An Ergodic Theory of Sovereign Default*

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Abstract

We develop a small open economy model in which atomistic households borrow abroad in the form of defaultable private debt, while the government chooses whether to enforce repayment or induce default. Because borrowing is decentralized, households do not internalize the impact of higher aggregate debt on the sovereign risk premium, leading to unstable or unsustainable debt dynamics in the absence of policy intervention. We propose a two-step nested fixed-point operator with a refined default punishment that delivers a stationary recursive equilibrium in a minimal state space. Under mild conditions, this equilibrium exists, is unique, and induces a globally stable and ergodic Markov process for income and net external assets. Default occurs when debt crosses an endogenous threshold and permanently shifts the economy to a new ergodic distribution, implying at most one default per stochastic steady state. Calibrating the model to Argentine data, we show that it aligns with long-run moments for private external debt and the current account, and rationalizes the differences between local and global moments around the 2001 default. Following Samuelson's Correspondence Principle, we connect the restrictions associated with the existence of an ergodic equilibrium with comparative statics results. We analyze an MIT shock: A 100 basis point increase in the world interest rate, which raises long-run leverage debt and default frequency.

Keywords: Default; Private external debt; Ergodicity; Stability.

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1 Introduction

Many recent crises have unfolded in economies where a large fraction of external liabilities is issued by the private sector. In these environments, as claimed by Arellano et al. (2016), policymakers retain the legal and regulatory tools to influence whether private external debts are ultimately honored. Yet, most of the literature on the effects of default risk in open economies focuses on public debt, and, with few exceptions, deals with dynamics surrounding a crisis rather than the long-run properties of the model economy. The table below shows that, in some countries, there are substantial differences in the values of fundamental macroeconomic variables prior to a default with respect to their long-run trend.

Table 1: Global and local moments for Argentina

CA/GDP		External Assets	
1983-2001	1960-2017	1983-2001	1960-2017
-2.4%	-0.7%	-36.3%	-25.7%

Note: CA/GDP stands for current account to GDP. Net external assets refer to the overall figure, which includes total public and private debt, as well as debt from international organizations. In the body of the paper, we present evidence in favor of these differences for private and defaultable external debt.

On top of these facts, the lack of constructive existence proofs, which connect rigorous theory and numerical efficiency, in the literature affects the accuracy of the computed solutions. As a result, the literature on default risk lacks a theory that simultaneously (i) treats private external debt as the main state variable, (ii) allows an altruistic government to induce default on that debt, and (iii) delivers a well-defined stochastic steady state from which long-run moments can be accurately computed and compared to the data.

This paper develops such a framework and uses it to articulate an "ergodic theory" of sovereign default on private external debt. We study a small open economy in which atomistic households issue one-period defaultable foreign debt, while a benevolent government decides whether to enforce repayment or to induce default on the privately issued stock. Foreign investors are competitive and risk-neutral, and price the bond so as to break even in expectation. Because borrowing is decentralized, private agents take the interest rate schedule as given and fail to internalize that higher aggregate debt increases the probability of future default and, consequently, the equilibrium interest rate. The associated pecuniary externality implies that, in the absence of government intervention, the private economy typically generates unstable or unsustainable debt paths.

Our main theoretical result is that default is essential to recover a globally stable and ergodic equilibrium in this environment. We constructively prove the existence of stationary recursive equilibria in which the government uses default to eliminate private external debt paths that will eventually become unsustainable or unstable in the near future. We show that authorities can identify these types of trajectories using current levels of GDP and indebtedness alone. In this way, this paper complements Pierri and Seoane (2025), proving the existence of a globally stable and ergodic equilibrium.

The key technical innovation is a two-step nested fixed-point operator: a Coleman–Reffett–type Euler operator solves the decentralized household problem, while an Aguiar–Amador–type dual operator determines the continuation values associated with repayment and default, as well as the corresponding bond pricing function. By refining the default punishment—which we treat as an endogenous equilibrium object rather than a primitive—we ensure that this operator is order-preserving and that the induced equilibrium interest rate is monotone in aggregate debt. Under mild conditions, this nested operator has stationary computable equilibria; under stronger curvature and lower-bound conditions, the equilibrium is unique.

We then demonstrate that, once the government is permitted to induce default, the unique stationary equilibrium is ergodic. In our baseline specification, income is i.i.d., and the state of the economy is summarized by current income and aggregate net external assets. We prove that, for a suitable endogenous punishment, the induced Markov process over this minimal state space admits a unique invariant probability measure and that all positive-probability paths remain in a compact "stable state space." A central implication is that each ergodic distribution can feature at most one default episode. Hence, default not only stabilizes short-run dynamics by ruling out explosive paths, but also permanently changes the long-run stochastic steady state of the economy.

The global dynamics implied by this equilibrium can be fully characterized in terms of demarcation curves for net external assets—a phase diagram that maps debt today into debt tomorrow for each income realization. This allows us to classify asset trajectories as locally stable, unstable, or outright unsustainable, and to identify "high-debt traps" in which the private economy converges toward very negative net external asset positions. In the absence of default, these traps correspond to unstable or diverging paths. The government prevents the process from exiting the stable region by enforcing a default once debt reaches an endogenous threshold. A particularly useful implication is that long-run sustainability of private external debt can be evaluated solely from the current debt–GDP pair: for low income realizations, all paths are unsustainable, and default is inevitable; for higher income realizations, when the economy has full access to international capital markets, there exists a critical debt level above which the economy is on an unstable trajectory and will default with high probability unless income improves.

The ergodic structure of the equilibrium also disciplines comparative statics. Thus, we can apply Samuelson's Correspondence Principle to MIT shocks. This principle states that the conditions for stability, in our case ergodicity, affect comparative statics results, which in our environment imply comparing two different distributions. Any permanent change in fundamentals—such as an increase in the international risk-free rate—must induce a new ergodic distribution consistent with global stability. In our model, a higher world interest rate operates through three channels. First, it raises the risk premium and the default probability. Second, it lowers the refined default punishment and default thresholds. Third, it tilts households' Euler equation toward saving when access to markets is available, rotating the demarcation curves for assets upward. Combining these channels, a higher risk-free rate shifts the support of the ergodic distribution to the left and fattens its left tail: long-run economies facing tighter global financial conditions are more indebted on average and spend more time in highly leveraged states, a result which follows from the conditions required to guarantee the ergodicity of both measures, before and after the shock. Thus, the conditions for stability affect the comparative statics results, which in this case implies that economies after a positive shock to the international risk-free rate will experience higher debt levels permanently.

We calibrate the model using Argentine private external debt and current-account statistics from 1983 to 2016, thereby linking the model to the data. This period includes one sovereign-induced wave of private external default (2001) but no additional default events, which is exactly the pattern implied by our ergodic theory: one default per ergodic distribution. We use the constructive algorithm associated with our two-step operator to jointly determine the refined default punishment and the equilibrium functions, and then match long-run ("global") moments for net external assets, debt service, and the current account. The calibrated model reproduces not only these unconditional moments but also the differences between short-run ("local") moments around the default episode and long-run moments computed over the entire sample.

We exploit the ergodic structure to quantify the long-run effects of a higher global interest rate. In a counterfactual experiment, we increase the international risk-free rate by 100 basis points and recompute the ergodic equilibrium. The stationary distribution of net external assets shifts to the left and becomes more left-skewed, with a substantially larger probability mass in states of very high indebtedness. These results highlight private defaultable debt as a powerful transmission mechanism for global financial conditions: tighter world interest rates, rather than inducing precautionary saving, can increase leverage and default risk when borrowing is decentralized, and the sovereign uses default on private liabilities as a stabilization instrument.

We also compare our framework with a stationary version of the model, which incorporates Markov income shocks and exogenous default costs, as in existing decentralized borrowing–centralized default models. Using the same parameters as in a leading specification with exogenous punishment, we show that imposing our endogenous punishment and constructiveness requirements changes the model's quantitative implications for local moments. With Markov shocks, the model with endogenous punishment tends to generate larger current-account deficits and higher debt than observed in the data when calibrated to local moments; however, it also replicates the sustained output expansion observed in Argentina during the years following default. This exercise underscores that ignoring the global stability and ergodicity constraints can lead to quantitatively and conceptually different conclusions about the role of private external debt and sovereign policy.

1.1 Relation with the literature

Our main contribution is to derive an ergodicity result for a small open economy with financial frictions. Because ergodicity is deeply connected with global stochastic stability, we can also provide a novel rationale behind default: it stabilizes an inherently unstable economy.

Ergodicity is not new in macro theory. For the standard RBC model, there is an extensive discussion in Lucas et al. (1989). However, these results depend on the continuity of the equilibrium. There are very few results when the stationary equilibrium may be discontinuous, either because there are multiple decentralized equilibria or because there is a planner that induces a discontinuous equilibrium rule. For general equilibrium models with incomplete markets, Duffie et al. (1994) show the existence of an ergodic equilibrium. However, those results depend on the existence of convexifying sunspots, which severely affects the computability of equilibrium. To our best knowledge, the only other paper that provides conditions to guarantee that a computable equilibrium is ergodic even though it may be discontinuous is Pierri and Reffett (2021), who rely on an expanded state space to obtain an ergodic representation. In contrast, we do not need additional state variables in our paper. Thus, it is the first to show that it is possible to derive an ergodic equilibrium in minimal state space, even in the presence of discontinuities.

Theoretical results in the sovereign default literature are rare apart from the notable exceptions of Auclert and Rognlie (2016), Aguiar and Amador (2019) and Feng and Santos (2021). The first paper shows that if there is an equilibrium in the Eaton and Gersovitz (1981) model, that equilibrium is unique. Then, Aguiar and Amador (2019) prove the constructive existence and uniqueness of the Markov Equilibria of the one-period-bond model as in Eaton and Gersovitz (1981). They show these properties by rewriting the model in a dual form that allows characterizing the Markov Equilibria as a fixed point of a contraction mapping. As their equilibrium operator is a contraction, it converges to the unique fixed point starting from any initial condition. Our operator requires two steps. We solve the decentralized problem in the first step, given the second step, which contains the government's decision. The constructiveness of the first step requires a direct application of the results in Coleman (1991), which are constructive for some initial conditions. Thus, contrary to the results in Aguiar and Amador (2019) and due to the presence of private optimization, to guarantee the consistency of iterations, we need to restrict the space of equilibrium functions by refining the space of suitable default punishments. Thus, in this paper, after-default households' income is an endogenous equilibrium object. Feng and Santos (2021) show a stationary equilibrium in a model with capital and labor. However, these papers are silent regarding the global stochastic dynamics and the long-run ergodic properties of those models and are based on the centralized default-centralized borrowing framework.

Regarding modeling choices, we are not the first to build a model with external decentralized debt. The most common friction in this literature follows the setup of Bianchi (2011), where the authors rely on a collateral constraint to stabilize debt levels. This constraint is subject to a pecuniary externality as the private agents do not internalize that issuing debt increases the probability of hitting the collateral constraint in the future. In our case the pecuniary externality affects the equilibrium interest rates, destabilizing debt services in the absence of default. Regarding foreign decentralized debt and centralized default, Kim and Zhang (2012) designs a model along those lines and similar to ours, where they assume that households issue private defaultable debt and do not internalize the impact of debt accumulation in the price of debt. However, they consider exogenous default costs along the lines of Aguiar and Gopinath (2006) and Arellano (2008) while we must carefully select the default costs to prove our theoretical results. Moreover, the authors use prices instead of interest rates, something that affects the definition of equilibrium that they can use in that model.[1] These differences prevent the authors from addressing the points we address here about the existence, uniqueness, and ergodicity of equilibrium.

1.2 Structure of the paper

The remainder of this paper goes as follows. In section 2, we present the model. Section 3 describes the main theoretical results. Section 4 the some theoretical implications of our model. Section 5 contains the quantitative implementation and the main numerical results. Section 6 concludes. All proofs are available in the appendix. The Supplementary Appendix contains additional material and details for Sections 3, 4, and 5.

2 The model

We consider a small open endowment economy populated by households and a government. Households are atomistic, risk-averse agents that issue a non-contingent net external asset in order to *optimize* consumption intertemporally. The benevolent government decides in each period whether to allow the private sector to repay its debt or to force it to default.[2] International investors are deep-pocketed, risk-neutral agents. Their objective is to obtain break-even returns on the risky asset in expectation, and they understand the mechanics of default risk. Below, we formally describe the economic environment.

[1] By assuming that bonds pay an interest rate instead of being purchased by a below-parity price, we can write the equilibrium in minimal state space (i.e., exogenous shocks and net external assets).

[2] The Government can prevent excessive borrowing in any state by forcing the private sector to default on its debt. The intuition is that, in emerging economies, this is typically achieved through a domestic currency depreciation, suspending access to foreign exchange markets, direct capital controls, and other measures.

2.1 The international investors

International investors are risk-neutral, deep-pocketed agents whose objective is to break even in expectation. Denoting the risk-free rate by R^* and using π for the transition probabilities of Markov shocks, the expected return of risky net external assets satisfies:

$$R(B', y) = R^* \left[\sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y', y) \mathbb{I} \left\{ V^c(B', y') > V^{def}(y') \right\} \right]^{-1}. \tag{1}$$

Here, $B' = B_+(B, y)$ is the aggregate debt in equilibrium and $\mathbb{I} \left\{ V^c(b_+(B, y), y') > V^{def}(y') \right\}$ is an indicator function that takes the value 1 in states in which the government does not induce default. The assumption of incomplete markets implies that investors cannot condition the return on debt on future states of nature. Hence, the equilibrium interest rate does not depend on either future income or on the individual's future debt choices. See Pierri and Seoane (2025) for a detailed discussion.[3]

We next provide a remark on this pricing equation. Ayres et al. (2018) derive the same pricing rule *but* in a model that is constructed specifically to generate multiple equilibria. Our framework is robust to these types of multiplicities. As we can prove that the equilibrium is compact, we can always take a measurable stationary selection. Even if the resulting equilibrium is discontinuous, we can still prove existence using the operator presented below. Moreover, by strengthening our assumptions, we can eliminate other sources of multiplicities, typically related to the theoretical structure used in our proofs (see, for instance, Mirman et al. (2008)). Thus, we will refine the stationary equilibria using a measurable selection, if necessary from Equation (1), and then show uniqueness using that selection.

[3] Moreover, the expected return depends on current income only if the income process is persistent.

2.2 The households

There is a continuum of identical households that receive a positive stochastic endowment y that follows an i.i.d. process.[4] In default, households have no choice and must consume their net-of-default-penalty endowment. If the economy is not in default, households can issue debt, $b_+ < 0$, at a non-contingent gross interest rate R. Now, we will say that the utility function is increasing, concave, and differentiable. Then, we will formally state these assumptions. As the default is centralized and the endowment is i.i.d., the interest rate is decreasing in aggregate assets and does not depend on y; hence, we denote the return on net asset issued today and maturing tomorrow by $R(B_+)$.[5] The recursive problem of the agent is:

$$V^c(b, B, y; h) = \max_{b_+ \geq -\bar{b}} u(y + bR(B) - b_+) + \beta \mathbb{E}[V(b_+, h(B, y), y'; h)], \tag{2}$$

where $\bar{b} > 0$ is a uniform upper bound on debt, which, as we will state below, guarantees the compactness of the stationary equilibrium. Here, $b_+^*(b, B, y; h)$ denotes the policy function and h is the aggregate law of motion for per-capita assets B_+. Moreover, we let V^c be the value function under repayment, and V be the value of the option between default and repayment. We define V and the value function in default in the next subsection. Taking $R(B)$ as given, since households are atomistic and debt is decentralized, the Euler equation for this problem can be written heuristically as:

$$u'(c) \geq \beta \mathbb{E} \Big\{ u'(c_+) \left[R(B_+) \times \mathbb{1}(\text{Repay}) + 0 \times (1 - \mathbb{1}(\text{Repay})) \right] \Big\}. \tag{3}$$

In equation (3), $\mathbb{1}(\text{Repay})$ is an indicator function that takes the value 1 for the y' states that imply repayment and zero otherwise. Centralized default decisions, therefore, affect the dynam-

[4]We only need the i.i.d. assumption to prove the ergodicity of the equilibrium. Nevertheless, we introduce this assumption here because it facilitates simplification of the notation. In section 5.2.2, we solve a stationary version of the model with Markov shocks. While the ergodic equilibrium allows us to target long-run moments, the stationary one is useful to match short-run stylized facts. As the introduction shows, both statistics differ significantly; thus, both equilibrium notions are empirically and theoretically relevant.

[5]As in Arellano (2008), incomplete markets imply that the interest rate does not depend on the future endowment. The preliminary remarks associated with the proofs for section three in the appendix contain details about the interest rate, the Euler equation, and the operator used to constructively prove the existence of equilibria.

ics of private debt. The arguments in Pierri and Seoane (2025) show that we can formalize the heuristic expression in (3).[6] Thus, we can write the Euler equation as follows:

$$u'(c(b,B,y;h)) \geq R(B_+)(h(B,y))\beta\mathbb{E}[u'(c_+(b_+(b,B,y;h),h(B,y),y';h))]. \tag{4}$$

Equation (4) may hold with strict inequality if the upper bound on debt is binding and $c(b,B,y;h) = y - b_+(b,B,y;h) + bR(B)$. As in Kim and Zhang (2012), we model a perceived law of motion h for aggregate states that maps $(B \times Y) \mapsto B_+$. Thus, because the integral in the expectation operator is taken with respect to y', $B_+ = h(B,y)$ does not depend on that variable and can be factored out of the Euler equation. This is an important difference relative to Arellano (2008): since debt is decentralized, the effects of individual borrowing on the interest rate are not internalized by the private agent. This feature is central to our departure from the centralized borrowing–centralized default literature, which is the standard assumption to introduce default risk in open economies. Moreover, note that the expectation is taken over y' only for those states in which the economy does not default. In default, the asset's return is zero, a fact that households also recognize.

2.3 Centralized default

The government is benevolent and either forces the private sector to repay or to default. To focus on a stationary equilibrium, we set $b = B$. As the government does not choose the level of debt but only the decision to induce a private default, the problem of the government is:

$$\text{Default if:} \quad V^c(B,y) \leq V^{def}(y). \tag{5}$$

Notice that in Equation (5), the Government's choice variable is neither B nor B_+. In Arellano (2008), authorities can freely choose the level of debt as they have a continuous choice variable, the level of equilibrium debt tomorrow. Here, they have only one instrument: the dichotomous default choice, which, ex post, sets the equilibrium level of debt to zero only if the Government decides to default. Thus, all the differentiability issues with the value function discussed in Clausen

[6]See Equation (21) in the subsection Preliminary Remarks in that paper.

and Strub (2020) do not arise. We let $V^c(B, y)$ represent the continuation value, i.e., the reward from repaying the outstanding debt, and it satisfies

$$V^c(B, y) = u(y - b_+(y, B; h) + BR(B)) + \beta \mathbb{E} \max \Big\{ V^c(b_+(y, B; h), y'), V^{def}(y')) \Big\}, \tag{6}$$

while $V^{def}(y)$ denotes the value of default and satisfies

$$V^{def}(y) = u(y^{def}(y)) + \beta \mathbb{E} \Big\{ \theta V^c(0, y') + (1 - \theta) V^{def}(y') \Big\}. \tag{7}$$

We use θ for the probability of regaining access to the market after default occurs. Note that, *if consumption and assets in the next period are both increasing in* B *for each* y, we obtain the following characterization of default sets:

$$\left\{ \begin{matrix} \text{Repay} & if\ B > \bar{B}(y) & \text{as this implies} & V^c(B, y) > V^{def}(y) \\ \text{Default} & if\ B \leq \bar{B}(y) & \text{as this implies} & V^c(B, y) \leq V^{def}(y) \end{matrix} \right\}. \tag{8}$$

This characterization is consistent with the findings of Arellano (2008) for centralized debt. It relies on the existence of a stationary equilibrium that defines h. In the next section, we show that there exists at least one set of functions $(c, R, V^c, V^{def}, \bar{B})$ that defines h as follows:

$$\left\{ \begin{matrix} if\ B \geq \bar{B}(y),\ h(B, y) = b_+(B, y; h)\ and\ c(B, y; h) = y + BR(B) - h(B, y) \\ if\ B < \bar{B}(y),\ \text{with probability } \theta,\ h(0, y) = b_+(0, y; h)\ and\ c(0, y; h) = y^{def}(y) - h(0, y) \\ if\ B < \bar{B}(y),\ \text{with probability } 1 - \theta,\ h(B, y) = 0\ and\ c(B, y; h) = y^{def}(y) \end{matrix} \right\} \tag{9}$$

Note that h is discontinuous even if it is unique. Therefore, the tools used to prove the existence of equilibrium must be robust to discontinuities. Fortunately, we will show that equation (4) induces an ordered structure, allowing us to use suitable theorems. In particular, the results in Coleman (1991), Mirman et al. (2008), and Aguiar and Amador (2019) serve this purpose. Moreover, the household does not internalize the default restrictions associated with (9). Thus, since $\bar{b}$

can be assumed to be arbitrarily large, we can prove the results using a standard Euler operator without explicitly considering inequality constraints. As is typical in the default literature, the model assumes that the government has an enforcement technology that keeps the private economy away from individual optimization (as described by equation (4) and formally captured by h when $B < \bar{B}(y)$) as long as re-entry is not possible.

3 Existence and characterization of equilibria

To constructively[7] prove the existence of equilibrium, we first need to define and characterize it. Our definition of equilibrium is standard and can be stated as follows:

Definition 1 (Recursive equilibria, H)**.** *A recursive equilibrium is a set of policy functions* (c, b_+, h)*, value functions* (V^c, V^{def}) *and prices* R*, with* $H \equiv (c, b_+, R, V^c, V^{def}, h)$*, such that:*

- *taking* R *as given, the policy functions* c *and* b_+ *solve the household problem in* (2)*;*
- R *satisfies the lender's zero-profit condition in* (1) *and is consistent with the government's default decision in* (5)*;*
- V^c *and* V^{def} *are given by* (6) *and* (7)*;*
- h *is consistent with the rational expectations condition in* (9)*.*

Since any set of elements H that satisfies Definition 1 is time-independent, we refer to such an object as a *stationary equilibrium*. We call a stationary equilibrium *ergodic* if it admits an ergodic invariant measure. Moreover, we say that the equilibrium is *stable* if it generates paths that are *recurrent* (i.e., they do not diverge to the boundary of the equilibrium set) and if the state space that characterizes it is *connected* (i.e., the process does not break into "islands," or, equivalently, it is irreducible).[8] Remarkably, the stochastic steady state of the economy is globally stable and, at

[7]A constructive existence proof generates a convergent algorithm through a sequence of successive approximations. As we use order theory to prove existence, convergence to the fixed point is not uniform (i.e., iterations do not converge from any initial condition). This differs from the standard practice in the literature, which is based on the contraction mapping theorem.

[8]The appendix containing the proofs of section 3.2 formally defines an accessible atom, which jointly ensures recurrence and irreducibility. As the atom is also helpful in showing ergodicity, this concept is deeply connected with the stability of paths.

the same time, is an invariant ergodic measure. We refer the reader to Pierri and Seoane (2025) for a detailed discussion on the relationship between global stochastic stability and ergodicity. In this paper, we focus on the numerical, thus quantitative, implications of ergodicity; leaving its qualitative implications for a separate paper.

To characterize the equilibrium, we impose two standard assumptions:

Assumption 1 (Finite i.i.d. endowments). *All $y \in \{Y_{LB}, ..., Y_{UB}\} \equiv Y$ with $Y_{LB} > 0$, $Y_{UB} < \infty$ and $\pi(y) > 0$, where π is a probability measure.*

Assumption 2 (Preferences). *$u : \mathbb{X} \to \mathbb{R}$, where $\mathbb{X}$ is the consumption space. The function u is once differentiable with derivative $u'(c)$, strictly increasing, strictly concave, unbounded below, and bounded above. Moreover, u' satisfies the Inada conditions: $lim_{c\to\infty} u'(c) = 0$ and $lim_{c\to 0} u'(c) = \infty$. Finally, $\beta R^* < 1$.*

Assumption 1 allows us to define a process $(\Omega, \Sigma, \mu_{y_0})$ with a typical element in the sequence space $\{y_0, y_1, ...\}$ and an associated process in the space of random variables $[c, b_+, R](\omega)$, which maps Ω to $\mathbb{R}^3$ (see Lucas et al. (1989), chapters 7 to 9). As shown below, under these assumptions c, b_+ and R are bounded almost everywhere with respect to μ_{y_0}, for all $y_0 \in Y$. It is worth noticing that Assumption 1 is only necessary to establish ergodicity. We can prove the existence of a stationary equilibrium under Markov shocks. In section 5.2.2, we solve the model with Markov shocks. Assumption 2 is standard. We use the requirement on the boundless of u to derive an upper bound on net assets. To satisfy it, it suffices to impose a CRRA instantaneous return function, with a mild restriction on the curvature parameter.

Lemma 1 (Bounds). *Under Assumptions 1 and 2, $[c, b_+, R](\omega) \in \mathbb{K}$ almost everywhere in Ω, where $\mathbb{K} \subset \mathbb{R}^3$ is compact. Moreover, $c(\omega)$ is bounded below almost everywhere in Ω by $\underline{c} > 0$.*

Proof. See Pierri and Seoane (2025). □

We can now characterize the policy function induced by equation (4).

Lemma 2 (Policy Functions)**.** *Under Assumptions 1 and 2, if $R(B)$ is decreasing in B, then $c(b, B, y; h)$ and $b_+(b, B, y; h)$ are both weakly increasing in b when $b = B$ for any $y \in Y$ and any h.*[9] *Moreover, either c or b_+ is strictly increasing.*

Proof. See Pierri and Seoane (2025). □

Lemma 2 not only provides qualitative properties of the policy functions, it also implies that a persistent recession may generate explosive debt paths that culminate in default. In other words, default episodes occur during economic downturns. Of course, there is a "limit" to the degree of instability that is compatible with a tractable equilibrium. To curb the instability inherent in the model, we assume $R\beta < 1$.

3.1 Existence of equilibrium

We now derive a novel two-step nested fixed-point operator to prove existence. We embed individual optimization (the first step) into the government's problem (the second step). Regarding the first step, we show that equation (4) induces a Coleman–Reffett operator on (c, b_+) that satisfies the properties in Lemmas 1 and 2 for any R that is decreasing in B. This result depends on V^c and V^{def}. We then show, using the results in Aguiar and Amador (2019), that these value functions have a unique fixed point for any triple (c, b_+, R). This is the second step. To obtain a constructive iterative procedure, we use the fixed point of (V^c, V^{def}) to update R and then use (4) to update (c, b_+). Because the bounds on policy functions and interest rates are uniform and depend only on primitive assumptions, equations (5), (6) and (7) preserve the monotonicity of R, which is critical for constructive existence. We now formally define the operator.

Definition 2 (Nested two-step fixed-point operator)**.** *We prove the existence of a stationary equilibrium using the following operator:*

- ***Coleman–Reffett.*** *Given R, equation (4) generates an operator A with $c_n \to Ac_n = c_{n+1}$.*

- ***Aguiar–Amador.***

[9] Both c and h are contained in a set of functions C defined in the appendix.

- *Equations (6) and (7) induce an operator $\mathbb{T}$ with $\left[V_j^c, V_j^{def}\right](c_n) \rightarrow \mathbb{T}\left[V_j^c, V_j^{def}\right](c_n) = \left[V_{j+1}^c, V_{j+1}^{def}\right](c_n)$.*
- *This operator has a fixed point $\left[V_*^c, V_*^{def}\right](c_n)$.*

- *Equations (5) and (1) update $R\left(\left[V_*^c, V_*^{def}\right](c_n)\right)$.*
- *The Coleman–Reffett operator updates c using $R\left(\left[V_*^c, V_*^{def}\right](c_n)\right)$.*
- *The procedure continues until convergence, with $R\left(\left[V_*^c, V_*^{def}\right](c_*)\right) \equiv R_*$ and c_* a fixed point of A.*

The first step in Definition 2 requires an initial interest rate R_0. Moreover, the convergence induced by the Coleman–Reffett operator depends on its initial condition c_0. In contrast to the contraction mapping theorem, convergence is not uniform across initial conditions for (V_0^c, V_0^{def}). Thus, we must impose a consistency requirement for the first iteration of the operator: given c_0, the pair (V_0^c, V_0^{def}) must generate R_0 as a market outcome. To this end, we introduce a refinement on the default cost. We refer to the refined cost as *endogenous* because it is consistent with the existence of a stationary equilibrium. In our context, this endogenous cost is chosen by the sovereign. Intuitively, Pierri and Seoane (2025) presents a closed-form two-period model showing that not all possible default punishments are consistent with the existence of equilibrium. We formalize this discussion in Assumption 3 below.

Assumption 3 (Endogenous stationary punishment)**.** *Let $\mathbb{C}$ be the space of all possible consumption functions*[10] *and let $\mathbb{B}$ be the compact set containing any B, both refined by Lemma 1. Let $c_0 \in \mathbb{C}$. Then $y^{def}(y)$ with $y \in Y$ satisfies:*

1. $V_0^c(B, y) = u(c_0(B, y)) + \beta\mathbb{E}\left\{V_0^c(y + R^*B - c_0(B, y), y')\right\}$;
2. $V_0^c(B, y) \geq V_0^{def}(y) = u(y^{def}(y)) + \beta\mathbb{E}\left\{(1-\theta)V_0^{def}(y') + (\theta)V_0^c(0, y')\right\}$ *for all* $(y, B) \in Y \times \mathbb{B}$;
3. c_0 *satisfies* $\bar{c}_0 = SUP(\mathbb{C})$ *or* $\underline{c}_0 = INF(\mathbb{C})$;
4. $y \geq y^{def}(y)$ *for all* $y \in Y$.

[10] $\mathbb{C}$ contains the candidates for a possible consumption function. It is bounded below by 0 and above by the total available resources. See the appendix for a formal definition of this set.

Remark 1 (Refined default punishment)**.** *As discussed in Pierri and Seoane (2025), there are some values of the default penalty that are not consistent with the existence of equilibrium. Assumption 3 presents a set of restrictions on the possible values of this penalty that allows us to show the existence of an equilibrium. We refer to the default penalty that satisfies these restrictions as the refined default punishment, as we propose a refinement of all possible punishments consistent with our existence proof.*

Assumption 3 allows for asymmetric default costs (e.g., $y^{def}(y) = \hat{y}$ if $y > \hat{y}$ and $y^{def}(y) = y$ if $y \leq \hat{y}$), in line with Arellano (2008). Moreover, since u is unbounded below by Assumption 2 and (V_0^c, V_0^{def}) have a fixed point under standard arguments, Assumption 3 is relatively mild. Given point 3, we can prove points 1 and 2, which implies that we only need to impose the requirement point 4 by assumption. Intuitively, condition (1) in Assumption 3 defines the value of repayment under c_0. This equation differs from (6): the latter includes the option between repayment and default, while the former does not. As a result, the equilibrium interest rate in the first iteration equals the risk-free rate R^*. Condition (2) then guarantees that this is indeed the case. Condition (3) is needed for convergence under the Coleman–Reffett operator. Condition (4) provides the standard qualitative characterization of default.

We now establish the existence of stationary equilibria in Definition 1. Under Assumptions 1, 2 and 3, Definition 2 allows us to compute the equilibrium *directly*, without relying on a heuristic updating rule for the interest rate.[11] Moreover, since the initial conditions in Definition 2 satisfy Assumption 3, the existence proof induces an iterative, convergent algorithm. We therefore call this equilibrium *computable*. Coupled with the ergodicity of equilibrium, which we state later, the existence of a stationary computable equilibrium implies that we can accurately characterize the model's long-run behavior using numerical simulations.

Theorem 1 (Existence of stationary computable equilibria (SCE))**.** *Under Assumptions 1–3, there exist at least two SCE, $H(\bar{c}_0)$ and $H(\underline{c}_0)$, with $c_*(\underline{c}_0)(B, y) \leq c_*(\bar{c}_0)(B, y)$ for all $B \in \mathbb{B}$ given y.*

Proof. See the appendix. □

Thus, we establish the existence of multiple ordered equilibria. We now turn to uniqueness and ergodicity.

[11] The proof of existence requires that every iteration preserves the monotonicity of interest rates. Hence, not every updating rule can be used.

3.2 Uniqueness and Ergodicity

In this section, we show two additional properties of stationary computable equilibria. First, the SCE is unique under a strengthening of Assumption 2.[12] Second, by imposing an additional restriction on Assumption 3,[13] the equilibrium is ergodic. Taken together, these results provide the main theoretical contribution of the paper: *to our knowledge, this is the first available proof of ergodicity in default models*. As shown in Table 1, this is important given the pronounced differences between short- and long-run behavior in default episodes. While the stationary equilibria help us to characterize the short-run behavior of the model, only with the ergodic equilibrium can we *also* study the long-run.

Assumption 4 (Pseudo-concavity of the utility function and lower bound for the interest rate in highly indebted economies.)**.** *In addition to Assumption 2, suppose that* $u'(c_1c_2) = u'(c_1)u'(c_2)$ *for all* $c_1, c_2 \in \mathbb{C}$ *and* $c_1, c_2 > \overrightarrow{0} \in \mathbb{R}^2$*. Moreover, assume that the upper bound on debt* $-\bar{b} > 0$*, defined in problem* (2)*, satisfies* $\beta R(\bar{b}) > 1$*.*

Note that the CRRA utility function $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 1$ satisfies Assumptions 2 and 4. Assumption 4 guarantees that equation (4) defines a pseudo-concave operator A. In particular, given some $\alpha \in (0, 1)$ and for all possible consumption functions c, we have $A(\alpha c) > \alpha A(c)$. In addition, the restriction on the upper bound for debt / lower bound for the interest rate guarantees that net external assets eventually increase. Since Assumption 2 already imposes $\beta R^* < 1$, we can rule out equilibria that lie near the boundary of the state space. Together with the uniform positive lower bound on consumption in Lemma 1 and the ordered structure of fixed points in Theorem 1, Assumption 4 is sufficient to establish uniqueness.

Theorem 2 (Uniqueness of stationary computable equilibria)**.** *Under Assumptions 1, 3 and 4, there is at most one SCE* $c_* > 0$*.*

Proof. See the appendix. □

Remark 2 (Uniqueness and possibly multiple interest rates.)**.** *Since Ayres et al. (2018), it is well known that Equation* (1)*, which characterizes equilibrium interest rates, may generate multiple equilibria.*

[12] For a model with centralized default and public debt, uniqueness is shown in Aguiar and Amador (2019). This paper provides the first proof of uniqueness for a model with private debt.

[13] The modified version of Assumption 3 still implies that the default punishment is endogenous.

However, Lemma 1 allows us to take a measurable selection out of a compact-valued correspondence. As our existence proofs are robust to the presence of discontinuous equilibria, we show uniqueness after taking this selection, acknowledging that multiple roots may exist due to the risk-neutral pricing assumption of international, deep-pocketed investors. Theorem 2 only proves the uniqueness of the Coleman-Reffett step in the operator presented in Definition 2.

So far, given a measurable selection of possible multiple roots in Equation (1), we have shown that an equilibrium exists and is unique. To complete the equilibrium analysis, we need to study its global dynamic properties. As discussed above, the decentralized economy may generate explosive paths. The government chooses to default to eliminate such paths. Moreover, as we will see in section 5, the government chooses to return to asset markets when GDP under free capital mobility is sufficiently high. Hence, the unique ergodic equilibrium has both long-run and post-default implications.

To show ergodicity, we must add an additional assumption. In this paper, we present the result, deferring the proof to a separate paper (see Pierri and Seoane (2025)).

Assumption 5 (Ergodic punishment)**.** *In addition to Assumption 3, assume that $y^{def}(y) = y^{def}$ for all $y \in Y$. Let $B^N(y_{LB}) \equiv B = b_{+,*}(B, y_{LB})$. Assume that $B^N(y_{LB}) < 0$.*

Note that the second part of Assumption 5 imposes restrictions on endogenous variables. We need to assume the existence of a non-stochastic steady state B^N for the lowest shock y_{LB}. A slightly weaker result can, however, be obtained using standard arguments, so this assumption is relatively mild. The following remark formalizes this point.

Remark 3 (Existence of a non-stochastic steady state B^N)**.** *Under Assumption 4, we can refine the existence result and show uniqueness in Theorem 2. Theorem 10 in Mirman et al. (2008) shows that the equilibrium of the private economy without default is continuous. Moreover, b_+ is increasing in y. Finally, equation (2) implies the existence of an upper bound on debt $-\bar{b}$. Then it follows that $B \geq b_{+,*}(B, y_{LB}) \geq \bar{b}$,[14] which in turn implies that we must impose $B = b_{+,*}(B, y_{LB})$ by assumption.*

[14]For any y: (i) $B \geq b_{+,*}(B, y) \geq \bar{b}$ if $b_{+,*}(B_0, y) < B_0$, which implies that the economy converges either to $B^N(y)$ or $\bar{b}$; (ii) $B \leq b_{+,*}(B, y) \leq B_{UB}$ if $b_{+,*}(B_0, y) > B_0$, in which case the economy converges to $B^N(y)$ or B_{UB}.

If we could prove that the policy function for net assets is concave, the existence of a non-stochastic steady state would follow immediately. Although we cannot prove this result analytically, the computed policy functions in section 5 are concave. Furthermore, the assumption $B^N(y_{LB}) < 0$ in Assumption 5 has a minimal consistency requirement: as we are modeling default, we require that in the non-stochastic steady state households hold negative net assets in the worst possible scenario (i.e., $y = y_{LB}$). We can state, without proof, the ergodicity of the unique computable stationary equilibrium.

Theorem 3. *Under Assumptions 1, 4 and 5, there exists y^{def} such that $B^N(y_{LB}) < \bar{B}(y^{def}) < 0$ and (Z, P_φ) has a unique ergodic probability measure μ_*^{def}.*

Proof. See Pierri and Seoane (2025). □

Theorem 3 establishes the existence of an ergodic stochastic steady state, which we characterize using a stationary measure. The standard law of large numbers implies that we can accurately approximate moments of this steady state using simulations.

4 Implications of the theory

The theoretical results presented in the previous section, when combined, generate a powerful toolkit for accurately studying the effects of default risk on small open economies. First, by using the constructive existence result, combining Theorems 1 and 2, we can derive an accurate algorithm and an appropriate default cost. Next, we need to characterize the model's behavior. Typically, the literature relies on simulations. Using Theorem 3, we can also accurately simulate the model, as the ergodicity of the steady state guarantees that simulations eventually reach it. Finally, there is another, less common, implication of our toolkit: we can globally characterize *every positive probability path*. Thus, we can classify future trajectories based on the current state of the economy. This is useful because *we can evaluate the sustainability of debt solely based on the current GDP and debt levels*. The easiest way to classify these paths is using a *phase diagram*, a graphical device connecting debt today and tomorrow iteratively. This section makes a graphical analysis of global equilibrium dynamics, providing an additional tool to the numerical analysis. Like

Azariadis and Lambertini (2003) and Brock and Hommes (1997), we use this tool to globally characterize the dynamical system generated by the model economy. Contrary to these authors, we have a stochastic steady state, which we characterize using an invariant (ergodic) measure. The supplementary appendix for this section contains more subtle implications of the formal results derived in section 3.

Figure 1 illustrates how we can characterize positive probability paths for the simplest possible case: an economy with 2 shocks. Consider the following exercise: start at point 0, where the initial states are B_0 and Y_{UB}, and the latter represents the big shock. In this point, the optimal debt issuance corresponds to $b_{+,*}(B_0, Y_{UB})$. Suppose now that the economy stays in Y_{UB}, which implies that households are accumulating debt until point 1. If a negative endowment shock hits the economy, we may observe an increase in debt to smooth consumption, which moves the economy to point 2. If a sequence of low shocks Y_{LB} occurs, the economy defaults in point 3. The figure illustrates that when returning to asset markets, the economy can transition to point 5.1 if a high shock, Y_{UB}, occurs or to point 4.1 if we observe Y_{LB}.

Notice that $b_+(., Y_{LB})$ is convex. Thus, at point A, the non-stochastic steady state generates stable paths. However, point A is unstable without default if this curve is concave. In the next section, we will see a version of Figure 1 for the calibrated economy where all demarcation curves are concave. Now suppose that there is no intersection between $b_{+,*}(., Y_{LB})$ and the $45°$ degree line, which may occur, for instance, when the demarcation curves are concave. In this case, in the absence of default, the economy may diverge, generating unsustainable trajectories. Theorem 3 shows that the government can eliminate unstable and unsustainable debt paths, regardless of the curvature of the policy. More to the point, the picture illustrates that the economy is stable without default if: a) both demarcation curves have non-stochastic steady states, b) $b_+(., Y_{UB})$ is concave and $b_+(., Y_{LB})$ is convex. As these properties cannot be traced back to primitives, we introduce a centralized default to guarantee the sustainability of debt.

Figure 1 shows that the process does not contain divergent paths because of properties a) and b) mentioned in the paragraph above. Still, it hits the upper bounds for debt associated with default $\overline{B}\left(Y^{DEF}\right)$ with positive probability starting from any initial condition. Thus, by definition, we are improving the ability of the model to reproduce any observed probability of default based on changes to deep parameters, as these changes affect y^{def} by Assumptions 3 and

5. Moreover, as the distribution is ergodic, we can match multiple empirical moments as f in $\sum f(z)/N \to \mathbb{E}(f(z); \mu_*^{def})$ is arbitrary as long as it is continuous.

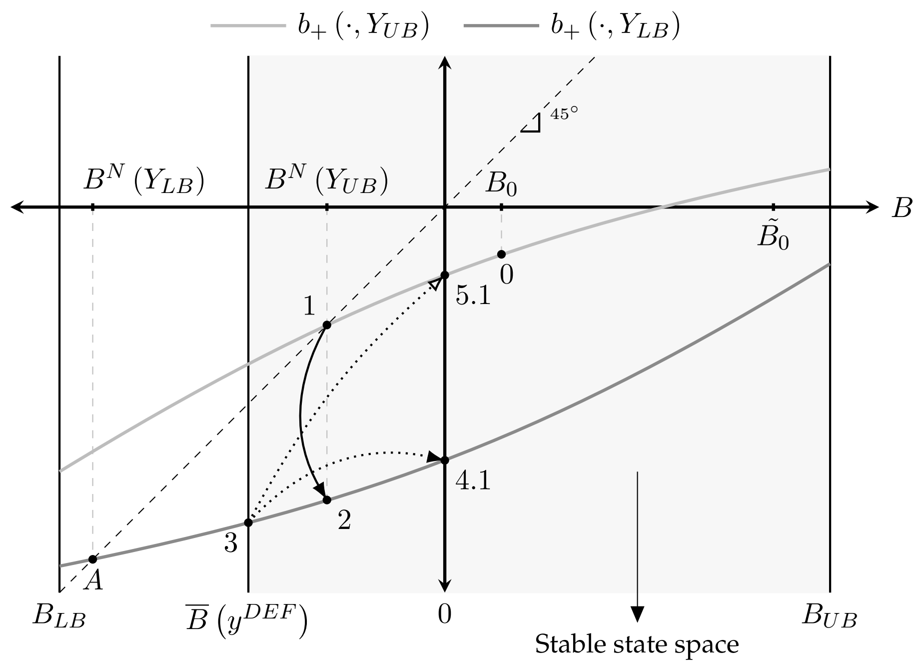


Figure 1: Transitions for an initial net debtor

Note: Start with the initial state B_0, Y_{UB} in point 0, an initial condition which implies $B_+ = b_+(B_0, Y_{UB})$. A good endowment shock increases country debt to point 1, where a bad endowment realization occurs, inducing the economy to issue debt to smooth consumption (point 2). If a sequence of Y_{LB} occurs, the economy goes straight to default in point 3. The figure illustrates that when returning to asset markets, the economy can transition to point 5.1 if Y_{UB} or 4.1 if Y_{LB}.

One of the main contributions of this paper is to provide a toolkit for *globally* characterizing all positive probability paths generated by an economy affected by default risk. Notice, remarkably, that demarcation curves may have more than one intersection with the $45°$ line. That is, we have more than one well-defined non-stochastic steady state. Because of their local nature, they can't be used to approximate the actual behavior of the economy. However, as discussed in the next section, non-stochastic steady states help classify stochastic paths as stable, unstable, and non-sustainable. It is possible that demarcation curves do not cross the $45°$ line or that these curves intersect each other, but the local equilibrium is unstable. In these cases, the private economy generates explosive debt paths without default. Theorem 3 proves that government action, through default, is welfare-improving as it eliminates explosive debt paths.

Note that, due to Theorem 3, point 3 in Figure 1 is to the right of point A. That is, in a recession, the economy is attracted to default. After default, the economy jumps, generating a discontinuity, to either points 4 or 5 *spending a finite number of periods in exclusion*, at $(0, 0)$ in Figure 1, according to θ. Moreover, the system between points 3 and 4/5 *behaves as an i.i.d process* as transition probabilities depend on θ, independent of B by assumption. It turns out that this is the distinctive characteristic of a discontinuous ergodic equilibrium.[15] Then, after we obtain a re-entry draw from θ, we can either go to 4 or 5 depending on y. In this sense, the zero at the vertical axis defines the appropriate initial condition for the economy after the re-entry. In section 5, this fact will give rise to a threshold value for GDP that determines re-entry to capital markets.

Equation (8) implies $\bar{B}(y_{UB}) < \bar{B}(y_{LB})$. The figure above suggests that if we set the lower bound of the state space to be $\bar{B}(y_{LB}) = \overline{B}\left(Y^{DEF}\right)$, then interest rates are bounded; a result which follows formally from Lemma 1. An implication of this fact is that we can target the current account and the trade balance. The literature typically focuses only on the latter.

Note that only one default, identified with $\overline{B}\left(Y^{DEF}\right)$, is possible for each ergodic distribution. This fact has two important consequences. First, *default has permanent effects as it changes the stochastic steady state of the economy.* Second, ergodicity influences our reading of the effects of parameter changes or *MIT shocks. As suggested by Echenique (2002), the intuition behind comparative statics is dynamic in nature. This follows directly from Samuelson's Correspondence Principle.*

Regarding the first point, the empirical evidence suggests significant differences in the values of descriptive statistics computed locally, around the default, and globally for the whole sample. The implications of Theorem 3 give us an explanation for this fact. For instance, Argentina and Ecuador experienced more than one default between 1960 and 2017; therefore, the pooled average across the whole sample may contain information on multiple different steady states. In the case of Argentina, the events in 1982 and 2001 resulted in significantly different GDP levels. Thus, y^{def} should reflect this fact. As there could be only 1 attraction or regeneration point for each ergodic distribution in discontinuous models, we obtain 2 different invariant measures μ_*^{82} *and* μ_*^{01},

[15] In the supplementary appendix for section 3.2 we introduce the notion of an atom: a point in which the conditional and the unconditional distribution are equal $P_\varphi(z_*, A) = \mu(A)$, generating transitions independent of the state space, which is critical to proving the stability of the dynamical system.

which in turn implies the cumulative average from 1960 to 2017 can't converge to $\mathbb{E}(z;\mu_*^{82})$ and to $\mathbb{E}(z;\mu_*^{01})$.[16] As long as y^{def} differs between default episodes, the state space is different, which in turn implies $\mathbb{E}(z;\mu_*^{82}) \neq \mathbb{E}(z;\mu_*^{01})$.

Regarding the relationship between comparative statics and stability, as noted by Acemoglu and Jensen (2015), any parameter change or MIT shock in these models generates a change in the ergodic distribution. The joint effects of these changes on demarcation curves and the default threshold not only affect the ergodic distribution but also the global dynamics of the model and thus all positive probability paths. In Figure 1, Assumption 5 implies that point A should always lie to the left of $\bar{B}(y^{def})$. Thus, any change in, for instance, the discount factor β or the international interest rate R^*, must respect these restrictions. Stated differently, the requirements associated with global stability affect comparative statics results, as suggested by Samuelson's Correspondence Principle. Intuitively, if, say, a change in R^* shifts $\bar{B}(y^{def})$ to the left, it should also move demarcation curves up. In Figure 1, this means that the economy *can sustain higher levels of debt as it not only accumulates more debt but also more assets due to the higher interest rate.* This implies that point A moves to the left as required and the mean value of net external assets in the new ergodic distribution is smaller, connecting global dynamics with comparative statics.

5 Taking the model to the data

We calibrate the model to match ergodic moments. We chose Argentina between 1982 and 2016 as this sample includes only 1 event, the default of 2001.[17]

Theorem 3 allows us to use observations *after* default, which is not frequent in the literature. In contrast to us, Arellano (2008) targets the 1983-2001 period. We refer to this approach as stationary or local. Instead, our strategy targets a longer sample (1983-2016), including observations during and after default, that will align with the model's ergodic long-run moments.

To calibrate the model, we use a novel algorithm based on Definition 2, refining the initial condition of the iterative process using Assumption 3. That is, we initialize the Nested Fixed Point

[16] The notation intends to make it clear that one stable distribution corresponds to the default in 1982 (μ_*^{82}) while the other corresponds to the default in 2001 (μ_*^{01}).

[17] Between 2014 and 2016, the country was affected by a court ruling that took Argentina out of the international capital markets. The results are similar if we choose 1983-2013 instead of 1983-2016.

Operator in Definition 2 in the supremum of the space of candidate functions $\bar{c}_0$ and compute the endogenous punishment $y^{def}(y)$ according to the procedure described in Assumption 3. We then take the minimum value of $y^{def}(y)$ and compute the ergodic punishment. Theorem 2 guarantees the uniqueness and constructive computability of the stationary equilibrium using the Nested Fixed Point Operator described in Definition 2. In case the equilibrium interest rate is not unique for some subset of the state space (i.e., Equation (1) has more than one root), we say that the equilibrium is unique after taking a measurable selection. This is possible due to Lemma 1. Theorem 3 guarantees the ergodicity of the equilibrium.

Table 2 shows that the short-run versus long-run targets differ substantially, which confirms the relevance of the results in this paper as we can study the stationary (short-run) and ergodic (long-run) equilibrium separately.

Table 2: Data with respect to GDP

Date /	Stock of net external assets			Net private debt services		Current account	
Percent	Total	Defaultable	Private	Capital	Interests	Mean	STD
83-01	-36.5%	-31.0%	-6.7%	-1.1%	-0.5%	-2.3%	0.8
83-16	-34.0%	-28.9%	-8.2%	-1.4%	-0.6%	-0.8%	3.6

Note: The second row contains the "local" sample between the default episodes of 1982 and 2002. The third row shows the "global" sample, which includes the default of 2002. The second column contains total external assets divided by the GDP. The third one shows private plus public external assets, excluding loans granted by international and multilateral organisms not subject to a haircut. The fourth column denotes private external assets only. The fifth and sixth columns show yearly capital payments (i.e., amortizations) and interests of private external debt. "Mean," and STD denote the average of the current account to GDP and its standard deviation divided by the mean, respectively.

From Table 2, it is clear that the structure of the model and the results in the theory section affect the value of the moments to be targeted for 2 reasons: i) fundamental macro variables behave differently around the default, which is a "local behavior", and in the whole sample. This is the case of the current account: the mean around the default implies a deficit almost 3 times bigger than in the whole sample and the dispersion is much lower. ii) The "refinement" process for debt statistics implies that the targeted level varies from -34.0% to -1.4%: first, we remove the multilateral organizations from the sample (the average for the whole sample goes from -34.0% to -28.9%). Then, we remove public debt, and the average goes down to -8.2%, and then we use the average duration of debt (6 years) to derive the yearly capital payments -1.4%. As the model only contains 1-period bonds, we follow Arellano (2008) and target yearly debt services.

The calibration results are standard and follow those presented in Pierri and Seoane (2025). To keep the paper self-contained, we present them in the Supplementary Appendix. We now focus on the calibrated dynamics of the model, the interactions between these dynamics and comparative statics, and the difference between short and long-run simulations.

5.1 The dynamics of the calibrated model

Figure 2 shows the global stochastic dynamics of net external assets for the estimated phase diagram. As before, the horizontal axis represents the stock of this variable "today," and the vertical axis represents it "tomorrow." The demarcation curves, one for each level of exogenous endowment, show the equilibrium dynamics. The arrows indicate whether an equilibrium is stable or unstable, and the intersection with the $45°$ degree line defines the non-stochastic steady states B^N. Remarkably, when $B^N < 0$, the equilibrium is unstable; *debt creates an unstable environment regardless of the GDP level*. $\bar{B}(Y^{DEF}, Y_{Dj})$ indicates the level of debt that will trigger a default if the j-th endowment is realized. Notice that, as shown in the figure, the default threshold $\bar{B}$ is positive if $Y \in [Y_{D1}, Y_{D5})$. These thresholds will define an "exclusion area" as they imply that, after default, the Government keeps the private sector out of the international capital markets unless the GDP is sufficiently high.

To begin with the description of global dynamics, we must first identify some additional preliminary elements in Figure 2. Demarcation curves are ordered based on different output levels y, from the highest Y_{D10} to the lowest Y_{D1}. We call these values deciles. Using standard results we can show that a higher decile implies that the demarcation curves b_+ shift to the north, as seen in Figure 2. We depict these curves only for $D1, D7$, and $D10$. However, as there is a monotonic increasing relationship, we know that "between" $b_+(., Y_{D7})$ and $b_+(., Y_{D10})$, we can find $b_+(., Y_{D9})$ which is above the former but below the latter. Contrary to what we saw in figure 1, some demarcation curves *does not* have an intersection with the $45°$ line, and others have more than 1. This result is due to the calibrated parameters. For instance, $b_+(., Y_{D7})$ has 2 intersections, one at $B = -0.08$ and the other at $B = 0.24$, and $b_+(., Y_{D1})$ does not have an intersections. These intersections are "non-stochastic steady states" (see Remark 3) and satisfy $B^N(y) \equiv B = b_+(B, y)$ for

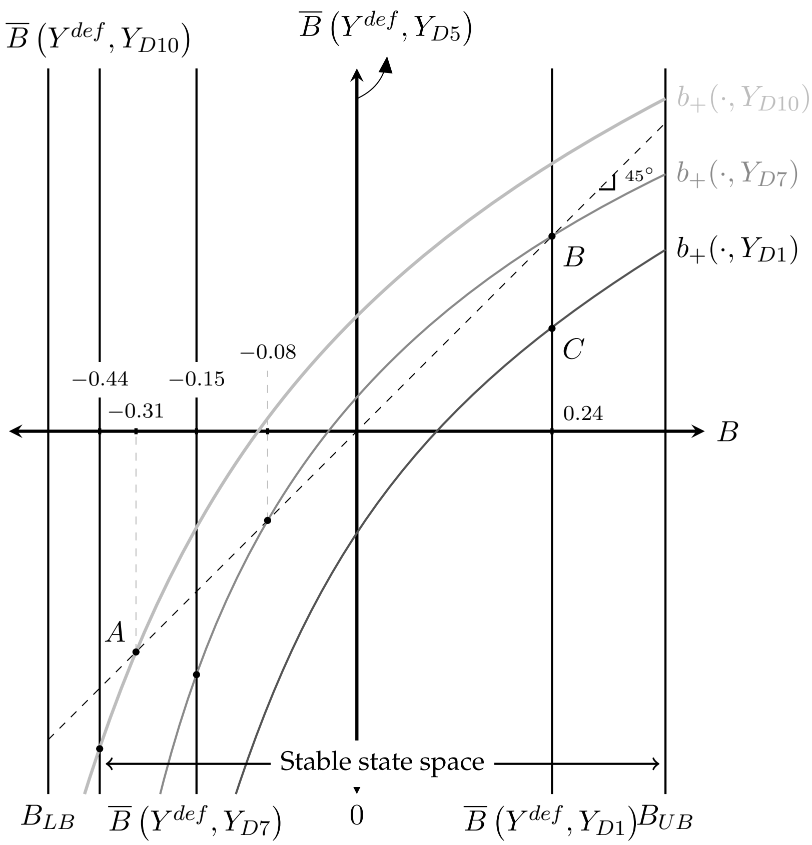


Figure 2: Calibrated phase diagram ($\mu = -2.0\%$, $STD = 0.08$)

Note: The phase diagram in this picture follows from the model calibrated to Argentina with the calibration described in Table 6, in the Supplementary Appendix.

some $y \in Y$. Finally, we will refer to a "high debt trap" as a negative non-stochastic steady state $B^N < 0$. For example, for Y_{D7} and Y_{D10}, $B^N(Y_{D7}) = -0.08$ and $B^N(Y_{D10}) = -0.31$ respectively.

Figure 2 contains 2 important implications:

i) We have 3 types of demarcation curves.

- Unstable curves. For high shocks (Y_{D10} and Y_{D9}) the intersection with the $45°$ line implies that the equilibrium is "unstable" (for instance point "A" in the figure). This also happens in deciles 6 to 8 but *only* when the economy has debt. When households are net creditors, demarcation curves generate "stable" paths for these GDP levels. For deciles 9 and 10, because private agents accumulate external assets at a fast pace, *except in the non-stochastic steady state*, in the absence of shocks and without default, the economy *would* converge to the boundary of the state space; outside the stable region. *Thus, high GDP values generate unstable dynamics.*

- Unsustainable curves. For low shocks (Y_{D1} to Y_{D5}) there is no intersections with the $45°$ line. We call these paths "non-sustainable": the economy will converge outside the stable state space for all points along these curves. For these paths, the planner also *stabilizes the economy by defaulting*. This happens when the economy hits, for instance, $\bar{B}(Y^{DEF}, Y_{D1})$. Notice that if the country has a positive external net asset position for low GDP values, the government may choose to confiscate them. This is because, otherwise, households would get into a dynamic, inefficient region. Behind the decision to confiscate assets, there are at least 2 reasons, one technical and the other intuitive. As regards the latter, as in Aguiar and Amador (2019), we would need to restrict endogenous variables, particularly value functions, to guarantee that default is only observed when the country is a net debtor. To keep assumptions on endogenous variables to the minimum, as we must endogenously compute the default penalty to guarantee the constructive existence of equilibria, we prefer not to impose additional limits on endogenous variables (i.e., those needed to guarantee constructive existence and default only when net external assets are negative). [18] There is also a powerful economic intuition behind the decision to confiscate assets: it only happens for those demarcation curves associated with unsustainable paths (i.e., those that do not intersect with the $45°$ line and thus do not have a non-stochastic steady state inside the stable state space). For instance, in point C in figure 2, the government defaults because the value of continuing to honor debt is affected by unsustainable paths. In the not-so-distant future, the government will be forced to default with a high probability as assets are on an explosive path.

- Curves with stable and unstable regions. For intermediate shocks (deciles 6 to 8), there is a "stable" (when the economy has assets and policy functions are "flat") and an "unstable" (when the economy has debt and policy functions are "steep") region. Notice that although there are 2 non-stochastic steady states for each curve, this economy has a unique equi-

[18] In the appendix, to keep the structure of the proofs similar to Aguiar and Amador (2019), we impose the necessary restrictions to guarantee that default is only observed when the country is a net debtor. However, our existence, uniqueness and ergodicity results do not depend on this assumption.

librium as there is only 1 interest rate per element in the state space. Thus, *instability and multiplicity are not necessarily related to each other*.

ii) There are 2 important concepts behind this figure: *a) debt traps*. The curvature of the policy function determines the pace of accumulation of assets. Because of the concavity of demarcation curves, for negative levels of net external assets, this function is *"steep"* and for positive levels is *"flat"*. *At high and intermediate levels of GDP (i.e., deciles 6 to 10), households want to avoid a "high debt trap", as represented by a negative non-stochastic steady state, by accelerating the pace at which they accumulate assets; a fact that introduces instability into the private economy. If debt is sufficiently high, decentralized private action is insufficient, and the planner has to restore stability by introducing default into the decentralized equilibrium*. This is the next fact: *b) The default acts as a stabilization policy.* The presence of default *stabilizes* the economy by returning paths inside the stable state space once the trajectory hits $\bar{B}(Y^{DEF}, Y_{D_i})$ for all i. For a detailed discussion on the relationship between default and stability, we refer the reader to Pierri and Seoane (2025).

The facts above have an important implication: *Based on the position of the economy, characterized solely by net external assets and GDP, we determine the long-run sustainability of debt.* If GDP is "low" (i.e., at or below decile 5), external debt is unsustainable and, regardless of the net external asset position of the county, the private sector will default. If GDP is "high" (i.e., at or above decile 6), there is a debt value that destabilizes the economy, the non-stochastic steady state $B^N(y)$. Notice that this value could be smaller or bigger than the default threshold $\bar{B}$. For instance, if GDP levels are at decile 7 and the net external assets are slightly below -0.08, the private economy still has access to international capital markets. However, unless the economy moves to a higher decile, a default will be observed. Thus, we can derive *early warning indicators when agents have fluent access to international capital markets* that alert us to the fact that the economy is on an unstable track and will eventually default, unless we observe an increase in aggregate income.

5.2 Long and short run implications of the model

Before we dealt exhaustively with positive probability paths. In this subsection, we explore the model using the ergodic laws of large numbers, which pools all these paths using a partial sum. First, we study the interaction between global dynamics and comparative statics, or MIT shocks, through a change in the international interest rate. Second, we characterize short-run or local and long-run or global dynamics separately. To this end, we solve the stationary model with Markov shocks and, as is customary in the literature, calibrate the model using "local" moments. Then, we compare the results obtained with those of Kim and Zhang (2012), who also solve a model with centralized default and decentralized debt, but without considering the restrictions associated with ergodicity.

5.2.1 The Correspondence Principle: Risk-free rate and defaultable private debt

This section presents the quantitative implications of an MIT shock to the model under the restrictions imposed by ergodicity, which is a form of global stability. The present the numerical details, borrowed from Pierri and Seoane (2025), in the Supplementary appendix.

The change in the international risk-free rate operates through 3 channels. First, it increases the risk premium and the probability of default, thereby increasing the mean indebtedness $\mu(B/Y)$ and the standard deviation of debt, as the latter statistic is not scale-invariant. Accordingly, the threshold value of debt, $\bar{B}(Y^{def}, Y_{D1})$ must decrease.

The second channel is connected to the effect of this shock on the *default punishment*. The arguments in the Supplementary Appendix to this section show that an increase in the international risk-free rate reduces the ergodic default punishment y^{def}. The increase in the risk premium mentioned in the first channel and the change in the default punishment generate a hike in the value of debt services, pushing consumption closer to the lowest possible value and thus decreasing the value of repayment in the long-run equilibrium: $V_*^c(\bar{B}, Y_{Di}) = V_*^{def}(Y^{def})$. Thus, $\bar{B}$ also falls for all i. Table 7 only reports $\bar{B}(Y_{D1})$ as this value defines the stable state space in Figure 2. As the upper bound for assets is exogenous, the reduction of $\bar{B}(Y_{D1})$ implies that the support of the ergodic distribution increases, *generating a longer left tail when compared with the benchmark economy*.

The third effect is that as the interest rate increases, the demarcation curves rotate upward, implying more assets tomorrow for the same level of assets today. This is the typical Euler equation effect, as we observe more savings. As during autarky, the country is not allowed to save, higher interest rates and more assets increase the value of continuation in the stationary equilibrium, V_*^c, pushing down $\bar{B}(Y_{D1})$ even further.

The interaction of the 3 mentioned effects (the increase in the risk premium, the change in the endogenous threshold, and the shift of demarcation curves) implies that point A in Figure 2 also moves to the left. This is consistent with Assumption 5, which guarantees the ergodicity of the new equilibrium. In this environment, as noted by Acemoglu and Jensen (2015), a MIT shock implies a change in the ergodic distribution. The results in this paper imply, as suggested by Samuelson's Correspondence Principle, that the differences between the new and the old ergodic distribution can be inferred based on the properties guaranteeing the stability of the equilibrium. As point A and all default thresholds move to the left in Figure 2 after the shock, the new equilibrium can sustain a higher debt level. To see this fact, note that the support of the new ergodic distribution moves to the left, and all demarcation curves move up, implying less unsustainable and more stable or unstable trajectories. As in unsustainable trajectories, the Government defaults on assets; after the shock, there are more positive probability paths with debt.

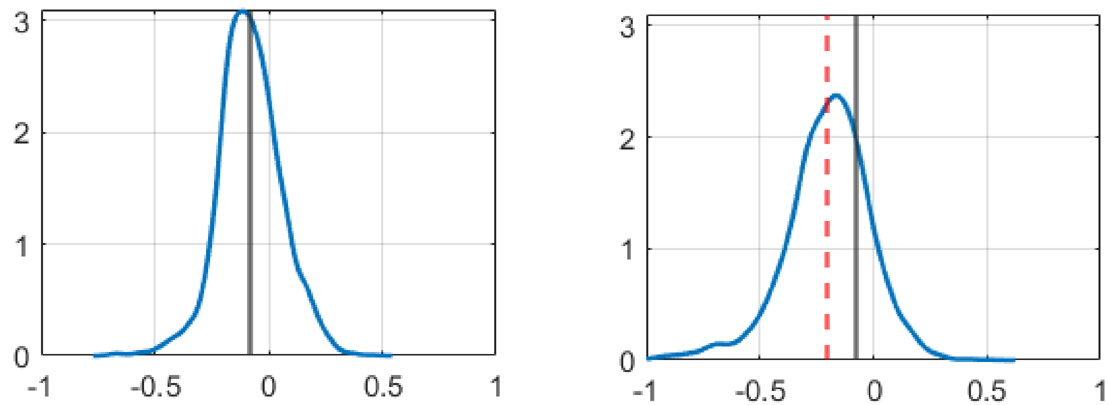
Table 3 contains the changes in the mass allocated to every quantile of the ergodic distribution for the benchmark calibration (BE). Figures 3 supplement this information by plotting the kernel densities of the debt-to-output ratio. The Supplementary Appendix to this section contains further details about the ergodic distribution.

Table 3: Comparative statics of kernels (change in mass)

STDs	-9.83	-4.83	0.30	5.30
Bin	$[-0.80, -0.40)$	$[-0.40, 0.01)$	$[0.01, 0.41)$	$[0.41, 0.81]$
P1-BE	1.13	11.51	-12.45	-0.19

$STDs$ stands for the number of standard deviations of the benchmark distribution between the mean of this distribution ($STD(B/Y) = 0.08$, $\mu(B/Y) = -0.02$) and the left border of each bin. $P1 - BE$ contains the difference in mass at each bin, expressed in percentage points, between the $P1$ distribution as characterized in table 7 and the benchmark BE.

A 100-basis-point hike in the international risk-free rate generates an increase of 11.51 percentage points in the mass associated with the bin that is 4.83 standard deviations to the left of the mean. The mass allocated to the left increases, resulting in a reduction in mean external assets and increased dispersion. Notice that the hike in mass associated with negative extreme values, related to net debt, is six times higher than the decrease in mass observed in the positive extreme values, associated with net assets. Further, the number of standard deviations to the "left" is higher when compared to the values that characterize the extreme "right". *Thus, the increase in the interest rate generates a fat tail associated with extremely high indebtedness.*



(a) Baseline

(b) High R^*

Figure 3: Debt to output ratio distribution

Debt to output ratio distributions for each of the calibrations in table 7. The black vertical line is the mean of the debt-to-output ratio in the baseline economy. The dotted red line is the mean of calibration P1. To compute each density, we removed exclusion periods in the simulations.

5.2.2 Short run simulations: Refined default penalty and Markov shocks

In this subsection, we test the implications of our theoretical results by comparing the model's predictions with those of the literature. In particular, we use the parameters from Kim and Zhang (2012) and solve the model using the operator described in Definition 2 under Assumption 3.

The table below contains the results of solving and simulating the model using operator 2 with Markov shocks.

Table 4 shows the effects of an endogenous punishment in a stationary model: simulations in our model over-estimate (i.e., more current account deficit and more debt than in data) local moments (i.e., between the defaults of 1982 and 2001) as the penalty does not truncate the high-end of the distribution of shocks, lowering the cost of default. Moreover, as shocks are Markov, they are persistent. Thus, the model with a refined default penalty remains in an expansion for a larger fraction of time, even under default, compared to an asymmetric penalty. This is not at

Table 4: Calibration with Markov shocks and an endogenous penalty

Results	Current account / GDP		Net external assets / GDP	
	83-01	83-16	83-01	83-16
Data	-2.4%	-0.8%	-36.5%	-34.0%
Stationary	-3.7%	-0.8%	-49.7%	-36.6%
Kim and Zhang (2012)	-1.2%	N/A	-22.5%	N/A

We borrow the parameters from Kim and Zhang (2012). The Markov process for GDP has an autoregressive coefficient (ρ) of 0.945, and the standard deviation of the residuals in this process (STD_e) equals 0.02. $\theta = 0.1$ and $\beta = 0.97$. N/A stands for Not Available as Kim and Zhang (2012) only targets local moments.

odds with the data, as Argentina, between 2002 and 2005, when the country was under default, grew at an average yearly rate of 8.7%, increasing in every period.

6 Concluding remarks

This paper develops a theory of sovereign default on private external debt that is globally accurate. In our small, open economy, atomistic households borrow abroad in the form of defaultable private claims, while the government decides whether to uphold or suspend repayment. Because borrowing is decentralized, private agents do not internalize the impact of their leverage on the sovereign risk premium, resulting in unstable or unsustainable debt dynamics in the absence of policy intervention. We demonstrate that, with an appropriately refined default punishment, a unique stationary recursive equilibrium exists in a minimal state space, and that the associated Markov process for income and net external assets is globally stable and ergodic. Default occurs when debt reaches an endogenous threshold, preventing the divergence of the debt process and permanently reshaping the stochastic steady state, such that each ergodic distribution features at most one default.

We use the phase diagram representation implied by this equilibrium to characterize global dynamics and identify regions of stable, unstable, and unsustainable debt paths. In this sense, default operates as an endogenous stabilization device that keeps the economy inside the stable state space, rather than as the mechanical endpoint of an explosive trajectory. Calibrating the model to Argentine private external debt and current-account data from 1983 to 2016, we demonstrate that it matches long-run moments for interest payments, default frequency, and the external position, and rationalizes the difference between local moments around the 2001 default episode and global moments computed over the entire sample. A permanent increase in the world interest rate shifts the ergodic distribution of net external assets to the left, raises long-run leverage and default frequency, and increases the time the economy spends in highly indebted states.

Our results suggest that sovereign default on private liabilities can be understood as a macroeconomic stabilization mechanism in environments with decentralized external borrowing, and that ignoring global stability and ergodicity can lead to misleading conclusions about external sustainability and crisis risk. Extending the analysis to richer settings—with production, heterogeneity, nominal frictions, or multiple defaults within a given ergodic distribution—and studying optimal policy tools such as macroprudential regulation or capital controls in this framework are promising directions for future research.

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Appendix

We will show the results in each subsection separately.

Preliminary remarks

The purpose of this subsection is to carefully derive the Euler equation (4), which is the same used by Kim and Zhang (2012). Regarding this equation, there are at least two major differences with respect to the centralized default literature (see, for instance, Arellano (2008)). First, the agent that issues debt/purchases assets does not internalize her portfolio decisions on market prices. This was already noted by Kim and Zhang (2012). Second, the Government chooses to default or repay but does not issue debt. Then, to compute the Euler equation, we must take the derivative with respect to the individual state b_+, not the aggregate state B_+, which only affects the value of repaying $V^c(b, B, y; h)$, not the option value $V(b, B, y; h) = max\{V^c(b, B, y; h), V^{def}(y)\}$. Thus, we can use the standard envelope theorem as we take the derivative with respect to b on $V^c(b, B, y; h)$ and then forward this expression 1 period. To take care of the interaction between default and the marginal value repaying, we use the fact that the interest rate depends only on the aggregate state. Thus, the discontinuities happened at the equilibrium (or aggregate) level.

For the sake of exposition, we first present the relevant value functions. The value for the household of the repaying option for the Government is given by:

$$V^c(b, B, y; h) = \max_{b_+ \geq -\bar{b}} u(F(b, B, y) - b_+) + \beta \mathbb{E}[V(b_+, h(B, y), y'; h)], \tag{10}$$

where $F(b, B, y) = y + bR(B)$ and h is the aggregate law of motion for assets B_+ in equation (9), note that due to the i.i.d structure in assumption 2, the interest rate is not a function of the exogenous shocks. This is clear from equation (1). However, even if we allow for Markov shocks or compute off (recursive) equilibrium paths using the operator presented in definition 2, we can factor the interest rate outside the expectation term in the Euler equation. As we assume the presence of incomplete markets, the return on net assets b_+ must be independent of y_+. Moreover, the interaction between default and private welfare is captured by the derivative of $V^c(b_+, B_+, y_+; h)$

with respect to b_+. The interest rate enters the Euler equation after replacing this derivative using the envelope theorem.

Formally, the general value (i.e., with Markov shocks and outside the recursive equilibrium) of the interest rate affected by default risk is given by:

$$R^*(B,y;h^n) \equiv \\ R^* \left[\sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y',y) \mathbb{I} \left\{ V^c(b_+(b,B,y;h^n), h^n(B,y), y') > V^{def}(y') \right\} \right]^{-1}, \tag{11}$$

where h^n is the n-th iteration using the operator presented in definition 2. Outside the recursive equilibrium, during iterations, we have $b_+(b,B,y;h^n) \neq h^n(B,y)$, but in a recursive equilibrium, we get by definition $b_+(b,B,y;h^*) = h^*(B,y)$. As we model a competitive equilibrium, individual states do not affect the interest rate. Thus, marginal changes in b_+ must not affect the default decision inside the expected option value $\mathbb{E}_y(V)$, which we present below as equation (12), but only the marginal value under repayment (as agents cannot save under default). To enforce this fact, we must set a strict inequality inside the indicator function (i.e., when the Government is indifferent, it defaults). Now, using this notation, we rewrite the right-hand side of equation (4):

$$\mathbb{E}_y[V(b_+(b,B,y;h^n), h^n(B,y), y'; h^n)] = \sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y',y) \\ [\mathbb{I} \left\{ V^c(b_+(b,B,y;h^n), h^n(B,y), y'; h^n) > V^{def}(y') \right\} V^c(b_+(b,B,y;h^n), h^n(B,y), y'; h^n) \\ \mathbb{I} \left\{ V^c(b_+(b,B,y;h^n), h^n(B,y), y'; h^n) \leq V^{def}(y') \right\} V^{def}(y')]. \tag{12}$$

Note that V^{def} and the first term in the second line of equation (12) do not change with b_+. The latter is based on the absence of partial default and the lack of savings during autarky, and the latter is based on the price-taking assumption. Then, we can rewrite equation 4 as:

$$
\begin{aligned}
&u'\left(F(b,B,y)-b_{+}(b,B,y;h^{n})\right)\geq \\
&\beta R^{*}(B,y;h^{n})\sum_{y'\in[Y_{LB},Y_{UB}]}\pi(y',y)\mathbb{I}\left\{V^{c}(b_{+}(b,B,y;h^{n}),h^{n}(B,y),y')>V^{def}(y')\right\} \\
&u'\left(F(b_{+}(b,B,y;h^{n}),h^{n}(B,y),y')-b_{+}(b_{+}(b,B,y;h^{n}),h^{n}(B,y),y';h^{n})\right).
\end{aligned}
\tag{13}
$$

The second and the third lines in equation (13) capture the interaction between default and the marginal utility of consumption, which follows after applying the envelope theorem to V^c. Critically, as we are in a model with centralized default but decentralized debt, changes in b_+ do not affect the indicator functions in $\mathbb{E}_y(V)$. Thus, the concerns in Clausen and Strub (2020) do not arise. Even though there are incomplete markets and the interest rate can be factored out of the expectation in the Euler equation, the number of future exogenous states the Government repays affects consumption decisions today through the marginal utility of future consumption in the Euler equation. However, as agents do not internalize the effects of their decisions on market prices, the model predicts over-borrowing, which gives a rationale for the intervention of a benevolent Government.

Finally, note that the second line in equation (13) can be written as:

$$
R^{*}\sum_{y'\in[Y_{LB},Y_{UB}]}\pi(y',y)\frac{\mathbb{I}\left\{V^{c}(b_{+}(b,B,y;h^{n}),h^{n}(B,y),y')>V^{def}(y')\right\}}{\sum_{y'\in[Y_{LB},Y_{UB}]}\pi(y',y)\mathbb{I}\left\{V^{c}(b_{+}(b,B,y;h^{n}),h^{n}(B,y),y')>V^{def}(y')\right\}}. \tag{14}
$$

To derive the existence of a recursive equilibrium, we must enforce, as in Coleman (1991), that $b_+ = B_+$. Thus, equation (14) must be restricted to satisfy:

$$
\begin{aligned}
&R^{*}\sum_{y'\in[Y_{LB},Y_{UB}]}\pi(y',y) \\
&\frac{\mathbb{I}\left\{V^{c}(b_{+}(b,B,y;h^{n}),b_{+}(b,B,y;h^{n}),y')>V^{def}(y')\right\}}{\sum_{y'\in[Y_{LB},Y_{UB}]}\pi(y',y)\mathbb{I}\left\{V^{c}(b_{+}(b,B,y;h^{n}),b_{+}(b,B,y;h^{n}),y')>V^{def}(y')\right\}},
\end{aligned}
\tag{15}
$$

In equation (15) we are imposing $B' = b_+(b, B, y; h^n)$. Below, when we prove the existence of a stationary equilibria, we guarantee that this holds for n sufficiently large. Using this equation, we can define the equilibrium interest rate $R(B', y')$ when $b = B$:

$$\begin{aligned} R(B', y') &\equiv R^* \pi(y', y) \\ &\frac{\mathbb{I}\left\{V^c(b_+(B, B, y; h^n), b_+(B, B, y; h^n), y') > V^{def}(y')\right\}}{\sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y', y) \mathbb{I}\left\{V^c(b_+(B, B, y; h^n), b_+(B, B, y; h^n), y') > V^{def}(y')\right\}}. \end{aligned} \tag{16}$$

Under the restriction in equation (15) and using the definition in equation (16), as V^c is increasing in b (which is a standard result due to Lucas et al. (1989)), $R(B', y')$ is decreasing in B' for any y'. This property will be essential to prove lemma 2 and to preserve the order structure in the operator presented in definition 2. Using $R(B', y`)$ we can write the Euler equation in (13) as:

$$\begin{aligned} u'\left(F(b, B, y) - b_+(b, B, y; h^n)\right) &\geq \sum_{y' \in [Y_{LB}, Y_{UB}]} \pi(y', y) R(B', y') \\ u'\big(F(b_+(b, B, y; h^n), B', y') &- b_+(b_+(b, B, y; h^n), B', y'; h^n)\big), \end{aligned} \tag{17}$$

Both Euler equations (13) and (17) are equivalent to the one found in Kim and Zhang (2012) and to equation (4) in the body of the paper.

It is clear from the definition of $R(B', y')$ that, even though there are incomplete markets, the return on assets is affected by shocks y as noted by Zame (1993). In particular, in a recession, y is low. Due to the Markov structure of π, it is possible that the probability of getting low shocks tomorrow y' would be high. Thus, the denominator in equation (16) goes down, raising the ex-ante return on net assets. Thus, default provides a natural hedge as in Zame (1993). In the i.i.d. case, we can't connect a recession today with an expected recession tomorrow. However, we can still use the same argument because in (16), shocks today y affect the value of repayment V^c: ex-ante, the return on net assets would be high when a recession occurs.

Proofs

The space of candidate consumption functions is given by Equation (18), presented below:

$$C(\mathbb{B} \times Y) = \begin{Bmatrix} 0 \leq C(B, y) \leq F(B, y) \\ 0 \leq C(B', y) - C(B, y) \leq F(B', y) - F(B, y) \; if \; B' \geq B \end{Bmatrix} \tag{18}$$

We must show that A maps $C(\mathbb{B} \times Y)$ into itself. As we can iteratively update b_+ using $F(B, y) - Ac(B, y)$, this will suffice to show that c and b_+ are both weakly increasing in b when $b = B$ for any $y \in Y$ and h, the first part of Lemma 2.

Take $c \in C(\mathbb{B} \times Y)$. Let $B'(B, y) = y + R(B)B - Ac(B, y)$. *We will refer to A as the Coleman-Reffett operator*. Thus, for any $\hat{c}, \tilde{c} \in C(\mathbb{B} \times Y)$, with $\hat{c} \leq \tilde{c}$, we must show that $A\hat{c} \leq A\tilde{c}$ and $\hat{B}' \leq \tilde{B}'$. To do so, notice that:

$$u'(A\hat{c}(B, y))) = \beta E \left[u' \left(\hat{c} \left(\hat{B}', y' \right) \right) R \left(\hat{B}', y' \right) \right] \geq \beta E \left[u' \left(\tilde{c} \left(\hat{B}', y' \right) \right) R \left(\hat{B}', y' \right) \right] \geq$$
$$\beta E \left[u' \left(\tilde{c} \left(\tilde{B}', y' \right) \right) R \left(\hat{B}', y' \right) \right] \geq \beta E \left[u' \left(\tilde{c} \left(\tilde{B}', y' \right) \right) R \left(\tilde{B}', y' \right) \right] = u'(A\tilde{c}(B, y))).$$

Note that the first inequality follows from $\hat{c} \leq \tilde{c}$, the second from the optimality of consumption, and the third because R decreases in B by assumption. Note that the last inequality implies $\hat{B}' \leq \tilde{B}'$ and the first and the last terms together imply $A\hat{c} \leq A\tilde{c}$ as desired. Thus, $AC(\mathbb{B} \times Y) \subseteq C(\mathbb{B} \times Y)$, which in turn implies that any fixed point of A is a good candidate for h.

It remains to show that either c or b_+ is strictly increasing. As the Government is benevolent, any increase in B implies that $R(B)B$ increases. [19] Thus, with more resources, households choose to increase current or future consumption, implying that either c or b_+ strictly increases.

Proofs for section 3

We now turn to the proof of theorem 1. We will state the proof for the case with no re-entry (i.e., $\theta = 0$). Then, we show that we can extend the results for the general case. We will need

[19] See section 2.1 for a related discussion

an additional mild assumption on $y^{def}(y)$. This is only for clarity as we want the proof structure to be as close as possible to the ones in Aguiar and Amador (2019) and Coleman (1991). Under this assumption, we can show that the default threshold is negative for all possible shocks (i.e., $\bar{B}(y) < 0$ for any $y \in Y$), a fact that allows us to write the Aguiar-Amador operator in a tractable way. We need a modified version of problem 2 to state the assumption:

$$V^{NA}(b, y; R^*) = Max_{b_+ \geq 0}\, u(y + R^* b - b_+) + \beta E[V^{NA}(b_+, y'; R^*)]. \tag{19}$$

We call the value function V^{NA} as it represents a situation in which households have no access (NA) to debt. Problem 19 is a standard savings problem. To guarantee that any possible debt threshold is negative, we impose the following assumption, which contains the mentioned additional restriction on y^{def}.

Assumption 6 (Negative debt thresholds)**.** *Assume that $\beta R^* < 1$ and additionally:*

$$V^{NA}(0, y; R^*) \geq u(y^{def}(y)) + E_1(\textstyle\sum_t \beta^t u(y^{def})) \text{ for all } y \in Y$$

Lemma 3 (Negative debt thresholds)**.** *Under assumptions 1, 2 and 6, $\bar{B} < \vec{0}$, where $\vec{0} \in \mathbb{R}^Y$.*

Proof. Follows immediately from lemma 1 (i) in Aguiar and Amador (2019). □

We are now in position to define formally the Aguiar-Amador operator.

Definition 3 (Utility maximization problem (UMP))**.**

$$V^c_{n+1,*}(B, y) = u(c_{n+1}(B, y)) + \beta E max \left\{ V^c_{n+1,*}(b_{+,n+1}(B, y), y'), V^{def}(y') \right\}$$

Subject to

$$b_{+,n+1}(B, y) + c_{n+1}(B, y) = y + BR^* \left[\mathbb{I}(B > 0) + \mathbb{I}(B \leq 0) \sum_{y \in Y} \pi(y) \mathbb{I}\left(V^c_{n+1,*}(B, y) \geq V^{def}(y) \right) \right]^{-1}.$$

Note that $c_{n+1} = Ac_n$ defines the connection between the Coleman-Reffett operator and the Aguiar-Amador operator to be defined below. We use lemma 3 to write the equilibrium interest rate at iteration $n+1$. We now define the dual of the UMP, the expenditure minimization problem. In Aguiar and Amador (2019) the duality relation $\nu = V^c_{n+1,*}(B_{n+1,*}(\nu, y), y)$ was stated without proof [20]. We proceed in the same way. However, we must explicitly write the EMP to show the equivalence between it and the Aguiar-Amador operator.

Definition 4 (Expenditure Minimization Problem (EMP))**.**

$$B_{n+1,*}(\nu, y) = [(b_{+,n+1} + c_{n+1})(\nu, y) - y]\, R^{-1}$$
$$\left[\mathbb{I}\left(B_{n+1,*}(\nu, y) > 0\right) + \mathbb{I}\left(B_{n+1,*}(\nu, y) \le 0\right) \sum_{s \in Y} \pi(s) \mathbb{I}\left(\nu(s) \ge V^{def}(s)\right)\right]$$

Subject to

$$\nu = V^c_{n+1,*}(B_{n+1,*}(\nu, y), y), \quad \nu(s) = V^c_{n+1,*}(b_{+,n+1}(\nu, y), s) \quad for \quad s \in Y \tag{20}$$

$$\nu = u(c_{n+1}(B_{n+1,*}(\nu, y), y)) + \beta E max\left\{V^c_{n+1,*}(b_{+,n+1}(\nu, y), y'), V^{def}(y')\right\} \tag{21}$$

The equivalence between the EMP and the UMP is automatic, given the results in lemma 2. It turns out that EMP is not a contraction. However, we prove that an equivalent representation to EMP exists, called *optimal contract (OC)*, which we will show is well defined. This operator will allow us to iterate in j and find a fixed point for the pair $(B_{n+1,*}, V^c_{n+1,*})$ using equation (20).

Definition 5 (Optimal Contract (OC) and the Aguiar-Amador operator ($\mathbb{T}$))**.**

$$\mathbb{T} f_j(\nu, y) = SUP_{\{g_+(y')\}_{y' \in Y}} [(b_{+,n+1} + c_{n+1})(\nu, y) - y]\, R^{-1}$$
$$\left[\mathbb{I}\left(B_{n+1,*}(\nu, y) > 0\right) + \mathbb{I}\left(B_{n+1,*}(\nu, y) \le 0\right) \sum_{s \in Y} \pi(s) \mathbb{I}\left(V^c_{n+1,*}(b_{+,n+1}(\nu, y), s) \ge V^{def}(y)\right)\right]$$

Subject to

[20] See page 850.

$$\nu = u(c_{n+1}(B_{n+1,*}(\nu, y), y)) + \beta E max \left\{ g_+(y'), V^{def}(y')) \right\} \tag{22}$$

$$b_{+,n+1}(\nu, y) = f_j(g_+(y'), y') \quad \textit{for all} \quad y' \in Y \quad \textit{such that} \quad g_+(y') \geq V^{def}(y') \tag{23}$$

A fix point $\mathbb{T}$, f, satisfies $f = B_{n+1,*}$ and by equation (21) we can recover $V^c_{n+1,*}$, which is given by the pre-image of $B_{n+1,*}(., y)$ for each $y \in Y$. Intuitively, definition 5 gives the Government an additional instrument g_+ in order to enforce minimum expenditure f. In this sense, the maximal elements $\hat{g}_+(y)$ for all $y \in Y$ of a fixed point of $\mathbb{T}$, f, is a promised utility that sustain $(B_{n+1,*}, V^c_{n+1,*})$. Assuming that $\mathbb{T}$ has a fixed point, the next lemma shows that it is equivalent to $B_{n+1,*}$, which in turn has a unique value associated value function for the UMP, $V^c_{n+1,*}$. We later show that $\mathbb{T}$ has at least one non-trivial fixed point.

Lemma 4 (Optimal contract and expenditure minimization problem)**.** *Under assumptions 1, 2 and 6, any fixed point* $\mathbb{T}f$=f*, if it exists, satisfies:* $f = B_{n+1,*}$.

Proof. We will show this lemma in 2 steps.

We first show that EMP is a fixed point of $\mathbb{T}$. Let $\mathbb{V}$ the set of possible values of $V^c_{n+1,*}$ for all $\nu, y \in \mathbb{V} \times Y$. Because of lemma 1 and equation (6), $\mathbb{V}$ is compact. Take and arbitrary pair $\nu_0, y_0 \in \mathbb{V} \times Y$. This pair defines in turn a triple $b_{+,n+1}(\nu_0, y_0), c_{n+1}(\nu_0, y_0)$ and $B_{n+1,*}(\nu_0, y_0)$ from the EMP. Set $\hat{g}_+(y') = V^c_{n+1,*}(b_{+,n+1}, y')$ for all $y' \in Y$. We claim that, given that the objective function of EMP and OC are the same, setting $f = B_{n+1,*}$ suffices to show that $b_{+,n+1}$ and c_{n+1} satisfies equations (22) and (23). Equation (22) is satisfied by the definition of $V^c_{n+1,*}$ in equation (20). Equation (23) follows from the recursive structure given by private optimization in equation (4) and the equivalence between EMP and UMP [21]. As in Aguiar and Amador (2019), when $V^c_{n+1,*} < V^{def}$, $\hat{g}$ is any feasible function in $\mathbb{V}$. As the preceding argument can be done for any $(\nu_0, y_0) \in \mathbb{V} \times Y$, $b_{+,n+1}$ and c_{n+1} are feasible in OC which then implies $\mathbb{T}f \geq B_{n+1,*}$ or equivalently OC $\supseteq$ EMP.

We now show that a fixed point of $\mathbb{T}$ *is an EMP*. As $\mathbb{T}$ is assumed to have a fixed point we can use it as a candidate for $B_{n+1,*}$. Note then that the objective functions of EMP and OC are equal so, we

[21]See Aguiar and Amador (2019) page 866.

must only verify equations (20) and (21). The objective function together with equation (20) form a system with $card(Y)$ unknowns for each ν given $\hat{g}_+(y')$ for some $y' \in Y$. As we are assuming that $\mathbb{T}$ has a fixed point, this system has at least 1 solution, so equation (20) is satisfied. Equations (22) and (23) together imply that (21) is satisfied. □

We now show that $\mathbb{T}$ has a fixed point which is an increasing function of ν, which in turn assures that: a) there is a well defined sequence of functions f_j generating a pair $(B_{n+1,j}, V^c_{n+1,j})$, b) $\mathbb{T}$ has a fixed point f which generates $(B_{n+1,*}, V^c_{n+1,*})$.

For that we need the following theorem.

Theorem 4 (Existence of a lower fixed point, Mirman, et. al. Proposition 5)**.** *Let $\mathbb{F}$ be a poset and $h : \mathbb{F} \longrightarrow \mathbb{F}$ be order continuous. Assume that there is an element $a \in \mathbb{F}$ such that i) $a \leq h(a)$ and ii) every countable chain in $\mathbb{F}$ has a supremum. Then, h has a fixed point and the sequence of elements in $\mathbb{F}$ generated iteratively using h and starting in a, converges to the infimum of the set of fixed points.*

Lemma 5 (Existence of a fixed point in the Aguiar-Amador operator)**.** *Under assumptions 1, 2 and 6, $\mathbb{T}$ has a fixed point $\mathbb{T}f$=f.*

Proof. As the monotonicity of $\mathbb{T}$ is straightforward and bounds are uniform, order continuity is rather immediate. The maximization clause is essential to guarantee that the operator maps a carefully selected initial condition up. We now prove this claim formally. To serve this purpose, we need the following iterative version of OC:

$$f_{j+1}(\nu, y) = SUP_{\{g_+(y')\}_{y' \in Y}}$$
$$[(b_{+,n+1} + c_{n+1})(\nu, y) - y]\, R^{-1}$$
$$\left[\mathbb{I}\left(f_j(\nu, y) > 0\right) + \mathbb{I}\left(f_j(\nu, y) \leq 0\right) \sum_{s \in Y} \pi(s) \mathbb{I}\left(V^c_{n+1,j}(f_j(\nu, y), s) \geq V^{def}(y)\right)\right]$$

Subject to

$$V^c_{n+1,j}(f_j(\nu, y), s) = u(c_{n+1}(f_j(\nu, y), s)) + \beta E max\left\{g_+(y'), V^{def}(y'))\right\} \quad s \in Y \tag{24}$$

$$b_{+,n+1}(\nu, y) = f_j(g_+(y'), y') \quad \text{for all} \quad y' \in Y \quad \text{such that} \quad g_+(y') \geq V^{def}(y') \tag{25}$$

Let $\mathbb{F}$ be the space of real-valued bounded measurable increasing functions mapping $\mathbb{V} \times Y$ to $\mathbb{R}$. This set is a poset, and every countable chain in it has a supremum [22]. Take any $f_j \in \mathbb{F}$ with $f_0 = INF(\mathbb{F})$ and $V^c_{1,0}$ the initial condition in assumption 3. We can guarantee the existence of an infimum of $\mathbb{F}$ using lemma 1. The results in Aguiar and Amador (2019) imply that $\mathbb{T}f_j$ is also increasing [23]. To show that $\mathbb{T}$ is order continuous, note that the objective function in OC is bounded by lemma 1. Then, we have: $SUP \quad \mathbb{T}f_0 \leq SUP \quad \mathbb{T}f_1 = SUP \quad \mathbb{T}^2 f_0 \leq SUP \quad \mathbb{T}f_2, ..., \quad lim_n \quad SUP \quad Tf_n = SUP \quad lim_n \mathbb{T}^n f_0 = SUP \quad T(lim_n \mathbb{T}^n f_0) = SUP \quad T(lim_n f_n)$. Thus, $lim_n \quad SUP \quad Tf_n = SUP \quad T(lim_n f_n)$ [24] implies that the operator is order continuous. By setting $a = INF(\mathbb{F})$, by the definition of $\mathbb{T}$ we know that $a \leq \mathbb{T}a$

The desired result then follows. □

We will now prove theorem 1. We will use definition 1 and the iterative procedure in definition 2. To complete the proof, we need an additional result borrowed from Coleman (1991):

Theorem 5 (Existence of an upper fixed point, Coleman (1991), page 1098)**.** *An order continuous monotone operator A mapping a non-empty, partially ordered compact set $\mathbb{C}$ into itself, with an element c_0 such that $A(c_0) \leq c_o$, has a fixed point which can be computed by successive approximations $A^n(c_0)$ and converges to a maximal fixed point in the set $(c \leq c_o, c \in \mathbb{C})$.*

Note that theorems 4 and 5 can be used to find $c_*(\underline{c}_0)$ and $c_*(\overline{c}_0)$ in theorem 1. We will prove the result using lemmas 2, 4, and 5.

Proof of Theorem 1. Note that if $c_0 = SUP(\mathbb{C})$, then lemma 1 imply that $c_0(B, y) = Y_{UB} + R_{UB}B_{UB} - B_{LB}$ for all $B, y \in \mathbb{B} \times Y$. By assumption 3, $R_0 = R^*$ and thus equation (4) characterizes a standard savings problem. As $card(Y) > 1$, we know that $c_1 = A(c_0) \leq c_0$. Moreover, as problem 2 is a maximization problem and preferences satisfy Inada, we know that if $c_0 = INF(\mathbb{C})$, we have $c_1 = A(c_0) \geq c_0$. Note that, as consumption is uniformly bounded below and away from zero

[22] This last property is easily achieved as long as shocks are finite. I want to thank Kevin Reffett for pointing this out to me.

[23] See lemma 8.

[24] By the definition of OC, as it is a supremum, there can be at most one element f_{j+1} for each ν, y. Thus, $lim_n \quad SUP \quad Tf_n = SUP \quad lim_n \mathbb{T}^n f_0$.

by Lemma 1 and V^{def} is finite, $A(c_0)$ is well defined in this case. So we can set c_0 in either the supremum or the infimum of $\mathbb{C}$.

Take V_0^c, V_0^{def} from assumption 3. As $c \in \mathbb{C}$, under standard results, equation 5 implies that R_1 is monotone. Then under Lemma 2, the Coleman-Reffett operator implies that $c_1, b_{+,1}$ are monotone. Then, using equations 6 and 7 and lemmas 4 and 5, $B_{1,*}, V_{1,*}^c$ are well defined. Moreover, as $B_{1,*}$ is a fixed point of $\mathbb{T}$, it is increasing. Thus, as $\nu = V_{1,*}^c(y, B_{1,*}(\nu, y)) = V_{1,*}^c(y, f_{1,*}(\nu, y))$, by equation 5, R_2 is also monotone. Continuing with this logic, we can construct a sequence of ordered functions $SUP\ Ac_0 \leq SUP\ Ac_1 = SUP\ A^2c_0 \leq SUP\ Ac_2, ..., .$ As $\mathbb{C}$ is compact by Lemma 1, we can use the same argument as in Lemma 5 to show that A is order continuous. As $\mathbb{C}$ is compact, we know that it is countably chain-complete. Thus, under theorems 4 and 5, A has 2 ordered fixed points, depending on the initial condition c_0.

Until now V^{def} was assumed to have the form: $V^{def}(y) = u(y^{def}(y)) + \beta E(V^{def}(y))$. That is, there is no re-entry (i.e., $\theta = 0$). However, equation (7) assumes that $\theta \in (0, 1)$. We now extend the argument for a model with re-entry. The outside option with and without re-entry are connected as follows [25]:

$$\tilde{V}^{def}(y) = V^{def}(y) + \gamma v_0, \quad \text{where } \gamma \equiv \frac{\theta\beta}{1-\beta(1-\theta)}$$

Where $v_0 \equiv E(V_{n+1,*}^c(0, y) - V^{def}(y))$. Then, the UMP has the form:

$$V_{n+1,*}^c(B, y; v_0) = u(c_{n+1}(B, y)) - (1-\beta)\gamma v_0 + \beta E max \left\{ V_{n+1,*}^c(b_{+,n+1}(B, y), y'; v_0), V^{def}(y')) \right\}$$

Subject to

$$b_{+,n+1}(B, y) + c_{n+1}(B, y) = y + BR^* \left[\mathbb{I}(B > 0) + \mathbb{I}(B \leq 0) \sum_{y \in Y} \pi(y) \mathbb{I}\left(V_{n+1,*}^c(B, y; v_0) \geq V^{def}(y) \right) \right]$$

[25] A detailed computation of the steps required to connect both equations is available under request.

The following argument based on a modified version of $\mathbb{T}$ shows that there is a unique $v_{0,*}$ which satisfies: $v_{0,*} = E(V^c_{n+1,*}(B,y;v_{0,*}) - V^{def}(y))$. Let f_0 be the adequate initial condition based on theorem 4. Let $a < b$ be 2 possible values for v_0. Let $\mathbb{T}(. \mid v_0)$ be given by:

$$\mathbb{T}(f_j(\nu,y) \mid v_0) = SUP_{\{g_+(y')\}_{y' \in Y}} \left[(b_{+,n+1} + c_{n+1})(\nu,y) - y\right] R^{-1}$$
$$\left[\mathbb{I}\left(f_j(\nu,y) > 0\right) + \mathbb{I}\left(f_j(\nu,y) \le 0\right) \sum_{s \in Y} \pi(s) \mathbb{I}\left(V^c_{n+1,j}(f_j(\nu,y),s) \ge V^{def}(s) + \gamma v_0\right)\right]$$

Subject to

$$V^c_{n+1,j+1}(f_j(\nu,y),s) = u(c_{n+1}(f_j(\nu,y),s)) - (1-\beta)\gamma v_0 + \beta E max\left\{g_+(y'), V^{def}(y'))\right\} \quad s \in Y \quad (26)$$

$$b_{+,n+1}(\nu,y) = f_j(g_+(y'),y') \quad \text{for all} \quad y' \in Y \quad \text{such that} \quad g_+(y') \ge V^{def}(y') \quad (27)$$

Note that $\mathbb{T}(. \mid a) \ge \mathbb{T}(. \mid b)$. Then, $f_{1,a} = \mathbb{T}(f_0 \mid a) \ge \mathbb{T}(f_0 \mid b) = f_{1,b}$. Then applying $\mathbb{T}(. \mid a)$ to both sides, we get: $f_{2,a} = \mathbb{T}^2(f_0 \mid a) = \mathbb{T}(f_{1,a} \mid a) \ge \mathbb{T}(f_{1,b} \mid a) \ge \mathbb{T}(f_{1,b} \mid b) = \mathbb{T}^2(f_0 \mid b) = f_{2,b}$, where the first inequality follows from the monotonicity of $\mathbb{T}(. \mid a)$ and the second one by the fact that $a < b$. Continuing with this logic, we obtain: $f_{*,a} \ge f_{*,b}$, which shows that any fixed point of $\mathbb{T}(. \mid v_0)$ is decreasing in v_0. Then, using the equivalence between EMP and UMP, the arguments in Aguiar and Amador (2019) [26] shows that $v_{0,*} = E(V^c_{n+1,*}(B,y;v_{0,*}) - V^{def}(y))$ as desired.

Now it remains to be shown that any fixed point can be used to construct a candidate policy h. Let $b_{+,*}(B,y) = y + R_*(B)B - c_*(B,y)$, where R_* is the interest rate using $V^c_{*,*}, V^{def}_{*,*}, c_*$ for the model with re-entry. Let θ_i a realization from a uniform $[0,1]$ distribution. Then, we have:

$$h(B,y) =$$
$$\mathbb{I}\left\{b_{+,*}(B,y) < \overline{B}(y)\right\} \left(\mathbb{I}\left\{\theta_i \le \theta\right\} b_{+,*}(0,y) + \mathbb{I}\left\{\theta_i > \theta\right\} 0\right) + \mathbb{I}\left\{b_{+,*}(B,y) \ge \overline{B}(y)\right\} b_{+,*}(B,y)$$

[26] See page 861.

$$c(B,y) = \mathbb{I}\left\{b_{+,*}(B,y) < \overline{B}(y)\right\}\left(y^{def}(y)\right) + \mathbb{I}\left\{b_{+,*}(B,y) \geq \overline{B}(y)\right\}\left(y + R_*(B)B - h(B,y)\right)$$

□

Proofs for section 3.2

To prove uniqueness, we rely on theorem 11 in Coleman (1991) (see page 1102). We need the two conditions in assumption 4. The restrictions on $\bar{b}$ and on $R(\bar{b})$ guarantee that there exists a strictly positive lower bound for consumption that is preserved after one iteration under the Coleman-Reffett operator. Formally, given any two different positive fixed points c_1, c_2, it is possible to choose two points, $\bar{b} \leq B_1 \leq B_0$ [27], together with a scalar $0 < t < 1$, such that $c_1(B,y) \geq tc_2(B,y)$, with equality for some (B,y), and $c_1(B,y) \geq A(tc_2)(B,y)$ for all $B \geq B_1$, and all y. This property is called B_0-monotonicity. Then, we will extend the pseudo-concavity of preferences to the Coleman-Reffett operator (i.e., $A(tc) > tA(c)$). Intuitively, pseudo-concavity implies that agents smooth consumption as they can improve a reduction in consumption tc with respect to a linear operator (i.e., $A(tc) = tA(c)$). We use these two properties (B_0-monotonicity and pseudo-concavity) to show, by contradiction, that there can be at most one fixed point. We rely on a standard result as in Coleman (1991). If there is a uniform and robust lower bound for consumption, which we guarantee by proving B_0-monotonicity, it suffices to add sufficient curvature to the policy function, using the pseudo-concavity of preferences, to prove uniqueness.

Proof of Theorem 2. Let $\underline{c}_* \leq \overline{c}_*$ be the 2 candidate fixed points in theorem 1. Take α such that: $\underline{c}_*(B,y) \geq \alpha\overline{c}_*(B,y)$ for all $B, y \in \mathbb{B} \times Y$ and $\underline{c}_*(B,y) = \alpha\overline{c}_*(B,y)$ for some B, y. Note that this equality is possible as consumption is bounded below and away from zero. Then, as assumption 4 implies that u is pseudo-concave and that we can eliminate (trivial) equilibria in the boundary of the set, theorem 11 in Coleman (1991) implies: $\underline{c}_*(B,y) = A(\underline{c}_*(B,y)) \geq A(\alpha\overline{c}_*(B,y)) > \alpha A(\overline{c}_*(B,y)) = \alpha\overline{c}_*(B,y)$ for all B, y. Note that the last equality implies $\underline{c}_*(B,y) > \alpha\overline{c}_*(B,y)$ for all B, y, which is a contradiction as $\underline{c}_*(B,y) = \alpha\overline{c}_*(B,y)$ for some B, y. □

[27] Note that, contrarily to Coleman (1991), we can choose the first inequality to be weak as we prove that R is finite for any B in the compact state space of the model.

Supplementary Appendix

Supplementary material for section 3.2

It turns out that if we restrict the number of possible distinct values that y can take to be finite, we can prove the existence of an ergodic probability measure. Using equation (9) and restricting assumption 3.4 such that $y^{def}(y) = y^{def}$ for all $y \in Y$ we can construct a *point* z_* which the process hits with positive probability starting from any initial condition. This point will be called *atom* and belongs to Z. The discussion below shows how z_* creates an orbit which endows the dynamical system with a recurrent and connected structure, which in turn implies that: i) there will be a unique (and thus ergodic) invariant measure for each atom, ii) the stochastic process represented by (Z, P_φ) is globally stable. Note importantly, this implies *that there could be at most 1 default for each stable distribution, associated with y^{def}, which in turn implies that that this type of events are so extreme that generate a change in the entire stable distribution of the economy.*

Once we find z_*, we construct a stable state space. That is, any meaningful (i.e. with positive measure) subset of this state space will be hit by the process in finite time. This property, called irreducibility, guarantees the uniqueness and ergodicity of the process together with the global stochastic stability of the process. If we allow for discontinuous equilibrium function φ, we can construct a phase diagram such that the process jumps to the atom every time there is a default. The results in Meyn and Tweedie (1993) give us the tools to prove all the intermediate steps required to go from the existence of a SCE to its ergodicity.[28]

Supplementary Appendix for section 4

This contains further implications of the theoretical results in the model.

I) Severity of the crisis and permanent effects on the stable distribution. If we measure the severity of the crises as the difference between the average detrended GDP (i.e., $\mathbb{E}(y)$) and the level of activity after default and during exclusion (i.e., y^{def}), the figure above can be used to illustrate the effects of default on the long run distribution of the model, and consequently on key observed unconditional moments. Let us compare 2 economies, i, j, that only differ in

[28] See Meyn and Tweedie (1993), chapters 5, 8 and 10 for a detailed discussion of the implications of the existence of an atom for the existence of an invariant probability measure.

the severity of the crises with $[\mathbb{E}(y_i) - y_i^{def}]/\mathbb{E}(y_i) > [\mathbb{E}(y_j) - y_j^{def}]/\mathbb{E}(y_j) > 0$. In economy i, the most affected one, point 3 will be closer to A in figure 1. Note that this last point is the same in both economies as definition 2, which is used to construct the non-stochastic steady state, is independent of the default decision. As the process is ergodic, all the points in the state space, characterized by $[\bar{B}(y_{LB}), B_{UB}]$, are hit with positive probability starting from any initial condition. If the crises is more severe, the support of the stable distribution increases, i.e. $[\bar{B}(y_{i,LB}), B_{i,UB}] \supset [\bar{B}(y_{j,LB}), B_{j,UB}]$. If this is case it is likely that the most affected country:

- has a smaller level of assets on average (i.e. $\mathbb{E}(B; \mu_{i,*}^{def}) < \mathbb{E}(B; \mu_{j,*}^{def})$). That is, a more serious crises is associated with a higher level of net external private debt, and,
- as the support is bigger, the variance of the distribution of debt increases. As the interest rate spread $R(B) - R^*$ is monotonic in B, we will observe a higher and more volatile spread *even after the default occurs*.

We study whether this intuition is right for a relevant calibration in the next section. As y_i^{def}, y_j^{def} are endogenous as well as point 3, we need to solve the model for different values of the deep parameters and compute the effect on the threshold $\bar{B}(y^{def})$, the contours $b_+(., y)$ and the ergodic statistics $\mathbb{E}(B; \mu_{i,*}^{def})$, $\mathbb{E}(B; \mu_{j,*}^{def})$), among others.

II) Modelling countries with no default but with default risk. As we show that μ_*^{def} is ergodic, we know that $\sum f(z)/N \to \mathbb{E}(z; \mu_*^{def})$ for any value of y^{def} even if it has never been observed. Thus, we can design a calibration or estimation procedure to recover the value of GDP that would be observed if the country decides to default, even if this event has never been observed recently in history. Thus, it is possible to use a default model to explain the risk premium during 2008 for countries like Spain or Portugal, which experienced a hike in this variable without actually defaulting.

Supplementary material for Section 5

We target yearly interest payments with respect to GDP, $(R(B)B)/Y$, and the frequency of default using β and θ. Table 5 shows the moments in the data and those implied by the model.

Table 5: Results

Variable	$(R(B)B)/Y$*	Def. freq.*	B/Y	CA/Y	$C.V.(CA/Y)$
Data	-0.6%	3.0%	-1.4%	-0.8%	3.6
Model	-0.6%	2.4%	-2.0%	-1.3%	4.2

Note: * denotes moments that are matched using the simulated method of moments. The rest of the statistics are non-targeted moments. $(R(B)B)/Y$ are (yearly) interest payments of private external debt with respect to GDP. "Def. freq." is the default frequency for events preceded by 19 years (between 1983 and 2001) of open access to the international credit markets. B/Y are yearly capital payments (i.e., amortizations) of foreign private debt over GDP. CA/Y is the current account of GDP, and C. V. is the coefficient of variation of CA/Y, its standard deviation divided by its mean.

Table 5 also shows three non-targeted moments: the yearly capital payments with respect to GDP, B/Y, the current account to GDP, CA/Y, and its volatility, $C.V.(CA/Y)$. [29] As can be seen, the model replicates these non-targeted moments well.

Note that matched moments are global (i.e., include observations after default). As discussed, the baseline calibration relies on i.i.d. instead of Markov shocks. As in the latter case the equilibrium is stationary but not ergodic, we can only target local moments using Markov shocks. As an extension, in subsection 5.2.2, we calibrate the model using Markov shocks.

Table 6 presents the parameter values calibrated in our model and those of the related references for comparison.

Table 6: Parameters

Parameter	Value	Kim and Zhang (2012)	Arellano (2008)	Description
σ	2.0	2.0	2.0	Risk aversion param.
θ*	0.0725	0.10	0.28	Re-entry prob.
β*	0.935	0.97	0.953	Discount factor
ρ_e	0.001	0.945	0.945	Persis. (endowment)
STD_e	0.02	0.02	0.02	St. dev. (endowment)
r^*	1.7%	1.7%	1.7%	Net risk-free interest rate

Note: the second column contains the values of the parameters used in this paper as a benchmark calibration. The third and fourth columns contain the analogous set of parameters in Kim and Zhang (2012) and Arellano (2008), respectively. * denotes parameters used in the simulated method of moments, except for the interest rate, as r^* is the standard notation for the international risk-free rate borrowed from the literature. The remaining parameters, as seen in columns 3 and 4, are borrowed from the literature. ρ_e and STD_e are the AR(1) process coefficients that were discretized using a grid of 15 points. $\beta R^* = \beta(1 + r^*) < 1$ as required by assumption 2.

[29] To measure the volatility, we compute the coefficient of variation, C.v., which is simply the standard deviation divided by the mean.

Supplementary material for Section 5.2.1

Table with the effect of the MIT shock The table below shows how a change in the international risk-free rate affects the endogenous variables in the model.

Table 7: Comparative statics of moments

Sim.	$\bar{B}(Y_{D1})$	Y^{def}	$\mu(B/Y)$	$STD(B/Y)$	Def. freq	$E((B/Y)^2)$	$E(B/Y)^2$	$R^* - 1$
Baseline (BE)	0.24	1.00	-2.0	7.7	2.4	0.006	0.001	1.7
High R^* (P1)	-0.05	0.98	-7.0	14.7	4.1	0.027	0.005	2.7

Note: The first row contains the benchmark calibration. $\mu(B/Y)$ is the long-run mean of the ratio of net external assets to GDP and is expressed in percentage points. Def. freq and $STD(B/Y)$ are also expressed in percentage points. $\bar{B}(Y_{D1}) \equiv \bar{B}(Y^{def}, Y_{D1})$ and Y^{def} are the threshold for debt for shocks at decile 1 and the value of GDP during default, respectively.

The International risk-free rate and the Ergodic punishment A hike in the international risk-free rate r^* induces a decrease in Y^{def} (defined in assumption 3), as the increase in the interest rate decreases V_0^c, the value of repayment in the *initial condition of the iterative process* presented in definition 2. As the convergence of the operator is not uniform starting from any initial condition, and the accuracy of simulations is closely tied to constructive proof, initial conditions are crucial. To understand why V_0^c falls, note that c_0 is the supremum of the space of consumption functions $\mathbb{C}$, defined in the appendix, and depends on $y + (1 + r^*)B$. [30] An increase in the interest rate negatively affects the low values of consumption (associated with $B < 0$) and positively impacts the high ones (related to $B > 0$). The instantaneous return function is strongly concave, bounded above, and unbounded below (see assumption 2), so the first effect dominates. Intuitively, this assumption on preferences implies that households dislike low consumption much more than they like high consumption. Because the value of repaying V_0^c falls, the value of autarky must also fall, which pushes the endogenous punishment Y^{def} down. Notice that the instantaneous return function implies that households prefer to stay out of low consumption values not only in the initial condition of the process but also in the ergodic equilibrium.

Further characteristics of the ergodic distribution: concentration. *Finally, we characterize the concentration of the ergodic distribution for B/Y.* Since we derive a stable process, we can study

[30] Maximal consumption also depends on maximal debt, $-\bar{b}$ in equation (2). However, as this value is independent of (y, B), we omit it from the argument.

the amount of time the economy will spend in a given subset of the stable state space. We choose to construct these subsets using the standard deviation and the mean (i.e., $[\mu_*(B/Y) - STD(B/Y), \mu_*(B/Y) + STD(B/Y)]$. For this purpose, we use the information in Table 7. For the case of the BE, the mass accumulated at $+/-1$ standard deviation, where the support takes values between $[-9.7\%, +5.6\%]$, is 89.0%. Recall that in the case of the standard normal distribution $N(0,1)$ this value is 68.2%. Thus, the mean is a powerful attraction point of the process. In other words, *as there is 1 default per stable distribution, the fact that almost* 90% *of the time net external assets to output ratio will fluctuate at most at 1 standard deviation away from the mean (as against nearly* 70% *in the normal distribution), implies that this type of events has a drastic impact on the performance of the economy*. As the mean is $B/Y = 2.0\%$ and default takes the economy to $B/Y = 0$, even with a 2.4% default frequency, the economy rarely leaves the neighborhood of the mean. This result is robust: if we increase the international risk-free rate by 100 basis points, the economy spends 89.7% of time $+/-1$ standard deviation away from the mean, taking values at $[-21.7\%, +7.1\%]$.