Simulations in Models with Heterogeneous Agents, Incomplete Markets and Aggregate Uncertainty

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Abstract

This paper present conditions to guarantee the convergence of simulations to a stochastic steady state, characterized by an invariant probability distribution, in an endowment economy with a finite number of heterogeneous agents, aggregate uncertainty and uncountable shocks. The results are robust to the presence of multiple discontinuous equilibria and do not require ad-hoc convexification techniques, like "sunspots". Thus, our results are numerically implementable. We work on a Markov environment with an enlarged state space, applied to an incomplete markets model, to characterize ergodic equilibria and differentiate them with respect to time-independent, and stationary ones. We show that, by imposing a mild restriction on the discontinuity set, *every* measurable time-independent selection can be used to approximate the stochastic steady state of the model. Considering the common practice of clustering agents according to, for instance, deciles of the wealth and assuming uncountable income shocks, the results in this paper can help to design calibration and estimation methods for heterogeneous agent models based on unconditional moments.

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1 Introduction

Frequently, researchers seek to investigate the long and short run effects of economic policies on dynamic equilibria. To achieve this objective, the variables in the model are often *simulated*. This is done to characterize the stochastic structure of the model and possibly its steady state. Unfortunately, in non-optimal economies, there is *no general method* which guarantee this sort of characterization. The commonly used procedures generate different outcomes (see Hatchondo, et. al., (2010), De Groot, et. al., (2013), among others) and the simulations obtained from them may not provide accurate representations of the economies depicted by the models (see for instance Feng, et.al., (2014)). With complete markets, the contraction mapping theorem provides a unified framework as it generates a constructive proof that shows the existence of a unique equilibrium with continuous a transition function. In this context, the stochastic steady state exists, and simulations converge to it, which is all guaranteed by standard results (see Stokey, Lucas and Prescott (1989)). In heterogenous agents models with incomplete markets uniqueness are rarely shown. In the absence of it, characterizing the relationship between simulations and the steady state can be very challenging due to the possible discontinuity of transition functions and the presence of multiple equilibria.

This paper present conditions to guarantee the convergence of simulations to a stochastic steady state, characterized by an invariant probability distribution, in an endowment economy with *a finite number of* heterogeneous agents, aggregate uncertainty, *and uncountable shocks*. The results are robust to the presence of multiple equilibria and discontinuous selections and do not require ad-hoc convexification techniques, like "sunspots". Thus, our results are numerically implementable. We show that, by imposing a mild restriction on the discontinuity set, *every measurable selection* can be used to approximate a stochastic steady state of the model. In this sense, we are solving jointly the problems associated with the lack of a unique equilibrium (i.e., the presence of multiple and discontinuous transitions) as we are characterizing those selections that approximate the stochastic steady state. Up to now, the literature showed the existence of at least one of these selections without a addressing the qualitative properties of it. Considering the common practice of clustering agents according to, for instance, deciles of the wealth and assuming uncountable income shocks, the results in this paper can help to design calibration and simulation-based estimation techniques for heterogeneous agent models.

Formally, we provide a *framework for the characterization and simulation* of non-optimal economies. Moreover, we give conditions which guarantee the ergodic behavior of endogenous variables for at least *some* selection. We work on a generalized (i.e., with an expanded state space), possibly discontinuous, Markov environment applied to an incomplete markets model with 1 period real assets offered in zero net supply to characterize ergodic equilibrium selections and differentiate them with respect to time-independent, and stationary ones. While ergodic equilibria are path-independent, stationary representations generate simulations affected by the history of shocks and the initial portfolio distribution. The existence of these 2 types of equilibria require a mild restriction on the discontinuity set and *every* time-independent selection under this assumption is *at least* stationary, as the requirements for stationarity are milder with respect to the sufficient conditions for ergodicity. Thus, the latter are easier to get but they are not useful to calibrate or estimate the model as they generate path-dependent simulations. To solve this problem, we show that "averaging" *appropriately* across paths is sufficient to ensure that stationary transitions generate simulations that can approximate the stochastic steady state of the model. That is, it is possible to use *an arbitrary* selection to characterize the long run behavior of the model.

Our results go beyond the existence of a stochastic steady state (as in Dufffie, at. al. (1994)) as we can identify qualitative properties of selections connected with ergodicity and, in the absence of them, we can still have a sharper characterization of the long run of the model. While stationarity follows from the compactness of equilibrium and a mild assumption on the discontinuity set, ergodicity requires additional restrictions on either: a) the number of possible endogenous states, or b) the type of selections that are admissible to construct the Markov kernel.

As the restrictions required to show ergodicty are specific to the model described in this paper and we need to refine the selection process substantially, our results suggest that in a more general setting heterogenous agent models will *only be stationary,* which in turn produce path dependent simulations. We mentioned that to eliminate this dependence it suffices to average multiple time paths. However, as the presence uncountable shocks is essential, the numerical implementation of this procedure is not immediate. Thus, the results in this paper indicate that simulations of *computed* heterogenous agent models with aggregate uncertainty are frequently sensitive to the initial conditions of the economy; *a fact which implies that heterogeneity may have long lasting effects and may explain the different outcomes found by distinct methods in practice.* In the absence of a sharp characterization of ergodic selections, the presence of multiple equilibria can explain the dependence of simulations on the method used because each one of them may capture a different selection, and no one could be ergodic; generating history-dependent paths. We show that, if the support of the stationary distribution is unknown, numerical simulations may depend on initial conditions and, thus, cannot approximate the steady state.

For expository purposes, we apply our results to a workhorse model in the literature. We use a stochastic endowment economy with incomplete markets, finitely many heterogeneous agents, 1 period real assets and uncountable exogenous states as in Mas-Colell and Zame (1996). It is a matter of further research to investigate if our results can be extended to another model with a finite number of heterogeneous agents, incomplete markets and uncountable shocks.

1.1 Literature Review

We provide a unified theoretical framework to characterize non-optimal heterogenous agent models with aggregate uncertainty. The necessity of it comes from the failure of methods frequently used (i.e., Kydland and Prescott, 1982, Krusell and Smith, 1998, Cooley and Quadrini, 2001, Chari, Kehoe and McGrattan, 2002, among others) in providing simultaneously: (i) an adequate representation of the stochastic steady state that is compatible with an empirical test for the model, (ii) a well-defined time-independent law of motion for the endogenous variables and (iii) a characterization of stationary and ergodic selections in the presence of multiple equilibria. This paper fills the gap in the literature by dealing with facts (i) to (iii) at the same time and provide identifiable conditions related to ergodicity. In optimal economies, all these pieces came together with the contraction mapping theorem. If the welfare theorems fail do not hold, Kubler and Schmedders (2002) and Santos (2002) showed that there is a tension between time-independence and continuity of the recursive equilibrium. This fact is associated with the presence of multiple equilibria. If it is not possible to show uniqueness, we need to give up on continuity, which in turn implies a specific machinery to derive an ergodic, stationary and/or time-independent recursive representation.

Duffie, et. al. (1994) and Blume (1982) showed the existence of an ergodic invariant measure for some non-optimal economies, but they did not characterize selections, which is the first step towards the computation of the equilibria. Feng, et. al. (2014) derived a time invariant recursive representation, but they did not prove ergodicity. The most closely related papers to this one in the literature are Santos and Peralta Alva (2013 and 2015), Brumm, et. al (2017), and Cao (2020) (see also Duggan (2012), and He and Sun (2017)). In these papers there are features of the model that are used to achieve either ergodicty of the Markovian equilibrium or the existence of a recursive representation. Assumptions relate to a continuum of households in

Cao (2020) or to restrictions on exogenous shocks in Duggan (2012), Santos and Peralta Alva (2013 and 2015), Brumm et. al. (2017) and He and Sun (2017).

With respect to Santos and Peralta (2013 and 2015), our paper identifies selections taken from the equilibrium correspondence and characterize them to obtain an ergodic, a stationary, and a time-independent equilibrium. In Santos and Peralta, although it is proved that such equilibrium exists, they do not characterize a particular ergodic selection nor differentiate it with respect to a stationary one. They show that there exist at least one ergodic selection and characterize the properties of the Markov kernel associated with it. We go further in this direction and found that ergodicity relates to the continuity of the selection with respect to exogenous shocks, allowing for a large discontinuity set with respect to endogenous states as it is found in the literature (see for instance Kubler and Schmedders 2002). We provide a selection mechanism associated with ergodicty and show that also a restriction in the number of available assets suffices to achieve it. We then characterize stationary selections and show how to eliminate the history dependence of stochastic path.

Cao (2020) showed the existence of an ergodic recursive equilibrium as a selection from an equilibrium correspondence. Neither the author can characterize the ergodic selections, nor differentiate them from stationary and non-stationary ones.

The existence of a recursive representation is a first step but not sufficient to guarantee the stability of simulations in the long run. This is the main difference with respect to Duggan (2012), Brumm et. al. (2017) and He and Sun (2017).

The strategies used for the proofs differ from previous results. One of the consequences of allowing for multiple equilibria is that the selected transitions may not be continuous. This fact causes a serious problem for the existence of an invariant measure. The literature has circumvented this problem by using a fixed-point theorem for correspondences. Unfortunately, this approach requires conditions which affect the computability of transitions (like the convexification technique used in Duffie, et. al. or the impossibility to identify an appropriate

selection in Blume). The strategy in this paper is to derive verifiable conditions on each computable transition that restore the continuity of the Markov process.
There is another branch of the literature that derived results to show the ergodicity of recursive equilibria with incomplete markets using an “irreducible atom”. The papers of Zhu (2020) and Pierri and Reffett (2021) assume the presence of a finite number of shocks and occasionally binding inequality constraints. Although these papers can circumvent the numerical problems associated with uncountable shocks, they are not useful for models with ex-ante heterogenous agents.

2. Recursive equilibrium in heterogenous agent models

This section uses an infinite horizon general equilibrium model with incomplete markets to introduce a recursive equilibrium concept, discussed its existence and several properties. We choose the simplest representation of an incomplete markets model with aggregate uncertainty and a finite number of heterogeneous agents. Despite its simplicity, this structure can contain both major obstacles to derive a stationary and ergodic representation: multiplicity and discontinuity.

2.1 Structure of the Economy

The model is a standard infinite horizon discrete time pure exchange economy. A Markov chain defines the law of motion for the exogenous state variable[2]. For every period t, a shock s_t occurs; $s_t \in S$ and $S = (1,2,\dots,S)$. To model the evolution of uncertainty, an event tree approach is assumed. Each tree $\mathfrak{T}$ has a unique root, $\sigma_0 =$

[2] The set of exogenous shocks is assumed to be finite in all the equilibrium concepts mentioned in this section. This is done because the conditions to guarantee the existence of the sequential equilibria are well known. The Time Homogeneous Markov Equilibrium in Duffie, et. al. (1994), Kubler and Schmedders' Markov equilibrium and Feng's Recursive equilibrium can be defined for an arbitrary set of exogenous shocks (see Duffie, et.al. page 749 and Santos and Peralta Alva page 6 respectively). The conditions for the existence of the sequential equilibria with an uncountable, atomless and iid shocks, which is essential for the results in sections 3 and 4, are presented in section 5.

s_0. A typical element will be denoted $\sigma_t = (s_0, s_1, \dots, s_t)$. Each σ_t has a unique predecessor $\sigma_t^* = (s_0, s_1, \dots, s_{t-1})$ and S sucessors, $\sigma_t s$, for each $s \in S$. Since the exogenous shocks follow a first order Markov process, when S is finite, the evolution of $\{s_t\}_{t=0}^{\infty}$ can be characterized by a transition matrix, $p = [p(s_i, s_j)]$. When postpone the details for uncountable shocks to section 3. For any given s_0, the probability of σ_t is $\mu_t(\sigma_t) = \prod_{t=1}^{t} p(s_{t-1}, s_t)$ and $\mu_0(\sigma_0) = \delta_{s_0}$, where δ_{s_0} is the Dirac measure at s_0.

The number of agents is assumed to be finite, with a typical element denoted $i \in I$. Each agent is endowed with $e^i(\sigma_t)$ units of the single consumption good. For simplicity, the endowment process is supposed to be iid: $e^i(\sigma_t s) = e^i(s)$, with $e^i: S \to \mathbb{R}_{++}$. The vector of endowments at any node is denoted $(\sigma_t) = \{e^i(\sigma_t)\}_{i=1}^{I}$. Each agent has an additively separable well behaved[3] utility function which is used to evaluate consumption streams, $c = \{c(\sigma_t)\}_{\sigma_t \in \mathfrak{T}}$:

$$U_i(c) = \sum_{t=0}^{\infty} (\beta^i)^t \sum_{\sigma_t^* s} [u_s^i(c^i(\sigma_t^* s))] \mu_t(\sigma_t^* s)$$

The asset structure is characterized by J one period numeraire real assets, offered in zero net supply, and traded at each node of the tree, $\sigma_t \in \mathfrak{T}$. An asset held by agent i is denoted $\theta_j^i(\sigma_t) \in \mathbb{R}$ and pays dividends $d_j(\sigma_t s) \in \mathbb{R}_+$, only at the S immediate successors of σ_t[4]. The portfolio of agent i at node σ_t will be denoted $\theta^i(\sigma_t) \in \mathbb{R}^J$. It is assumed that the dividend process is also iid: $d_j(\sigma_t^* s) = d_j(s)$, where $d_j: S \to \mathbb{R}_+$[5]. Further, the $J \times S$ payoff matrix, d, is supposed to have full row rank

[3] To the conditions stated in Duffie, et. al. (1994) page 765, Kubler and Schmedders (2002) implicitly added the assumption that u_s^i has uniformly bounded gradients. This is done to satisfy a terminal condition on the discounted expected marginal utility (see equation 1 in page 288) which in turn is used to obtain a definition of equilibria based on first order and market clearing conditions. This last definition is essential for the recursive equilibrium literature as can be seen in sections 2.3, 2.5 and the appendix.

[4] Agents are allowed to short sale every asset θ_j. To define a Time Homogeneous Markov Equilibrium, Duffie, et. al. assumed that there are no short sales and a different asset structure (J Lucas trees). However, Braido (2013) showed that the equilibrium concepts in Duffie, et. al. still holds if short sales are permitted for a general asset structure, which includes one period real assets offered in zero net supply, provided that marginal utilities are uniformly bounded above.

[5] Except in section 2.4, where the equilibrium has closed form, for economies with $\#S < \infty$, it will be assumed that the dividend structure has a riskless bond as in assumption A.6 in Magill and Quinzii (1994) (i.e., $d_j(s) = 1$ for any $s \in S$ and $j \in \{1, \dots, J\}$.

and a column of d will be denoted $d(\sigma_t)$. Consequently, market incompleteness follows directly from $J < S$. Finally, the price of security θ_j at node σ_t will be denoted $q_j(\sigma_t) \in \mathbb{R}_+$, asset prices will be collected at the vector $q(\sigma_t) \in \mathbb{R}_+^J$.

2.2 Sequential Competitive Equilibrium[6]

An economy $\mathcal{E}$ is characterized by the endowments, payoffs, the structure of preferences and the initial distribution of assets: $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^I, \{\theta_-^i\}_{i=1}^I\right]$. A sequential equilibrium for $\mathcal{E}$ can then be defined as follows,

<u>Definition 1</u>. A <u>sequential competitive equilibrium</u> for $\mathcal{E}$ is a collection of:

i) *consumption vectors* $\left[\{c^i(\sigma_t)\}_{i=1}^I\right]_{\sigma_t \in \mathfrak{T}}$,

ii) *portfolio holdings* $\left[\{\theta^i(\sigma_t)\}_{i=1}^I\right]_{\sigma_t \in \mathfrak{T}}$,

iii) *asset prices* $[\, q(\sigma_t)]_{\sigma_t \in \mathfrak{T}}$

Such that for $s_0 \in S$ *and* $\{\theta_-^i\}_{i=1}^I$ *satisfy:*

a) (Feasibility) For all $\sigma_t \in \mathfrak{T}$, $\sum_{i=1}^I \theta^i(\sigma_t) = \vec{0}$, *where* $\vec{0} \in \mathbb{R}^J$.

b) (Optimality) For each agent $i \in I$ *and prices* $[\, q(\sigma_t)]_{\sigma_t \in \mathfrak{T}}$:

$$\left[c^i(\sigma_t), \theta^i(\sigma_t)\right]_{\sigma_t \in \mathfrak{T}} \in argmax\,\{\, U^i(c) \text{ subject to } c(\sigma_t) = w^i(\sigma_t) - \theta^i(\sigma_t).q(\sigma_t) \text{ for all } \sigma_t \in \mathfrak{T} \text{ and } sup_{\sigma_t \in \mathfrak{T}}\left|\theta^i(\sigma_t).q(\sigma_t)\right| < \infty\}.$$

As the payoff matrix does not depend on prices, its (row) rank is constant for any period $0 \leq t \leq \infty$. Consequently, the excess demand function of all agents can be shown to be continuous[7]. To establish the existence of equilibria, an implicit debt constrained is added in condition b). Magill and Quinzii (1994) showed that this

[6] This concept is analogous to the Financial Market Equilibrium in Magill and Quinzii (1996), page 228, extended to an infinite horizon economy.

[7] See Magill and Quinzii 1996, exercise 3, page 276 for a counterexample for the case of long-lived assets.

condition rules out Ponzi schemes, it is never binding and is sufficient for existence.

2.3 Correspondence Based Recursive Equilibria

The modern recursive literature allows for multiple equilibria and requires a correspondence in order to capture the first order dynamic behavior of the economy. There are 3 seminal papers in this branch of the literature: Duffie, et. al. (1994), Kubler and Schmedders (2003) and Feng, et. al. (2014). All these papers show the existence of a time independent first order recursive structure under mild assumptions.

Section 2.3.1 introduces the results in Duffie, et. al. and discusses its usefulness and limitations for the purposes of this paper. As the recursive structure in Kubler and Schmedders (2003) uses Duffie, et. al.'s results, it shares the same properties and thus will be omitted[8]. Section 2.3.2 discusses the recursive equilibrium in Feng, et. al., which is the starting point of the results in this paper. As before, for the sake of concreteness, details are left to the appendix.

2.3.1 Duffie's et. al. (1994) Time Homogeneous Markov Equilibria

This section illustrates how Duffie, et. al.'s results can be used to: i) show the existence of a sequential equilibrium (fact 2.3.2), a result that will be used in applications in section 5; ii) derive a time invariant recursive structure and to generate a stationary Markov process[9]; iii) simulate the process (fact 2.3.1).

[8] One of the main contributions of Kubler and Schmedders (2003) is a correspondence based recursive structure, called Markov Equilibria, with minimal state space. As this paper is not concern with the numerical properties of the algorithms involved in the computation of the recursive structure, Kubler and Schmedders' results could be replaced with Feng, et. al.'s which are not affected by the problems in Duffie, et. al. but may have a larger state space. It would be interesting to derive Kubler and Schmedders' Markov equilibria from Feng, et. al.'s structure.

9 See definition 4 in the appendix, page 18.

This section also discusses the limitations of the results in Duffie, et. al., most of them concern simulations. These facts are essential to understand how the results in Feng, et. al. (2014) fit the purposes of this paper as they solve all the problems in Duffie, et. al. but preserves all its useful properties.
Two facts are worth mentioning from Duffie, et. al.'s equilibrium notion:

> Fact 2.3-1): the state space J is the smallest[10] set that can be used to define an equilibrium correspondence as it contains all initial states of any infinite horizon sequential competitive equilibrium and is time independent. It can be used to iterate forward a *first order dynamic stochastic process with a time invariant state space*. This property is frequently called "self-justification" and is weaker than its analogous in games with endogenous states called "self-generation"[11]. If the equilibrium is uniformly compact, then J is the cartesian product of rectangles formed by the upper and lower bounds of the variables in the state space.
>
> Fact 2.3-2): The existence of J requires the existence of a temporary equilibrium for a truncated economy with finite time and all these equilibria must be uniformly compact. Then the former is typically extended to infinity by induction (see lemmas 3.4 and 3.5, page 768), the latter follows from the existence of uniform bounds on endogenous variables. The non-emptiness of the temporary equilibria is directly connected to the "consistency" requirement in the game theory as, for instance, in Phelan and Stacchetti (2001).

These 2 facts combined with an argument on the optimality of the sequences generated from J using the equilibrium correspondence (see section 3.4 in Duffie,

[10] In the temporary equilibrium framework of Hildenbrand and Grandmont (1974) it is possible to set $J = Z$ as overlapping generation agents only live 2 periods. In this type of economies, agents live infinitely many periods and thus it is possible that the backward induction procedure implied by equations 1 and 2 converges to an empty set. Fact 2.5.1) show that this is not the case for economies with compact K and $C_j \neq \emptyset$ for $j \geq 1$.
[11] I would like to thank K. Reffett for pointing this out to me. See Phelan and Stacchetti (2001).

et. al.) can be used to show the existence of a sequential competitive equilibrium. Although this result has already been applied to other incomplete market economies for the case of finite shocks (see Kubler and Schmedders, 2003, Lemma 2), it is not generally used in economies where S is assumed to be uncountable and compact. For the results in this paper, the last structure of exogenous shocks turns out to be important[12]. Thus, this type of existence proof will be discussed in section II of the appendix which involves applications. In the model presented in section 2.1 and 2.2, the Mas-Colell and Zame (1996) framework allows showing the existence of J and the compactness of equilibria. The optimality argument in section 3.4 of Duffie, et. al. can be straightforwardly extended in that model to the case of uncountable shocks.

A time invariant Markov process is constructed using 2 building: a state space and a Markov operator. In the context of Duffie, et. al., the state space is J [13]. The Markov operator is denoted π and is a selection of the equilibrium correspondence G (denoted $\pi \sim G$) such that $\pi: J \to \mathcal{P}(J)$, where $\mathcal{P}$ is the space of measures generated by J. A pair (J, π) is called Time homogeneous Markov Equilibria (THME). Even though the results in Duffie, et. al. can be used to guarantee the existence of a recursive structure, a THME *is not a computable representation of the* sequential competitive equilibrium as *the time invariant transition functions of the recursive equilibrium depend on unobservable variables, which are part of the selection devise*. This fact is illustrated by the following lemma.

Suppose that the state space, $J \subseteq Z_D$, can be written as $Z_D = S \times \hat{Z}$, where $\hat{Z} = \left\{[\theta_-, c, q, \theta] \in \mathbb{R}^{IJ} \times \mathbb{R}^I \times \mathbb{R}^J \times \mathbb{R}^{IJ} \middle| \sum_{i=1}^I \theta_-^i = \vec{0}, \sum_{i=1}^I \theta^i = \vec{0}\right\}$. Considering the exogeneous

[12] Fact 2.5.2) implies that any truncated economy ($j \leq T < \infty$) which has uniformly bounded endogenous variables (contained in K) can be used to prove the existence of a sequential infinite horizon equilibria. That is, any recursive equilibrium is a sequential equilibrium. However, there may be some sequential equilibria that are not recursive or that do not have a terminal debt level equal to 0. So, fact 2.5.2) can be used to prove the existence of a subset of all possible sequential equilibria. I would like to thank A. Manelli for pointing this out to me.

[13] It is standard to assume that Z is a Borel Space. As the Cartesian product of a finite set and a finite dimensional Euclidean space is a complete, separable and metric space, the product space is a Polish space. Thus, Z is a measurable subset of a Polish space. If $\mathcal{B}_{[Z]}$ is the Borel sigma-algebra generated from Z, $(Z, \mathcal{B}_{[Z]})$ is a Borel Space. Consequently, measurable will always mean Borel measurable and any measure will be a Borel measure.

nature of $S,$ this assumption can be imposed without loss of generality. The equilibrium correspondence G maps $Z_D \mapsto \mu$ with $\mu \in \mathcal{P}(Z_D)$. Then,

Lemma 1: If (J, π) constitute a THME any realization of a process $\{z_t\}$ satisfies as $\hat{z}_{t+1} = f(s_{t+1}, \alpha_{t+1}, z_t)$ where f is a measurable function and $\alpha_{t+1} \in [0,1]$ is uniformly distributed and i.i.d.

Proof: See Lemma 2.22 page 34 In Kallenberg (2006).

Duffie, et. al. (1994) interprets α_{t+1} as sunspots. Note that lemma 1 implies that for each state z_t, any exogenous shock s_{t+1} could be associated with a continuum of possible continuation states in $\hat{Z}$, each one of them derived from the realization of an *unobservable variable* (α_{t+1}). Consequently, a tree structure would not be an appropriate representation of $\{z_t\}_{t=0}^{\infty}$.

A THME has limited predictive power about the evolution of the process. Thus, a "refinement" is required to obtain a computable object. Duffie, et. al. also provided sufficient conditions for the existence of this refined equilibria. If S is a *finite set*, a subset of G, denoted g, is also an equilibrium correspondence. g induces a compact state space [14] for a Markov process that has realizations without sunspots.

For the purposes of this and Duffie, et. al.'s papers, this refinement is insufficient as the Markov process associated with (J, π) may not be *stationary nor ergodic.*

The concept of conditionally spotless THME was introduced to address this topic and to derive a notion of steady state, called *ergodic measure*[15]. This equilibrium refines the notion of self-generation in fact 2.3.1 to transitions without sunspots. An invariant measure guarantees that the Markov process associated with the THME is stationary and the ergodicity of the steady state *implies that the process generates convergent sample paths.*

[14] The expectation correspondence $g \subset G$ is obtained by restricting μ in equation 2 to the set $\mathcal{P}_F(S \times \hat{Z}) \subset \mathcal{P}(Z)$ defined in setion A.1.1 in the appendix. The set of conditions on K and C_j mentioned above can still be used to guarantee the existence of a self-justified set for g.

[15] See Theorem 1.1 and Proposition 1.3 in Duffie, et. al., page 750 and 757.

The authors argued that this last type of equilibrium, called conditionally spotless THME, implies that any sequence $\{z_t\}_{t=0}^{\infty}$ in J can be described by a random variable representation of the expectations correspondence: a function f that satisfies $\hat{z}_{t+1} = f(s_{t+1}, \alpha_t, z_t)$ for any t, where $\alpha_t \in [0,1]$ is uniformly distributed, i.i.d and represents another type of sunspot which is used to convexify the equilibrium correspondence. As f is not a computable object, because of the presence of sunspots at t, it is inappropriate for this paper. However, as described in Duffie, et. al.[16], facts 2.3-1) and 2.3-2) can still be used to characterize sunspots-free paths $\hat{z}_{t+1} = f(s_{t+1}, z_t)$.

2.3.2 Feng, et. al.'s Recursive Equilibria

The virtue of Feng, et. al. approach is to derive a recursive structure that exists even in the presence of multiple equilibria as in Duffie, et. al. but at the same time generates computable time invariant transitions. These facts imply that the results in Feng, et. al. can be used to derive laws of motion for the endogenous variables that do not depend neither on unobservable variables nor time. This is a consequence of the definition of the expectation correspondence, which now maps to the space of random variables directly. Besides, this structure sometimes has a lower dimensional state space when is compared to Duffie, et. al.'s, a property that is desirable from a numerical point of view.

In order to obtain these results, it is necessary to restrict the number of possible recursive equilibria and derived a correspondence $\Phi: \tilde{Z} \times S \rightrightarrows \tilde{Z}$ that maps $(\tilde{z}_t, s_{t+1}) \longmapsto \tilde{z}_{t+1}$, which can be used to construct the analogous of Duffie, et. al.'s equilibrium correspondence, G[17] but with a different image. The state space, denoted Z_F and defined in the appendix, is composed by a subset of the state space in Duffie, et. al, $Z_F \subseteq Z_D$, and an auxiliary variable m. This additional variable relates

[16] See sections 3.1 and 3.2.

[17] The procedure to derive the analogous of G in Duffie, et. al. from Φ will be presented at the beginning of section 3.

to the derivative of the value function in the sequential problem and under some conditions can be shown to be envelopes of a recursive problem, connecting both types of equilibrium directly through the Euler equation. Thus, $\tilde{z} \in \tilde{Z}$ is of the form $[z_F, m(z_F)]$. Typically, $[z_F, m(z_F)]$ can be mapped to a subset of the state space in Duffie, et. al. using the definition of an envelope. As the image of Φ is in the space of realizations, an additional state variable (S) must be added to the domain of the correspondence.

A procedure like the one described in Duffie, et. al. can be used to refine the state space from $\tilde{Z}$ to $\tilde{J}$, where $\tilde{z} \in \tilde{Z}$ contain all the possible initial variables, $\tilde{z}_0 = [s_0, q_0, \theta_0, m_0]$, each of them associated with a different SCE. As noted by Duffie, et. al., under compactness this procedure is not necessary. The state space in Feng, et. al. does not contain neither portfolios "yesterday" nor consumption but includes envelopes. These facts increase the numerical efficiency of the algorithms and helps describing the global dynamics of the economy as m is easily characterized in terms of exogenous shocks, prices and portfolios using the definition of the envelope, the utility function and the compactness of the state space.

Below we present 4 relevant facts for this equilibrium:

Fact 2.3-3): The realizations of the process depend only on observable variables.

Fact 2.3-4): The equilibria in Feng, at. al. is a subset of those in Duffie, et. al.

Fact 2.3-5): For *some* models which fit definition 1, there is a selection $\varphi \sim \Phi$ which can be chosen to be continuous in s_+ for each $\tilde{z} \in \tilde{J}$. This fact does not imply the uniqueness of the equilibrium and will be extensively discussed in section 4.1.

Fact 2.3-6): The state space in Feng, et. al. is smaller than the one in Duffie, et. al.

3. Stationarity and Ergodicity

To obtain empirically meaningful simulations some notion of stationarity is required. Time homogeneity is desirably, but it is not enough. A reliable procedure requires an invariant measure. This section formally proves first the existence of an invariant measure and later its ergodicity for a computable recursive structure.

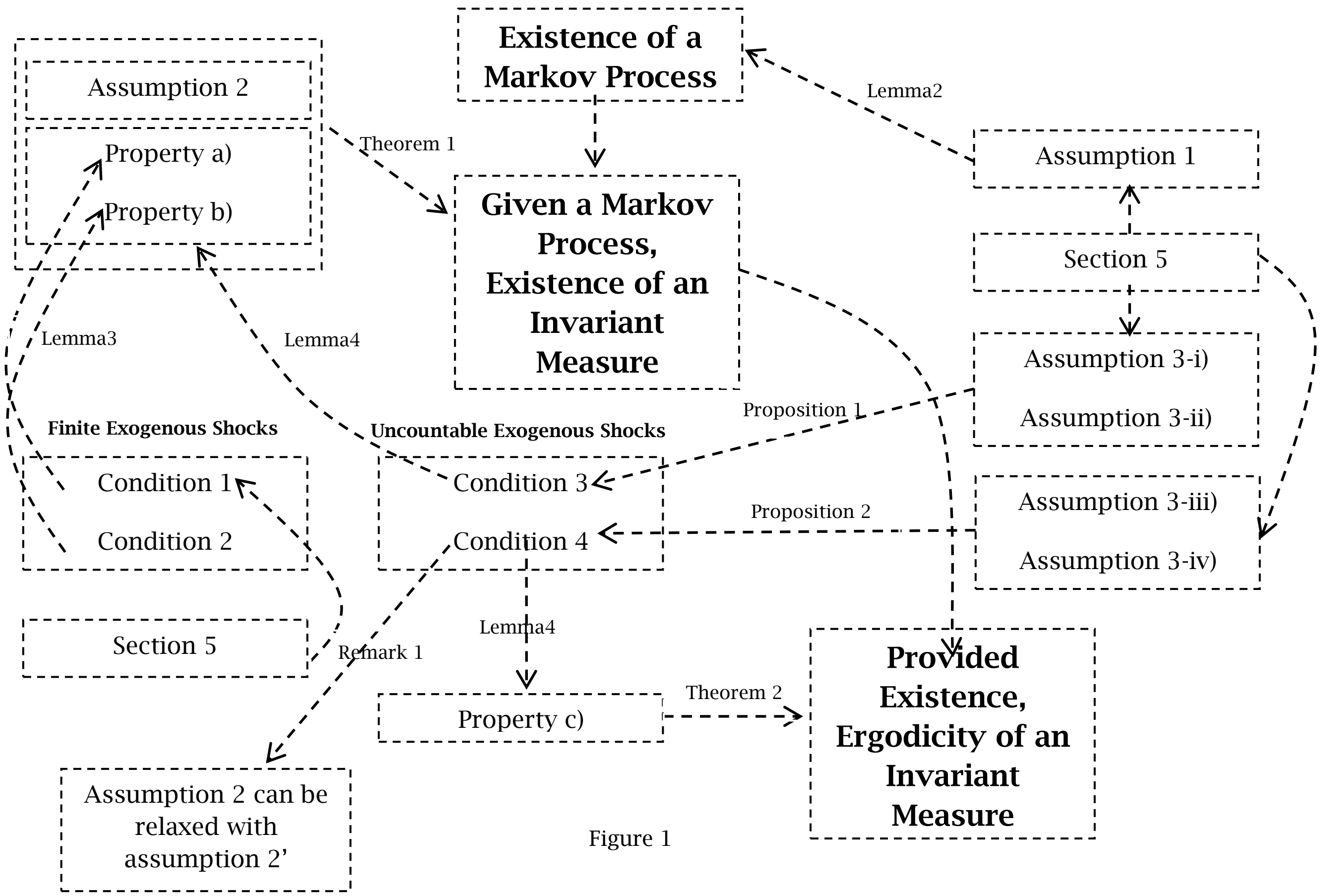

Figure 1

Sections 3.1 and 3.2 together establish the existence of an invariant measure for models with at most a finite number of exogenous shocks that fit into Feng, et. al's framework. Sections 3.1 and 3.3 show the existence of an invariant (theorem 1) and of an ergodic (theorem 2) measure for models with an uncountable number of shocks[18]. Section 3.4 provides sufficient conditions for stationarity (associated with an invariant measure, proposition 1) and ergodicity (proposition 2).

The figure above implies that theorems are proved given some properties, which are associated with conditions. The relationship between conditions and properties are stated in lemmas. Conditions and properties are stated in terms of endogenous variables and operators, respectively. Then, assumptions are based on primitives and connected with conditions through propositions. Stationarity and ergodicity follow from primitive assumptions 3-i), 3-ii) and 3-iii), 3-iv) respectively. Conditions are associated with numbers (1, 2, ..., etc.), properties with lowercase letters (a, b, ..., etc.) and assumptions with roman numbers (i, ii, ..., etc.). To complement figure 1, we present the following table.

Result	Implications	Requirements
Theorem 1	Existence of an invariant Measure	Assumptions 1) and 2). Properties a) and b) for atomless measures.
Theorem 2	Existence of an ergodic invariant Measure	Assumptions 1) and 2). Properties a), b) and c) for absolutely continuous measures.
Lemma 3 (finite shocks)	Properties a) and b) for atomless measures.	Assumption 1). Conditions 1) and 2)
Lemma 4 (Uncountable Shocks)	Properties a) and b) for atomless measures. Properties a), b) and c) for abs. cont. measures	Assumptions 1) and 2). Condition 3), associated with properties a) and b) for atomless measures. Condition 4), associated with properties a), b) and c) for abs. cont. measures.
Proposition 1	Condition 3	Assumptions 1, 3-i) and 3-ii)
Proposition 2	Condition 4	Assumptions 1, 3-iii) and 3-iv)

Table 1

[18] Economies with an infinite but countable number of shocks are intentionally left out as they represent a particularly challenging case for the purpose of this paper: the existence of equilibria requires the same strength of assumptions as in the case of uncountable shocks (see Mas Collel and Zame, 1996) and the existence of an invariant measure is as difficult to show as the case of a finite number of shocks.

3.1 Theorems 1 and 2: Existence of an Invariant Measure and Ergodicity

The starting point of this section is a Markov operator for exogenous shocks, $p(s,A) \geq 0$ defined for all $s \in S$ and $A \in \mathcal{B}_S$, where S is compact and $\mathcal{B}_S$ denotes the Borel sets in S, together with the equilibrium correspondence in Feng, et. al., discussed in section 2.3.2. This correspondence is assumed to satisfy:

<u>Assumption 1:</u> Let $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ be the equilibrium correspondence in section 2.5.2. Then, $\tilde{J}$ is compact and Φ is upper hemi continuous and compact valued.

These properties can be obtained from mild assumptions on the primitives of the model discussed in section 2, both for finite (Magill and Quinzii, 1994, page 858, assumption 1 to 5) and infinite (Araujo, et. al. 1996, page 122, assumptions 1 and 3) shocks. A detailed discussion is postponed to section II in the appendix.
Assumption 1 together with the following lemma allows defining a Markov operator.

<u>Lemma 2</u>: Let Φ satisfy assumption 1. Then, $\varphi \sim \Phi$ is a $\mathcal{B}_{\tilde{J}\times S}$ -measurable selection of Φ and $P_\varphi(\tilde{z}, A) \geq 0$ is a Markov operator on $(\tilde{J}, \mathcal{B}_{\tilde{J}})$, where P_φ is given by:

5) $P_\varphi(\tilde{z}, A) = p(s, \{s' \in S | \varphi(\tilde{z}, s') \in A\})$, *where* $\tilde{z} = [s, \hat{z}]$

Proof: See Lemma 1 in Hildenbrand and Grandmont (1974), page 260, and section 4.1.

Lemma 2 implies the existence of a $\mathcal{B}_{\tilde{J}\times S}$ - measurable function φ, which is the natural candidate to construct the time invariant transition function of the process defined by $(\tilde{J}, P_\varphi)$ with typical realization $\{\tilde{z}_t\}_{t=0}^{\infty}$ as it satisfies $\tilde{z}_{t+1} = \varphi(\tilde{z}_t, s_{t+1})$ for any initial condition[19]. We postponed a detailed discussion of this issue to section 4.1. Remarkably, fact 2.3-2) presented above coupled with the structure of the sequential equilibrium, inherited from 1 period real securities and only 1 consumption good, allows us to extend the existence of a measurable selection for temporary equilibrium models in Hildenbrand and Grandmont (1974) to infinite horizon economies. As discussed in Feng and Hoelle (2017), there is an "indeterminacy" problem associated with a continuum of possible selections of the equilibrium correspondence in Feng, et. al. (2014). The stationary state space $\tilde{J}$ is constructed using all these possible selections. The uniform compactness of equilibrium and the results in Duffie, et. al (see lemmas 3.4 and 3.5, page 768), allows us to construct stationary selections that represent an infinite horizon equilibrium out of the truncated economy by induction. Then, we derive a selection mechanism that is directly related with ergodicity. Feng and Hoelle (2017) proposed a similar refinement for overlapping generation models, connecting each selection with a different steady state. We go a step further and show that it is possible to characterize these steady states using simulations.

Let $B(\tilde{J})$ and $\mathcal{P}(\tilde{J})$ be the space of bounded $\mathcal{B}_{\tilde{J}}$-measurable functions and the space of probability measures on $\tilde{J}$ respectively. Let $\hat{P}_\varphi: B(\tilde{J}) \to B(\tilde{J})$ and $P_\varphi^*: \mathcal{P}(\tilde{J}) \to \mathcal{P}(\tilde{J})$ be the semigroup and adjoint operators defined by $\hat{P}_\varphi f(\tilde{z}) = \int f(\tilde{z}')P_\varphi(\tilde{z}, d\tilde{z}')$ and $P_\varphi^*\mu(A) = \int \mu(d\tilde{z})P_\varphi(\tilde{z}, A)$. Standard results[20] imply that $\hat{P}_\varphi f(\tilde{z}) \in B(\tilde{J})$ and $P_\varphi^*\mu(A) \in \mathcal{P}(\tilde{J})$ provided that $f \in B(\tilde{J})$ and $\mu \in \mathcal{P}(\tilde{J})$, respectively.

Theorem 1 establishes *properties* which guarantee that the Markov process $(\tilde{J}, P_\varphi)$ has an invariant measure, $\mu \in \mathcal{P}(\tilde{J})$ with $\mu = P_\varphi^*\mu$, provided that under assumption 1 *φ may not be continuous.* An invariant, not necessarily ergodic, measure is a fixed point of P_φ^* and implies the stationarity of $(\tilde{J}, P_\varphi)$.

[19]A careful definition of the stochastic process associated with $(\tilde{J}, P_\varphi)$ will be given in section 4.

[20] Stokey, Lucas and Prescott, 1989, page 213 to 216

The discontinuity of φ is at the heart of the problem as it breaks the continuity of the adjoint operator. *Theorem 1 restores this property by restricting* μ *and* φ *such that the discontinuities in the transition function are negligible in an appropriate sense*. The following assumption formally states the mentioned restriction on φ.

<u>*Assumption 2:*</u> *Let* $\varphi \sim \Phi$ *be a* $\mathcal{B}_{\tilde{J}\times S}$ – measurable selection of the correspondence defined in assumption 1 and $\Delta\varphi$ its discontinuity set*. Then,* $\Delta\varphi$ *is a collection of at most a countable number of points.*

Under assumption 1, the range of φ is uniformly bounded. Thus, assumption 2 allows φ having at most a countable number of jump discontinuities. To argue in favor of the mildness of this last assumption, we borrow from Hildenbrand (1974) and Santos and Peralta Alva (2015). From the former we use the following results which characterizes the discontinuities of selections in upper-hemi continuous compact valued correspondences. Let $C(\varphi,\varepsilon)$ be the set of implosions and explosions of size $\varepsilon > 0$:

$$1)\quad C(\varphi,\varepsilon) = \left\{(\tilde{z},s) \in \tilde{J}\times S \;\middle|\; \exists\, U_{(\tilde{z},s)}\ with\ SUP_{x\in U_{(\tilde{z},s)}}\delta\big(\varphi(x),\varphi(\tilde{z},s)\big) < \varepsilon\right\}$$

Where $U_{(\tilde{z},s)}$ is a neighborhood and δ is the Hausdorff distance. If $x \in C(\varphi,\varepsilon)$ for all $(\tilde{z},s) \in \tilde{J}\times S$ and $\varepsilon > 0$ then φ is continuous. Assumption 1 implies that φ is only upper-hemi continuous. Fortunately, under this assumption, $C(\varphi,\varepsilon)$ is a "big" set:

<u>*Lemma I (Hildenbrand (1974), page 31)*</u>: Let φ map a metric space $(\tilde{J}\times S)$ into a totally bounded metric space $(\tilde{J})$. φ is compact valued and upper-hemi continuous. Then, for all $\varepsilon > 0$ $C(\varphi,\varepsilon)$ is an open and dense subset of $\tilde{J}\times S$.

Moreover, using a result in Santos and Peralta Alva (2005), we can ignore this dense set as far as the continuity of the Markov kernel is concerned:

Claim I (Santos and Peralta Alva (2005) page 1942): Let $\tilde{z}_j \to \tilde{z}$. Then:

2) $\int f(\tilde{z}')P_\varphi(\tilde{z}_j, d\tilde{z}') = \int f\left(\varphi(\tilde{z}_j, s')\right)p(s, ds') \to \int f(\varphi(\tilde{z}, s'))p(s, ds') = \int f(\tilde{z}')P_\varphi(\tilde{z}, d\tilde{z}')$

This assumption is *also* satisfied in the presence of discrete jumps and further discontinuities of $\varphi(., s')$ that are smoothed out after integrating over s'.

These results are relevant for the weak convergence of measures, which in turn is at the heart of the existence proofs for an invariant and ergodic measure. As $p(s, ds')$ is a density, lemma I and claim I imply that, if assumption 2 is violated, then there must be a positive measure discontinuity set which does not belong to $C(\varphi, \varepsilon)$ and is in the *frontier of* $\tilde{J} \times S$. *These facts imply that if the results in this paper are applied to a price-dependent occasionally binding inequality constraints problem, which typically hits the constraint with positive probability, assumption 2 is relevant.* However, in the model described in section 2, inequality constraints are price-independent short sale restrictions, which in turn imply that we can set them to be arbitrarily large and may not bind frequently. Inequality constraints are not essential for this model, and they are imposed only to guarantee the compactness of the equilibrium. Based on lemma I and claim I, in section 4.1, we will describe a selection mechanism for the model presented in section 2 which illustrates that assumption 2 is only relevant in the frontier of the state space. We will show that it is always possible the construct a selection of Φ that is continuous along s' if we are in the interior of $\tilde{J} \times S$.

Note that the Markov process defined by $(\tilde{J} \times S, P_\varphi)$ may not be irreducible. That is, the state space which defines it, $\tilde{J} \times S$, is not entirely stable and may contain islands and unstable (transient) points. If the process is irreducible, we know that $\tilde{J} \times S$ is convex as it is formed by rectangles. Thus, as convex sets have a frontier with zero measure, we can dispense with assumption 2 under lemma I and claim I[21]. For the purpose of this paper, assumption 2 is useful as it provides an alternative way to characterize non-irreducible ergodic chains. As can be seen in Pierri and Reffett (2021), irreducibility may be obtained in representative agent incomplete market models but are rather hard to prove once we add heterogeneity into the picture.

Now it is possible to state one of the main results in this paper:

<u>*Theorem 1 (Existence of an Invariant Measure-Stationarity of the Process)*</u>*:*
Let $\varphi \sim \Phi$ satisfies assumptions 1 and 2. Suppose additionally that a) $P_\varphi^*: \mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ and b) $\mathcal{P}_0(\tilde{J})$ is weak closed, where $\mathcal{P}_0(\tilde{J})$ is the set of atomless measures in $\mathcal{P}(\tilde{J})$. Then there is a measure $\mu \in \mathcal{P}_0(\tilde{J})$ such that $\mu = P_\varphi^* \mu$.*
Proof: see the online the appendix.

Note that a) and b) are "properties" of the process $(\tilde{J}, P_\varphi)$ and together imply that the discontinuity set of φ is negligible. That is, $\mu_n \longrightarrow_{weak*} \mu$ and $\mu(\Delta\varphi) = 0$. Sections 3.2 to 3.4 relate these properties with verifiable "conditions" on P_φ, φ and S. If property a) is satisfied, it suffices to assume that the set $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_0(\tilde{J})\}$ is *weak** closed. This is the strategy taken here and is concerned with the variability of the image of the transition functions, which cannot accumulate mass at any given point. This property requires variability in the image of a possible vector valued function as we move through the coordinates in the domain (see the supplementary material for section 5.1 in the appendix).
Let $IM(\varphi, \mathcal{P}_1) = \{\mu \in \mathcal{P}_1(\tilde{J}) | \mu = P_\varphi^* \mu\}$, where $\mathcal{P}_1(\tilde{J}) \subseteq \mathcal{P}_0(\tilde{J})$. That is, $IM(\varphi, \mathcal{P}_1)$ is a set of invariant measures of $(\tilde{J}, P_\varphi)$ which belong to $\mathcal{P}_1(\tilde{J})$, the set of absolutely continuous measures with respect to the Lebesgue measure on $\tilde{J}$, denoted θ. Under

[21] I would like to thank Juan Pablo Rincón Zapatero for pointing out this to me.

assumptions 1) and 2), if properties a) and b) hold for $\mathcal{P}_1(\tilde{J})$, the non-emptiness of $IM(\varphi,\mathcal{P}_1)$ can be assured using theorem 1 as long as $\mu \in \mathcal{P}_1(\tilde{J})$, $\mu(\Delta\varphi) = 0$ *and* μ *is the* weak* limit of a sequence of measures generated by P_φ^*. To show that $IM(\varphi,\mathcal{P}_1)$ is compact, which is essential for the existence of an ergodic measure, it is necessary to impose stronger conditions on P_φ, and consequently on φ, than the ones that are required for from theorem 1. Once this strengthening has been made, the closedness of $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_1(\tilde{J})\}$ follows from the same result used for $\mathcal{P}_0$ which insures the non-emptyness of $IM(\varphi,\mathcal{P}_1)$. This result cannot be applied to show the compactness of $IM(\varphi,\mathcal{P}_1)$. A suitable proof of this result is available in the appendix. A discussion of these issues can be found in the supplementary material to sections 3.1, 3.3 and 3.4 in the appendix.

A set $A \in \mathcal{B}_{\tilde{J}}$ is called invariant under P_φ if $P_\varphi(z, A) = 1$ for any $z \in A$. Let $IM(\varphi)$ be the set of invariant measures associated with selection φ. We say that $\mu \in IM(\varphi)$ is ergodic if either $\mu(A) = 0$ or $\mu(A) = 1$ for any invariant set under P_φ. The next theorem presents properties of $IM(\varphi)$ which guarantee that there exists an ergodic measure.

Theorem 2 (Existence of an Ergodic Measure-Ergodicity of the Process):

Let $\varphi \sim \Phi$ satisfies assumptions 1 and 2. Suppose additionally that $IM(\varphi,\mathcal{P}_1) \neq \emptyset$, where $\mathcal{P}_1$ is the set of absolutely continuous measures with respect to the Lebesgue measure. If c) $IM(\varphi,\mathcal{P}_1)$ is closed, then $IM(\varphi,\mathcal{P}_1)$ contains an ergodic measure.

Proof: The closedness of the set implies its compactness from proposition 2.8 in Futia (1982, page 385). As $IM(\varphi,\mathcal{P}_1)$ is convex, the Krein-Milman theorem (see Simon, 2011, theorem 8.14, page 128) implies that the set of extreme points of $IM(\varphi,\mathcal{P}_1)$, denoted $\mathcal{E}(IM(\varphi,\mathcal{P}_1))$, is non-empty. Remark 6.3 in Varadhan (2001, page 190) implies that if $\mu \in \mathcal{E}(IM(\varphi,\mathcal{P}_1))$, then μ is ergodic.

Theorems 1 and 2 are the first attempt to show separately the existence of an invariant and an ergodic measure for a *computable* correspondence based recursive equilibrium. For the case of uncountable shocks, we found sufficient conditions for stationarity and ergodicity by characterizing a *particular selection* and connecting it with primitive conditions of the model (see section 3.4 and fact 2.3-5 in section 2.3 with the associated supplementary appendixes). These facts ensure that our results can be used for computation and estimation of heterogenous agent models with aggregate uncertainty and incomplete markets.
Sections 3.2 and 3.3 identify conditions on P_φ which guarantee properties a), b) and c) associated with theorems 1 and 2. These conditions will be traced back to the primitives of certain type of economies in sections 3.4.

3.2 The case of a finite number of shocks

Theorem 1 requires 2 properties. Namely, that the adjoint operator associated with some Markov process $(\tilde{J}, P_\varphi)$ maps the set of atomless measures, $\mathcal{P}_0(\tilde{J})$, into itself (property a) and that $\mathcal{P}_0(\tilde{J})$ is closed (property b). The relationship between these properties and certain conditions of the Markov operator P_φ allows connecting the existence of an invariant measure with primitive assumptions in the model (i.e., restrictions on preferences, shocks, etc) as they affect $\varphi \sim \Phi$ and thus P_φ.
This section takes the first step towards that direction by restricting S, the set which contain the exogenous shocks, to be of finite cardinality. Let $\mu_{n,\theta}$ be a sequence of measures generated by applying P_φ^* iteratively on some $\theta \in \mathcal{P}(\tilde{J})$. Then, the following lemma states conditions on P_φ which guarantee properties a) and b).

<u>Lemma 3 (Conditions for stationarity in models with finite shocks)</u>:
Let Φ satisfy assumption 1 and $\#S < \infty$. Then, the measurable space $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ has an atomless measure θ. Let $\{a\}$ be any point in $\tilde{J}$. Suppose that for some $\varphi \sim \Phi$:

1) $\theta(\{a\}) = 0$ *implies* $P_\varphi(z,\{a\}) = 0$ θ*-almost everywhere*

2) $Sup_n \mu_{n,\theta}(\{a\}) = 0$*.*

Then, properties a) and b) in theorem 1 are satisfied.

Proof: see the online appendix.

Note that lemma 3 requires *conditions* 1 and 2 to hold *simultaneously* to guarantee properties a) and b). Condition 1 is associated with property a) and condition 2 with property b). While the supplementary material for section 5.1, in the appendix, presents mild sufficient conditions on the primitives of the economy presented in section 2.1 which guarantee condition 1, it is still an open question how to assure that condition 2 holds in a general equilibrium non-optimal economy with heterogenous agents[22]. Thus, a strong assumption on endogenous variables, condition 2, is required to assure the weak*-closedness of $\mathcal{P}_0(\tilde{J})$ when the state space is of the form $\tilde{J} = S \times \hat{Z}$, S is finite and $\hat{Z}$ is uncountable.

3.3 The case of an infinite number of shocks

This section presents conditions on the Markov operator P_φ for economies with an uncountable number of shocks s. Lemma 4 below is analogous to lemma 3 for this type of models. However, there are 3 important differences with respect to the case presented in section 3.2. First, the existence of an invariant measure follows only from 1 requirement, a strengthening with respect to condition 1) in lemma 3. Second, it is possible to define conditions on P_φ which guarantee the ergodicity of the invariant measure separately (i.e., condition 4). Third, we can connect properties a), b) and c) in theorems 1 and 2 respectively with assumptions on the set of shocks, its distribution and $\varphi \sim \Phi$. This last fact will be proved in section 3.4.

[22] Ito (1964, page 177) gave an example of a discontinuous function $\varphi \sim \Phi$ satisfying conditions 1)-2). However, it is not clear how to derive sufficient conditions on the primitives or how to characterize selections to ensure that condition 2) holds. However, this can be done if we assume an uncountable number of shocks as we do in section 3.3 and 3.4.

Lemma 4 (Conditions for stationarity and ergodicity with uncountable shocks):

Let Φ satisfy assumption 1 and 2. Further, suppose that S be an uncountable compact set. Then, the measurable space $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ has an atomless measure θ. Let $\{a\}$ and B be, respectively, any point and a Borel measurable set in $\tilde{J}$. Suppose that for some $\varphi \sim \Phi$:

3) *Stationarity:* $\theta(\{a\}) = 0$ *implies* $P_\varphi(z, \{a\}) = 0$ *for any* $z \in \tilde{J}$ *and* $z \notin \Delta\varphi$.

4) *Ergodicity:* $\forall \varepsilon > 0, \exists \delta > 0$ *such that* $\theta(B) < \delta$ *implies* $P_\varphi(z, B) < \varepsilon$ *for any* $z \in \tilde{J}$.

If condition 3) holds, then properties a) and b) in theorem 1 are satisfied.

If condition 4) holds, then property c) in theorem 2 is satisfied.

Proof: see the online appendix.

Remark 1: Condition 4) implies condition 3). Further, lemma 4 showed that $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_1(\tilde{J})\}$ *is weak* closed. Thus, assumption 2) can be replaced with the following, milder version:*

Assumption 2'): Let $\varphi \sim \Phi$ be a $\mathcal{B}_{\tilde{J}\times S}$-measurable selection of the correspondence in assumption 1 and $\Delta\varphi$ its discontinuity set. Then, $\Delta\varphi$ has zero Lebesgue measure[23].

The discussion in section 3.1 implies that this assumption is only relevant for the frontiers of the stable state space.

Condition 3) states that $P_\varphi(z,.)$ is an atomless measure for any z in the state space which does not belong to the discontinuity set $\Delta\varphi$[24]. Note that condition 1) in lemma 3 only requires $P_\varphi(z,.)$ to be atomless almost everywhere. Thus, condition 3)

[23] As any point in $\tilde{J}$ has zero Lebesgue measure, the result follows from Billingslley (1995, see equation 32.4 in page 422). The possibility to replace assumption 2 by 2' once condition 4 has been imposed follows from the fact that $\mu(\Delta\varphi) = 0$ if μ is the *Weak** limit of $\{\mu_n | \mu_n = P_\varphi^* \mu_{n-1}, \mu_0 \in \mathcal{P}_1(\tilde{J})\}$ and $\Delta\varphi$ has zero Lebesgue measure.

[24] The equilibrium correspondence in Feng, et. al. has an image in a separable finite dimensional space. Thus, it suffices to consider at most a countable set of selections (see Hildenbrand and Grandmont, 1974). This fact in turn implies that $P_\varphi(z, \{a\}) = 0$ for any $z \in \Delta\varphi$ as $\Delta\varphi$ is a finite set and there is an uncountable number of exogenous shocks. The existence of an atomless measure θ in models where condition 4 is not guaranteed to hold is shown in the proof of lemma 3 in section IV of the appendix.

is stronger than 1). The supplementary material for this section presents an example of a Markov process which satisfies condition 1) but the adjoint operator is not closed in the space of atomless measures. In that sense, condition 2) is essential. However, in this example condition 3 is violated. As shown in lemma 3, condition 3 is sufficient to show that the adjoint operator is closed in the space of atomless measures, ensuring the existence of an invariant measure under the assumption of uncountable shocks. The supplementary material for section 5, in the appendix, illustrates how to verify this condition on the economy defined in section 2. Condition 4) states that $P_{\varphi}(z,)$ is absolutely continuous w.r.t. θ uniformly in $z \in \tilde{J}$. As it is discussed in the supplementary appendix for this section, condition 4 is milder than the Doeblin condition (see for instance Stokey, Lucas and Prescott (1989)) and, as will be discussed in sections 3.4 and 4.1, can be traced back to restrictions on the selection of the equilibrium correspondence or on the state space.

The discussion below highlights a tension between the existence of a time invariant recursive equilibrium, the stationarity and the ergodicity of the associated Markov process. With finite shocks, the existence of the sequential equilibria can be proved by imposing mild requirements on the primitives, but the existence of an invariant measure involves the absence of heterogeneous agents. With uncountable shocks, ergodicity follows from the requirements presented in section 3.4 below.

The difference between conditions 1) and 3) (i.e., $\theta(\{a\}) = 0$ implies $P_{\varphi}(z, \{a\}) = 0$ θ-almost everywhere and uniformly in all continuity point respectively) has 2 important consequences. First, condition 1) allows S being a finite set. The existence of a sequential equilibrium follows from mild assumptions for this type of economies. This is the bright side. On the other hand, proving the existence of an invariant measure requires condition 2), which is very challenging to derive from primate conditions. If $\#S < \infty$, the existence of the sequential equilibrium and of the recursive structure in Feng, at. al. can be derived from primitive assumptions of the model. As can be seen in Zhu (2020) or Pierri and Reffett

(2021), the assumptions needed to prove the existence of an invariant measure when $\#S < \infty$ imply the existence of a representative agent. As this paper deals with heterogeneity, those results cannot be applied. Second, condition 3) allows proving the existence of an invariant measure imposing only this additional requirement to assumptions 1) and 2). Under this strengthening, condition 2 can be replaced by the closedness of the set of atomless measures under the adjoint operator, which is proved in the online appendix. This condition follows from assuming that S is uncountable and from a mild requirement on its distribution, as it only requires variability along one coordinate of the selection φ (see the supplementary appendix of section 5). However, showing the existence of a sequential equilibrium and of an appropriate recursive structure requires restrictions on endogenous variables. This last fact is discussed in sections 2.5.1 (see fact 2) and 5.

In summary, there is a tradeoff between the mildness of the assumptions required to prove the existence of a sequential equilibrium and to prove the existence of an invariant measure. In this sense, if the goal is to show the existence of a recursive stationary structure, we must impose strong assumptions to show the existence of sequential equilibria and then recursiveness and stationarity comes almost for free.

From the preceding discussion the crucial step in the existence of an invariant measure and its ergodicity is to ensure that the non-atomicity / absolute continuity of a sequence of measures is preserved under *weak** limits. This can be seen by noting that properties b) and c) in theorems 1 and 2 requires, respectively, the closedness of $\mathcal{P}_0$ and $\mathcal{P}_1$ and that, as was shown in lemmas 3 and 4, these properties impose restrictions on the Markov operator. Section 3.4 discussed how these restrictions reflect on φ and the primitives of the model. The example in the supplementary appendix to this section illustrate the problem at hand.

3.4 Sufficient conditions for stationarity and ergodicity

The conditions stated in lemma 4 allow to guarantee that the properties associated with the existence of an invariant and an ergodic measure (properties a) to c) in theorems 1 and 2) hold. However, they are based on the restrictions on endogenous variables. This section goes a step forward and connects these conditions (numbered 3) and 4) in lemma 4) with primitives in the model (assumptions 3-i,ii and iv) and with a mild continuity requirement on selections (assumption 3-iii).

<u>Assumption 3</u>: Let S be the set containing the exogenous shocks, $p(s,.)$ its distribution, $\Phi\colon \tilde{J} \times S \rightrightarrows \tilde{J}$ the equilibrium correspondence presented in definition 5 (see the technical appendix of section 2.3.2) and $\Delta\varphi$ the discontinuity set of $\varphi \sim \Phi$. Assume that:

- *i) S is uncountable and compact*
- *ii) $p(s,.)$ is atomless $\forall s \in S$*
- *iii) Suppose that assumption 2 holds. Let $(\tilde{z}, s') \in \Delta\varphi$ and $\{s_n\}$ a sequence with $s_n \to s$. In addition, suppose that $\lim\limits_{(\tilde{z}, s'_n) \to (\tilde{z}, s')} \varphi(\tilde{z}, s'_n) = \varphi(\tilde{z}, s')$ $\forall \tilde{z} \in \tilde{J}$*
- *iv) $p(s,.) = U[\underline{s}, \overline{s}]$ $\forall s \in S$, where $U[\underline{s}, \overline{s}]$ is the uniform distribution on $[\underline{s}, \overline{s}]$, a closed bounded interval of $\mathbb{R}$.*

Assumption 3-iii) allows for some path $(\tilde{z}_n, s'_n)$ to be discontinuous. For any $(\tilde{z}, s') \in \Delta\varphi$ there may exist $(\tilde{z}_n, s')$ with $lim_{(\tilde{z}_n, s') \to (\tilde{z}, s')} \varphi(\tilde{z}_n, s') \neq \varphi(\tilde{z}, s')$ but continuity is required on S for each $\tilde{z} \in \tilde{J}$. This assumption allows us to connect rectangles in the range of $\varphi(\tilde{z},.)$ with closed sets in S. Then, in proposition 2 below, the countable union of these rectangles will be associated with a small measure set to derive condition 4. As it was discussed in section 3.1, for the model described in section 2, assumption 3-iii), the continuity on s_+ for each $\tilde{z} \in \tilde{J}$, follows from mild restrictions on the recursive equilibrium in Feng, et. al. The procedure in Feng, et.

al. can be used to construct a selection φ which satisfies assumption 3-iii). Section 4 explains how to choose the selection to satisfy this assumption for the model with multiple assets.

The next 2 propositions connect assumption 3 with conditions 3 and 4.

Proposition 1 (Sufficient Condition for Stationarity):

Suppose that assumption 1, 3-i) and 3-ii) hold. Then, condition 3) is satisfied: $\theta(\{a\}) = 0$ *implies* $P_{\varphi}(z, \{a\}) = 0$ *for any* $z \in \tilde{J}$ *for an arbitrary point* $\{a\} \in \tilde{J}$

Proof: see the online appendix.

Proposition 2 (Sufficient Condition for Ergodicity):

Suppose that assumption 1, 3-iii) and 3-iv) hold. Then, condition 4) is satisfied. That is, $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $\theta(B) < \delta$ *implies* $P_{\varphi}(z, B) < \varepsilon$ *for any* $z \in \tilde{J}$.

Proof: see the online appendix.

Clearly, proposition 2 calls for stronger assumptions than proposition 1. This is because it involves verifying not only the non-atomicity of $P_{\varphi}(z,.)$, which requires only taking care of points in $\tilde{J} \subset \mathbb{R}^K$, but also its absolute continuity, which demands proving that sets of the form $\{a_1\} \times [a_2, b_2] \times \ldots \times [a_K, b_K]$ also have zero Lebesgue measure. Each of these sets can be "matched" with a sequence of rectangles which can be traced back to $p(s,.)$ under assump. 3-iii).

Remark 2: proposition 2 holds under a different version of assumption 3-iv).

Assumption 3-iv'): Let $(s,.) = U[LB(s), UB(s)]$ $\forall s \in S$, *where* $U[LB(s), UB(s)]$ *is the uniform distribution on* $[LB(s), UB(s)]$.

See the online appendix for a discussion.

Assumption 3-iv’) is weaker than 3-iv) as it allows the exogenous states to follow a Markov process instead of being i.i.d. However, the theorem which guarantees existence of an equilibrium correspondence for an economy with uncountable shocks requires assumption 3-iv). This result will be shown latter in section 5.

4 Characterization of the recursive economy and of simulations

In this section we further describe the equilibrium correspondence in framework of Feng, et. al. and characterize the selections which satisfy the critical requirement to achieve ergodicity (i.e., assumption 3-iii). Moreover, we explain the implications of stationarity and ergodicty for simulations. In the first case, for models that satisfy assumption 2 and the requirements in proposition 1, we show that the frequently used cumulative average will converge to random variable, which realizations depend on the entire observed path. Thus, not only the time limit of any cumulative average is history-dependent but also it is affected by the initial distribution of portfolios. To connect simulations with a well-behaved steady state for this case, we show that it is possible to “average” across paths to relate simulations with a history-independent stochastic steady state (i.e., an invariant measure for stationary models that is shown to exist in theorem 1). In the second case, for models that satisfy assumption 2 and the requirements in proposition 2, we show that it is possible to extend stationarity to ergodicty and, thus, cumulative averages will be history-independent, and each time limit will relate to a well-behaved steady state (i.e., an invariant measure which exists under theorem 2).

4.1 Characterization of recursive equilibria

The state space, $\tilde{Z}$, can be decomposed in 2 parts: payoff relevant variables Z_F and auxiliary variables m. In particular, let $Z_F \equiv \{[s, q, \theta] \in S \times \mathbb{R}^J \times \mathbb{R}^{IJ} \big| \sum_{i=1}^{I} \theta^i = \vec{0}\}$, $m^{i,j} \equiv$

$d^j(s)\left(u_s^i(c^i)\right)'$, where m is the vector of shadow values of the marginal return to investment for all assets and all agents. Assume, additionally to the hypothesis stated in section 2.1, that there exists a short sale constraint $\bar{B} > 0$ such that $\theta^{i,j} \geq -\bar{B}$. Using the budget constraint, equation 1, it is possible to define a correspondence V that maps $(z) \mapsto m$ as follows: for each $z \in Z_F$, $c^i \in [e^i(s) + \theta^i d(s) - I\bar{B}q, e^i(s) + \theta^i d(s) + I\bar{B}q]$ defines a selection $m \sim V(z)$ which is obtained by taking some $\theta_+^{i,j} \geq -\bar{B}$ for all $i \in I$ and $j \in J$. Provided, as discussed in section 2.5.1, that all endogenous variables in the model are (uniformly) contained in a compact set K, V is compact valued and $Gr(V)$ is compact.

Then, as in the previous subsection, it is possible to derive a time invariant compact state space, which is analogous to Duffie, et. al.'s self-justified set. Let $\widetilde{K} \subset K$ and $\widetilde{K} \equiv Gr(V_0)$. The first order conditions of the model can be written as:

3) $c^i = e^i(s) + \theta^i d(s) - \theta_+^i q$

4) $\left[q\left(u_s^i(c^i)\right)' - \beta E_{p(s,.)}(m_+^i)\right]\left[\theta_+^i - \bar{B}\right] = \vec{0}$

Where $E_{p(s,.)}$ is the expectation with respect to $p(s,.)$, the conditional distribution of s_+ given s, and $m_+^i(s_+) \sim V(\theta_+, q_+, s_+)$. Thus, 4) is defined using the expected value with respect to $p(s,.)$ over $[\theta_+, q_+](s_+)$.

4.1.1 Time independent State Space

Let $Gr(V_j) = C_j$. Iterating on $Gr(V_1)$, it is possible to derive a sequence of nested sets $\{C_j\}$ for $j \geq 1$ where C_j contains all $\tilde{z}_0$ of any j-period economy. Note that this procedure defines an operator $G_K: Gr(V) \to Gr(V)$, where V is some set … V_j … V_0. The non-emptiness and compactness of each C_j follows from the arguments in section 2.5.1 as, respectively, equations 3) and 4) are identical to the optimality conditions implied by the definition of "equilibrium with explicit debt constraint" in Magill and Quinzii (page 862) and the recursive equilibria in Feng, at. al. are a subset of those in Duffie, et. al.[25]

As G_K maps compact sets to compact sets, Feng, et. al. showed (theorem 2.1 in page 6) that $V_n \to V^*$, where V^* is the analogous of Duffie, et. al.'s self justified set. Thus $Gr(V^*) = \tilde{J}$ contains all possible first period payoff relevant variables $\tilde{z}_0(\sigma_0)$ for the sequential competitive equilibrium in definition 1.

Finally, $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ is defined as follows: take any $\tilde{z} = [\tilde{s}, \tilde{\theta}, \tilde{q}, \widetilde{m}] \in \tilde{J}$, $\tilde{z}_+ \in \Phi(\tilde{z}, \tilde{s}_+)$ if $\tilde{z}_+ \in \tilde{J}$ and $(\tilde{z}, \tilde{z}_+)$ satisfy equations 3) and 4) with $m(s_+) \sim V^*(\theta_+, q_+, s_+)(s_+)$. The following definition summarizes this discussion:

<u>*Definition 2 (Feng, et. al.'s recursive equilibrium):*</u> *Let $\tilde{J} = Gr(V^*)$ and $\tilde{J} \subseteq \widetilde{K}$. $\Phi: \tilde{J} \times S \rightrightarrows \tilde{J}$ is an equilibrium correspondence if $\tilde{z}_{t+1} \in \Phi(\tilde{z}_t, s_{t+1})$ and $\{\tilde{z}_t\}_{t=0}^{\infty}$ satisfy the optimality conditions in equations 3)-4) and the feasibility restrictions in the definition of Z.*

The procedure described above can be repeated an infinite number of times as $\tilde{J}$ contain all possible initial conditions $\tilde{z}_0(\sigma_0)$ for any $T \in \mathbb{N}$ period economy. A *time invariant transition function* is obtained by taking a selection of Φ, denoted $\varphi \sim \Phi$. This function is measurable, as Φ has closed graph and is compact valued (see Stokey, Lucas and Prescott, page 60 theorem 3.4 and 184 theorem 7.6), and does not depend on unobservable variables.

4.1.2 Selection Mechanism (Continuity along $\mathbf{s}_+$)

The functions θ_+ and q_+, mapping $s_+ \mapsto \theta_+$ and $s_+ \mapsto q_+$ respectively, can be chosen to be *continuous* provided that S in an uncountable set. Each of these functions is associated with a predecessor in $\tilde{Z}$.

[25] Section 5.1 will provide some additional details about these facts.

Now it is possible to define the analogous of a "self-justified set" in Feng, et. al. framework. To begin with, the set of all states, $\tilde{z} \in \widetilde{K}$, of any 2-period economy is contained in:

$$Gr(V_1) = \left\{ \tilde{z} \in \widetilde{K} \middle| \exists\, \tilde{z}_+ \in Gr(V_0)\ with\ \tilde{z}, \tilde{z}_+\ satisfying\ eq. 3)\ and\ 4) \right\}$$

That is, $[s, q, \theta, m] \in Gr(V_1)$ if $c^i(\theta_+^i)$ obtained from 3) for all $i \in I$ satisfy equation 4) for some $m_+^i(s_+) \sim V(\theta_+, q_+, s_+)$ with $[\theta_+, q_+, s_+] \in V_0$[26]. For any arbitrary iteration j, notice that for each $s_+ \in S$ there could be more than 1 possible pair (θ_+, q_+). However, as θ_+ is chosen at time "t", to satisfy the restrictions of the SCE, it must be s^t-measurable, where s^t is a branch of the tree $\mathfrak{T}$ defined in section 2.1. Thus, $\theta_+(s_+)$ can be chosen to be constant and thus *continuous for each* $\tilde{z} \in \widetilde{K}$. Moreover, any possible discontinuity in $q_+(s_+)$ can be ruled out by appropriately changing θ_{++} in $[e^i(s_+) + \theta_+^i d(s_+) - I\bar{B}q_+, e^i(s_+) + \theta_+^i d(s_+) + I\bar{B}q_+]$ with $\theta_{++} \in [-I\bar{B}, I\bar{B}]$. Suppose without loss of generality that $lim_{s_n' \to s'} q_+(s_n') < q_+(s')$. This is possible as the equilibrium is compact. Then:

$$lim_{s_n' \to s'}\, m_+^i(s_n') = u'\left[lim_{s_n' \to s'} e^i(s_n') + \theta_+^i lim_{s_n' \to s'} d(s_n') - \vec{\theta}_{++}^i lim_{s_n' \to s'} q_+(s_n')\right]$$

Where $lim_{s_n' \to s'} \theta_{++}^i(s_n') = \vec{\theta}_{++}^i$ is defined by the original selection $m_+^i(s_+) \sim V(\theta_+, q_+, s_+)$ as there must be at most 1 value for each $s_+ \in S$[27]. Then, it suffices to set $\varepsilon > 0$ such that $\left[lim_{s_n' \to s'} q_+(s_n')\right]\varepsilon = q_+(s')$ and multiply $\vec{\theta}_{++}^i$ by $1/\varepsilon$. Thus, we preserve budget feasibility (equation 3), optimality (equation 4) and market feasibility (as ε can be chosen to be uniform across agents).

As discussed in section 3.1, equations 1 (in lemma I) and 2 (in claim I) implies that for interior points the discontinuities along s' can be ignored. The paragraphs above are simply an application of those results. That is, the upper hemi-continuity

[26] As V_0 is a correspondence, $[\theta_+, q_+, s_+] \in V_0$ refers to elements in the domain of V_0. For the sake of simplicity and as there is no confusion, we prefer this notation to $[\theta_+, q_+, s_+] \in dom(V_0)$.

[27] As the equilibrium is compact, we know that $lim_{s_n' \to s'}[\theta_{++}^i(s_n') q_+(s_n')] = lim_{s_n' \to s'} \theta_{++}^i(s_n') lim_{s_n' \to s'} q_+(s_n')$.

and compact valuedness of V implies that "explosions and implosions" are small enough to be ruled out by any perturbation in $[-I\bar{B}, I\bar{B}]$. Not surprisingly, it is only possible to ensure the continuity of $[\theta_+, q_+](s_+)$ for an interior path for any $\tilde{z} \in V^*$.

Note that this procedure can be defined for each $[\theta_+, q_+, s_+] \in V_0$ but, as we may have $V^* \subset V_0$, it is possible that the required "perturbed" portfolio $\vec{\theta}^i_{++}/\varepsilon$ may not belong to V^*. Moreover, this restriction concerns only to ergodic selections. These facts has 2 implications, one involving ergodic and the other stationary selections. As regards the latter, as in Duffie, et. al. (1994), if the equilibrium is compact, we can choose $V^* = V_0$. As assumption 3-iii) is imposed on time-independent selections, as it must guarantee condition 4) in lemma 4 that involves the Markov kernel, it suffices to verify the continuity of q_+ on s_+ for V_0. However, in dynamic applications, as the stochastic process is not irreducible, it is possible that $V^* \subset V_0$. In this case, the selection mechanism described may require to "pick" assets such that $\theta^i_{++}(s) \notin V^*$[28]. In this case, the selection belongs to an ε-equilibrium which, as discussed in Kubler and Schmedders (2003), is a suitable stationary equilibrium concept for models with collateral constraints. In this sense, we go a step further with respect to these authors and show that the ε-equilibrium can generate an ergodic equilibrium. With respect to stationary selections, notice that θ_{++} can be chosen to depend only on s_+. As definition 2 implies that the Markov process iterates from $[\theta, q, s, s_+]$ to $[\theta_+, q_+, s_+]$, unless we require some regularity on q_+ as in the ergodic equilibrium, we are free to chose θ_{++}. Thus, the measurability requirement of the stationary selections, provided the uniform compactness of the equilibrium, follows from standard results. This may not be the case in more general models as noted by Feng and Hoelle (2017) who derive their results for overlapping generation models. The discipline imposed by the infinite dimension

[28] Heuristically, in an irreducible process, every positive measure set is "hit" by the chain with positive probability starting from any initial condition. Thus, $V^* = V_0$. During the construction of the ergodic selection described above because the process may not be irreducible, it may happen that $\vec{\theta}^i_{++}/\varepsilon \notin V^*$ as $V^* \subset V_0$. Thus, we are dealing with an ε-equilibrium as defined in Kubler and Schmedders.

of the optimization problem in definition 1 is critical to extend the stationarity of the temporary equilibrium to an infinite horizon problem. This is noted by Duffie, et. al.[29] who use the temporary equilibrium framework and uniform compactness to show the optimality of a sunspot-free stochastic process generated by a stationary transition of a generalized version of the equilibrium correspondence in definition 2. More to the point, the “indeterminacy” described in Feng and Hoelle (2017) is generated by an equilibrium characterized by a system with more unknowns than equations. This generates a *non-stationary* “indeterminacy” as it is associated with the system of equations that characterized the sequential equilibrium. In this paper the “indeterminacy” is associated with the equilibrium correspondence described in definition 2. That is, the “indeterminacy” is generated by a *stationary* system of equations and thus solved after picking a measurable selection. After selecting $m_+^i(s_+) \sim V(\theta_+, q_+, s_+)$ and thus $\theta_{++}(s_+)$, the system has the same number of unknows than equations as there is 1 Euler equation for each asset in θ and 1 market clearing condition for each price q.

4.2 Convergent Simulations.

The main result in this section is a direct application of Birkhoff’s ergodic theorem and the ergodic decomposition theorem for Markov process. Thus, the results will be stated without proof. This section follows closely chapter 6 of Varadhan (2001). The technical details are contained in the appendix of this section.

As in Santos and Peralta-Alva (2013), Kamihigashi and Stachurski (2015) or chapter 14 of Stokey, Lucas and Prescott, a simulation is *convergent* if it obeys a strong law of large numbers. In contrast to what is stated in those papers, convergence will be achieved only for a subset of all possible initial conditions. This is because the assumptions necessary to guarantee convergence starting from an arbitrary initial

[29] See proposition 3.2 in page 768.

condition are too strong for the purpose of this paper (see the appendix for details).

The conditions for stationarity (i.e., the existence of an invariant not necessarily ergodic measure) are milder with respect to the ones required to achieve ergodicity, can be traced back to primitives and do not require to construct a tailor-made selection or a restriction to the number of assets as in the case of an ergodic equilibrium. These facts imply that in practice we may *not* found this last class of equilibrium, which in turn has an important implication as regards the predictions of the model: because the strong law of large numbers for this class of processes (see Meyn and Tweedie, 1993, chapter 17) implies that simulations will not converge to time-invariant expected values, they will converge to a history dependent random variable; which is different for any possible initial condition. Contrarily, ergodic Markov processes hit the time invariant mean computed using the ergodic measure. We will first assume in this section that assumptions 1), 2'), 3-iii) and 3-iv') hold which ensure that the process is ergodic. Then, we will address the stationary case separately. Remark 2 allows $\{s_t\}$ to be generated by a Markov process (S, p) if S is an uncountable compact set of $\mathbb{R}$.

Let $\boldsymbol{P}_{\varphi,z_0}$ and $\boldsymbol{P}_{\varphi,\mu}$ the measures defined in the technical appendix of section 4.2. The following facts follow from Varadhan (2001, pages 179 and 187-192):

Fact 4.2-i): $\mu \in IM(\varphi)$ then $\boldsymbol{P}_{\varphi,\mu}$ is stationary and the process $(\tilde{J}, P_\varphi)$ is stationary.

Fact 4.2-ii): μ is ergodic if and only if $\boldsymbol{P}_{\varphi,\mu}$ is ergodic

Fact 4.2-iii): $\boldsymbol{P}_{\varphi,\mu} = \int \boldsymbol{P}_{\varphi,v} Q(dv)$, where v is an ergodic measure in $IM(\varphi)$ and $Q{:}\,\mathcal{P}(\tilde{J}) \rightarrow [0,1]$ a measure on $\mathcal{E}\big(IM(\varphi)\big)$, the set of extreme points of $IM(\varphi)$.

Fact 4.2-iv): $lim_{n\to\infty}[\sum_{t=1}^{n} f(z_t)]n^{-1} = \int f(z)\mu(dz)$ for almost every $\{z_t\}$ with respect to $\boldsymbol{P}_{\varphi,z_0}$ if z_0 belong to a set of positive μ-measure and $\mu \in IM(\varphi)$.

Fact 4.2-iv) follows from the previous 2 facts: as the ergodicity of μ is equivalent to the ergodicity of $\boldsymbol{P}_{\varphi,\mu}$ (fact 4.2-ii), theorem 2 suffices to show the existence of a Markov ergodic process $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$. Then fact 4.2-iii), the ergodic decomposition theorem for Markov processes, implies that Birkhoff's ergodic theorem can be applied to any initial condition in a positive μ-measure with $\mu \in IM(\varphi)$. That is, μ can be assumed to be ergodic w.l.o.g. $IM(\varphi)$ must have an ergodic measure, v, which is guaranteed by the compactness of the set of invariant measures.

The stochastic process derived directly from a sequential equilibrium may not be stationary. Fact 4.2-i) illustrates the importance of the results in section 3: even if an invariant measure cannot be shown to be ergodic, it suffices to prove the existence of a stationary process associated with the sequential equilibrium, which typically follows from mild requirements. This is because assumptions 3-i) and 3-ii) can be verified from primitive conditions of the model. However, the results in Durret (2019, see section 7.2) imply that the convergence in fact 4.2-iv) cannot be achieved.

If the process is stationary but not ergodic, the cesaro average will converge to a random variable which realizations depends on the initial condition of the model. In particular, $Z_0, Z_1, \ldots$ is stationary if $\{Z_0, Z_1, \ldots, Z_m\}$ and $\{Z_k, Z_{k+1}, \ldots, Z_{k+m}\}$ has the same distribution for each m and $k > 0$. A Markov process with an invariant measure μ and Z_0 distributed according to it is stationary[30]. Let $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$ be the state space, sigma-algebra and measure presented before, where μ is an invariant not necessarily ergodic measure. A measure preserving map h satisfies $\boldsymbol{P}_{\varphi,\mu}(h^{-1}(A)) = \boldsymbol{P}_{\varphi,\mu}(A)$ with $A \in \mathcal{F}$. If h is the shift operator (i.e., $h(\omega_0, \omega_1, \ldots) =$

[30] see Durret 2019, page 279 example 7.12

$\omega_1, \omega_2, \ldots)$ and $(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu})$ is stationary, then h is a measure preserving map. Then we have the following result: we say that a set $A \in \mathcal{F}$ is invariant if $\boldsymbol{P}_{\varphi,\mu}(A \Delta h^{-1}(A)) = 0$, where Δ is the symmetric difference between 2 sets[31]. Moreover, if $\boldsymbol{P}_{\varphi,\mu}(A) > 0$ and μ is an invariant measure, A is an invariant set. Let $\omega \in \Omega$. Then, $Z_t : \Omega \to \tilde{J}$, $\omega(t) = z_t = Z_t(\omega)$ and $Z(h^m(\omega)) = Z_m(\omega)$. Then, we have the following result:

$$lim_{n\to\infty} \left[\sum_{t=1}^{n} f(z_t)\right] n^{-1} = lim_{n\to\infty} \left[\sum_{t=1}^{n} f\left(Z(h^n(\omega))\right)\right] n^{-1} \to_{\boldsymbol{P}_{\varphi,\mu}\ a.e.} E[f(z)|\mathbb{I}]$$

Where $E[f(z)|\mathbb{I}]$ is an expectation conditional on the sigma-algebra of invariant sets $\mathbb{I}$ as h is measure preserving and Z is measurable with respect to $\mathbb{I}$[32]. Intuitively, the measure preserving property implies that the only relevant sets almost everywhere are invariant. $E[f(z)|\mathbb{I}]$ is a random variable which realizations depend on a particular event $\omega \in A \in \mathbb{I}$, with $\omega_0 \in A_0$ and $\mu(A_0) > 0$. Thus, any history ω and initial condition ω_0 implies that the cesaro average $[\sum_{t=1}^{n} f(z_t)]n^{-1}$ converge to a different value. To solve this problem, it is possible to use the law of iterated expectations. In particular, $E_{\boldsymbol{P}_{\varphi,\mu}}[E[f(z)|\mathbb{I}]] = E_{\boldsymbol{P}_{\varphi,\mu}}[f(z)] = E_{\mu}[f(z)]$, where the first equality follows from the fact that the trivial sigma-algebra $\{\emptyset, \Omega\}$ is included in $\mathbb{I}$ and the second follows from μ being an invariant measure.

These results have an important takeaway point: either if a model is ergodic, if it satisfies assumptions 1, 2, 3-iii) and 3-iv), or only stationary, if it satisfies the milder set of assumptions 1, 2, 3-i) and 3-ii), it is possible to hit the stochastic steady state of the model. In the last case, we need to average across histories:

[31] The symmetric difference between A and B is the collection of elements in A and B but not in both sets. It is also called the disjoint union.
[32] See Durret (2019) page 281, exercise 7.11.

$$lim_{j\to\infty}\left[lim_{n\to\infty}\left[\sum_{t=1}^{n} f\left(Z\left(h^{n}(\omega^{j})\right)\right)\right]n^{-1}\right]j^{-1}$$
$$= lim_{j\to\infty}\left[lim_{n\to\infty}\left[\sum_{t=1}^{n} f(z_n^j)\right]n^{-1}\right]j^{-1}$$

Using this procedure we can eliminate the effect of the lack of ergodicity on simulations. The intuition goes as follows: the difference between stationary and ergodic selections comes from the behavior of densities in the limit. As the set of invariant measures in an ergodic equilibrium must be weakly compact, densities must be bounded. To guarantee that property, we need to impose an additional assumption on selections to show that the absolute continuity of the Markov kernel can be preserved under limits. Even if this assumption is not satisfied, as we can show under milder requirements that the set of invariant measures is not empty, weak compactness is not far away. As stationarity implies that the steady state of the model is history dependent, by averaging across histories, it is possible to restore the ergodicity of the equilibrium. Because there is an uncountable number of shocks and invariant measures are atomless, the law of iterated expectations allows us to prove that the set of steady states with unbounded densities has zero-measure. In other words: unbounded densities in a compact sequential equilibrium can only be generated by the convergence of the process to an absorbing state, a singleton, which has zero measure under the equilibrium distribution of the process in the stationary case $\boldsymbol{P}_{\varphi,\mu}$.

5 Applications

We apply the theoretical results presented before to a concrete parametrization of the economy in section 2. Following figure 1, the requirements to achieve the existence of an ergodic measure can be categorized in 3: properties (a-c), conditions (1-4) and assumptions (1-3). Section 3.2 and 3.3 connected conditions, mostly on the Markov operator P_φ, with properties of the associated process $(\tilde{J}, P_\varphi)$. Section 3.4 shows that conditions 1-4 can be generated by assumptions for the case of uncountable shocks. While most of these assumptions, 3-i), 3-ii) and 3-iv), are stated in terms of the primitives of the model, there are 2 which are still stated in terms of endogenous variables: 2 and 3-iii). Section 3.1 and 4.1 relate these 2 assumptions and show how they interact between each other, generating mild restrictions on the model presented in section 2.

The first step in the applications section is to prove the existence of a compact sequential equilibria in definition 1. This fact leads to the existence of an appropriate recursive structure in the sense of Feng, et. al. as stated in assumption 1 and definition 2. As in Mas-Collel and Zame (1996), the presence of uncountable shocks requires imposing additional assumptions with respect to the canonical model with incomplete markets. This assumption requires total wealth (i.e., $e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t)$, $\sigma_t \in \mathfrak{T}$) to be bounded away from zero. Due to the presence of short sale constraints, this requirement is mild.

Once this additional hypothesis holds, by carefully refining selections from the recursive equilibrium correspondence in definition 2 as shown in section 4.1, there is an important gain in terms of the predictive power of the model as the theory developed in this paper allows showing not only that the model has a well-behaved steady state (theorem 1) but also that it is ergodic (theorem 2) and that simulations converge to a time independent stochastic steady state without for any time path. Refining selections implies knowing the discontinuity set. To avoid this problem, we show that it is possible to satisfy assumption 3-iii) by restricting the number of real assets in the model. As the results in section 4.2 show, it is possible to eliminate the time-dependence of simulations by averaging across multiples histories. Considering the discussion which follows lemma I and claim I in section 3.1, assumption 2 is mild. Thus, following proposition 1, lemma 4 and theorem 1, we can extend our results to *any* measurable selection of the equilibrium correspondence in definition 2. To complete the description of the model, we present 2 additional subsections containing all the requirements necessary to show the existence of a compact sequential equilibrium, something that is taking as given by assumption 1.

5.1 Finite shocks and implicit function theorem for condition 3

The model is the same as the one described in section 2.1. Following figure 1, the first step to prove the stationarity of the model is to derive a recursive representation for the sequential equilibria. As discussed in section 2.5.2 and 4.1, *the existence of a recursive structure is guaranteed by the existence of the sequential competitive equilibria and the compactness of the equilibrium set.* In the present framework, these properties will be shown to be implied by the assumptions listed in this subsection. Moreover, for the sake of completeness, all the assumptions required for the existence of an invariant measure are presented below.

Assumptions 4.1-i) to 4.1-v) ensure the existence of a non-empty compact equilibrium set which will be shown to be sufficient to derive a Markov representation of equilibria. Provided this representation, to show the existence of an invariant measure, it suffices to impose assumption 2, property a) and property b) (presented in section 3.1, theorem 1). The first and the last are stated as a hypothesis below (assumptions 4.1-vi and 4.1-vii respectively) and the second one will be derived from primitive conditions of the model which are implicit in assumptions 4.1-i) to 4.1-v).

Assumption 4.1). *Suppose an incomplete market economy as the one described in section 2.1. To that structure add the following assumptions:*

i) *The utility function in the optimality condition of definition 1 is:*

$$U_i(c) = \sum_{t=0}^{\infty}(\beta)^t \sum_{\sigma_t^* s}\left[u_s^i(c^i(\sigma_t^* s))\right]\mu_t(\sigma_t^* s)$$

Where $u_s^i\left[c^i(\sigma_t^* s)\right] = 1 - e^{-\lambda c^i(\sigma_t^* s)}$ *with* $\lambda > 0$.

ii) *The realizations of the exogenous shock* s_t *lie in set* S *of finite cardinality for any time period* $= 0,1,\dots$.

iii) *Endowments satisfy:* $e^i(\sigma_t) > 0$ *and* $\sum_{i=1}^{I} e^i(\sigma_t) < K$ *with* $K > 0$ *for any agent* $i \in \{1,\dots,I\}$ *and node* σ_t *. Idiosyncratic endowments are strictly positive and aggregate endowments are uniformly bounded. There is aggregate and idiosyncratic uncertainty.*

iv) *There is a finite number,* J *, of numerarie short lived assets with (uniformly) bounded dividends and short sale constraints. That is, for each agent* i *and any node* σ_t *the portfolio is given by* $\theta^i(\sigma_t) \geq -B,\ B \in \mathbb{R}_+^J$*, the associated dividends by* $d(\sigma_t s) \in M \subset \mathbb{R}_+^J$*, where* M *is uniformly bounded, and the budget equation by*

$$c^i(\sigma_t) = e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t) - \theta^i(\sigma_t).q(\sigma_t)$$

Where $q(\sigma_t)$ *is the price of the portfolio in terms of the numerarie for every node* σ_t *and* σ_t^* *is the predecessor of* σ_t.

v) *There is a riskless bond. There is an asset* l *which has associated dividends given by* $d_l(\sigma_t s) = 1$ *for any* $s \in S$ *and any node* σ_t.

vi) *Assumption 2 holds (i.e., the discontinuity set of any measurable selection of the equilibrium correspondence has at most finite cardinality).*

vii) *Condition 2 holds (i.e., provided that the adjoint operator maps the set of atomless measures into itself, this set is weakly closed).*

Except the assumption on u_s^i, the short sale constraints, 4.1-vi) and 4.1-vii), the rest are standard in the literature. The results in Magill and Quinzii (1994) imply that under assumptions 4.1-i) to 4.1-v), excluding the restriction on u_s^i, the economy describe in section 2.1 has a non-empty compact equilibrium set[33].

[33] See assumption A.1 to A.6 and the discussion that follows in pages 858-60.

The chosen instantaneous return function u^i on assumption 4.1-i) guarantees that marginal utility is bounded on the entire feasible consumption set which, because of assumption 4-iii), is given by $[0, K]$. Kubler and Schmedders (2002) shows that assumptions 4.1-i) to 4.1-v), including the restriction on the return function but excluding the short sale constraints, imply that any sequence of consumption bundles $\{\{c^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$, portfolios $\{\{\theta^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$ and prices $\{q(\sigma_t)\}_{\sigma_t\in\mathfrak{T}}$ which satisfy the feasibility requirement $\sum_{i=1}^{I}\theta^i(\sigma_t)=\vec{0}$, where $\vec{0}\in\mathbb{R}^J$ for any $\sigma_t\in\mathfrak{T}$, and the Kuhn Tucker conditions listed in equation 3 and 4 (see the technical appendix to section 2.5.2) meet the optimality and feasibility conditions in definition 1 and thus constitutes a sequential equilibrium. The compactness of the equilibrium set follows from Magill and Quinzii (1994).

Short sale constraints are standard in the recursive literature since Duffie, et. al. (1994). Braido (2013) showed that a recursive equilibrium in the sense of Duffie, et. al. exists even if explicit short sale constraints are removed. This is possible as Magill and Quinzii (1994) showed that there is a uniform bound on assets even in the absence of short sale constraints. However, the theoretical results in this paper depend on Feng, et. al. recursive equilibria which, as discussed in section 2.5.2, are a subset of all possible recursive equilibria in Duffie, et. al. It is not clear that Braido's results hold in Feng, et. al.'s framework. Thus, short sale constraints are imposed to guarantee the existence of an appropriate (sunspots free) recursive equilibrium.

As seen in section 2.5.2 (see also Feng, et. al. section 2.2), if the equilibrium set is compact and can be generated by the set of equations implied by the Kuhn Tucker and feasibility conditions, the equilibrium correspondence Φ in definition 2 satisfies the assumptions in lemma 2 and thus P_φ is a well-defined Markov operator. These facts imply that $(\tilde{J}, P_\varphi)$ defines a (compact) Markov process with typical state $\tilde{z}=[s,\theta,q,m]\in\tilde{J}$ and $m_j^i=d^j(s)\big(u_s^i(c^i)\big)'$.

While assumptions 4.1-i) to 4.1-vi) are relatively mild, assumption 4.1-vii) is strong as it directly implies the weak-closedness of $\mathcal{P}_0(\tilde{J})$ (i.e., property b). Further, this assumption cannot relate to primitive conditions of the model. Fortunately, it is possible to obtain properties a) and b) jointly by strengthening condition 1. This is done by lemma 4, that requires only condition 3, which strengthens condition 1 by requiring it to hold *uniformly in all continuity points*. Proposition 1 shows that condition 3 holds if the model is allowed to have uncountable exogenous shocks s. Considering the distinctive nature of this type of economies, they must be treated separately. Section 5 in the body of the paper and II.2 below addresses this point.

5.2 Uncountable Shocks

The discussion in the preceding section sets a tradeoff: to get rid of unverifiable assumptions like property b), the structure of exogenous shocks must be modified. Unfortunately, proving the existence of the sequential equilibria (and thus the existence of an appropriate recursive structure in the sense of Feng, et. al.) with uncountable shocks requires imposing an additional assumption on 4.1-i) to 4.1-v). This assumption, labeled 4.2-ii) below, was extensively discussed in the literature (see for instance Mas-Colell and Zame, 1996, or Araujo, et. al. 1996). Assumption 4.2-ii) implies the existence of a positive wealth in each node. Given the presence of short sale constraints, the boundedness of dividends and endowments, in the present context, it is rather mild.

Assumption 4.2 contained all the sufficient conditions to show the existence of an ergodic invariant measure in the model discussed in section 2, except assumption 3-iii) which will be treated separately in a lemma below.

<u>Assumption 4.2).</u> Suppose an incomplete market economy as the one described in section 2.1. To that structure add the following assumptions:

i) *Assumptions 6.1-i), 6.1-iii) and 6.1-iv) hold.*

ii) $e^i(\sigma_t) + \theta^i(\sigma_t^*).d(\sigma_t) > 0$, $\sigma_t \in \mathfrak{T}$

iii) *Assumptions 3-i) and 3-iv) hold (i.e., the set of exogenous shocks is* $S = [\underline{S}, \overline{S}] \subset \mathbb{R}$ *and* $p(s,.) = U[\underline{S}, \overline{S}]$*, where* U *is the uniform distribution).*

iv) *Assumption 2' holds (i.e., the discontinuity set is at most of zero lebesgue measure).*

Assumptions 4.2-i) to 4.2-iii) guarantees the existence of the sequential equilibria. The proof follows immediately by extending the induction argument in Mas-Colell and Zame (1996) for $T = \infty$ as in Duffie, et. al. (1994, see fact 2.5-2 in section 2.5.1). Theorem 4.1 in Mas-Colell and Zame allows proving the non-emptiness C_j for $1 \leq j \leq T < \infty$, where C_j is the set of initial states of a $j + 1$ period economy defined in section 4.1. The compactness of K, the set that includes all payoff relevant states, follows from theorem 4.2 also in Mas-Collel and Zame. The induction argument in section 5 of that paper can be used to set $T = \infty$. The optimality argument in Duffie, et. al. (section 3.4) can be immediately extended to the Mas-Colell and Zame framework as theorem 4.1 and 4.2 hold $\mu_s^\infty(s_0,.)$-a.e. for $s_0 \in S$ and θ_-^i satisfying assumption 4.2-ii), where $\left(\Omega, \mathcal{F}, \mu_s^\infty(s_0,.)\right)$ is the stochastic process defined in section 4.2 but restricting the state space Ω to contain only an infinite sequences of exogenous shocks $\{s_t\}$.

The compactness of K and the upper hemi continuity (in z_+) of the system of equations defined by 3), 4) and the feasibility of assets guarantees that the equilibrium correspondence, Φ in definition 2, satisfies the assumptions required by lemma 2. Thus, there is at least 1 measurable selection $\varphi{\sim}\Phi$ and $\left(\tilde{J}, P_\varphi\right)$ defines a Markov process.

Once an appropriate Markov process have been shown to exist, proposition 2 implies that assumptions 4.2-iii), 4.2-iv) and 3.iii) are sufficient to show the ergodicity of the process $\left(\tilde{J}, P_\varphi\right)$. The following lemma shows that if there is only 1 asset or the recursive equilibrium notion in Feng, et. al. is appropriately restricted (see fact 2.5-5 in section 2.5.2), assumption 3-iii) can be omitted.

<u>Lemma 5</u>: Suppose that fact 2.5-5 holds or $J = 1$ (i.e., there is 1 asset). Then, under assumptions 4.2-i) to 4.2-iv), $(\tilde{J}, P_{\varphi})$ has an ergodic invariant measure.

Proof: see the online appendix.

6 Conclusions and directions for future research

This paper develops the theoretical foundations for the characterization of the dynamic behavior through simulations in incomplete markets models with aggregate uncertainty and heterogeneous agents. The results in this paper are relevant as they provide a set of assumptions which ensure that empirically relevant models can be taken to data. The parameters obtain are then reliable to perform policy experiments which could be welfare enhancing.

The paper provides a set of results which allow characterizing incomplete markets general equilibrium models beyond existence. Further, it distinguishes between the predictive performances of models with different degree of uncertainty as measured by the cardinality of the set which contains exogenous shocks. Also, this article presents a set of sufficient conditions and procedures that guarantee that simulations reflect the long and short run behavior of general equilibrium models.

Although the assumptions rather mild, because the results are specific to the model described in this paper, there is scope for future research both in models with a finite number or with uncountable shocks. For the former, condition 2, which ensures the existence of an invariant measure, must relate to primitive conditions. The results in Zhu (2020) or Pierri and Reffett (2021) are not appropriate as they rely either on the partial equilibrium nature of the model, the lack of aggregate uncertainty or the existence of a representative agent. Further, condition 2 must also be modified to guarantee the ergodicity of the measure as theorem 2 requires even stronger assumptions than theorem 1 as illustrated by properties b) and c). For the case of uncountable shocks, an extensive numerical test must be performed as the typical "discretization" of the state space may not be adequate.

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Appendix

Supplementary Material for section 3.3

<u>*Example 1 (non-uniform boundness of densities)*</u>*:* Let $P: S \times \mathcal{B}_S \to [0,1]$ be a transition function with $S = [0,1]$, $P(s, \{s/2\}) = 1$ and $\theta = U[0,1]$. Condition 1 is satisfied as $P(s, \{a\}) = 0$ except for $s = 2a$ with $\theta(\{2a\}) = 0$. Thus, under lemma 3, $P_\varphi^*: \mathcal{P}_0([0,1]) \to \mathcal{P}_0([0,1])$, where $\varphi(s) = s/2$. Note then that property a) in theorem 1 holds. However, property b) will not be satisfied. Let $\mu_1 = P_\varphi^* \theta$ and $A = [0, a]$ with $0 < a < 1$. Then $\mu_1(A) = 2a$, that is, $\mu_1 = U[0,1/2]$ which has a density of 2. In general, $\mu_n = U[0,1/2^n]$ with $\mu_n = P_\varphi^* \mu_{n-1}$. Thus, $\{\mu_n\}$ has an associated sequence of densities of $\{2^n\}$, which is not a uniformly bounded sequence of functions. Kempton and Persson (2015, page 11) show that absolutely continuity is preserved under *weak** limits if the sequence of densities associated with $\{\mu_n\}$ is uniformly bounded.

This paper proved that that absolutely continuity is preserved under *weak** limits by imposing condition 4), that is slightly weaker than the uniform integrability of densities (see Diestel, 1991 for a detailed discussion), which is in turn weaker than the mentioned uniform boundness.

Example 1 shows that $\mathcal{P}_0([0,1])$, the subset of atomless measures in $\mathcal{P}([0,1])$ generated under the action of P_φ^*, is not closed as it contains a sequence of measures weakly converging to a Dirac measure at 0. Note that condition 3) is not satisfied in example 1 as $P_\varphi(z, \{a\}) = 0$ must hold uniformly in z, not a.e.

Condition 2), by lemma 3, and condition 3), by lemma 4, guarantee the closedness of $\mathcal{P}_0$ for the case of finite and uncountable shocks respectively.

As lemma 4 shows, condition 4) assures the closedness of $\mathcal{P}_1$. This condition implies that the family of measures $\{P_\varphi(z, .) | z \in \tilde{J}\}$ is absolutely continuous w.r.t. θ

and that sets with a θ-measure smaller than δ have $P_\varphi(z,.)$-measure uniformly bounded by ε, where uniformity means that ε is independent of δ. This last condition is weaker than the uniformly integrability of densities, denoted by $\bar{p}_\varphi(z,z')$, as the latter requires $\int_B |\bar{p}_\varphi(z,z')|\theta(dz') < \varepsilon$ while the former only implies $\int_B \bar{p}_\varphi(z,z')\theta(dz') < \varepsilon$ (see Diestel, 1991). Although the distinction is subtle, it has important consequences: if $\int_B \bar{p}_\varphi(z,z')\theta(dz') < \varepsilon$ implies $\int_B |\bar{p}_\varphi(z,z')|\theta(dz') < \varepsilon$ for any $z \in \tilde{J}$, then $\bar{p}_\varphi(z,z')$ is bounded away from zero in $\tilde{J} \times \tilde{J}$. But in this case, exercise 11.4 in Stokey, Lucas and Prescott implies that P_φ satisfies the Doeblin condition (i.e. $\theta(B) < \delta$ implies $\int_B \bar{p}_\varphi(z,z')\theta(dz') < 1 - \varepsilon$ for any $z \in \tilde{J}$), which is a sufficient for the existence of an ergodic invariant measure (see page 345-8 for a discussion). A similar result holds if $\bar{p}_\varphi(z,z')$ is uniformly bounded above in $\tilde{J} \times \tilde{J}$.

By the discussion in example 1 and in the preceding paragraph, in this paper it will not be assumed that densities are neither bounded nor uniformly integrable as it suffices to restrict the Markov operator only to condition 4.

Note that assumption 2', like assumption 2, represents an upper bound on the genericity of the multiple equilibria problem discussed in section 2.4. Condition 4 is stronger than condition 3. Thus, as any invariant measure under condition 4 is absolutely continuous with respect to the Lebesgue measure, the constraint imposed by $\mu(\Delta\varphi) = 0$ in theorem 1 is now less restrictive: $\Delta\varphi$ can be an uncountable set if it has zero Lebesgue measure.

The strategy in lemma 4 is different from the one used to show the closedness of $\mathcal{P}_0$ under the adjoint operator. Since Futia (1982), ergodicity requires the compactness of the set of invariant measures. As $IM(\Phi,\mathcal{P}_0)$ may contain unbounded density functions, compactness can't be guaranteed[34]. Lemma 4 shows how condition 4 implies that small θ-measure sets have arbitrary small μ_n-measure,

[34] Condition 4) implies that the Markov operator is absolutely continuous which, it is well known, implies the existence of densities $\bar{p}_\varphi(z,z')$, which are bounded uniformly in z (as required by condition 4) and a.e. in z'. These facts cannot be assured by imposing condition 3) as the Radon–Nikodym theorem cannot be applied.

where $\{\mu_n\}$ is any sequence in $IM(\Phi, \mathcal{P}_1)$, and that this latter property guarantees that absolute continuity is preserved under *weak** limits of $\{\mu_n\}$.

Supplementary Material for section 3.3

Remark on the local convergence of the Law of Large Numbers

Santos and Peralta Alva (2013) require that condition 4) holds for any selection of Φ_j and Φ. Kamihigashi and Stachurski (2015) requires that φ be continuous and Breiman's theorem in Stokey, Lucas and Prescott require a unique ergodic measure. In contrast, *the results in this paper require only that condition 4 holds for some selection $\{\varphi_j\}$ and φ. Moreover, we characterize these selections and connect the restrictions on the Markov kernel with characteristics of the selections; proving a refinement mechanism. Further, theorems 1 and 2 allow φ to be discontinuous and $(\tilde{J}, P_\varphi)$ to have multiple ergodic measures.* In this kind of setting, there are no results that guarantee the global almost sure convergence of simulations. Thus, a local theorem, like Birkhoff's, must be used.

Details of the Stochastic Process

To present the results for this section some additional definitions are required. Let $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ be a measurable space and $\left(\tilde{J}^t, \mathcal{B}_{\tilde{J}}^t\right) = (\tilde{J} \times \ldots \times \tilde{J}, \mathcal{B}_{\tilde{J}} \times \ldots \times \mathcal{B}_{\tilde{J}})$ the associated product space. Let $A = A_1 \times \ldots \times A_t$ be a measurable rectangle (see Stokey, Lucas and Prescott page 195 for a definition) in $\mathcal{B}_{\tilde{J}}^t$. Let $\varphi \sim \Phi$ and $z_0, \ldots, z_t \in \tilde{J}$. As long as t is finite, by virtue of the Caratheodony and Hahn theorems and theorem 7.13 in Stokey, Lucas and Prescott (1989), the measure $\mu^t(z_0, A)$, defined by $\mu^t(z_0, A) = \int \cdot_{A_1} \ldots \int \cdot_{A_t} P_\varphi(z_{t-1}, dz_t) \ldots P_\varphi(z_0, dz_1)$, can be uniquely extended to a probability measure in any set of $\mathcal{B}_{\tilde{J}}^t$, where $\int \cdot_{A_i}$ denotes integration w.r.t. $P_\varphi(z_{i-1}, dz_i)$.

Analogously, let $B = A_1 \times ... \times A_T \times \tilde{J} \times ...$be a finite measurable rectangle (see page 221 of Stokey, Lucas and Prescott for a definition) and $\mathcal{L}$ its power set. Let $\mathcal{M}$ be the algebra generated by finite unions in $\mathcal{L}$ and $\mathcal{F} = \mathcal{B}_{\mathcal{M}}$ (i.e., $\mathcal{F}$ is the sigma field generated by $\mathcal{M}$). Then $\mu^{\infty}(z_0, B) = \int_{A_1} ... \int_{A_T} P_{\varphi}(z_{T-1}, dz_T) ... P_{\varphi}(z_0, dz_1)$ can be shown to be extended to $\mathcal{F}$ in 2 steps. First, using the Caratheodony and Hahn theorems it is possible to extend $\mu^{\infty}(z_0, B)$ to $\mathcal{M}$ and then to $\mathcal{F}$. Later, using standard arguments for processes with a finite dimension distribution (see Shiryaev 1996, Ch. 9), $\mu^{\infty}(z_0, B)$ can be shown to be countably additive.

Standard results (see for instance exercise 8.6 in Stokey, Lucas and Prescott) imply that $\left(\Omega, \mathcal{F}, \mu^{\infty}(z_0, .)\right)$ is a Markov process with stationary transitions P_{φ}. Let $\Omega = \tilde{J} \times \tilde{J} \times ...$ with typical realization $\omega \in \Omega$. As Ω is the space of sequences, it is natural to define a $\mathcal{F}_t$-measurable random variable $z_t : \Omega \to \tilde{J}$, where $\omega(t) = z_t = z_t(\omega)$ denotes a typical realization and $\{\mathcal{F}_t\}$ is a sequence of nested sigma algebras on $\{\times_{i=1}^{t} \tilde{J}(i)\}$, where $\tilde{J}(i) = \tilde{J}$ for $i \geq 1$. The shift operator is denoted by $T : \Omega \to \Omega$. A set $A \in \mathcal{F}$ is called T-*invariant* if $TA = A$[35].

Let $\mu^{\infty}(z_0, B) \equiv \boldsymbol{P}_{\varphi, z_0}(B)$. Under the same assumptions, $\boldsymbol{P}_{\varphi_j, z_0}(B)$ can be analogously defined if $\tilde{J}$ is replaced by K, which was supposed to be compact in assumption 4-i). Further, $\boldsymbol{P}_{\varphi,\mu} \equiv \int_{A_0} \boldsymbol{P}_{\varphi, z_0} \mu(dz_0)$ can be used to define a stochastic process $\left(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu}\right)$ which allows to randomize z_0 as μ is a measure on $(\tilde{J}, \mathcal{B}_{\tilde{J}})$. $\left(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu}\right)$ is said to be *stationary* if $\boldsymbol{P}_{\varphi,\mu}[C(t,n)] = \boldsymbol{P}_{\varphi,\mu}[C(t',n)]$ for all $n \geq 0$ and $t \neq t'$ with $C(t,n) = \{\omega \in \Omega : [z_{t+1}(\omega), .., z_{t+n}(\omega)] \in C\}$. $\left(\Omega, \mathcal{F}, \boldsymbol{P}_{\varphi,\mu}\right)$ is said to be ergodic if $\boldsymbol{P}_{\varphi,\mu}(A) \in \{0,1\}$, where A is a T-invariant set.

[35] Exercise 6.2 in Varadhan shows that this definition can be used w.l.o.g.

Supplementary Material for section 5.1

Given the existence of a Markovian representation $(\tilde{J}, P_\varphi)$, theorem 1 implies that to prove the existence of an invariant measure, it suffices to impose assumption 2, condition a) and condition b). The first and the last are listed in assumptions 4.1-vi) and 4.1-vii).

Property a), namely that the adjoint operator associated with P_φ maps the space of atomless measures into itself, holds if the implicit function theorem can be applied to the system of equations defined by equations 3, 4 and $\sum_{i=1}^{I} \theta^i = \vec{0}$ in a full lebesgue measure set[36]. Let $z = [s, \theta, q]$ and $F(z, z_+) = \vec{0}$ be the system of $J + J \times I$ equations that can be obtained by replacing equation 3 into 4 and considering only interior solutions[37]. Section V.1 in this appendix will show that, under assumptions 4.1-i) to 4.1-v), $D_{z_+}F(z, z_+)$ has full rank a.e. in z, where $D_{z_+}F$ is the Jacobian matrix of F with respect to z_+.

Once this property has been established, it suffices to apply lemma 3. That is, lemma 3 connects condition 1 (i.e., $\mu(\{a\}) = 0$ *implies* $P_\varphi(z, \{a\}) = 0$ *z-a.e. with respect to an atomless measure* μ) with property a) (i.e., $P_\varphi^*: \mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ where P_φ^* is the adjoint operator, $\tilde{J}$ is the state space of the process and $\mathcal{P}_0(\tilde{J})$ the space of atomless measures in $\mathcal{P}(\tilde{J})$). The arguments below show that the full rank of $D_{z_+}F(z, z_+)$ is sufficient to guarantee condition 1.

We now argue that under assumptions 4.1-i) and 4.1-vi) the implicit function theorem can be applied to the system of equations that is equivalent to the sequential competitive equilibrium in definition 1. The result can be applied a.e. and uniformly in any continuity point of the state space. Jump discontinuities

[36] See the discussion in the preliminary remark of lemma 3 in section III of this appendix implies for details.
[37] The discussion in section A.2.1 in the appendix connects Φ with F and $\tilde{z}$ with z. Once Φ is defined, it suffices to note that $\tilde{z} = [z, m]$ and m is defined by the additional equation given above.

under assumption 2, because of lemma I and claim I, cannot generate "flat" selections and, thus, can be excluded of the analysis.

The results in Magill and Quinzii (1994) and Kubler and Schmedders (2003) imply that under assumptions 4.1-i) to 4.1-v) the following system of equations defines a sequence of consumption bundles $\{\{c^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$, portfolios $\{\{\theta^i(\sigma_t)\}_{i\in I}\}_{\sigma_t\in\mathfrak{T}}$ and prices $\{q(\sigma_t)\}_{\sigma_t\in\mathfrak{T}}$ which satisfy the feasibility and optimality requirements in definition 1:

A.1) $\sum_{i=1}^{I}\theta_+^i=\vec{0}$ with $\vec{0}\in\mathbb{R}^J$

A.2) $q_j u_s^i(e^i(s)+\theta^i d(s)-\theta_+^i q)'-\beta\sum_{s_+\in S}d_j(s_+)p(s,s_+)u_s^i(e^i(s_+)+\theta_+^i d(s_+)-\theta_{++}^i q)'=0,\ j\in J, i\in I$

Let $z=[s,\theta,q]$ with $\sum_{i=1}^{I}\theta^i=\vec{0}$ and $m^i=d(s)u_s^i(e^i(s)+\theta^i d(s)-\theta_+^i q)'$. Also let $F(z,z_+)=\vec{0}$ be the system of equations defined by A.1) and A.2), where $\vec{0}\in\mathbb{R}^{J+J\times I}$.

The discussion in section 2.5.2 and the definition of an expectation correspondence, definition 2, imply that under assumptions 4.1-i) to 4.1-v) $[z_+,m_+]\in\tilde{J}$ if $[z,m]\in\tilde{J}$, where $\tilde{J}$ is the state space in definition 2. Moreover, the same results imply that each θ_{++} implicit in m_+ define a different selection $m_+\sim V^*(z_+)$, where $\tilde{J}=Gr(V^*)$. Thus, as A.1) and A.2) can be used to define a particular selection $\varphi\sim\Phi$, θ_{++} can be assumed to be constant throughout the analysis.

Further, because: a) $s,s_+\in S$ and $\#S<\infty$ and condition 1 is required to hold a.e. in an atomless measure μ and b) for the case of uncountable shocks, condition 3 must hold uniformly in all continuity points, the discussion in the preliminary remark of lemma 3 in the online appendix implies that it suffice to show that $D_{z_+}F(z,z_+)$ has full rank. That is, if the mentioned full-rank condition holds a) μ -a.e. in z for the case of finite shocks and b) for all continuity points in $\tilde{J}$ for the case of uncountable shocks, it implies that we get $\mu(D)=0$; where $D=\{[z,m]\in\tilde{J}: P_\varphi([z,m],\{a\})>0\ if\ \mu(\{a\})=0\}$ is the set containing the "flat" parts of the selection which in turn implies that the adjoint operator may fail to map the space of atomless measures into itself. Moreover, assumption 4.1-vi) guarantees that $D_{z_+}F(z,z_+)$ is well defined μ -a.e. in z and for all continuity points as the discontinuity set of φ is allowed to have up to finite cardinality and F is defined for interior solutions only. To complete the proof, it suffices to write $D_{z_+}F(z,z_+)$ explicitly to note that this matrix has full rank under assumptions 4.1-i) and 4.1-v) if there is more than 1 asset[38].

Notice that the implicit function theorem is required to hold a.e. in z in condition 1 and in every z which does not belong to the discontinuity set $\Delta\varphi$. Thus, there is no contradiction between this property and the possible discontinuity of φ as, considering assumption 2, the discontinuity set of φ has finite cardinality. This property implies that: a) $\Delta\varphi$ has zero measure on μ and thus can be excluded from the states z in condition 1, b) we can exclude $\Delta\varphi$ from $\tilde{J}$ in condition 3 without loss of generality as the equilibrium correspondence can be fully characterized by a countable number of selections (constructed for a given point in the domain) as its image is contained in a finite dimensional compact set (see Hildenbrand and Grandmont, 1974) and claim I and lemma I implies that we are dealing with jump discontinuities which are a "non-constant" part of the selection by definition. Thus, if $z \in \Delta\varphi$ and $P_{\varphi}(z,B)$, then B has at most a countable number of elements and thus has zero atomless measure.

[38] $D_{z_+}F(z,z_+)$ is available under request.

Online Appendix: Proofs and Related Comments

Theorem 1

Heuristic description of Theorem 1

Let $C(\tilde{J})$ be the space of continuous functions on $\tilde{J}$. P_φ has the Feller property if the semigroup operator maps $C(\tilde{J})$ into itself. Lemma 9.5 in Stokey, Lucas and Prescott (page 261) shows that if $f \in C(\tilde{J})$, $\hat{P}_\varphi f(\tilde{z}) \in C(\tilde{J})$. The absence of the Feller property also affects the continuity of the adjoint operator, which is critical to guarantee the existence of a fixed point of it. As P_φ^* is defined over an infinite dimensional space, to discuss its continuity, it is necessary to select an adequate topology. The *weak** topology, the coarsest topology that makes the linear functional $\{\mu \mapsto \int f d\mu, f \in C(\tilde{J})\}$ continuous, is frequently chosen. This is because P_φ^* generate sequences of *weak** convergent measures under mild assumptions[39]. Under assumption 1, $\tilde{J}$ is a compact subset of a finite dimensional Euclidean space. Thus, Helly's theorem (Stokey, Lucas and Prescott, page 374) implies the existence of a *weak** -convergent subsequence in $\mathcal{P}(\tilde{J})$, which is the starting point of most existence theorems.

As discussed in Aliprantis and Border (2006, page 47), the choice of a weak topology implies a tradeoff: there are a lot of weakly convergent sequences but there are few weakly continuous functionals. Thus, the Feller property is used to guarantee the *weak** continuity of P_φ^*: $\mu_n \rightarrow_{Weak*} \mu$ implies $P_\varphi^* \mu_n \rightarrow_{Weak*} P_\varphi^* \mu$ if $\hat{P}_\varphi$ has the Feller property (see Stokey. Lucas and Prescott, page 376).

If φ can be shown to be continuous, under assumption 1, Theorem 2.9 in Futia (1982, page 383) would imply the existence of an invariant measure for P_φ^*. It only

[39] This is not the case of the *strong* topology, which is the topology generated by the total variation norm. Stokey, Lucas and Prescott (page 335 to 337) provides an example of a Markov process that generates sequences that converge in the *weak** topology but not in the strong (norm) topology.

suffices to take a sequence of measures generated by applying P_φ^* iteratively on some $\mu_0 \in \mathcal{P}(\tilde{J})$ that is robust to cyclical behavior and fits into the framework of Helly's theorem. Let $\mu_{n_k} \to_{Weak*} \mu$ be the subsequence generated by Helly's theorem. The continuity of P_φ^* implies $P_\varphi^* \mu_{n_k} \to_{Weak*} P_\varphi^* \mu$. Subtracting both subsequences, the desired result follows. *Theorem 1 in this paper shows the existence of an invariant measure for* $(\tilde{J}, P_\varphi)$ *even if* φ *is allowed to have (a certain type of) discontinuities.*

The strategy of the proof for Theorem 1 goes along the lines of Hildenbrand and Grandmont (1974). It borrows from theorem 12.10 in Stokey, Lucas and Prescott (1989) (page 376), theorem 3.5 in Molchanov and Zuyev (2011, page 15) and proposition 1 in Ito (1964, see page 155). The following subsection contains a detailed description of the procedures used *up to* now to prove the existence of an invariant measure and the reasons that make them unsuitable for this paper.

Using proposition 1 in Ito and theorem 3.5 in Molchanov and Zuyev it is possible to restore the continuity of P_φ^* without the Feller property. As P_φ^* and $\hat{P}_\varphi$ can be interchanged (see for instance Stokey, Lucas and Prescott page 216), if $\mu_{n_k} \to_{Weak*} \mu$, for some $f \in C(\tilde{J})$:

$$\int f(\tilde{z}) P_\varphi^* \mu_{n_k}(d\tilde{z}) = \int \hat{P}_\varphi f(\tilde{z}) \mu_{n_k}(d\tilde{z}) \nrightarrow \int f(\tilde{z}) P_\varphi^* \mu(d\tilde{z}) = \int \hat{P}_\varphi f(\tilde{z}) \mu(d\tilde{z})$$

As $\hat{P}_\varphi f(\tilde{z})$ may not be continuous. $\hat{P}_\varphi f(\tilde{z})$ is bounded and $\mathcal{B}_{\tilde{J}}$-measurable. Theorem 3.5 in Molchanov and Zuyev implies that $\int \hat{P}_\varphi f(\tilde{z}) \mu_{n_k}(d\tilde{z}) \to \int \hat{P}_\varphi f(\tilde{z}) \mu(d\tilde{z})$ if $\mu\big(\Delta \hat{P}_\varphi f\big) = 0$, where $\Delta \hat{P}_\varphi f$ is the *set of discontinuities of* $\hat{P}_\varphi f$.

Thus, it only suffices to show that the discontinuity set generated by φ *is sufficiently small under the limiting measure.* To achieve this property, proposition 1 in Ito is used to show that P_φ^* maps the set of *atomless measures, which will be denoted* $\mathcal{P}_0(\tilde{J}) \subset \mathcal{P}(\tilde{J})$, *into itself. The proof will be complete if it can be shown that* $\mu \in \mathcal{P}_0(\tilde{J})$ *and* $\mu\big(\Delta \hat{P}_\varphi f\big) = 0$. As a measure is atomless if and only if $\mu(\{a\}) = 0$, $\{a\} \in \tilde{J}$ (i.e. it assigns zero measure to points, see Hildenbrand and Grandmont 1974, page 45), it suffice to restrict the cardinality of $\Delta \hat{P}_{\varphi} f$ to be at most countable and to show that $\mathcal{P}_0(\tilde{J})$ is closed. The latter property will be insured by imposing conditions on P_{φ} in section 3.2 and 3.3. The former will be guaranteed by restricting the discontinuity set of φ as in assumption 2 and 2'.

Related results on the existence of Invariant measures.

Because the expectation correspondence in Duffie, et. al. G is defined as map $z \mapsto \mu$, with $\mu \in \mathcal{P}(Z)$, using the arguments presented in section 2.5.1 and 3.1 it is easy to show that a THME (J, π) can be used to define a sequence of measures $\{\lambda_t\}_{t=0}^{\infty}$ in $\mathcal{P}(J)$ such that $\lambda_{t+1} = \pi^* \lambda_t$, where π^* is the adjoint operator associated with (J, π).

Grandmont and Hildenbrand showed that the continuity of π^* is sufficient to show the existence of an invariant measure λ, provided that J is a compact set and G is constructed from an equilibrium correspondence: every $\pi \sim G$ satisfies $\pi = \pi_{\varphi}$ with $\varphi \sim \Phi$ and $\Phi: J \times S \to J$. Provided that assumption 1 holds, the existence of π_{φ} follows from Lemma 2. As discussed in the supplementary appendix of section 3.1, π^* is continuous iff $\hat{\pi}$ has the Feller property, where $\hat{\pi}$ is the semigroup operator associated with (J, π). The authors could not show that $\hat{\pi}$ has this property and had to assume it (see Lemma 2 in Grandmont and Hildenbrand, page 263).

As the selection φ may not be continuous, thus the result in Hildenbrand and Grandmont was considered unsatisfactory. Blume (1982) dispense with this assumption and took a rather different approach. Given a Markovian structure with time homogeneous transitions, the author used Fan's fixed-point theorem to show the existence of an invariant measure for $G_B: \mathcal{P}(Z) \to \mathcal{P}(Z)$, where $G_B = \{\pi_{\varphi}^*: \varphi \sim \Phi\}$. As G_B was assumed to be nonempty, to have closed graph and Z to be compact, the required upper-hemicontinuity followed immediately. However, to apply Fan's theorem, G_B must be convex valued[40]. To guarantee this latter property, Blume assumed that S is characterized by an atomless, not necessarily absolutely continuous, measure. Clearly, if S is a finite set, this last assumption is not realistic. The arguments in section 3.2 try to fill this gap. Even if S is a compact and uncountable set, and $p(s,.)$ is atomless $\forall s \in S$, as discussed at the end of section 3.1, the results in Blume only shows that G_B has a fixed point, which is equivalent to $IM(\Phi) \neq \emptyset$ but weaker than $IM(\varphi) \neq \emptyset$ for any $\varphi \sim \Phi$ satisfying assumptions 1, 2 and the additional hypothesis presented in section 3.3.

The results in Blume highlighted the necessity of a "convexified" correspondence, G_B, to prove the existence of an invariant measure. This was the approach taken by Duffie, et. al. (1994), theorem 1.1, to show the existence of an *ergodic* measure. As discussed in section 2.5.1 and its supplementary appendix, provided the existence of time-invariant state space and that G is convex valued, Duffie, et. al. (1994) showed that a refinement of a THME, called conditionally spotless, has an ergodic measure. The following definition states this notion of equilibrium formally:

<u>Definition A1 (Conditionally Spotless THME)</u>:

Let $\mathcal{P}_F(S \times \hat{Z}) = \{\mu \in \mathcal{P}(S \times \hat{Z}) | \exists\, h, h: S \to \hat{Z}, with\ Supp(\mu) = Gr(h)\}$. A THME (J, π) is <u>spotless</u> if $\pi(z) \in \mathcal{P}_F(S \times \hat{Z})$ for all $z \in J$. A THME (J, π) is called <u>conditionally spotless</u> if for all $z \in J$, $\exists\, M \subset \mathcal{P}_F(S \times \hat{Z}) \cap G(z)$, $\eta \in \mathcal{P}(M)$, $\pi(z) = \int v \, d\eta(v)$ and G is convex valued.

Note that a spotless THME removes the possibility of sunspots discussed in Lemma 1: given $z_t \in J$, there is a measure $\mu_{z_t} \in G(z_t) \cap \mathcal{P}_F(S \times \hat{Z})$, which gives the conditional distribution of z_{t+1}, $\hat{z}_{t+1} = h(s_{t+1})$ and $\mu_{z_t}(Gr(h)) = 1$. Intuitively, each pair (z_t, s_{t+1}) is associated with a unique $\hat{z}_{t+1}$ or equivalently $\hat{z}_{t+1} = h_{\mu_{z_t}}(s_{t+1})$ and $\hat{z}_{t+1}$ satisfy the optimality and feasibility requirements contained in the definition of G. Note that it

[40] G_B is convex-valued if $\lambda_1', \lambda_2' \in G_B(\lambda)$, with $\lambda_1' = \pi_{\varphi_1}^* \lambda$, $\lambda_2' = \pi_{\varphi_2}^* \lambda$ and $\varphi_1, \varphi_2 \sim \Phi$,then $\lambda' \in G_B(\lambda)$ with $\lambda' = (\alpha)\pi_{\varphi_1}^* \lambda + (1 - \alpha)\pi_{\varphi_2}^* \lambda$, $(\alpha)\pi_{\varphi_1}^* + (1-\alpha)\pi_{\varphi_2}^* \in G_B$ and $\alpha \in [0,1]$.

is possible to refine even more a spotless THME by letting $\hat{z}_{t+1} = h_{z_t}(s_{t+1})$, where the measurability of f must be shown and $z_{t+1} \sim \mu \in \mathcal{P}(S \times \hat{Z})$ must be defined accordingly. *The results in section 3 and 4 hold for this last type of equilibria.*

To show the existence of an ergodic invariant measure for a spotless THME the authors proceeded in 2 steps. First, they applied Fan's fixed point theorem to $T \equiv E \circ m_2 \circ m_1^{-1}: \mathcal{P}(J) \to \mathcal{P}(J)$, where $m_1: \mathcal{P}\big(Gr(G_J)\big) \to \mathcal{P}(J)$, $m_2: \mathcal{P}\big(Gr(G_J)\big) \to \mathcal{P}\big(\mathcal{P}(J)\big)$ give the marginals of $\mathcal{P}\big(Gr(G_J)\big)$ and $E\eta \equiv \int \mu d\eta(\mu), \eta \in \mathcal{P}\big(\mathcal{P}(J)\big)$ is the mean of η, which is uniquely defined by the Riesz representation theorem for continuous function[41]. T is a continuous linear functional and G_J is upper hemi-continuous. This was assumed in Duffie, et. al. In the context of this paper, a similar property follows from theorem 3.1 in Blume under assumption 1 provided that G_J is constructed from Φ using Lemma 2. However, this last procedure only captures a subset of all possible recursive equilibria. Under these 2 properties, T is also upper hemi-continuous[42]. As J is a self-justified set (i.e. is a stationary state space in terms of Duffie, et. al.), G_J is nonempty. T is convex valued as G assumed to be so. As $\mathcal{P}(J)$ is nonempty, (weakly) compact and convex, T has a fixed point. Second, the authors showed that any λ with $\lambda = T(\lambda)$ also satisfies $\lambda = \pi \cdot \lambda$. To derive this result, they defined a transition function $P: J \to \mathcal{P}\big(\mathcal{P}(J)\big)$ and showed that $E \circ P(z) \in G_J(z)$ λ-a.e. Thus, $\pi(z) = \int v \, d\eta(v)$ almost everywhere for $\eta \in \mathcal{P}\big(\mathcal{P}(J)\big)$.

To obtain an ergodic measure for a conditionally spotless THME, which is defined for economies with a *finite number of shocks*, $\mathcal{P}(J)$ should be replaced with $G_J(z) \cap \mathcal{P}_F\big(S \times \hat{Z}\big)$. This implies that G_J is convex valued: definition A1 assures that for any $z \in J$, there exist an expectation correspondence $\hat{g}$ which is convex valued as it contains any possible randomization $\mathcal{P}(M)$ over spotless transitions $M \subseteq \mathcal{P}_F\big(S \times \hat{Z}\big) \cap G(z)$ for any $z \in J$. A selection $\pi(z) \sim \hat{g}(z)$ is constructed by changing $\eta \in \mathcal{P}(M)$ and

[41] See Theorem 14.12 in Aliprantis and Border (2006, page 496).
[42] See Grandmont (1983, page 158).

computing $\pi(z) = \int v \, d\eta(v)$. The assumption that G is convex valued can be done w.l.o.g. once transitions f are allowed to depend on "contemporaneous" sunspots (α_t), which select among randomized spotless transitions. Sections 3.3, 3.4 and 4.1 shows that, if we restrict the class of models with respect to Duffie, et. al, it is possible to construct an equilibrium correspondence that contains a stationary and an ergodic equilibrium for uncountable shocks.

The discussion above implies that the transition functions generated by a conditionally spotless THME are affected by sunspots; a fact that affects the computability of the recursive structure. The authors did not prove the existence of an ergodic measure for a spotless THME, which generate sunspots free transition function. This paper shows this result for a refinement of all possible spotless THME (i.e., those generated from Feng, et. al.'s recursive structure).

Santos and Peralta Alva (2013) show that $IM(\Phi, \mathcal{P}_1) = \{\varphi \sim \Phi, \mu \in \mathcal{P}_1 | \mu = P_\varphi^* \mu\} \neq \emptyset$. Unfortunately, there are some concerns about the Santos and Peralta Alva (2013) framework. First, it is not clear if S is a finite set. If S can be characterized by a Markov process with an atomless Markov operator (i.e., $p(s,.)$ is atomless for all $s \in S$), the non-emptiness of the set of invariant measures $IM(\Phi)$ follows immediately from theorem 3.1 in Blume (1982). *This paper provides conditions which guarantee the non-emptiness of* $IM(\varphi)$ *for any* $\varphi \sim \Phi$ *that satisfies assumption 1 and 2 which is slightly stronger than* $IM(\Phi) \neq \emptyset$*. It is also convenient in applications as frequently it is desirably to compute only an approximation of* φ*.* Second, the conditions which guarantee the existence of an ergodic measure in $IM(\Phi)$ have not been established, at least separately from those that guarantee $IM(\Phi) \neq \emptyset$. *Theorem 1 and 2 establishes, respectively, the properties of* $(\tilde{J}, P_\varphi)$ *associated with the existence of an invariant and an ergodic measure. The first set of conditions are milder and thus do not require to construct a "tailor-made selection" as we did in section 2.5.2.* Third, the critical assumptions in Santos and Peralta Alva (2013), assumption 2.3 and remark 6.2, have been stated in terms of P_{φ}, not on primitives, and the procedure to compute an ergodic selection is not available. Thus, it may be difficult to identify these assumptions in certain applications.

Preliminary Remark on $\tilde{J}$

As theorem 1 will show that there exist $\mu \in \mathcal{P}_0$ with $\mu = P_{\varphi}^{*}\mu$ (i.e., an invariant measure exists and it is atomless), it is necessary for the state space of the process defined by $(\tilde{J}, P_{\varphi})$ to be uncountable. This is because the candidate measure μ_N, with $\mu_{N_k} \rightarrow_{Weak*} \mu$, satisfies $Supp(\mu_K) \subseteq \tilde{J}$ as it is constructed applying iteratively P_{φ}^{*}.

Fortunately, the results used to guarantee the non-emptiness of C_j for $j \geq 1$ (i.e., the set which contains all initial states, $\tilde{z}_0$, of any j-period economy) which were discussed in sections 2.5.1 (fact 2), 2.5.2 and 5 can be used to guarantee the desired result. In particular, Theorems 25.1 in Magill and Quinzii (1996) and theorem 4.1 together with section 5 in Mas-Colell and Zame (1996) for economies with finite and infinite number of shocks respectively can be used to show the existence of a sequential competitive equilibrium (see Definition 1) for a truncated economy $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^{I}, T\right]$, with $T < \infty$. The optimality conditions in Definition 1 for this economy are:

OA1) $c^i = e^i(s) + \theta^i d(s) - \theta_+^i q$

OA2) $\left[q\left(u_s^i(c^i)\right)' - \beta E_{p(s,.)}(m_+^i)\right]\left[\theta_+^i - \bar{B}\right] = \vec{0}$

Where short sale constraints $\bar{B}$ are assumed to hold and $\theta_+^i = 0$ if $\theta^i = \theta^i(\sigma_{T-1})$. In the sequential economic literature, if $\theta_+^i = \theta^i(\sigma_0)$, it is customary to assume that $\theta_-^i \equiv \theta^i = 0$ and $\sigma_0 \equiv s_0$ is supposed to be fixed. However, in the recursive literature, both θ_-^i and σ_0 are allowed varying as $\tilde{z}_0 = [s_0, \theta_-^i, \hat{z}_0]$, where $\hat{z}_0$ contains the rest of the state space.

Moreover, the existence of equilibria for $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^{I}, T\right]$ requires that $e^i(s_0) > 0$ (see assumption A.2 in Magill and Quinzii, page 858). Thus, provided the rest of the assumptions mentioned in sections 2.1, 2.5.1 and 5 hold, as noted by Duffie, et. al. (Lemma 3.4), θ_-^i and s_0 can be chosen arbitrarily as long as $e^i(s_0) + \theta_-^i d(s_0) > 0$, which can be considered the initial endowment of goods if the exogenous state is s_0. Formally, it suffices to assume that:

Definition OA1: The initial distribution of assets θ_- will be called admissible and denoted $\theta_- \in \Lambda$ if is feasible and satisfies $Min_{i \in I, s \in S} e^i(s) + \theta_-^i d(s) > 0$.

Remark OA1: $\tilde{J} = S \times \Lambda \times \hat{Z}$, where $\Lambda \times \hat{Z}$ is uncountable because and has no isolated points: i) Λ is uncountable and has no isolated points according to definition A2, ii) under the assumptions made in sections 2.1, 2.5.2 and 5, $C_j \neq \emptyset$ independently of the cardinality of S (i.e., an equilibrium for $\mathcal{E} = \left[e, d, \{U^i\}_{i=1}^{I}, \theta_-\right]$ exists independently of the cardinality of S) for any $\theta_- \in \Lambda$ (i.e., for any admissible θ_-).

Remark OA1 is frequent in applications: see for instance Duffie, et. al. (1994) section 3 and Kubler and Schmedders (2003) page 1777. Vector θ_- describes any predetermined level of asset holdings or the capital stock. Consequently, in numerical approximations θ_- is supposed to be contained in an uncountable subset of $\mathbb{R}$ and its properties (i.e., compactness) can be defined independently of those characterizing $\hat{Z}$ as (s, θ_-) are initial conditions of some sequential competitive equilibrium. Thus, Λ is compact if and only if it is closed. This last property is easily verifiable as can be seen in Kubler and Schmedders (2003) (see lemma 1, page 1776). As will be seen in the proof of lemma 3, the crucial property of Λ, besides its cardinality, is the lack of isolated points. This property follows w.l.o.g. from definition OA1.

In all the proofs, except if it is mentioned explicitly, it will be assumed that the state space can be written as $\tilde{J} = S \times \Lambda \times \hat{Z}$ and that Λ is admissible.

Preliminary Remark on theorem 1

Theorem 3.5 in Molchanov and Zuyev (2011) only requires the discontinuity set to have zero measure under the limiting measure (i.e., $\mu_n \rightarrow_{Weak*} \mu$ and $\mu(\Delta\varphi) = 0$). Thus, it is only necessary, under assumption 2, for μ to be atomless. The arguments in sections 3.2 and 3.3 illustrate the usefulness of properties a) and b) to achieve this purpose. Proposition 1 in Ito (1964) holds under quite mild assumptions on the primitives and assures property a). The critical property is then b), which hold under rather different assumptions depending on the cardinality of S.

Theorem 3.5 in Molchanov and Zuyev restores the continuity of the adjoint operator by extending the set of adequate functions for the *weak** topology from continuous to Borel measurable if the limiting measure is atomless and assumption 2 holds. The example below illustrates the importance of the atomless assumption when dealing with function which is only measurable.

<u>Atomic measures and tight spaces</u> [43]*:* Let $P: S \times \mathcal{B}_S \rightarrow [0,1]$ be a transition function with $S = [0,1]$ and $P(s, \{s/2\}) = 1$. Let $\{\lambda_n\}$ be a sequence of Dirac measures with $\lambda_n = \delta_{(1/2)^n}$. Thus, $\lambda_n \rightarrow \delta_0$, where the convergence is in distribution. Define the bounded Borel measurable function $f(s) = \{1 \; if \; s = 0 \, ; 0 \; otherwise\}$ and $\delta_0 \equiv \lambda$. Then $\int f(s)\, \lambda_n(ds)$ =0 and $\int f(s)\, \lambda(ds) = 1$ which in turn implies that $\lambda_n \nrightarrow_{weak^*} \lambda$. The reason behind the lack of $weak^*$ convergence is the impossibility to reduce the measure of the discontinuous part of f.

[43] This example borrows from Stokey, Lucas and Prescott (1989), page 336. Note that $\{\lambda_n\}$ satisfies $\lambda_n = P \cdot \lambda_{n-1}$. That is, it is possible to generate a sequence of non-atomic measures out of the action induced by P. I would like to thank Prof. R. Fraiman for pointing this out to me.

Proof of theorem 1

Let Φ be an equilibrium correspondence in definition 2 which satisfies assumption 1. By Lemma 2 $P_\Phi = \{P_\varphi : \varphi \sim \Phi\} \neq \emptyset$ and upper hemi continuous (see for instance proposition 2.2. in Blume, 1982). If P_Φ is convex valued, an ergodic invariant measure can be shown to exist using proposition 1.3 in Duffie, et. al. (1994) (page 757).

If P_Φ is not convex valued, suppose that assumption 2 together with properties a) and b) in theorem 1 hold. Choose any $\lambda_0 \in \mathcal{P}(\tilde{J})$ and construct a non-oscillating sequence of measures $\{\mu_N\}$ with $\mu_N = h(\{\lambda_n\})$, where h averages the first N-1 elements of $\{\lambda_n\}$ and λ_n satisfies $\lambda_n = P_\varphi^* \lambda_{n-1}$. The dependence of $\{\mu_N\}$ on λ_0 can be omitted w.l.o.g. as the initial condition is arbitrary.

As $\mu_N \in \mathcal{P}(\tilde{J})$ for any N, Helly's theorem (see Stokey, Lucas and Prescott (1989) page 372 and 374) implies that $\{\mu_N\}$ has a weakly convergent subsequence. That is, $\{\mu_{N_k}\} \to_{weak*} \mu$.

For notational simplicity $P_\varphi^* \lambda$ and $\hat{P}_\varphi f$ will be replaced by $\pi \cdot \lambda$ and $\pi \cdot f$ as P_φ with $\varphi \sim \Phi$ will be held constant throughout the proof.

For any $f \in C(\tilde{J})$ note that:

$$\left| \int f(z)\mu(dz) - \int (\pi \cdot f)(z)\mu(dz) \right|$$
$$\leq \left| \int f(z)\mu(dz) - \int f(z)\mu_{N_k}(dz) \right| + \left| \int f(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu_{N_k}(dz) \right|$$
$$+ \left| \int (\pi \cdot f)(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu(dz) \right| \quad (OA.3)$$

From the corollary of theorem 8.1 in Stokey, Lucas and Prescott (1989) (page 215), $(\pi \cdot f): Z \to \mathbb{R}$ is a bounded $\mathcal{B}_{[\tilde{J}]}$-measurable function. Further, from property a) and b), μ is atomless. Under assumption 2, $\mu(\Delta\varphi) = 0$. Then, from theorem 3.5 in Molchanov and Zuyev (2011, fact f), the third term in OA3 can be made arbitrarily small. Further, noting that $\{\mu_{N_k}\} \rightarrow_{weak*} \mu$ and $f \in C(\tilde{J})$, the first and the third term in A.3 can be made arbitrarily small.

Following the same reasoning as in Stokey, Lucas and Prescott (1989) page 377, the second term satisfies:

$$\left|\int f(z)\mu_{N_k}(dz) - \int (\pi \cdot f)(z)\mu_{N_k}(dz)\right| \leq 2\|f\|/N \quad (OA.4)$$

Where $\|.\|$ is the sup-norm. Thus, for an N arbitrarily large, $\int f(z)\mu(dz) = \int (\pi \cdot f)(z)\mu(dz) = \int f(z)(\pi \cdot \mu)(dz)$, where the last equality follows from theorem 8.3 in Stokey, Lucas and Prescott (1989) (see page 216). Thus, $\int f(z)\mu(dz) = \int f(z)(\pi \cdot \mu)(dz)$. As f was arbitrary, by virtue of corollary 2 of theorem 12.6 in Stokey, Lucas and Prescott (1989) (page 364) $\mu = \pi \cdot \mu$, QED.

■

Lemma 3

Preliminary remark

The proof of this lemma requires π to be θ-nonsingular. A transition function is said to be θ-nonsingular if for any measurable set B, $\theta(B) = 0$ implies $\pi(z, B) = 0$ θ-a.e. As θ is atomless this is equivalent to say that the set D, defined below, is a finite set.

$$D = \{z \in \tilde{J}: \pi(z, B) > 0 \; if \; \theta(B) = 0\} \quad (OA.5)$$

Additionally, B was restricted to be a point. For those transition functions defined by lemma 2, Ito (1964) show that any *non-constant possibly discontinuous many-to-*

one function $\varphi \sim \Phi$ will generate a θ-nonsingular transition function. This can be seen by written π_φ in lemma 2 as

$$\pi_\vartheta(z,B) = p\{s|s' \in S: \varphi(z,s') = a\} = p\{s|s' \in S: \{s_i'\} \cap \tilde{\varphi}^{-1}(z,.)(a_{\hat{z}})\} \quad (OA.6)$$

Where $z = [s,\hat{z}]$, $p(s|.)$ is the s^{th} row of the transition matrix which defines the evolution of the exogenous process $\{s_t\}$, $B = \{s_i'\} \times B_{\hat{z}}$ was restricted to a point $a = \{s_i'\} \times a_{\hat{z}}$, $\varphi(z,s') = [s', \tilde{\varphi}(z,s')]$ is a vector valued function and $\tilde{\varphi}^{-1}(z,.)(B_{\hat{z}})$ is the z-section of the pre-image of $\tilde{\varphi}$ on $B_{\hat{z}}$.

From OA.5 and OA.6, under assumption 2, $\#D < \infty$ provided that $\tilde{\varphi}(.,s')$ is non-constant in z for all $s' \in S$. The supplementary material to section 5.1 in the appendix shows that the implicit function theorem can be used to show that the model defined in section 2.1 generates θ-nonsingular transition functions.

Proof of lemma 3

Let $(\tilde{J}, \mathcal{B}_{\tilde{J}}, m)$ be a measure space. By assumption 1, $\tilde{J}$ is compact and by remark OA1 this set could be written as $\tilde{J} = S \times \Lambda \times \hat{Z}$, where Λ contain all admissible states and $\Lambda \times \hat{Z}$ is uncountable and has no isolated points. Further, note that any measure in $(\Lambda, \mathcal{B}_\Lambda)$, denoted m_Λ, is a Radon measure as Λ is a Hausdorff metric space and m_Λ is: i) defined over a Borel sigma-algebra ($\mathcal{B}_\Lambda$), ii) regular as it is a measure on a Hausdorff (compact) metric space (Λ), iii) $\mathcal{B}_\Lambda$-finite as it is a probability measure. Thus, as Λ has no isolated points, $(\Lambda, \mathcal{B}_\Lambda)$ has an atomless measure m_Λ^A (see Bogachev 2007, page 136) which in turn implies by remark OA1 that there is a measure m^A in $(\tilde{J}, \mathcal{B}_{\tilde{J}})$ that is also atomless. The first part of the lemma is completed by setting $m^A \equiv \theta$.

Let $\mathcal{P}_0(\tilde{J}) \subset \mathcal{P}(\tilde{J})$ be the set of atomless measures in $\mathcal{P}(\tilde{J})$ generated by π, starting from θ. It follows from proposition 1 in Ito (1964) that π maps $\mathcal{P}_0(\tilde{J}) \to \mathcal{P}_0(\tilde{J})$ as π is

θ-nonsingular by condition 1. Finally, condition 2 is just the definition of a *weak*-closed* set applied to $\mathcal{P}_0(\tilde{J})$.

■

<u>Example of an atomless measure $\theta \in \mathcal{P}(\tilde{J})$.</u> The reference measure θ could be a mixed joint density: $\theta(s \times A) = P(s = \{s\}, \hat{z} \in A) = \int_A p_{S,\hat{Z}}(s, \hat{z}) d\hat{z}$ where $p_{S,\hat{Z}}(s, \hat{z}) = \theta(s \times \{\hat{z}\}) = 0$ is a density function on $\hat{Z}$ which may vary with any $s \in S$. From fact 14 page 45 in Hildenbrand and Grandmont (1974), θ is atomless.

Lemma 4

Preliminary Remark

The implication of condition 4) requires showing the *weak** closedness of $IM(\varphi, \mathcal{P}_1)$. The proof below shows that $IM(\varphi, \mathcal{P}_1)$ is *weak** sequentially compact: that every *bounded* sequence in $IM(\varphi, \mathcal{P}_1)$ has a *weak** convergent subsequence. As $\mathcal{P}_1$ can be endowed with the Prohorov metric (see Hildenbrand and Grandmont 1974, page 49), sequential compactness implies that $IM(\varphi, \mathcal{P}_1)$ is not only closed but also compact.

Proof of lemma 4

For the existence of an atomless measure on $\tilde{J} = S \times \Lambda \times \hat{Z}$ with S uncountable and compact, let θ be the uniform measure on $\tilde{J}$.

For property a), note that condition 3) implies that P_φ is θ-nonsingular. Thus, proposition 1 in Ito (1964) applies just as in the proof of lemma 3.

To prove property b), note that any point $\{a\} \in \tilde{J}$ has zero Lebesgue measure. Thus, under condition 3):

$$\mu_n(\{a\}) = \int P_\varphi(z,\{a\})\,\mu_{n-1}(dz) = 0$$

Where the second equality follows from condition 3) and implies that the desired result follows automatically.

Property c) will be proved in 3 parts: i) $IM(\varphi,\mathcal{P}_1) \neq \emptyset$. As $\tilde{J}$ is compact, Helly's theorem implies the existence of a weak* converging subsequence in $IM(\varphi,\mathcal{P}_1)$ denoted w.l.o.g. $\mu_n \rightarrow_{weak*} \mu$. It will be shown that: ii) μ is absolutely continuous w.r.t θ, iii) $\mu \in IM(\varphi,\mathcal{P}_1)$.

In what follows it will be assumed w.l.o.g. that $\theta(dz) = dz$. This is done for expositional purposes only.

i) Standard results (See Billingsley 1968, page 422) imply that condition 4) is equivalent to the following statement: for any measurable set B, $\theta(B) = 0$ implies $SUP_{z\in \tilde{J}}[\pi_\vartheta(z,B)] = 0$. Thus, π_ϑ is θ -nonsingular. By proposition 1 in Ito (1964), $\pi_\vartheta{:}\,\mathcal{P}_1 \rightarrow \mathcal{P}_1$. Also, under condition 4), an argument identical to the one used to prove property b) implies that $\mathcal{P}_1$ and the adjoint operator generates a *weak** closed set. Under assumption 2, theorem 1 implies that $IM(\varphi,\mathcal{P}_1) \neq \emptyset$.

ii) By the characterization of absolutely continuity in Billingsley (1968, page 422), it suffices to show that for any $\varepsilon > 0$, $\exists\delta > 0$ such that $\theta(B) < \delta$ implies $\mu(B) < \varepsilon$. Condition 4) implies that $\pi_\vartheta(z,.)$ is absolutely continuous w.r.t. θ for any $z \in \tilde{J}$. That is, $\pi_\vartheta(z,dz')$=$\bar{\pi}_\vartheta(z,z')dz'$ where $\bar{\pi}_\vartheta(z,.)$ is the density associated with $\pi_\vartheta(z,dz')$. Take any sequence $\{\hat{\mu}_n\} \in IM(\varphi,\mathcal{P}_1)$. Note that $\{\pi_\vartheta\hat{\mu}_n\}$ is a family of measures that satisfies the hypothesis of Helly's theorem and $\{\pi_\vartheta\hat{\mu}_n\} \in IM(\varphi,\mathcal{P}_1)$.

Let $\pi_\vartheta\hat{\mu}_n \equiv \mu_n$ and note that $\{\mu_n\} \in IM(\varphi,\mathcal{P}_1)$ and has a *weak** limit denoted (passing to a subsequence if necessary) μ.

Note that $\mu_n(B) = \int_B h_n(z')\theta(dz')$ where $h_n(z') = \int \bar{\pi}_\vartheta(z,z')\,\mu_n(dz)$. But now note that $\mu_n(B)$ could be written as:

$$\mu_n(B) = \int_B h_n(z')\theta(dz') = \int \left[\int_B \bar{\pi}_\vartheta(z,z')dz'\right]\mu_n(dz)$$

Condition 4) implies that $\left[\int_B \bar{\pi}_\vartheta(z,z')dz'\right] < \varepsilon$ uniformly in z. Thus $\mu_n(B) < \varepsilon$. The arguments in the first part of lemma 3 imply that $\{\mu_n\}$ and μ are regular measures. Thus, B can be assumed to be open w.l.o.g. Now, the definition of *weak** convergence implies (see theorem 12.3-c in Stokey, Lucas and Prescott, page 358) $\mu(B) \le liminf_n \mu_n(B)$. To complete the proof, by the preliminary remark of this lemma, it suffices to note that $liminf_n \mu_n(B) < \varepsilon$.

iii) It remains to show that $\mu \in IM(\varphi, \mathcal{P}_1)$.

Take $\mu_n \rightarrow_{weak*} \mu$. Note that for any $f \in C(\tilde{J})$:

$$\lim_n \int f(z)\mu_n(dz) = \int f(z)\mu(dz)$$

$$= \lim_n \int f(z)[\pi\mu_n](dz) = \lim_n \int [\pi f](z)\mu_n(dz) = \int [\pi f](z)\mu(dz)$$

$$= \int f(z)[\pi\mu](dz) \quad (OA.7)$$

Where the first equality in OA.7 follows from the definition of *weak** convergence of $\mu_n \rightarrow_{weak*} \mu$, the second from $\{\mu_n\} \in IM(\varphi, \mathcal{P}_1)$, the third from theorem 8.3 in Stokey, Lucas and Prescott, the forth from theorem 3.5 in Molchanov and Zayev as μ is absolutely continuous w.r.t. θ (and thus atomless) and the last equality from theorem 8.3 in Stokey, Lucas and Prescott again. Note that A.7 implies $\int f(z)\mu(dz) = \int f(z)[\pi\mu](dz)$. As $f \in C(\tilde{J})$ is arbitrary, the proof is complete.

■

Proposition 1

Proof of Proposition 1

Under assumption 1, lemma 2 implies that π is well defined (i.e. is a Markov operator). Under assumptions 3-i) and 3-ii) the result follows from equation A.6) by noting that $\{s_i'\} \cap \tilde{\varphi}^{-1}(z,.)(a_{\hat{z}})$ is either a point in S or $\emptyset$ for any $z \in \tilde{J}$.

■

Proposition 2

Preliminary remark

Arbitrarily selecting $\in \tilde{J}$, it will be shown that $\forall \varepsilon(z) > 0, \exists \delta(z) > 0$ such that $\theta(B) < \delta(z)$ implies $\pi(z,B) < \varepsilon(z)$. As $\tilde{J}$ is compact and $\varepsilon(z), \delta(z)$ are finite (real) numbers, it suffices to take $max_{z \in \tilde{J}}\, \varepsilon(z) = \varepsilon$ and $max_{z \in \tilde{J}}\, \delta(z) = \delta$.

For the first part of the proof the following fact will be useful: let θ be the Lebesgue measure and $R \subseteq \tilde{J} \subset \mathbb{R}^K$ a rectangle and μ^V its volume. That is, $R = [a_1, b_1] \times \ldots \times [a_K, b_K]$ and $\mu^V(R) = [b_1 - a_1] \ldots [b_K - a_K]$. Then, $\theta(B) = 0$ if $\forall \gamma > 0$, $\exists \{R_i\}_{i=1}^{\infty}$ with $B \subseteq \bigcup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$. The proof of the first part the proposition will be completed if it can be shown that for each $\varepsilon(z) > 0$, there exist an $\gamma > 0$ such that $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$ implies $\sum_{i=1}^{\infty} \pi(z, R_i) \leq \varepsilon(z)$ because $\theta(B) = 0$ as long as $\sum_{i=1}^{\infty} \mu^V(R_i) < \gamma$.

Proof of proposition 2

Note that any positive $\pi_{\varphi}(z,.)$-measure rectangle, R_i, could be written as

$$R_i = \left[\varphi_1\left(z, s_{1,i}' - 2^{-1}h_{1,i}\right), \varphi_1\left(z, s_{1,i}' + 2^{-1}h_{1,i}\right)\right] \times \ldots \times \left[\varphi_K\left(z, s_{K,i}' - 2^{-1}h_{K,i}\right), \varphi_K\left(z, s_{K,i}' + 2^{-1}h_{K,i}\right)\right]$$

where the first coordinate is just $\left[s_{1,i}' - 2^{-1}h_{1,i}, s_{1,i}' + 2^{-1}h_{1,i}\right]$, φ_k and $s_{k,i}'$ denote any coordinate of φ for $1 \leq k \leq K$ and the elements of S that generates coordinate k of rectangle i.

Note assumption 3-iii) implies that $\varphi_k(z,.)$ is allowed to oscillate continuously, not necessarily forming a straight line, between $\varphi_k(z,x)$ and $\varphi_k(z,y)$ where $x = s'_{k,i} - 2^{-1}h_{k,i}$ and $y = s'_{k,i} + 2^{-1}h_{k,i}$. Thus, by theorem 2.27 in Aliprantis and Border (2006), $h_{k,i}$ is the length of the interval in the pre-image of $\varphi_k(z,.)$, where $\varphi_k(z,x)$ and $\varphi_k(z,y)$ are exactly the endpoints of the k^{th} coordinate of rectangle R_i.
Now equation OA.6) implies that:

$$\pi(z,R_i) \leq p\left(s,\bigcap_{k=1}^{K}\left[s'_{k,i} - 2^{-1}h_{k,i}, s'_{k,i} + 2^{-1}h_{k,i}\right]\right) = p\left(s,\bigcap_{k=1}^{K}\left[0,h_{k,i}\right]\right)$$

Where the inequality follows from the preceding discussion and the equality from assumption 3-iv) after normalizing $p(s,.)$ to be in the unit interval.
Now note that assumption 1 implies that $\mu^V(R_i)$ is finite as the range of any $\varphi{\sim}\Phi$ is bounded, and $\sum_{i=1}^{\infty}\mu^V(R_i) < \gamma$ implies $lim_{i\to\infty}\mu^V(R_i) = 0$. Thus,

$$\pi_\varphi(z,R_i) \leq \varepsilon(z)_i 2^{-i} \quad \text{OA.8)}$$

Where $\varepsilon(z)_i = min_k h_{k,i}$.

Also, from $lim_{i\to\infty}\mu^V(R_i) = 0$, equation OA.8) implies that $lim_{i\to\infty}(\varepsilon(z)_i)$ is finite. Thus, $SUP_i\varepsilon(z)_i = max_i\varepsilon(z)_i = \varepsilon(z)$ and $\sum_{i=1}^{\infty}\pi(z,R_i) \leq \varepsilon(z)$, as $\sum_{i=1}^{\infty}2^i = 1$.

Now to prove the dependence of γ on $\varepsilon(z)$, let $R_{i,k}$ be the k^{th} coordinate of rectangle R_i. Note that assumption 3-iii) implies, by theorem 2.34 in Aliprantis and Border, that for all $i, \exists k$ with $R_{i,k} = [\varphi_1(z,x), \varphi_1(z,y)]$ and $[x,y]$ has length smaller or equal to $\varepsilon(z)$. Consequently, $\varepsilon(z)$ could be made arbitrarily small as desired and there will always be an associated γ such that OA.8) holds. As z is arbitrary, the proof is complete.

■

Comment on remark 2: the result follows from replacing $p(s, \bigcap_{k=1}^{K}[0, h_{k,i}2^{-i}])$ by $p(s, \bigcap_{k=1}^{K}[LB(s), h_{k,i}2^{-i}])$ in equation OA.8) and noting that $\varepsilon(z)_i = min_k \frac{h_{k,i-LB(s)}}{UB(s)-LB(s)}$, where z is a vector of the form $z = [s, \hat{z}]$, is a finite number for all $z \in \tilde{J}$.

Lemma 5

Preliminary remark of Lemma 5

As discussed in section 3.4), the existence of an ergodic invariant measure can be shown under a slightly weaker assumption than 3-iv). The results hold under assumption 3.iv') which allows $p(s,.)$, the distribution of exogenous shocks, to depend on s. Assume further that,

<u>*Assumption OA.1)*</u>*: $p(s,.)$ satisfies assumption 3-iv') and it has the Feller property.*

The proof below assumes that $p(s,.)$ satisfies assumption OA.1) provided the existence of a recursive structure Φ. The results in Mas-Colell and Zame (1996) imply that assumption 3.iv) is required to insure the existence Φ in definition 2. However, the proof will be done imposing the less restrictive assumptions in case Φ can be derived under milder restrictions for a different type of economy.

Under assumptions 4.2-i) to 4.2-iv) and 3-iii) the result in lemma 5 follows from proposition 1 and 2 and theorems 1 and 2. Thus the proof of the lemma will only take care of the case of only 1 asset which allows to show the continuity imposed by assumption 3-iii). It will be shown that there exist a selection $\varphi \sim \Phi$, with $\varphi(\tilde{z}, s_+) = [s_+, \theta_+(\tilde{z}, s_+), q_+(\tilde{z}, s_+), m_+(\tilde{z}, s_+)]$, that is continuous in each coordinate in s_+. Moreover, considering the incomplete markets nature of the model, $\theta_+(\tilde{z}, s_+)$ will be assumed to be constant. That is, $\theta_+(\tilde{z}, s_+) = \theta_+(\tilde{z})$ for each $s_+ \in [LB(s), UB(s)]$. Once the continuity of $q_+(\tilde{z}, s_+)$ has been shown below, the continuity of $m_+(\tilde{z},.)$ follows from its definition.

Proof of lemma 5

Assume that $\theta_+(\tilde{z}, s_+)$ is constant in in s_+ for any given $\tilde{z} \in \tilde{J}$. In order to complete the proof, it suffices to show that $q(\tilde{z}, s_+)$ is continuous in s_+ for any given $\tilde{z} \in \tilde{J}$.

Under assumptions 6.2-i) to 6.2-iii) any equilibria in this economy exists satisfies equation A.12, the feasibility requirement, together with

OA.9) $q_j u_s^i(e^i(s) + \theta^i d(s) - \theta_+^i q)' - \beta K(s) \int d_j(s_+) u_s^i(e^i(s_+) + \theta_+^i d(s_+) - \theta_{++}^i q)' ds_+ = 0,\ j \in J, i \in I$

Where $K(s)$ is the constant associated with the uniform distribution in assumption 3-iv').

Now suppose that assumption OA.1 holds. Then, as mentioned in the preliminary remark, $p(s,.)$ has the Feller property. Then:

OA.10) $\lim_{s^n \to s^1} \beta K(s^n) \int m_{++}^{i,j}(x) dx = \beta K(s^1) \int m_{++}^{i,j}(x) dx = q_+^j(s^1) u(e^i(s^1) + \theta_+^i d(s^1) - \theta_{++}^i q_+(s^1))'$

The last equality in OA.10) follows because, under assumption 4.2-i) to 4.2-iii), there is a sequential competitive equilibrium for each s^1 which satisfies equation OA.9).

Under the special form $u_s^i = u$ in assumption 4.2-i), equation OA.9 and OA.10 implies:

OA.11) $\lim_{s^n \to s^1} \frac{\beta K(s^n) \int m_{++}^{i,j}(x) dx}{u\left(e^i(s^n) + \theta^i d(s)\right)'} = \lim_{s^n \to s^1} q_+^j(s^n) u(-\theta_{++}^i q_+(s^n))' = q_+^j(s^1) u(-\theta_{++}^i q_+(s^1))'$

Note that equation OA.9 implies the first equality in OA11) under u in assumption 4.2-i). Then, as $u\left(e^i(s^n) + \theta^i d(s^n)\right)'$ is bounded above and bounded away from zero for any admissible value of $e^i(s^n) + \theta^i d(s^n)$ under assumptions 4.2-i), equation OA.10 implies the last equality.

Now, setting $\lambda = 1$ in u w.l.o.g., the continuity of ln implies

$$\text{A.12)} \quad \underbrace{\lim_{s^n \to s^1}[-\theta^i_{++}q_+(s^n)] + \theta^i_{++}q_+(s^1)}_{A} + \underbrace{ln\,[\lim_{s^n \to s^1} q^j_+(s^n)] - ln\,[q^j_+(s^1)]}_{B} = 0$$

If $B = 0$, then as $\theta^i_{++} \neq 0$ w.l.o.g., A implies $lim\ q_+(s^n) = q_+(s^1)$ as desired.

Suppose that $B \neq 0$. The compactness of the equilibrium set implied by theorem 4.2 in Mas-Colell and Zame (1996) under assumptions 4.2-i) to 4.2-iii) implies that $B \in \mathbb{R}$. Then A.12) under $J = 1$ (i.e., there is only 1 asset) implies:

$$q^j_+(s^1) = \frac{B}{\theta^i_{++}(1 - exp(B))}$$

Note that A.9) implies that $q^j_+(s^1) \geq 0$ and that $\theta^i_{++} > 0$ w.l.o.g. as there are heterogenous agents and the asset is offered in zero net supply. Then, as B is a finite number and it was assumed to be different from zero, then $q^j_+(s^1) < 0$; implying a contradiction with $B \neq 0$.

■