Basic Properties of the Exponential Series

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1. The Exponential Function

$$\exp : \mathbb{R} \to (0, \infty)$$

is defined by

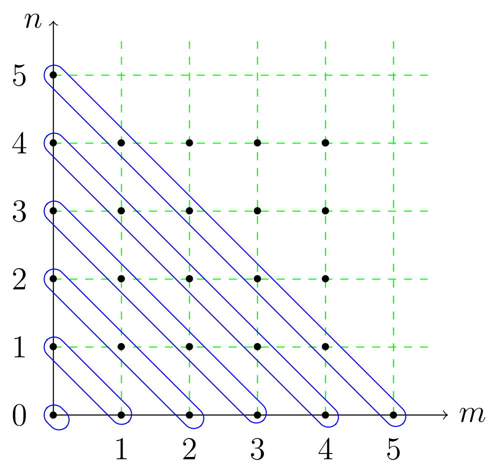
$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

By the ratio test, the series converges for all $x \in \mathbb{R}$.

2. Homomorphism Property

$$\exp(x)\exp(y) \equiv \exp(x+y)$$

Proof.



$$\exp(x)\exp(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^n}{n!}\frac{y^m}{m!}$$

Since the series is absolutely convergent the sum can be evaluated over diagonals $n+m = k, \quad k \in \mathbb{N}$ (see drawing). Along the diagonals:

$$\frac{1}{n!m!} = \frac{1}{k!}\binom{k}{n}$$

Hence,

$$\begin{aligned}
\exp(x)\exp(y) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^n}{n!}\frac{y^m}{m!} \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \binom{k}{n}\left(\frac{x^n y^{k-n}}{k!}\right) \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=0}^{k} \binom{k}{n} x^n y^{k-n} \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}(x+y)^k \\
&= \exp(x+y)
\end{aligned}$$

3. Corollary. Let $\ln = \exp^{-1}$, then

$$\exp(n\ln(x)) = x^n, \quad n \in \mathbb{N}$$

Proof. Easy induction exercise.

4. Definition. Let $a > 0$. Then

$$a^x = \exp(x \ln a), \quad x \in \mathbb{R}$$

5. Corollary. If $e := \exp(1)$ then

$$e^x = \exp(x)$$

6. For $t \in \mathbb{R}$ define:

   (a) $e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}$

   (b) $\cos(t) = \left(1 - \frac{1}{2}t^2 + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right)$

   (c) $\sin(t) = \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)$

   Notice $e^{it} = \cos(t) + i\sin(t)$.

7. Corollary

$$\cos(s+t) = \cos(s)\cos(t) - \sin(s)\sin(t)$$

Proof. Equate the real parts of

$$\exp(it)\exp(is) = \exp(it + is).$$

[1] http://pennance.us