assume, additionally to the hypothesis stated in section 2.1, that there exist a short sale constraint $\bar{B} > 0$ such that $\theta^{i,j} \geq -\bar{B}$. Using the budget constraint, equation 1, it is possible to define a correspondence V that maps $(z) \mapsto m$ as follows: for each $z \in Z_F$, $c^i \in [e^i(s) + \theta^i d(s) - I\bar{B}q, e^i(s) + \theta^i d(s) + I\bar{B}q]$ defines a selection $m \sim V(z)$ which is obtained by taking some $\theta_+^{i,j} \geq -\bar{B}$ for all $i \in I$ and $j \in J$. Provided, as discussed in section 2.5.1, that all endogenous variables in the model are (uniformly) contained in a compact set K, V is compact valued and $Gr(V)$ is compact.

Then, as in the previous subsection, it is possible to derive a time invariant compact state space, which is analogous to Duffie, et. al.'s self justified set. Let $\widetilde{K} \subset K$ and $\widetilde{K} \equiv Gr(V_0)$. The first order conditions of the model can be written as:

3) $c^i = e^i(s) + \theta^i d(s) - \theta_+^i q$

4) $\left[q\left(u_s^i(c^i)\right)' - \beta E_{p(s,.)}(m_+^i)\right][\theta_+^i - \bar{B}] = \vec{0}$

Where $E_{p(s,.)}$ is the expectation with respect to $p(s,.)$, the conditional distribution of s_+ given s, and $m_+^i(s_+) \sim V(\theta_+, q_+, s_+)$. Thus, 4) is defined using the expected value with respect to $p(s,.)$ over $[\theta_+, q_+](s_+)$.

The functions θ_+ and q_+, mapping $s_+ \mapsto \theta_+$ and $s_+ \mapsto q_+$ respectively, can be chosen to be *continuous* provided that S in an uncountable set. Each of these functions is associated with a predecessor in $\tilde{Z}$.

Following equation 2), in the technical appendix associated with section 2.5.1, the expected value should have been taken with respect to any possible distribution of z_+, μ. Thus, *equation 4) captures a subset of all possible* z *for any given* z_+[8].

Now it is possible to define the analogous of a "self justified set" in Feng, et. al. framework. To begin with, the set of all states, $\tilde{z} \in \widetilde{K}$,of any 2 period economy are contained in $Gr(V_1) = \{\tilde{z} \in \widetilde{K} \,|\, \exists\, \tilde{z}_+ \in Gr(V_0)\ with\ \tilde{z}, \tilde{z}_+\ satisfying\ eq. 3)\ and\ 4)\}$. That is, $[s,q,\theta,m] \in Gr(V_1)$ if $c^i(\theta_+^i)$ obtained from 3) for all $i \in I$ satisfy equation 4) for some $m_+^i(s_+) \sim V(\theta_+, q_+, s_+)$ with $[\theta_+, q_+, s_+] \in V_0$. For any arbitrary iteration j, notice that for each $s_+ \in S$ there could be more than 1 possible pair (θ_+, q_+). However, as θ_+ is chosen at time "t", in order to satisfy the restrictions of the SCE, it has to be s^t-measurable, where s^t is a branch of the tree $\mathfrak{T}$ defined in section 2.1. Thus, $\theta_+(s_+)$ can be chosen to be constant and thus *continuous for each* $\tilde{z} \in \widetilde{K}$. Moreover, any possible discontinuity in $q(s_+)$ can be ruled out by appropriately changing θ_{++} in $[e^i(s_+) + \theta_+^i d(s_+) - I\bar{B}q_+, e^i(s_+) + \theta_+^i d(s_+) + I\bar{B}q_+]$ with $\theta_{++} \in [-I\bar{B}, I\bar{B}]$. That is, if $m_+^i(s_+) \sim V(\theta_+, q_+, s_+)(s_+)$ with $[\theta_+, q_+, s_+] \in V_j$ contain a discontinuity in the q_+-coordinate, it is possible to choose $[\theta_+, q_+, s_+] \in \widetilde{K}$ as $Gr(V_{j+1}) \in \widetilde{K}$ by the uniform compactness of the equilibrium. The upper hemi-continuity and compact valuedness of V implies that "explosions and implosions" are small enough to be ruled out by any perturbation in $[-I\bar{B}, I\bar{B}]$ (see Hildenbrand and Grandmont (1974, page 31

[8] Duffie, et. al. also restricts $[\theta_+, q_+]$ to be a function of s_+ and that $s_+ \sim p(s,.)$. However, it is still possible to find a distribution, $z_+ \sim \mu$, which satisfy this restrictions and $E_{p(s,.)} \neq E_\mu$.

theorem 4). Not surprisingly, it is only possible to insure the continuity of $[\theta_+, q_+](s_+)$ for an interior path.

Let $Gr(V_j) = C_j$. Iterating on this procedure, it is possible to derive a sequence of nested sets $\{C_j\}$ for $j \geq 1$ where C_j contains all $\tilde{z}_0$ of any j-period economy. Note that this procedure defines an operator $G_K: Gr(V) \to Gr(V)$. The non-emptiness and compactness of each C_j follows from the arguments discussed in section 2.5.1 as, respectively, equations 3) and 4) are identical to the optimality conditions implied by the definition of "equilibrium with explicit debt constraint" in Magill and Quinzii (page 862) and the recursive equilibria in Feng, at. al. are a subset of those in Duffie, et. al.[9]

As G_K maps compact sets to compact sets, Feng, et. al. showed (theorem 2.1 in page 6) that $V_n \to V^*$, where V^* is the analogous of Duffie, et. al.'s self justified set. Thus $Gr(V^*) = \tilde{J}$ contains all possible first period payoff relevant variables $\tilde{z}_0(\sigma_0)$ for the sequential competitive equilibrium in definition 1. Note *that, given the uniform compactness of the SCE, the continuity of* $\varphi \sim \Phi$ *in* s_+ *(i.e. fact 2.5-5) may imply* $\tilde{J} = \tilde{K}$ at the end of the iterative process. In this case, the self-generation of the Markov equilibria (i.e. fact 2.5-1) and the existence argument in fact 2.5-2) are still satisfied as suggested by Duffie, et. al.

Finally, $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ is defined as follows: take any $\tilde{z} = [\tilde{s}, \tilde{\theta}, \tilde{q}, \widetilde{m}] \in \tilde{J}$, it will be said that $\tilde{z}_+ \in \Phi(\tilde{z}, \tilde{s}_+)if\tilde{z}_+ \in \tilde{J}$ and$(\tilde{z}, \tilde{z}_+)$ simultaneously satisfying equations 3) and 4) with $m(s_+) \sim V^*(\theta_+, q_+, s_+)(s_+)$ and $\widetilde{m}_+ \sim V^*(\theta_+, q_+, s_+)(\tilde{s}_+)$. The following definition, which will be addressed carefully in the appendix (see the remark before theorem 4), summarizes the preceding discussion:

[9] Section 5.1 will provide some additional details about these facts.

Definition 5: *Let* $\tilde{J} = Gr(V^*)$ *and* $\tilde{J} \subseteq \tilde{K}$ *.* $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ *is an* *equilibrium correspondence* *if* $\tilde{z}_{t+1} \in \Phi(\tilde{z}_t, s_{t+1})$ *and* $\{\tilde{z}_t\}_{t=0}^{\infty}$ *satisfy the optimality conditions in equations 3)-4) and the feasibility restrictions in the definition of* Z*.*

The procedure described above can be repeated an infinite number of times as $\tilde{J}$ contain all possible $\tilde{z}_0(\sigma_0)$, which are appropriate initial conditions for any $T \in \mathbb{N}$ period economy. A *time invariant transition function* is obtained by taking a selection of Φ, denoted $\varphi \sim \Phi$. This function is measurable, as Φ has closed graph and is compact valued (see Stokey, Lucas and Prescott, page 60 theorem 3.4 and 184 theorem 7.6), and does not depend on unobservable variables, thus *it constitutes the starting point of the theoretical results in section 3.*

Technical appendix of section 3.1

Comment on Theorem 1

Let $C(\tilde{J})$ be the space of continuous functions on $\tilde{J}$. It will be said that P_φ has the Feller property if the associated semigroup operator maps $C(\tilde{J})$ into itself. Lemma 9.5 in Stokey, Lucas and Prescott (page 261) can be modified to show that, if $f \in C(\tilde{J})$ but $\notin C(\tilde{J} \times S)$, $\hat{P}_\varphi f(\tilde{z}) \notin C(\tilde{J})$.

The absence of the Feller property also affects the continuity of the adjoint operator, which is critical to guarantee the existence of a fixed point of it. As P_φ^* is defined over an infinite dimensional space, in order to discuss its continuity, it is necessary to select an adequate topology. The *weak** topology, the coarsest topology that makes the linear functional $\{\mu \mapsto \int f d\mu,\ f \in C(\tilde{J})\}$ continuous, is frequently chosen in this framework. This is because P_φ^* generate sequences of *weak** convergent measures under mild assumptions[10]. In particular, under assumption 1, $\tilde{J}$ is a compact subset of a finite dimensional Euclidean space. Thus, Helly's theorem (Stokey, Lucas and Prescott, page 374) implies the existence of a *weak** - convergent subsequence in $\mathcal{P}(\tilde{J})$, which is the starting point of most existence theorems.

As discussed in Aliprantis and Border (2006, page 47), the choice of a weak topology implies a trade off: there are "a lot" of weakly convergent sequences but there are "few" weakly continuous functionals. Thus, frequently, the Feller property is used to guarantee the *weak** continuity of P_φ^*. That is, $\mu_n \rightarrow_{Weak*} \mu$ implies $P_\varphi^* \mu_n \rightarrow_{Weak*} P_\varphi^* \mu$ if $\hat{P}_\varphi$ has the Feller property (see Stokey. Lucas and Prescott, page 376).

If φ can be shown to be continuous, under assumption 1, Theorem 2.9 in Futia (1982, page 383) would imply the existence of an invariant measure for P_φ^*. It only suffice to take a sequence of measures generated by applying P_φ^* iteratively on some $\mu_0 \in \mathcal{P}(\tilde{J})$ that is robust to cyclical behavior and fits into the framework of Helly's theorem. Let $\mu_{n_k} \rightarrow_{Weak*} \mu$ be the subsequence generated by Helly's theorem. The continuity of P_φ^* implies $P_\varphi^* \mu_{n_k} \rightarrow_{Weak*} P_\varphi^* \mu$. Subtracting both subsequences, the desired result follows. *Theorem 1 in this paper shows the existence of an invariant measure for $(\tilde{J}, P_\varphi)$ even if φ is allowed to have (a certain type of) discontinuities.*

The strategy of the proof for Theorem 1 goes along the lines of Hildenbrand and Grandmont (1974). It borrows from theorem 12.10 in Stokey, Lucas and Prescott (1989) (page 376), theorem 3.5 in Molchanov and Zuyev (2011, page 15) and proposition 1 in Ito (1964, see page 155). The following subsection contains a detailed description of the procedures used *up to* now to prove the

[10] This is not the case of the *strong* topology, which is the topology generated by the total variation norm. Stokey, Lucas and Prescott (page 335 to 337) provides an example of a Markov process that generates sequences that converge in the *weak** topology but not in the strong (norm) topology.

existence of an invariant measure and the reasons that make them unsuitable for addressing the question at hand.

Using proposition 1 in Ito and theorem 3.5 in Molchanov and Zuyev it is possible to restore the continuity of P_φ^* in the absence of the Feller property. In particular, as P_φ^* and $\hat{P}_\varphi$ can be interchanged (see for instance Stokey, Lucas and Prescott page 216), if $\mu_{n_k} \to_{Weak*} \mu$, for some $f \in C(\tilde{J})$, $\int f(\tilde{z}) P_\varphi^* \mu_{n_k}(d\tilde{z}) = \int \hat{P}_\varphi f(\tilde{z}) \mu_{n_k}(d\tilde{z}) \nrightarrow \int f(\tilde{z}) P_\varphi^* \mu(d\tilde{z}) = \int \hat{P}_\varphi f(\tilde{z}) \mu(d\tilde{z})$ as $\hat{P}_\varphi f(\tilde{z})$ is may not be continuous. However, $\hat{P}_\varphi f(\tilde{z})$ is bounded and $\mathcal{B}_{\tilde{J}}$-measurable. Theorem 3.5 in Molchanov and Zuyev implies that $\int \hat{P}_\varphi f(\tilde{z}) \mu_{n_k}(d\tilde{z}) \to \int \hat{P}_\varphi f(\tilde{z}) \mu(d\tilde{z})$ if $\mu(\Delta \hat{P}_\varphi f) = 0$, where $\Delta \hat{P}_\varphi f$ is the *set of discontinuities of* $\hat{P}_\varphi f$.

Thus, it only suffices to show that the discontinuity set generated by φ *is sufficiently small under the limiting measure.* In order to achieve this property, proposition 1 in Ito is used to show that P_φ^* maps the set of *atomless measures, which will be denoted* $\mathcal{P}_0(\tilde{J}) \subset \mathcal{P}(\tilde{J})$, *into itself. The proof will be complete if it can be shown that* $\mu \in \mathcal{P}_0(\tilde{J})$ *and* $\mu(\Delta \hat{P}_\varphi f) = 0$. As a measure is atomless if and only if $\mu(\{a\}) = 0$, $\{a\} \in \tilde{J}$ (i.e. it assigns zero measure to points, see Hildenbrand and Grandmont 1974, page 45), it suffice to restrict the cardinality of $\Delta \hat{P}_\varphi f$ to be at most countable and to show that $\mathcal{P}_0(\tilde{J})$ is closed. The latter property will be insured by imposing conditions on P_φ in section 3.2 and 3.3. The former will be guaranteed by restricting the discontinuity set of φ as in assumption 2.

Related results on the existence of Invariant measures.

Because the expectation correspondence in Duffie, et. al. G is defined as map $z \mapsto \mu$, with $\mu \in \mathcal{P}(Z)$, using the arguments presented in section 2.5.1 and 3.1 it is easy to show that a THME (J, π) can be used to define a sequence of measures $\{\lambda_t\}_{t=0}^{\infty}$ in $\mathcal{P}(J)$ such that $\lambda_{t+1} = \pi^*\lambda_t$, where π^* is the adjoint operator associated with (J, π).

Grandmont and Hildenbrand showed that the continuity of π^* is sufficient to show the existence of an invariant measure λ, provided that J is a compact set and G is constructed from an equilibrium correspondence similar to the one presented in definition 6: every $\pi \sim G$ satisfies $\pi = \pi_\varphi$ with $\varphi \sim \Phi$ and $\Phi: J \times S \to J$. Provided that assumption 1 holds, π_φ follows from Lemma 2. As discussed in section 3.1, π^* is continuous iff $\hat{\pi}$ has the Feller property, where $\hat{\pi}$ is the semigroup operator associated with (J, π). Unfortunately, the authors could not show that $\hat{\pi}$ has this property and had to assume it (see Lemma 2 in Grandmont and Hildenbrand, page 263).

The arguments discussed in section 2.4 imply that φmay not be continuous, thus the result in Hildenbrand and Grandmont was considered unsatisfactory. Blume (1982) dispense with this assumption and took a rather different approach. Given a Markovian structure with time homogeneous transitions, the author used Fan's fixed point theorem to show the existence of an invariant measure for $G_B: \mathcal{P}(Z) \to \mathcal{P}(Z)$, where $G_B = \{\pi_\varphi^*: \pi = \pi_\varphi, \varphi \sim \Phi\}$. As G_B was assumed to be nonempty, to have closed graph and Z to be compact, the required upper-hemicontinuity followed immediately. However, to apply Fan's theorem, G_B has to be convex valued. That is, if $\lambda_1', \lambda_2' \in G_B(\lambda)$, with $\lambda_1' = \pi_{\varphi_1}^*\lambda$, $\lambda_2' = \pi_{\varphi_2}^*\lambda$ and $\varphi_1, \varphi_2 \sim \Phi$, then $\lambda' \in G_B(\lambda)$ with $\lambda' = (\alpha)\pi_{\varphi_1}^*\lambda + (1-\alpha)\pi_{\varphi_2}^*\lambda$, $(\alpha)\pi_{\varphi_1}^* + (1-\alpha)\pi_{\varphi_2}^* \in G_B$ and $\alpha \in [0,1]$. To guarantee this latter property, Blume assumed that S is characterized by an atomless measure. Clearly, if S is a finite set, this last assumption is not realistic. The arguments in section 3.2 try to fill this gap. Even if S is a compact, uncountable set and $p(s,.)$ is atomless $\forall s \in S$, as discussed at the end of section 3.1, the results in Blume only shows that G_B has a fixed point, which is equivalent to $IM(\Phi) \neq \emptyset$ but weaker than $IM(\varphi) \neq \emptyset$ for any $\varphi \sim \Phi$ satisfying assumptions 1, 2 and the additional hypothesis presented in section 3.3. As discussed in section 4, this last fact has important numerical implications as it allows approximating φ instead of Φ.

The results in Blume highlighted the necessity of a "convexified" correspondence, G_B, in order to prove the existence of an invariant measure. This was the approach taken by Duffie, et. al. (1994), theorem 1.1, to show the existence of an *ergodic* invariant measure. As discussed in section 2.5.1, provided the existence of self-justified set and that G is convex valued, Duffie, et. al. (1994) showed that a refinement of a THME, called conditionally spotless, has an ergodic invariant measure. The following definition states this notion of equilibrium formally:

Definition A1: Let $\mathcal{P}_F(S \times \hat{Z}) =$
$\{\mu \in \mathcal{P}(S \times \hat{Z}) | \exists\ a\ function\ h, h: S \to \hat{Z}, with\ Supp(\mu) = Gr(h)\}$. *A THME* (J, π) *is spotless if* $\pi(z) \in \mathcal{P}_F(S \times \hat{Z})$ *for all* $z \in J$. *A THME* (J, π) *is called conditionally spotless if for all* $z \in J$, $\exists\ M \subset \mathcal{P}_F(S \times \hat{Z}) \cap G(z)$, $\eta \in \mathcal{P}(M)$, $\pi(z) = \int v\, d\eta(v)$ *and* G *is convex valued.*

Note that the existence of a spotless THME removes the possibility of sunspots discussed in Lemma 1: given $z_t \in J$, there is a measure $\mu_{z_t} \in G(z_t) \cap \mathcal{P}_F(S \times \hat{Z})$, which gives the conditional distribution of z_{t+1}, $\hat{z}_{t+1} = h(s_{t+1})$ and $\mu_{z_t}(Gr(h)) = 1$. Intuitively, each pair (z_t, s_{t+1}) is associated with a unique $\hat{z}_{t+1}$ or equivalently $\hat{z}_{t+1} = h_{\mu_{z_t}}(s_{t+1})$. Note that it is possible to refine even more a spotless THME by letting $\hat{z}_{t+1} = h_{z_t}(s_{t+1})$, where the measurability of f has to be shown and $z_{t+1} \sim \mu \in \mathcal{P}(S \times \hat{Z})$ has to be defined accordingly. *The results in section 3 and 4 hold for this last type of equilibria.*

To show the existence of an ergodic invariant measure for a spotless THME the authors proceeded in 2 steps. First, they applied Fan's fixed point theorem to $T \equiv E \circ m_2 \circ m_1^{-1}: \mathcal{P}(J) \to \mathcal{P}(J)$, where $m_1: \mathcal{P}(Gr(G_J)) \to \mathcal{P}(J)$, $m_2: \mathcal{P}(Gr(G_J)) \to \mathcal{P}(\mathcal{P}(J))$give the marginals of $\mathcal{P}(Gr(G_J))$ and $E\eta \equiv \int \mu d\eta(\mu), \eta \in \mathcal{P}(\mathcal{P}(J))$ is the mean of η, which is uniquely defined by the Riesz representation theorem for continuous function[11]. As T is a continuous linear functional and G_J is upper-hemicontinuous. This was assumed in Duffie, et. al. In the context of this paper, a similar property follows from theorem 3.1 in Blume under assumption 1 provided that G_J is constructed from Φ using Lemma 2. However, as discussed in section 2.5.2, this procedure only captures a subset of all possible recursive equilibria. Under these 2 properties, T is also upper hemi-continuous[12]. As J is a self-justified set, G_J is nonempty. T is convex valued as G assumed to be so. Finally, as $\mathcal{P}(J)$ is nonempty, (weakly) compact and convex, T has a fixed point. Second, the authors showed that any λ with $\lambda = T(\lambda)$ also satisfies $\lambda = \pi \cdot \lambda$. In order to derive this result, Duffie, et. al. (1994) defined a transition function $P: J \to \mathcal{P}(\mathcal{P}(J))$ and showed that $E \circ P(z) \in G_J(z)$ λ-a.e. This last fact implies $\pi(z) = \int v \, d\eta(v)$ almost everywhere for $\eta \in \mathcal{P}(\mathcal{P}(J))$.

To obtain an ergodic invariant measure for a conditionally spotless THME, which is defined for economies with a finite number of exogenous shocks, $\mathcal{P}(J)$ should be replaced with $G_J(z) \cap \mathcal{P}_F(S \times \hat{Z})$. This implies that G_J is convex valued: definition A1 assures that for any $z \in J$, there exist an expectation correspondence $\hat{g}$ which is convex valued as it contains any possible randomization $\mathcal{P}(M)$ over spotless transitions $M \subseteq \mathcal{P}_F(S \times \hat{Z}) \cap G(z)$ for any $z \in J$. A selection $\pi(z) \sim \hat{g}(z)$ is constructed by changing M, $\eta \in \mathcal{P}(M)$ and computing $\pi(z) = \int v \, d\eta(v)$. The assumption that G is convex valued can be

[11] See Theorem 14.12 in Aliprantis and Border (2006, page 496).
[12] See Grandmont (1983, page 158).

done w.l.o.g. provided that it can be replaced by $\hat{g}$ once transitions f are allowed to depend on "contemporaneous" sunspots (α_t) which select among randomized spotless transitions.

Unfortunately, the discussion above implies that the transition functions generated by a conditionally spotless THME are affected by sunspots; a fact that affects the computability of the recursive structure. The authors did not prove the existence of an ergodic invariant measure for some spotless THME (definition 4), which generate sunspots free stationary transition function. The purpose of this paper is to show this result for a restriction of all possible spotless THME (i.e. those generated from Feng, et. al.'s recursive structure).

Technical appendix of section 3.3

The difference between conditions 1) and 3) has 2 important consequences. First, condition 1) allows S being a finite set. This fact follows from equation 5) and is discussed extensively in the preliminary remark of lemma 3 in section A.1.2 (see equations A.5 and A.6 in the preliminary remark of the proof of lemma 3). As was argued in section 2, the existence of a sequential equilibrium follows from mild assumptions for this type of economies. This is the bright side. On the other hand, however, proving the existence of an invariant measure requires condition 2), which is very challenging to derive from primate conditions. That is, if $\#S < \infty$, the existence of the sequential equilibrium and of the recursive structure in Feng, at. al. can be derived from primitive assumptions of the model. As can be seen in Martinez and Pierri (2018) and Pierri and Reffett (2018), the assumptions needed to prove the existence of an invariant measure when $\#S < \infty$ imply restrictions on S, φ and the number of agents. Second, condition 3) allows proving the existence of an invariant measure imposing only this additional requirement to assumptions

1) and 2). Under this strengthening, condition 2 can be replaced by a recently proved result (see proposition 2.3 in Santos and Peralta Alva, 2013). Also, as will be explained in section 3.4, this condition follows from assuming that S is uncountable and from an additional mild requirement on its distribution, $p(s,.)$. However, showing the existence of a sequential equilibrium and, as stated in section 2.5.1 (see fact 2), of an appropriate recursive structure requires strong restrictions on endogenous variables. This last fact will be discussed in section 5.2.

In summary, there is a tradeoff between the strength of the conditions which guarantee the existence of a recursive structure and its stationarity or, similarly, between the mildness of the assumptions required to prove the existence of a sequential equilibrium and to prove the existence of an invariant measure.

From the preceding discussion it is clear that the crucial step in the existence of an invariant measure and its ergodicity is to insure that the non-atomicity / absolute continuity of a sequence of measures is preserved under *weak** limits. This can be seen by noting that properties b) and c) in theorems 1 and 2 requires, respectively, the closedness of $\mathcal{P}_0$ and $\mathcal{P}_1$ and that, as was shown in lemmas 3 and 4, these properties impose restrictions on the Markov operator. Section 3.4 discussed how these restrictions reflect on φ and the primitives of the model. The following example illustrate the problem at hand.

Example 1 (non-uniform boundness of densities): Let $P: S \times \mathcal{B}_S \to [0,1]$ be a transition function with $S = [0,1]$, $P(s, \{s/2\}) = 1$ and $\theta = U[0,1]$. Note that condition 1 is satisfied as $P(s, \{a\}) = 0$ except for $s = 2a$ with $\theta(\{2a\}) = 0$. Thus, under lemma 3, $P_\varphi^*: \mathcal{P}_0([0,1]) \to \mathcal{P}_0([0,1])$, where $\varphi(s) = s/2$. Note then that property a) in theorem 1. However, property b) will not be satisfied. Let $\mu_1 = P_\varphi^* \theta$ and $A = [0, a]$ with $0 < a < 1$. Then $\mu_1(A) = 2a$, that is, $\mu_1 = U[0,1/2]$ which has a density of 2. In general, $\mu_n = U[0,1/2^n]$ with $\mu_n = P_\varphi^* \mu_{n-1}$. Thus, $\{\mu_n\}$ has an associated sequence of densities of $\{2^n\}$, which is not a uniformly bounded sequence of functions. It is not surprising then that Kempton and Persson (2015, page 11) show that absolutely continuity is preserved under *weak** limits if the sequence of densities associated with $\{\mu_n\}$ is uniformly bounded.

This paper proved that that absolutely continuity is preserved under *weak** limits by imposing condition 4), that is slightly weaker than the uniform integrability of densities (see Diestel, 1991 for a detailed discussion), which is in turn weaker than the mentioned uniform boundness.

The example above shows that $\mathcal{P}_0([0,1])$, the subset of atomless measures in $\mathcal{P}([0,1])$ generated under the action of P_φ^*, is not closed as it has a sequence of measures in it weakly converging to a Dirac measure at 0.

Condition 2), by lemma 3, and condition 3), by lemma 4, guarantee the closedness of $\mathcal{P}_0$ for the case of finite and uncountable exogenous shocks respectively. The latter result relies on lemma 2.3 in Santos and Peralta Alva (2013) that exploits the iterative nature of μ_n, generated by applying P_φ^* successively, under a slightly stronger assumption on P_φ than condition 3).

As it is also shown in lemma 4, condition 4) assures the closedness of $\mathcal{P}_1$. This condition implies that the family of measures $\{P_\varphi(z,.) | z \in \tilde{J}\}$ is absolutely continuous w.r.t. θ and that small θ-measure sets have $P_\varphi(z,.)$-measure uniformly bounded by ε. This last condition is weaker than the uniformly integrability of densities, denoted by $\bar{p}_\varphi(z,z')$, as the latter requires $\int_B |\bar{p}_\varphi(z,z')| \theta(dz') < \varepsilon$ while the former only implies $\int_B \bar{p}_\varphi(z,z') \theta(dz') < \varepsilon$ (see Diestel, 1991). Although the distinction is subtle, it has important consequences: if $\int_B \bar{p}_\varphi(z,z') \theta(dz') < \varepsilon$ implies $\int_B |\bar{p}_\varphi(z,z')| \theta(dz') < \varepsilon$ for any $z \in \tilde{J}$, then $\bar{p}_\varphi(z,z')$ is bounded away from zero in $\tilde{J} \times \tilde{J}$. But in this case, exercise 11.4

in Stokey, Lucas and Prescott implies that P_φ satisfies the Doeblin condition (i.e. $\theta(B) < \delta$ implies $\int_B \bar{p}_\varphi(z,z')\theta(dz') < 1 - \varepsilon$ for any $z \in \tilde{J}$), which is a sufficient for the existence of an ergodic invariant measure (see page 345-8 for a detailed discussion). A similar result holds if $\bar{p}_\varphi(z,z')$ is bounded above in $\tilde{J} \times \tilde{J}$.

Consequently, by the discussion in example 1 and in the preceding paragraph, in this paper it will not be assumed that densities are neither bounded nor uniformly integrable as it suffice to restrict the Markov operator only to condition 4.

Note that assumption 2', like assumption 2, represents an upper bound on the genericity of the multiple equilibria problem discussed in section 2.4. As remark 1 suggests, condition 4 is stronger than condition 3. Thus, as any invariant measure under condition 4 is absolutely continuous with respect to the Lebesgue measure, the constraint imposed by $\mu(\Delta\varphi) = 0$ in theorem 1 is now less restrictive. Thus, $\Delta\varphi$ can be an uncountable set as long as it has zero Lebesgue measure. Section 5.2 will discuss this issue in the context of a concrete application.

Finally, the strategy in lemma 4 is different from the one used in lemma 2.3 in Santos and Peralta Alva as $IM(\Phi,\mathcal{P}_1)$ does not contain sequences of the form $\mu_n = P_\varphi^* \mu_{n-1}$. In turn, lemma 4 shows how condition 4 implies that small θ-measure sets have arbitrary small μ_n-measure, where $\{\mu_n\}$ is any sequence in $IM(\Phi,\mathcal{P}_1)$, and that this latter property guarantees that absolute continuity is preserved under *weak** limits of $\{\mu_n\}$.

Technical appendix of section 4.2

In order to present the results for this section some additional definitions are required. Let $(\tilde{J},\mathcal{B}_{\tilde{J}})$ be a measurable space and $(\tilde{J}^t,\mathcal{B}_{\tilde{J}}^t) = (\tilde{J} \times \ldots \times \tilde{J}, \mathcal{B}_{\tilde{J}} \times \ldots \times \mathcal{B}_{\tilde{J}})$

the associated product space. Let $A = A_1 \times \ldots \times A_t$ be a measurable rectangle (see Stokey, Lucas and Prescott page 195 for a definition) in $\mathcal{B}_{\tilde{J}}^t$. Let $\varphi \sim \Phi$ and $z_0, \ldots, z_t \in \tilde{J}$. As long as t is finite, by virtue of the Caratheodony and Hahn theorems and theorem 7.13 in Stokey, Lucas and Prescott, $\mu^t(z_0, A)$, defined by $\mu^t(z_0, A) = \int \cdot_{A_1} \ldots \int \cdot_{A_t} P_\varphi(z_{t-1}, dz_t) \ldots P_\varphi(z_0, dz_1)$, can be uniquely extended to a probability measure in any set of $\mathcal{B}_{\tilde{J}}^t$. Note that$\int \cdot_{A_i}$ denotes integration with respect to $P_\varphi(z_{i-1}, dz_i)$.

Analogously, let $B = A_1 \times \ldots \times A_T \times \tilde{J} \times \ldots$be a finite measurable rectangle (see page 221 of Stokey, Lucas and Prescott for a definition) and $\mathcal{L}$ its power set. Let $\mathcal{M}$ be the algebra generated by finite unions in $\mathcal{L}$ and $\mathcal{F} = \mathcal{B}_{\mathcal{M}}$, that is $\mathcal{F}$ is the sigma field generated by $\mathcal{M}$. Further, $\mu^\infty(z_0, B) = \int \cdot_{A_1} \ldots \int \cdot_{A_T} P_\varphi(z_{T-1}, dz_T) \ldots P_\varphi(z_0, dz_1)$ can be shown to be extended to $\mathcal{F}$ in 2 steps. First, using the Caratheodony and Hahn theorems it is possible to extend $\mu^\infty(z_0, B)$ to $\mathcal{M}$ and then to $\mathcal{F}$. Later, using standard arguments for processes with a finite dimension distribution (see Shiryaev 1996, Ch. 9), $\mu^\infty(z_0, B)$ can be shown to be countably additive.

Standard results (see for instance exercise 8.6 in Stokey, Lucas and Prescott) imply that $(\Omega, \mathcal{F}, \mu^\infty(z_0, .))$ is a Markov process with stationary transitions P_φ. Let $\Omega = \tilde{J} \times \tilde{J} \times \ldots$ with typical realization $\omega \in \Omega$. As Ω is the space of sequences, it is natural to define a $\mathcal{F}_t$-measurable random variable $z_t: \Omega \to \tilde{J}$, where $\omega(t) = z_t = z_t(\omega)$ denotes a typical realization and $\{\mathcal{F}_t\}$ is a sequence of nested sigma algebras on $\{\times_{i=1}^t \tilde{J}(i)\}$, where $\tilde{J}(i) = \tilde{J}$ for $i \geq 1$. The shift operator is denoted by $T: \Omega \to \Omega$. A set $A \in \mathcal{F}$ is called T-*invariant* if $TA = A$[13].

Let $\mu^\infty(z_0, B) \equiv \boldsymbol{P}_{\varphi, z_0}(B)$ and note that under the same assumptions $\boldsymbol{P}_{\varphi_j, z_0}(B)$ can be analogously defined if $\tilde{J}$ is replaced by K, which was supposed to be

[13] Exercise 6.2 in Varadhan shows that this definition can be used w.l.o.g.

compact in assumption 4-i). Further, $\boldsymbol{P}_{\varphi,\mu} \equiv \int_{A_0} \boldsymbol{P}_{\varphi,z_0}\mu(dz_0)$ can be used to define a stochastic process $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$ which allows to randomize z_0 as μ is a measure on $(\tilde{J}, \mathcal{B}_{\tilde{J}})$. $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$ is said to be *stationary* if $\boldsymbol{P}_{\varphi,\mu}[C(t,n)] = \boldsymbol{P}_{\varphi,\mu}[C(t',n)]$ for all $n \geq 0$ and $t \neq t'$ with $C(t,n) = \{\omega \in \Omega: [z_{t+1}(\omega), .., z_{t+n}(\omega)] \in C\}$. $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$ is said to be ergodic if $\boldsymbol{P}_{\varphi,\mu}(A) \in \{0,1\}$, where A is a T-invariant set.

A.1.2) Proofs

Preliminary Remark on $\tilde{J}$

As theorem 1 will show that there exist $\mu \in \mathcal{P}_0$ with $\mu = P_{\varphi}^{*}\mu$ (i.e. an invariant measure exists and it is atomless), it is necessary for the state space of the process defined by $(\tilde{J}, P_{\varphi})$ to be uncountable. This is because the candidate measure μ_N , with $\mu_{N_k} \rightarrow_{Weak*} \mu$, satisfies $Supp(\mu_K) \subseteq \tilde{J}$ as it is constructed applying iteratively P_{φ}^{*}.

Fortunately, the results used to guarantee the non emptiness of C_j for $j \geq 1$ (i.e. the set which contains all initial states, $\tilde{z}_0$, of any j-period economy) which were discussed in sections 2.5.1 (fact 2), 2.5.2 and 5.1 can be used to guarantee the desired result. In particular, Theorems 25.1 in Magill and Quinzii (1996) and theorem 4.1 together with section 5 in Mas-Colell and Zame (1996) for economies with finite and infinite number of shocks respectively can be used to show the existence of a sequential competitive equilibrium (see Definition 1) for a truncated economy $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^{I}, T\right]$, with $T < \infty$. The optimality conditions in Definition 1 for this economy are:

A1) $c^i = e^i(s) + \theta^i d(s) - \theta_+^i q$

A2) $\left[q\left(u_s^i(c^i)\right)' - \beta E_{p(s,.)}(m_+^i)\right]\left[\theta_+^i - \bar{B}\right] = \vec{0}$

Where short sale constraints $\bar{B}$ are assumed to hold (see sections 2.5.2 and 5.1) and $\theta_+^i = 0$ if $\theta^i = \theta^i(\sigma_{T-1})$. In the sequential economic literature, if $\theta_+^i = \theta^i(\sigma_0)$, it is customary to assume that $\theta_-^i \equiv \theta^i = 0$ and $\sigma_0 \equiv s_0$ is supposed to be fixed. However, in the recursive literature, both θ_-^i and σ_0 are allowed varying as $\tilde{z}_0 = [s_0, \theta_-^i, \hat{z}_0]$, where $\hat{z}_0$ contains the rest of the state space.

Moreover, the existence of equilibria for $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^I, T\right]$ requires that $e^i(s_0) > 0$(see assumption A.2 in Magill and Quinzii, page 858). Thus, provided the rest of the assumptions mentioned in sections 2.1, 2.5.1 and 5.1 hold, as noted by Duffie, et. al. (Lemma 3.4), θ_-^i and s_0 can be chosen arbitrarily as long as $e^i(s_0) + \theta_-^i d(s_0) > 0$, which can be considered the initial endowment of goods if the exogenous state is s_0. Formally, it suffices to assume that:

Definition A2: *The initial distribution of assets θ_- will be called admissible and denoted $\theta_- \in \Lambda$ if is feasible and satisfies $Min_{i\in I, s\in S} e^i(s) + \theta_-^i d(s) > 0$.*

Remark A1: $\tilde{J} = S \times \Lambda \times \hat{Z}$, where $\Lambda \times \hat{Z}$ is uncountable because and has no isolated points: i) Λ is uncountable and has no isolated points according to definition A2, ii) under the assumptions made in sections 2.1, 2.5.2 and 5.1, $C_j \neq \emptyset$ independently of the cardinality of S (i.e. an equilibrium for $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^I, \theta_-\right]$ exists independently of the cardinality of S) for any $\theta_- \in \Lambda$ (i.e. for any admissible θ_-).

Remark 1 is frequent in applications: see for instance Duffie, et. al. (1994) section 3 and Kubler and Schmedders (2003) page 1777. Typically θ_- describes individual wealth, any predetermined level of asset holdings or the capital stock. Consequently, in numerical approximations θ_- is supposed to be contained in an uncountable subset of $\mathbb{R}$ and its properties (i.e. compactness) can be defined independently of those characterizing $\hat{Z}$ as (s, θ_-) are initial conditions of some sequential competitive equilibrium. Thus, Λ is compact if and only if it is closed. This last property is easily verifiable as can be seen in Kubler and Schmedders (2003) (see lemma 1, page 1776). As will be seen in the proof of lemma 3, the crucial property of Λ, besides its cardinality, is the lack of isolated points. This property follows w.l.o.g. from definition A2.

In all the proof in the appendix, except that it is mentioned explicitly, it will be assumed that the state space can be written as $\tilde{J} = S \times \Lambda \times \hat{Z}$ and that Λ is admissible.

Theorem 1

Preliminary Remark

As discussed in section 3.1, theorem 1 will fail if any selection $\varphi \sim \Phi$ has an uncountable discontinuity set. Fortunately, there are no examples in economics where such a function characterizes the (recursive) equilibrium set. In fact, the literature (see for instance Santos, 2002) has only found examples with jump discontinuities. As will be claimed in the preliminary remark of theorem 2, there are no available methods to numerically approximate a function with an uncountable discontinuity set. Consequently, if a model does not satisfy assumption 2 it can be said to be *non-computable.*

Theorem 3.5 in Molchanov and Zuyev (2011) only requires the discontinuity set to have zero measure under the limiting measure (i.e. $\mu_n \to_{Weak*} \mu$ and $\mu(\Delta\varphi) = 0$). Thus, it is only necessary, under assumption 2, for μ to be atomless. The arguments in sections 3.2 and 3.3 illustrate the usefulness of properties a) and b) to achieve this purpose. In particular, proposition 1 in Ito (1964) holds under quite mild assumptions on the primitives and assures property a). The critical property is then b), which hold under rather different assumptions depending on the cardinality of S.

As discussed in section 3.1, theorem 3.5 in Molchanov and Zuyev restores the continuity of the adjoint operator by extending the set of adequate functions in the *weak** topology from continuous to Borel measurable if the limiting function is atomless and assumption 2 holds. The following example illustrates the importance of the atomless assumption when dealing with a Borel measurable function in the $weak^*$ topology.

<u>*Example A.1 (atomic measures and tight spaces)*</u>[14]:Let $P: S \times \mathcal{B}_S \rightarrow [0,1]$ be a transition function with $S = [0,1]$ and $P(s, \{s/2\}) = 1$. Let $\{\lambda_n\}$ be a sequence of Dirac measures with $\lambda_n = \delta_{(1/2)^n}$. Thus, $\lambda_n \rightarrow \delta_0$, where the convergence is in distribution. Define the bounded Borel measurable function $f(s) = \{1\ if\ s = 0\,; 0\ otherwise\}$ and $\delta_0 \equiv \lambda$. Then $\int f(s)\,\lambda_n(ds)$ =0 and $\int f(s)\,\lambda(ds) = 1$ which in turn implies that $\lambda_n \nrightarrow_{weak^*} \lambda$. The reason behind the lack of $weak^*$ convergence is the impossibility to reduce the measure of the discontinuous part of f.

Proof of theorem 1

Let Φ be an equilibrium correspondence according to definition 6 which satisfies assumption 1. By Lemma 2 $P_\Phi = \{P_\varphi : \varphi \sim \Phi\} \neq \emptyset$ and upper hemi continuous (see for instance proposition 2.2. in Blume, 1982). If P_Φ is convex valued, an ergodic invariant measure can be shown to exist using proposition 1.3 in Duffie, et. al. (1994) (see page 757).

If P_Φ is not convex valued, suppose that assumption 2 together with properties a) and b) in theorem 1 hold. Choose any $\lambda_0 \in \mathcal{P}(\tilde{J})$ and construct a non-oscillating sequence of measures $\{\mu_N\}$ with $\mu_N = h(\{\lambda_n\})$, where h averages

[14]This example borrows from Stokey, Lucas and Prescott (1989), page 336. Note that $\{\lambda_n\}$ satisfies $\lambda_n = P \cdot \lambda_{n-1}$. That is, it is possible to generate a sequence of non-atomic measures out of the action induced by P. I would like to thank Prof. R. Fraiman for pointing this out to me.

the first N-1 elements of $\{\lambda_n\}$ and λ_n satisfies $\lambda_n = P_\varphi^* \lambda_{n-1}$. The dependence of $\{\mu_N\}$ on λ_0 can be omitted w.l.o.g. as the initial condition is arbitrary.

As $\mu_N \in \mathcal{P}(\tilde{J})$ for any N, Helly's theorem (see Stokey, Lucas and Prescott (1989) page 372 and 374) implies that $\{\mu_N\}$ has a weakly convergent subsequence. That is, $\{\mu_{N_k}\} \to_{weak*} \mu$.

For notational simplicity $P_\varphi^* \lambda$ and $\hat{P}_\varphi f$ will be replaced by $\pi \cdot \lambda$ and $\pi \cdot f$ as P_φ with $\varphi \sim \Phi$ will be held constant throughout the proof.

For any $f \in C(\tilde{J})$ note that:

$$\left| \int f(z)\mu(dz) - \int (\pi \cdot f)(z)\mu(dz) \right| \leq \left| \int f(z)\mu(dz) - \int f(z)\mu_{N_k}(dz) \right| + \left| \int f(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu_{N_k}(dz) \right| + \left| \int (\pi \cdot f)(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu(dz) \right| \quad (A.3)$$

From the corollary of theorem 8.1 in Stokey, Lucas and Prescott (1989) (page 215), $(\pi \cdot f): Z \to \mathbb{R}$ is a bounded $\mathcal{B}_{[\tilde{J}]}$-measurable function. Further, from property a) and b), μ is atomless. Under assumption 2, $\mu(\Delta\varphi) = 0$. Then, from theorem 3.5 in Molchanov and Zuyev (2011, fact f), the third term in A3 can be made arbitrarily small. Further, noting that $\{\mu_{N_k}\} \to_{weak*} \mu$ and $f \in C(\tilde{J})$, the first and the third term in A.3 can be made arbitrarily small.

Following the same reasoning as in Stokey, Lucas and Prescott (1989) page 377, the second term satisfies:

$$\left| \int f(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu_{N_k}(dz) \right| \leq 2\|f\|/N \quad (A.4)$$

Where $\|.\|$ is the sup-norm. Thus, for an N arbitrarily large, $\int f(z)\mu(dz) = \int(\pi \cdot f)(z)\mu(dz) = \int f(z)(\pi \cdot \mu)(dz)$, where the last equality follows from theorem 8.3 in Stokey, Lucas and Prescott (1989) (see page 216). Thus, $\int f(z)\mu(dz) = \int f(z)(\pi \cdot \mu)(dz)$. As f was arbitrary, by virtue of corollary 2 of theorem 12.6 in Stokey, Lucas and Prescott (1989) (page 364) $\mu = \pi \cdot \mu$, QED.

■

Lemma 3

Preliminary remark

The proof of this lemma requires π to be θ-nonsingular. A transition function is said to be θ-nonsingular if for any measurable set B, $\theta(B) = 0$ implies $\pi(z, B) = 0$ θ-a.e. As θ is atomless this is equivalent to say that the set D, defined below, is a finite set.

$$D = \{z \in \tilde{J}: \pi(z, B) > 0 \; if \; \theta(B) = 0\} \quad (A.5)$$

Additionally B was restricted to be a point. For those transition functions defined by lemma 2, Ito (1964) show that any non-constant possibly discontinuous many-to-one function $\varphi \sim \Phi$ will generate a θ-nonsingular transition function. This can be seen by written π_φ in lemma 2 as

$$\pi_\vartheta(z, B) = p\{s|s' \in S: \varphi(z, s') = a\} = p\{s|s' \in S: \{s_i'\} \cap \tilde{\varphi}^{-1}(z, .)(a_{\hat{z}})\} \quad (A.6)$$

Where $z = [s, \hat{z}]$, $p(s|.)$ is the s^{th} row of the transition matrix which defines the evolution of the exogenous process $\{s_t\}$, $B = \{s_i'\} \times B_{\hat{z}}$ was restricted to a point $a = \{s_i'\} \times a_{\hat{z}}$, $\varphi(z, s') = [s', \tilde{\varphi}(z, s')]$ is a vector valued function and $\tilde{\varphi}^{-1}(z, .)(B_{\hat{z}})$ is the z-section of the pre-image of $\tilde{\varphi}$ on $B_{\hat{z}}$.

From A.5 and A.6, it is clear that under assumption 2, $\#D < \infty$ provided that $\tilde{\varphi}(., s')$ is non-constant in z for all $s' \in S$. In section 5, the implicit function theorem is used to show that the model defined in section 2.1 generates θ-nonsingular transition functions.

Proof of lemma 3

Let $(\tilde{J}, \mathcal{B}_{\tilde{J}}, m)$ be a measure space. By assumption 1, $\tilde{J}$ is compact and by remark A1 this set could be written as $\tilde{J} = S \times \Lambda \times \hat{Z}$, where Λ contain all admissible states and $\Lambda \times \hat{Z}$ is uncountable and has no isolated points. Further, note that any measure in $(\Lambda, \mathcal{B}_\Lambda)$, denoted m_Λ, is a Radon measure as Λ is a Hausdorff metric space and m_Λ is: i) defined over a Borel sigma-algebra $(\mathcal{B}_\Lambda)$, ii) regular as it is a measure on a Hausdorff (compact) metric space (Λ), iii) $\mathcal{B}_\Lambda$-finite as it is a probability measure. Thus, as Λ has no isolated points, $(\Lambda, \mathcal{B}_\Lambda)$ has an atomless measure m_Λ^A (see Bogachev 2007, page 136) which in turn implies by remark A1 that there is a measure m^A in $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ that is also atomless. The first part of the lemma is completed by setting $m^A \equiv \theta$.

Let $\mathcal{P}_0(\tilde{J}) \subset \mathcal{P}(\tilde{J})$ be the set of atomless measures in $\mathcal{P}(\tilde{J})$ generated by π, starting from θ. It follows from proposition 1 in Ito (1964) that π maps $\mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ as π is θ-nonsingular by condition 1. Finally, condition 2 is just the definition of a *weak*-closed* set applied to $\mathcal{P}_0(\tilde{J})$.

■

Example A.2 ($\theta \in \mathcal{P}(\tilde{J})$ and θ is atomless). The reference measure θ could be a mixed joint density: $\theta(s \times A) = P(s = \{s\}, \hat{z} \in A) = \int_A p_{S,\hat{Z}}(s, \hat{z}) d\hat{z}$ where $p_{S,\hat{Z}}(s, \hat{z}) = \theta(s \times \{\hat{z}\}) = 0$ is a density function on $\hat{Z}$ which may vary with any $s \in S$. From fact 14 page 45 in Hildenbrand and Grandmont (1974), θ is atomless.

Lemma 4

Preliminary Remark

The implication of condition 4) requires to show the *weak** closedness of $IM(\varphi, \mathcal{P}_1)$. The proof below shows that $IM(\varphi, \mathcal{P}_1)$ is *weak** sequentially compact (i.e. that every bounded sequence in $IM(\varphi, \mathcal{P}_1)$ has a *weak** convergent subsequence). As $\mathcal{P}_1$ can be endowed with the Prohorov metric (see Hildenbrand and Grandmont 1974, page 49), sequential compactness implies that $IM(\varphi, \mathcal{P}_1)$ is not only closed but also compact.

Proof of lemma 4

For the existence of an atomless measure on $\tilde{J} = S \times \Lambda \times \hat{Z}$ with S uncountable and compact, let θ be the uniform measure on $\tilde{J}$.

For property a), note that condition 3) implies that P_φ is θ-nonsingular. Thus, proposition 1 in Ito (1964) applies just as in the proof of lemma 3.

In order to prove property b), note that any point $\{a\} \in \tilde{J}$ has zero Lebesgue measure. Thus, under condition 4), proposition 2.3 in Santos and Peralta Alva (2013, page 8) can be used to guarantee the desired result.

Property c) will be proved in 3 parts: i) $IM(\varphi, \mathcal{P}_1) \neq \emptyset$. As $\tilde{J}$ is compact, Helly's theorem implies the existence of a weak* converging subsequence in $IM(\varphi, \mathcal{P}_1)$ denoted w.l.o.g. $\mu_n \rightarrow_{weak*} \mu$. It will be shown that: ii) μ is absolutely continuous w.r.t θ, iii) $\mu \in IM(\varphi, \mathcal{P}_1)$.

In what follows it will be assumed w.l.o.g. that $\theta(dz) = dz$. This is done for expositional purposes only.

i) Standard results (See Billingsley 1968, page 422) imply that condition 4) is equivalent to the following statement: for any measurable set B, $\theta(B) = 0$ implies $SUP_{z \in \tilde{J}}[\pi_\vartheta(z, B)] = 0$. Thus, π_ϑ is θ-nonsingular. By proposition 1 in Ito (1964), $\pi_\vartheta: \mathcal{P}_1 \rightarrow \mathcal{P}_1$. Under the same condition, lemma 2.3 in Santos and Peralta Alva (2013) also holds, which implies that $\mathcal{P}_1$ is *weak** closed. Under assumption 2, theorem 1 implies that $IM(\varphi, \mathcal{P}_1) \neq \emptyset$.

ii) By the characterization of absolutely continuity in Billingsley (1968, page 422), it suffice to show that for any $\varepsilon > 0$, $\exists \delta > 0$ such that $\theta(B) < \delta$ implies $\mu(B) < \varepsilon$. Condition 4) implies that $\pi_\vartheta(z,.)$ is absolutely continuous w.r.t. θ for any $z \in \tilde{J}$. That is, $\pi_\vartheta(z, dz') = \bar{\pi}_\vartheta(z, z')dz'$ where $\bar{\pi}_\vartheta(z,.)$ is the density associated with $\pi_\vartheta(z, dz')$. Take any sequence $\{\hat{\mu}_n\} \in IM(\varphi, \mathcal{P}_1)$. Note that $\{\pi_\vartheta \hat{\mu}_n\}$ is a family of measures that satisfies the hypothesis of Helly's theorem and $\{\pi_\vartheta \hat{\mu}_n\} \in IM(\varphi, \mathcal{P}_1)$.

Let $\pi_\vartheta \hat{\mu}_n \equiv \mu_n$ and note that $\{\mu_n\} \in IM(\varphi, \mathcal{P}_1)$ and has a *weak** limit denoted (passing to a subsequence if necessary) μ.

Note that $\mu_n(B) = \int_B h_n(z')\theta(dz')$ where $h_n(z') = \int \bar{\pi}_\vartheta(z, z')\, \mu_n(dz)$. But now note that $\mu_n(B)$ could be written as:

$$\mu_n(B) = \int_B h_n(z')\theta(dz') = \int \left[\int_B \bar{\pi}_\vartheta(z, z')dz'\right] \mu_n(dz)$$

Condition 4) implies that $\left[\int_B \bar{\pi}_\vartheta(z, z')dz'\right] < \varepsilon$ uniformly in z. Thus $\mu_n(B) < \varepsilon$. The arguments in the first part of lemma 3 imply that $\{\mu_n\}$ and μ are regular measures. Thus, B can be assumed to be open w.l.o.g. Now, the definition of *weak** convergence implies (see theorem 12.3-c in Stokey, Lucas and Prescott, page 358) $\mu(B) \leq liminf_n \mu_n(B)$. In order to complete the proof, by the preliminary remark of this lemma, it suffice to note that $liminf_n \mu_n(B) < \varepsilon$.

iii) It remains to show that $\mu \in IM(\varphi, \mathcal{P}_1)$. Take $\mu_n \to_{weak*} \mu$. Note that for any $f \in C(\tilde{J})$:

$$\lim_n \int f(z)\mu_n(dz) = \int f(z)\mu(dz)$$

$$= \lim_n \int f(z)[\pi\mu_n](dz) = \lim_n \int [\pi f](z)\mu_n(dz) = \int [\pi f](z)\mu(dz)$$

$$= \int f(z)[\pi\mu](dz) \quad (A.7)$$

Where the first equality in A.7 follows from the definition of *weak** convergence of $\mu_n \to_{weak*} \mu$, the second from $\{\mu_n\} \in IM(\varphi, \mathcal{P}_1)$, the third from theorem 8.3 in Stokey, Lucas and Prescott, the forth from theorem 3.5 in Molchanov and Zayev as μ is absolutely continuous w.r.t. θ (and thus atomless) and the last equality from theorem 8.3 in Stokey, Lucas and Prescott again. Note that A.7 implies $\int f(z)\mu(dz) = \int f(z)[\pi\mu](dz)$. As $f \in C(\tilde{J})$ is arbitrary, the proof is complete.

■

Proof of Proposition 1

Under assumption 1, lemma 2 implies that π is well defined (i.e. is a Markov operator). Under assumptions 3-i) and 3-ii) the result follows from equation A.6) by noting that $\{s_i'\} \cap \tilde{\varphi}^{-1}(z,.)(a_{\hat{z}})$ is either a point in S or $\emptyset$ for any $z \in \tilde{J}$.

■

Proposition 2

Preliminary remark

Arbitrarily selecting $\in \tilde{J}$, it will be shown that $\forall \varepsilon(z) > 0, \exists \delta(z) > 0$ such that $\theta(B) < \delta(z)$ implies $\pi(z, B) < \varepsilon(z)$. As $\tilde{J}$ is compact and $\varepsilon(z), \delta(z)$ are finite (real) numbers, it suffices to take $max_{z \in \tilde{J}}\, \varepsilon(z) = \varepsilon$ and $max_{z \in \tilde{J}}\, \delta(z) = \delta$.

For the first part of the proof the following fact will be useful: let θ be the Lebesgue measure and $R \subseteq \tilde{J} \subset \mathbb{R}^K$ a rectangle and μ^V its volume. That is, $R = [a_1, b_1] \times \ldots \times [a_K, b_K]$ and $\mu^V(R) = [b_1 - a_1] \ldots [b_K - a_K]$. Then, $\theta(B) = 0$ if $\forall \gamma > 0$, $\exists \{R_i\}_{i=1}^{\infty}$ with $B \subseteq \cup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$. The proof of the first part the proposition will be completed if it can be shown that for each $\varepsilon(z) > 0$, there exist an $\gamma > 0$ such that $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$ implies $\sum_{i=1}^{\infty} \pi(z, R_i) \leq \varepsilon(z)$ because $\theta(B) = 0$ as long as $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$.

Proof of proposition 2

Note that any positive $\pi_\varphi(z,.)$-measure rectangle, R_i, could be written as

$$R_i = \left[\varphi_1\left(z, s'_{1,i} - 2^{-1}h_{1,i}\right), \varphi_1\left(z, s'_{1,i} + 2^{-1}h_{1,i}\right)\right] \times \ldots \times \left[\varphi_K\left(z, s'_{K,i} - 2^{-1}h_{K,i}\right), \varphi_K\left(z, s'_{K,i} + 2^{-1}h_{K,i}\right)\right]$$

where the first coordinate is just $\left[s'_{1,i} - 2^{-1}h_{1,i}, s'_{1,i} + 2^{-1}h_{1,i}\right]$, φ_k and $s'_{k,i}$ denote any coordinate of φ for $1 \leq k \leq K$ and the elements of S that generates coordinate k of rectangle i.

Note assumption 3-iii) implies that $\varphi_k(z,.)$ is allowed to oscillate continuously, not necessarily forming a straight line, between $\varphi_k(z,x)$ and $\varphi_k(z,y)$ where $x = s'_{k,i} - 2^{-1}h_{k,i}$ and $y = s'_{k,i} + 2^{-1}h_{k,i}$. Thus, by theorem 2.27 in Aliprantis and Border (2006), $h_{k,i}$ is the length of the interval in the pre-image of $\varphi_k(z,.)$, where $\varphi_k(z,x)$ and $\varphi_k(z,y)$ are exactly the endpoints of the k^{th} coordinate of rectangle R_i.

Now equation A.6) implies that $\pi(z, R_i) \leq p\left(s, \cap_{k=1}^{K}\left[s'_{k,i} - 2^{-1}h_{k,i}, s'_{k,i} + 2^{-1}h_{k,i}\right]\right) = p\left(s, \cap_{k=1}^{K}\left[0, h_{k,i}\right]\right)$, where the inequality follows from the preceding discussion and the equality from assumption 3-iv) after normalizing $p(s,.)$ to be in the unit interval.

Now note that assumption 1 implies that $\mu^V(R_i)$ is finite as the range of any $\varphi \sim \Phi$ is bounded, and $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$ implies $lim_{i\to\infty}\mu^V(R_i) = 0$. Thus,

$$\pi_\varphi(z, R_i) \leq \varepsilon(z)_i 2^{-i} \quad \text{A.8)}$$

where $\varepsilon(z)_i = min_k h_{k,i}$.

Also from $lim_{i\to\infty}\mu^V(R_i) = 0$, equation A.8) implies that $lim_{i\to\infty}(\varepsilon(z)_i)$ is finite. Thus, $SUP_i\varepsilon(z)_i = max_i\varepsilon(z)_i = \varepsilon(z)$ and $\sum_{i=1}^{\infty}\pi(z, R_i) \leq \varepsilon(z)$, as $\sum_{i=1}^{\infty}2^i = 1$.

Now to prove the dependence of γ on $\varepsilon(z)$, let $R_{i,k}$ be the k^{th} coordinate of rectangle R_i. Note that assumption 3-iii) implies, by theorem 2.34 in Aliprantis and Border, that for all $i, \exists k$ with $R_{i,k} = [\varphi_1(z,x), \varphi_1(z,y)]$ and $[x,y]$ has length smaller or equal to $\varepsilon(z)$. Consequently, $\varepsilon(z)$ could be made arbitrarily small as desired and there will always be an associated γ such that A.8) holds. As z is arbitrary, the proof is complete.

■

Proof of remark 2: the result follows from replacing $p(s, \cap_{k=1}^{K}[0, h_{k,i}2^{-i}])$ by $p(s, \cap_{k=1}^{K}[LB(s), h_{k,i}2^{-i}])$ in equation A.8) and noting that $\varepsilon(z)_i = min_k \frac{h_{k,i-LB(s)}}{UB(s)-LB(s)}$, where z is a vector of the form $z = [s, \hat{z}]$, is a finite number for all $z \in \tilde{J}$.

Theorem 3

Preliminary Remark

As in the case of theorem 1, theorem 3 has to be applied to economies that has at least 1 selection $\varphi\sim\Phi$ with at most a zero Lebesgue measure discontinuity set. This restriction must be extended to all approximated economies which are characterized by $\varphi_j\sim\Phi_j$. This is because, as discussed in section 3.1, $IM\{\varphi_j, \mathcal{P}_1\}$ may be empty if the discontinuity set is allowed to have positive Lebesgue measure.

The discussion in section 2.4 suggests that the cardinality of the discontinuity set is associated with the number of possible equilibria. Thus, even though it is theoretically possible to have a discontinuity set with positive Lebesgue measure, *the endogenous laws of motions in this economy may not be computable even using state of the art procedures*.

In particular, the algorithm in Feng, et. al. (2013) computes an outer approximation of the equilibrium correspondence (i.e. $Gr(\Phi_j) \supseteq Gr(\Phi)$). Thus, assumption 4-iii) may not hold for this procedure. Further, in this procedure it is not clear how to impose assumption 3-iii) because the interpolation method used to convexify the computed equilibrium correspondence is not specified in the paper (see page 11 for an outline of the algorithm and pages 39 to 41 for details). The procedure in Kubler and Schmedders (2003) circumvent some of these problems as it provides a convenient spline-based interpolation method. Unfortunately, the sequence of approximating functions is assumed to be continuous and to converge in the sup-norm on K. Both facts taken together imply that the limiting function is continuous on K (see page 1782), which may be inadequate in the context of this paper.

There are spline based procedures which allows computing functions with an uncountable discontinuity set (see for instance Silanes, et. al. 2001). These procedures converge uniformly on $(K \times S)\backslash\Delta\varphi$. Unfortunately, the arguments in the proof of theorem 3 will show that this type of convergence is inadequate under assumption 4-ii) if $\Delta\varphi$ has positive Lebesgue measure.

It is worth noticing that in an algorithm that approximates $\tilde{J}$ using a sequence of correspondences or sets, theorem 3.5 in Santos and Peralta Alva (2013) can be used to prove the desired upper hemi-continuity and compact valuedness of Φ_j (assumption 1 applied to theorem 3). This is the case of the recursive equilibrium algorithm in Feng, et. al. However, as mentioned before, this procedure generates a sequence of correspondence Φ_j with $Gr(\Phi_j) \supseteq Gr(\Phi)$, which may be inadequate under assumption 4-iii). Finally, it is possible to construct φ_j using a policy function $\varrho_j: S \times Z_1 \to \tilde{Z}$ with $Z = S \times Z_1 \times \tilde{Z}$ as in Kubler and Schmedders (2003). This procedure lowers the dimension of the state space and thus the computational burden, measure in CPU time, of the algorithm. The authors provided a detailed spline procedure, but they did not take care of $\Delta\varphi$. It is a matter of future research to establish if the spline procedure in Silanes, et. al., which addresses $\Delta\varphi$ appropriately, fits into the framework of theorem 3. In particular, assumption 4-iii) should be carefully enforced as it involves all zero Lebesgue measure sets which, because of assumption 2'), do not belong to $(K \times S)\backslash\Delta\varphi$.

Finally, the other known recursive algorithms (see for instance Raad, 2013) may not suitable for simulations as it is not clear how to fit those procedures into the theoretical framework outlined in this paper or in Santos and Peralta Alva (2013).

Consequently, if all stationary laws of motion (i.e. all $\varphi \sim \Phi$) have a positive Lebesgue measure set of discontinuities, this economy may not be accurately computable and thus is beyond the scope of this paper.

The proof of this theorem will proceed in 3 steps: first it will be shown that there is a sequence of absolutely continuous measures $\{\mu_j\}$, with $\mu_j = \pi_{\varphi_j}\mu_j$, which has a *weak** limit μ that is also absolutely continuous. Second, using the first result, it will be shown that the evaluation map, $Ev(\varphi, \mu) \equiv \pi_\varphi \mu$, is jointly continuous when φ is endowed with the sup-norm topology and μ with the weak topology. Finally, using the second result, it will be shown that $\mu = \pi_\varphi \mu$.

Proof of theorem 3

i) Assumptions 1, 2'), 3-iii) and 3-iv) applied to $\{\varphi_j\}$ implies, by theorem 2, that $IM\{\varphi_j, \mathcal{P}_1\} \neq \emptyset$ for all j. Assumption 4-i) implies that the sequence

$\{\mu_j\}$, with $\mu_j = \pi_{\varphi_j}\mu_j$, satisfies the hypothesis of Helly's theorem as $\mathcal{P}_1 \subset \mathcal{P}(K)$. Thus, $\{\mu_j\}$ has a subsequence weakly converging to μ. As assumption 3-iii) and 3-iv) hold for φ, proposition 2 implies $\pi_\varphi(z,A) < \varepsilon$ for any open set A with $\theta(A) < \delta$. Assumption 4-iii) implies that $lim_{j\to\infty}\pi_{\varphi_j}(z,A) \leq \pi_\varphi(z,A)$, which in turn implies that $lim_{j\to\infty}\pi_{\varphi_j}\mu_j(A) < \varepsilon$. The same arguments used in lemma 4-ii) implies that μ is absolutely continuous w.r.t. θ as desired.

ii) Let $\mu_j \to_{Weak*} \mu$. It has to be shown that $Ev(\varphi_j,\mu_j) \to_{Weak*} Ev(\varphi,\mu)$. The arguments in Blume (1982, page 63) implies that it suffice to take an arbitrary test function in the unit ball generated by the sup-norm on $C(\tilde{J})$. Thus, the proof will be completed if it can be shown that:

$$\left|\int f(z)(\pi_{\varphi_j}\mu_j)(dz) - \int f(z)(\pi_\varphi\mu)(dz)\right| \to 0 \text{ as } j \to \infty \quad \text{A.9)}$$

Using theorem 8.3 in Stokey, Lucas and Prescott, A.9) could be written as:

$$\left|\int (\pi_{\varphi_j}f)(z)\mu_j(dz) - \int (\pi_\varphi f)(z)\mu(dz)\right|$$
$$= \left|\int \left[\int f\big(\varphi_j(z,s')U(ds')\big]\mu_j(dz) - \int \left[\int f(\varphi(z,s')U(ds')\right]\mu(dz)\right|$$

Adding and subtracting $\int[\int f(\varphi(z,s')U(ds')]\mu_j(dz)$ and using the triangle inequality the above expression could be written as

$$\left|\int \left[\int f\big(\varphi_j(z,s')U(ds')\big]\mu_j(dz) - \int \left[\int f(\varphi(z,s')U(ds')\right]\mu(dz)\right|$$
$$\leq \left|\int \left[\int f\big(\varphi_j(z,s')\big)U(ds') - \int f(\varphi(z,s')U(ds')\right]\mu_j(dz)\right|$$
$$+ \left|\int \left[\int f(\varphi(z,s')U(ds')\right]\mu_j(dz) - \int \left[\int f(\varphi(z,s')U(ds')\right]\mu(dz)\right|$$

Because of assumption 2') is supposed to hold for $\{\varphi_j\}$ and φ, the first term is bounded above by $SUP_{(K\times S)\setminus\Delta\varphi}\left\|\varphi_j(z,s') - \varphi(z,s')\right\|_\infty$, which converges to zero by assumption 4-ii). The arguments in the proof of theorem 1 implies that the second term also converges to zero as μ is absolutely continuous w.r.t. θ and assumption 2') holds on φ. These 2 facts taking together implies $Ev(\varphi_j,\mu_j) \to_{Weak*} Ev(\varphi,\mu)$ as f is arbitrary.

iii) Equation A.9) and $\mu_j = \pi_{\varphi_j}\mu_j$ for any j implies

$$\left|\int f(z)\mu_j(dz) - \int f(z)(\pi_\varphi\mu)(dz)\right| \to 0 \text{ A.10)}$$

Also $\mu_j \to \mu$ implies

$$\left|\int f(z)\mu_j(dz) - \int f(z)\mu(dz)\right| \to 0 \text{ A.11)}$$

Now, taking $\left|\int f(z)\mu(dz) - \int f(z)(\pi_\varphi\mu)(dz)\right|$ and adding and subtracting $\int f(z)\mu_j(dz)$, the triangle inequality implies

$$\left|\int f(z)\mu(dz) - \int f(z)(\pi_\varphi\mu)(dz)\right|$$
$$\leq \left|\int f(z)\mu_j(dz) - \int f(z)\mu(dz)\right|$$
$$+ \left|\int f(z)\mu_j(dz) - \int f(z)(\pi_\varphi\mu)(dz)\right|$$

Finally, equation A.10) and A.11) implies
$\left|\int f(z)\mu(dz) - \int f(z)(\pi_\varphi\mu)(dz)\right| \to 0$ which proves the last part of the theorem.

■

Proof of Theorem 4 (LLN)

Preliminary Remark on the equilibrium correspondence Φ (definition 5)

In section 2.5.2 $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ was defined as containing any $\tilde{z}_+ = [\tilde{s}_+, \tilde{\theta}_+, \tilde{q}_+, \tilde{m}_+], \tilde{z} = [\tilde{s}, \tilde{\theta}, \tilde{q}, \tilde{m}] \in \tilde{J}$ simultaneously satisfying equations A.1) and A.2); implying $\tilde{z}_+ \in \Phi(\tilde{z}, \tilde{s}_+)$ with $m(s_+) \sim V^*(\theta_+, q_+, s_+)(s_+)$ and $\tilde{m}_+ \sim V^*(\theta_+, q_+, s_+)(\tilde{s}_+)$. This remark explores this claim in detail as it is essential to understand the meaning of $\{\tilde{z}_t\}$ as a realization $\omega \in \Omega$ of the process $(\Omega, \mathcal{B}_\Omega, P_\mu)$ defined in section 4.2.

For simplicity take a 5 period economy with only 2 exogenous shocks $S = \{s_1, s_2\}$ as will suffice to illustrate the iterative procedure that generates $\{\tilde{z}_t\}$. The figure below illustrates a sequence of $\{c_j\}$, as defined in section 2.5.2, obtained from equations A.1) and A.2):

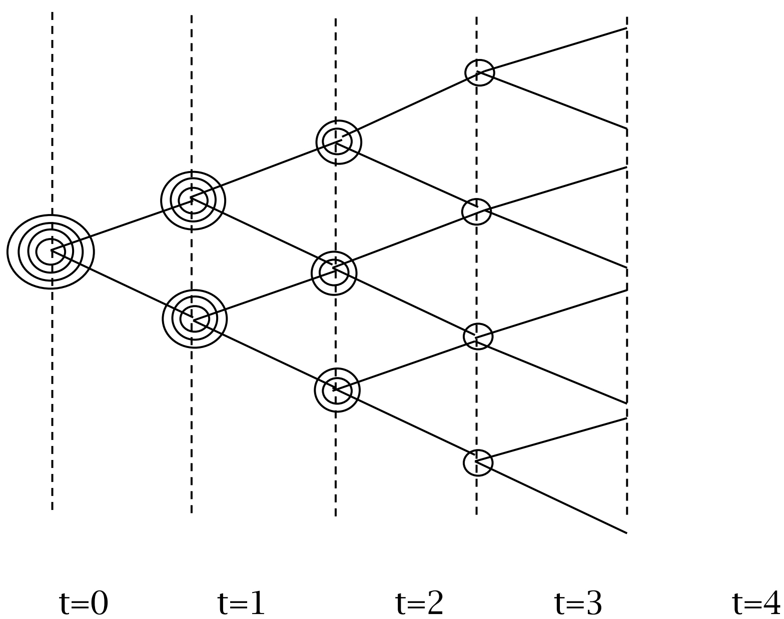

Figure A.1

The nodes with at least 1 circle belong to C_1 (i.e. the set that contains the initial conditions of all 2 period economies in figure A.1), the nodes with at least 2 circles belong to C_2 and so on. Remarkably, note that the only node at t=0 has 4 circles. Thus, *any element* of C_4 has the initial conditions *not only of* *a 5 period economy but also some of all possible initial conditions of any other economy depicted by the figure.* Further, $\tilde{J} = \cap_{j=1}^{4} C_j$.

Take any pair of elements in $\tilde{J}$, $[\tilde{z}^i, \tilde{z}^l]$ and let $c_j^i \in C_j$. Note that $[\tilde{z}^i, \tilde{z}^l] = [c_3^i, c_4^l] = [\tilde{z}_+, \tilde{z}]$ where the first equality follows from the definition of $\tilde{J}$ and the second from the definition of C_3 and C_4. Now, w.l.o.g., let $\tilde{z}_+ = [s_1, \tilde{\theta}_+, \tilde{q}_+, \widetilde{m}_+]$. From definition 5, $\tilde{z}_+ \in \Phi(\tilde{z}, s_1)$ if $[\tilde{z}_+, \tilde{z}]$ satisfies equations A.1) and A.2). But this fact follows from theorem 1.2 in Duffie, et. al. as, following the discussion in section 2.5.2, the recursive equilibrium in Feng, et. al. are a subset of all possible THME (see definition 4) implying $G(\tilde{z}) \cap \mathcal{P}(\tilde{J}) \neq \emptyset$, $G(\tilde{z}) = \{P_\varphi(\tilde{z}, .): \varphi \sim \Phi\}$ and $\tilde{z} \in \tilde{J}$ as it can be seen from fact 1) in section 2.5.1. Note that theorem 1.2 in Duffie, et. al. requires G to be closed graph. This property follows from standard results in Blume (1982) under assumption 1.

In order to iterate the process forward using Φ in definition 5, take $\{\tilde{s}, \tilde{q}, \tilde{\theta}, \widetilde{m}\} \in \tilde{J}$ and $\tilde{s}_+$. Given $\{\tilde{s}, \tilde{q}, \tilde{\theta}, \widetilde{m}\}$ use $m^{i,j} \equiv d^j(s)\left(u_s^i(c^i)\right)'$ and equation A.1 to compute c and $\tilde{\theta}_+$. Take a sequence $\{m_+(s_+)\}_{s_+}$ with $m_+(s_+) \sim V^*(\theta_+, q_+, s_+)(s_+)$ and $\tilde{\theta}_+ = \theta_+(s_+)$. If c and $\{m_+(s_+)\}_{s_+}$ satisfy equation A.2, then $\{\tilde{s}_+, q_+(\tilde{s}_+), \tilde{\theta}_+, m_+(\tilde{\theta}_+, \tilde{s}_+)\}$ is the next state.

The existence of $\{m_+(s_+)\}_{s_+}$ satisfying these properties is guaranteed by proposition 1.3 in Duffie, et. al. The fact that m_+ is a function of s_+ follows from the definition of Spotless THME (see definition A.1) applied to this type of economies as the vector $[\theta_+, q_+]$ is allowed to depend measurably on s_+ (see Duffie, et. al. page 767). Note that in this case, $\theta_+(s_+) = \tilde{\theta}_+$ once $\tilde{s}, \tilde{q}, \tilde{\theta}, \widetilde{m}$ has been fixed. Thus θ_+ is continuous on s_+, as required by assumption 3-iii).

Proof of theorem 4

The assumptions of theorem 3 implies that $\mathcal{E}\left(IM(\varphi_j,\mathcal{P}_1)\right) \neq \emptyset$ for any j sufficiently large. Let z_0^j be an initial condition which satisfies assumption 5. Then, fact 4.2-iv) implies that $\boldsymbol{P}_{\varphi_j, z_0^j}$ -almost surely, $lim_{N\to\infty}[\sum_{t=1}^{N} f(z_i^j(z_0^j,\omega,\varphi_j))]N^{-1} = \int f(z)\mu_j(dz)$ with $\mu_j \in IM(\varphi_j,\mathcal{P}_1)$.

Finally, By the assumptions in theorem 3, $\mu_j \to_{weak*} \mu$ and $\mu \in IM(\varphi,\mathcal{P}_1)$ or equivalently $\left|\int f(z)\mu_j(dz) - \int f(z)\mu(dz)\right| = 0$ for j sufficiently large. Then, $\left|[\sum_{t=1}^{N} f(z_i^j(z_0^j,\omega,\varphi_j))]N^{-1} - \int f(z)\mu(dz)\right| = 0$ for N, j sufficiently large as desired.

■

A.2) Section II.

A.2.1) Finite Shocks

This section proves that under assumptions 6.1-i) and 6.1-vi) the implicit function theorem can be applied to the system of equations that is equivalent to the sequential competitive equilibrium in definition 1.

The results in Magill and Quinzii (1994) and Kubler and Schmedders (2002) imply that under assumptions 6.1-i) to 6.1-v) the following system of equations defines a sequence of consumption bundles $\left\{\{c^i(\sigma_t)\}_{i\in I}\right\}_{\sigma_t\in\mathfrak{T}}$, portfolios $\left\{\{\theta^i(\sigma_t)\}_{i\in I}\right\}_{\sigma_t\in\mathfrak{T}}$ and prices $\{q(\sigma_t)\}_{\sigma_t\in\mathfrak{T}}$ which satisfy the feasibility and optimality requirements in definition 1:

A.12) $\sum_{i=1}^{I}\theta_+^i = \vec{0}$ with $\vec{0} \in \mathbb{R}^J$

A.13) $q_j u_s^i\left(e^i(s) + \theta^i d(s) - \theta_+^i q\right)' - \beta \sum_{s_+\in S} d_j(s_+) p(s,s_+) u_s^i\left(e^i(s_+) + \theta_+^i d(s_+) - \theta_{++}^i q\right)' = 0,\ j \in J, i \in I$

Let $z = [s, \theta, q]$ with $\sum_{i=1}^{I} \theta^i = \vec{0}$ and $m^i = d(s) u_s^i \left(e^i(s) + \theta^i d(s) - \theta_+^i q\right)'$. Also let $F(z, z_+) = \vec{0}$ be the system of equations defined by A.12) and A.13), where $\vec{0} \in \mathbb{R}^{J + J \times I}$.

The discussion in section 2.5.2 and 6.1 imply that under assumptions 6.1-i) to 6.1-v) $[z_+, m_+] \in \tilde{J}$ if $[z, m] \in \tilde{J}$, where $\tilde{J}$ is the expanded equilibrium state space in definition 5. Moreover, the same results imply that each θ_{++} implicit in m_+ define a different selection $m_+ \sim V^*(z_+)$, where $\tilde{J} = Gr(V^*)$. Thus, as A.12) and A.13) can be used to define a particular selection $\varphi \sim \Phi$, θ_{++} can be assumed to be constant throughout the analysis.

Further, because $s, s_+ \in S$ and $\#S < \infty$ and condition 1 is required to hold a.e. in an atomless measure μ, the discussion in the preliminary remark of lemma 3 implies that it suffice to show that $D_{z_+} F(z, z_+)$ has full rank μ -a.e. in z as this implies that $\mu(D) = 0$, where $D = \left\{[z, m] \in \tilde{J} : P_\varphi([z, m], \{a\}) > 0 \; if \; \mu(\{a\}) = 0\right\}$ was defined in equation A.6). Moreover, assumption 6.1-vi) guarantees that $D_{z_+} F(z, z_+)$ is well defined μ -a.e. in z as the discontinuity set of φ is allowed to have up to finite cardinality and F is defined for interior solutions only.

To complete the proof it suffice to write $D_{z_+} F(z, z_+)$ explicitly in order to note that this matrix has full rank under assumptions 6.1-i) and 6.1-v) provided that there is more than 1 asset[15].

A.2.1) Uncountable Shocks

Preliminary remark of Lemma 5

[15] $D_{z_+} F(z, z_+)$ is available under request.

As Discussed in section 3.4), the existence of an ergodic invariant measure can be shown under a slightly weaker assumption than 3-iv). The results holds under assumption 3.iv') which allows $p(s,.)$, the distribution of exogenous shocks, to depend on s. Assume further that,

Assumption A.1) : Let $p(s,.)$ satisfy assumption 3-iv'). Then, $p(s,.)$ has the Feller property.

The proof below assumes that $p(s,.)$ satisfies assumption A.1) provided the existence of a recursive structure Φ. The discussion in section 5.2 and the results in Mas-Colell and Zame (1996) imply that assumption 3.4) is required to insure the existence Φ in definition 5. Of course, 3.4) implies A.1). However, the proof will be done imposing the less restrictive assumptions in case Φ can be derived under milder restrictions for a different type of economy.

Under assumptions 6.2-i) to 6.2-iv) and 3-iii) the result in lemma 5 follows from proposition 1 and 2 and theorems 1 and 2. Thus the proof of the lemma will only take care of the case of only 1 asset which allows to dispense with assumption 3-iii). It will be shown that there exist a selection $\varphi \sim \Phi$, with $\varphi(\tilde{z}, s_+) = [s_+, \theta_+(\tilde{z}, s_+), q_+(\tilde{z}, s_+), m_+(\tilde{z}, s_+)]$, that is continuous in each coordinate in s_+. Moreover, taking into account the incomplete markets nature of the model, $\theta_+(\tilde{z}, s_+)$ will be assumed to be constant. That is, $\theta_+(\tilde{z}, s_+) = \theta_+(\tilde{z})$ for each $s_+ \in [LB(s), UB(s)]$. Once the continuity of $q_+(\tilde{z}, s_+)$ has been shown below, the continuity of $m_+(\tilde{z},.)$ follows from definition.

Proof of lemma 5

Assume that $\theta_+(\tilde{z}, s_+)$ is constant in in s_+ for any given $\tilde{z} \in \tilde{J}$. In order to complete the proof it suffice to show that $q(\tilde{z}, s_+)$ is continuous in s_+ for any given $\tilde{z} \in \tilde{J}$.

Under assumptions 6.2-i) to 6.2-iii) any equilibria in this economy exists satisfies equation A.12 together with

A.14) $q_j u_s^i\big(e^i(s)+\theta^i d(s)-\theta_+^i q\big)' - \beta K(s)\int d_j(s_+) u_s^i\big(e^i(s_+)+\theta_+^i d(s_+)-\theta_{++}^i q\big)' ds_+ = 0,\ j\in J, i\in I$

Where $K(s)$ is the constant associated with the uniform distribution in assumption 3-iv').

Now suppose that assumption A.1 holds. Then, as mentioned in the preliminary remark, $p(s,.)$ has the Feller property. Then:

A.15) $\lim_{s^n\to s^1}\beta K(s^n)\int m_{++}^{i,j}(x)dx = \beta K(s^1)\int m_{++}^{i,j}(x)dx = q_+^j(s^1)u\big(e^i(s^1)+\theta_+^i d(s^1)-\theta_{++}^i q_+(s^1)\big)'$

The last equality in A.15) follows because, under assumption 6.2-i) to 6.2-iii), there is a sequential competitive equilibrium for each s^1 which satisfies equation A.14).

After the discussion in section 5.2 above, the last equality follows from theorem 4.1, 4.2 and section 5 in Mas-Colell and Zame (1996). Further, under the special form $u_s^i = u$ in assumption 6.2-i), equation A.14 and A.15 implies:

A.16) $\lim_{s^n\to s^1}\frac{\beta K(s^n)\int m_{++}^{i,j}(x)dx}{u\left(e^i(s^n)+\theta^i d(s)\right)'} = \lim_{s^n\to s^1} q_+^j(s^n)u\big(-\theta_{++}^i q_+(s^n)\big)' = q_+^j(s^1)u\big(-\theta_{++}^i q_+(s^1)\big)'$

Note that equation A.14 implies the first equality in A16) under u in assumption 6.2-i). Then, as $u\left(e^i(s^n)+\theta^i d(s^n)\right)'$ is bounded above and bounded away from zero for any admissible value of $e^i(s^n)+\theta^i d(s^n)$ under assumptions 6.2-i), equation A.15 implies the last equality.

Now, setting $\lambda = 1$ in u w.l.o.g., the continuity of ln implies

$$\text{A.17)}\quad \underbrace{\lim_{s^n \to s^1}[-\theta^i_{++}q_+(s^n)] + \theta^i_{++}q_+(s^1)}_{A} + \underbrace{ln\,[\lim_{s^n \to s^1} q^j_+(s^n)] - ln\,[q^j_+(s^1)]}_{B} = 0$$

Suppose that $B = 0$, then as $\theta^i_{++} \neq 0$ w.l.o.g., A implies $lim q_+(s^n) = q_+(s^1)$ as desired.

Suppose that $B \neq 0$. The compactness of the equilibrium set implied by theorem 4.2 in Mas-Colell and Zame (1996) under assumptions 6.2-i) to 6.2-iii) implies that $B \in \mathbb{R}$. Then A.17 under $J = 1$ (i.e. there is only 1 asset) implies:

$$q^j_+(s^1) = \frac{B}{\theta^i_{++}(1 - exp(B))}$$

As B depends on s^1 for each $s^1 \in [LB(s), UB(s)]$, the equation above implies that $q^j_+(.)$ is continuous in s^1; implying a contradiction with $B \neq 0$ as $\theta^i_{++} \neq 0$ w.l.o.g.

■

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