JOURNAL OF APPLIED HYDROLOGY
Vol. XXIX No. 1 to 4, Jan. - Dec. 2016, pp. 60 - 66

A MATHEMATICAL ANALYSIS OF GROUNDWATER HYDRAULICS IN RANDOMLY HETEROGENEOUS SOIL

Amartya Kumar Bhattacharya
23, Biplabi Ambika Chakraborty Sarani, Kolkata – 700029, West Bengal, India
e-mail: dramartyakumar@gmail.com

ABSTRACT

The paper presents a detailed mathematical analysis of groundwater hydraulics in a randomly heterogeneous soil. As the soil medium investigated is randomly heterogeneous, the hydraulic conductivity follows the same pattern. In view of the fact that very little quantitative research has been conducted on groundwater flow in randomly heterogeneous soil, the present study assumes importance. The study develops a mathematical formulation for the velocity potential, velocity components and the stream function in in randomly heterogeneous soil. This mathematical formulation is complex and impossible to interpret physically. Therefore, the study continues its exploration by taking mildly random heterogeneity only for which a definite mathematical result has been found. This mathematical result is easy to interpret and the same has been done in the study.

Introduction

The paper presents a detailed mathematical analysis of groundwater hydraulics in a randomly heterogeneous soil. The hydraulic conductivity at a point is also randomly heterogeneous, where little quantitative research has been conducted on groundwater flow. The first author has worked into the problem of mathematical analysis of groundwater hydraulics in randomly heterogeneous soil in Bhattacharya (2005) and in Bhattacharya and Kumar (2012).

The Navier-Stokes equations for a generalised fluid flow are highly non-linear and are not amenable to solution by the Finite Element Method. Only certain reduced forms of the Navier-Stokes equations can be analysed using the Finite Element method - laminar groundwater flow governed by Darcy's law being one of them.

Mathematical Analysis

Laminar groundwater flow is governed by Darcy's law. Darcy's law is given below:

$$V = Ki$$

and is valid for R_e = Reynolds number = Vd/v < about 1 to 10

where,

K= hydraulic conductivity
V=velocity of flow per unit area of soil including area of soil pores
i = hydraulic gradient represented by sine of slope angle of the hydraulic grade line
d = mean grain size of the soil
v = kinematic viscosity of water

Hydraulic conductivity depends on permeability of the soil and the kinematic viscosity of water and is defined as

$$K = kg/v$$

where, k = permeability of the soil and is defined as

$$k = Cd^2$$

where, d = mean grain size of the soil.

V is called the superficial velocity because it is the discharge divided by the total cross-sectional area of flow. It is to be noted that since solid soil particles occupy a portion of this total cross-sectional area, the true seepage velocity is higher. The porosity, or the void ratio which can be derived from the porosity, of the soil determines the ratio of the superficial velocity and the true seepage velocity.

The true seepage velocity of water through the soil is given by

$$v = \frac{V}{\Phi}$$

where, Φ = porosity defined as ratio of volume of void to the total volume

The actual path of flow of groundwater through the pores in the soil is highly irregular and not amenable to a rigorous mathematical analysis. It is customary, therefore, to use the superficial velocity to mathematically analyse groundwater problems.

The groundwater velocity admits of a velocity potential and a flow net of equipotential lines and flow lines. Such a flow net can be studied analytically or numerically. The Finite Element method is a commonly used numerical method is such flow cases. The parameter to be determined at each node may be the velocity potential from which the velocity components may be easily determined.

If has been used finite elements of different configurations such as 3-noded triangular finite elements and 6-noded triangular finite elements. Other geometrical configurations are also possible. Similarly, either Laplacian or Hermitian Interpolation Functions can be chosen - choosing a Laplacian Interpolation Function makes the interpolation function simpler but also less accurate. To increase the accuracy of a Finite Element solution, one can (1) take smaller Finite Elements, (2) take a larger number of nodes per element and (3) use a Hermitian Interpolation Function.

It is to be noted that the present paper uses Finite Elements in a generic manner and the results are unconditionally valid for all Finite Element configurations and Interpolation Functions.

The unsteady groundwater flow equation for the three-dimensional case through an anisotropic aquifer can be expressed as

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial \phi}{\partial x}\right) + Q + C\frac{\partial \phi}{\partial t} = 0$$

where K_x, K_y, K_z represent the hydraulic conductivities in the three directions, Q is a specified inflow or outflow and □ is the velocity potential. The x and z axes are in mutually perpendicular horizontal directions and the y is in the vertical direction.

The above equation is derived with the following assumptions:
1. Darcy's law is valid throughout the flow domain.
2. The soil is fully saturated.
3. Both the soil and the water are incompressible.

Assuming the flow to be 2-dimensional and steady and then taking a vertical slice, the above equation transforms to the following form:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial y}\right) = 0$$

If the aquifer is assumed to be isotropic, then the superficial seepage velocity components in this vertical plane are v_x and v_y, where v_x and v_y are the velocity components in the horizontal and vertical directions respectively.

$$v_x = \frac{\partial \square}{\partial x}; v_y = \frac{\partial \square}{\partial y}$$

satisfies Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

In a homogeneous aquifer, the hydraulic conductivity, K, is the same at all points in the flow domain. In actual practice, due to non-homogeneity of soil properties like porosity, mean grain diameter etc., K varies from point to point in a manner that is not deterministic. If for an aquifer, the mean value of K is K and the standard deviation of K is σ_k , the hydraulic conductivity field may be considered to be weakly-random if σ_k << K.

Because it is difficult to solve for the velocity potential analytically in a vast range of situations, numerical methods like the Finite Element Method (Mondal, 2001; Maji et al, 2002) have been employed to compute the velocity potential. In the above two works, three-noded triangular finite elements with Lagrangian Interpolation have been used. This has been extended to six-noded triangular finite elements with Lagrangian Interpolation by Choudhury (2002), Manna et al (2003) and Bhattacharya et al (2016). Whatever be the exact nature of the element being utilised, ultimately the matrix equation developed comes out to be of the form

$$\{\phi(K)\} = [G(K)]^{-1}\{P(K)\}$$

where,

[G(K)] is the global constitutive matrix

{ □(K)} is the matrix of the nodal velocity potentials, and

{P(K)} is the equivalent of the load matrix in solid mechanics

If the hydraulic conductivity is deterministic, a definite pattern of equipotential lines is obtained for a definite flow geometry. If the hydraulic conductivity at different points is not deterministic, the equipotential lines and flow lines become uncertain. The present work addresses this situation.

$$\{\phi(K)\} = [G(K)]^{-1}\{P(K)\}$$

or,

$$[G(K)]\{\phi(K)\} = \{P(K)\}$$

Now, differentiating equation (4) with respect to K, one gets

$$[G(K)]\frac{\partial}{\partial K}\{\phi(K)\} = \frac{\partial}{\partial K}\{P(K)\} - \frac{\partial}{\partial K}[G(K)]\{\phi(K)\}$$

or,

$$\frac{\partial}{\partial K}[\phi(K)] = [G(K)]^{-1}\left\{\frac{\partial}{\partial K}P(K) - \frac{\partial}{\partial K}[G(K)]\{\phi(K)\}\right\}$$

where, $\frac{\partial}{\partial K}\{\phi(K)\}$ is the sensitivity of with respect to K.

$$\{\xi(K)\} = [G(K)]^{-1}\{P(K)\}$$

or,

$$\{\xi(K)\} = \frac{\partial}{\partial K}\{\phi(K)\}$$

and

$$\{P(K)\} = \frac{\partial}{\partial K}\{P(K)\} - \frac{\partial}{\partial K}[G(K)]\{\phi(K)\}$$

Now, undertaking a Neumann expansion,

$$[G(K)] = [\overline{G}(K)] + [G'(K)]$$

where, $[\overline{G}(K)]$ is the deterministic component of [G (K)]

and $[G'(K)]$ is the residual component.

$$[G(K)]^{-1} = \left([\overline{G}(K)] + [G'(K)]^{-1}\right) = ([I] + [H])^{-1}[\overline{G}(K)]^{-1}$$

$$= \left([I] - [H] + [H]^2 - [H]^3 + ...\right)[\overline{G}(K)]^{-1}$$

$$[G(K)]^{-1} = \left(\sum_{n=0}^{\infty}[-H]^n\right)[\overline{G}(K)]^{-1}$$

$$[H] = [\overline{G}(K)]^{-1}[G'(K)]$$

Therefore, the velocity potential matrix can be written as

$$\{\phi(K)\} = \left(\sum_{n=0}^{\infty}[-H]^n\right)\left[\overline{G}(K)\right]^{-1}\{P(K)\}$$

$$= \left([I]-[H]+[H]^2-[H]^3+....\right)\left\{\overline{\phi}(K)\right\}$$

where, $\left\{\overline{\phi}(K)\right\}$ is the mean of {(K)}.

This can be written as

$$\{\phi(K)\} = \left\{\overline{\phi}(K)\right\} - \{\phi_1(K)\} + \{\phi_2(K)\} - \{\phi_3(K)\} + \left\{\overline{\phi}(K)\right\} = \left[\overline{G}(K)\right]^{-1}\{P(K)\}$$

The above is a generalised equation for velocity potential in randomly heterogeneous soil, mild or otherwise. It may be safely said that the equipotential lines for a randomly heterogeneous soil differ considerably from that for a homogeneous soil. v_x and v_y may be obtained by differentiating the above equation.

So,

$$v_x = \frac{\partial}{\partial x}[G(K)]^{-1}\{P(K)\}$$

and

$$v_y = \frac{\partial}{\partial y}[G(K)]^{-1}\{P(K)\}$$

The Cauchy – Riemann Equations are

$$\frac{\partial \phi}{\partial x} = \frac{\delta \psi}{\delta y}$$

and

$$\frac{\delta \phi}{\delta y} = \frac{-\delta \psi}{\delta x}$$

where, ϕ = velocity potential, and

ψ = stream function.

Again,

$$\frac{\partial \phi}{\partial x} = \frac{\delta \psi}{\delta y} = v_x$$

$$\frac{\delta \phi}{\delta y} = \frac{-\delta \psi}{\delta x} = v_y$$

$$\psi = \int v_x dy$$

Since

$$v_x = \frac{\partial [G(K)]^{-1}\{P(K)\}}{\partial x}$$

Therefore,

$$\psi = \int \left[\frac{\partial [G(K)]^{-1}\{P(K)\}}{\partial x} \right] dy$$

Using complex variables and complex functions, ϕ and ψ can be recast in terms of a single complex potential ω (z)

$$\omega = \phi + i\psi$$

where, z is complex variable

$$z = x + iy$$

where, $i = \sqrt{-1}$ and (r,θ) plane polar cordinate

To obtain an interpretable result, it is necessary to assume that the heterogeneity of soil is mildly random. This entails assuming that the random matrix is much smaller than the averaged matrix.

Let now a weakly-random hydraulic conductivity field be considered.

Then

$$[\overline{G}(K)] >> [G'(K)]$$

and [H] is small.

It is permissible, then, to write

$$[G(K)]^{-1} \approx ([I]-[H])[\overline{G}(K)]^{-1}$$

$$\{\phi(K)\} \approx ([I]-[H])\{\overline{\phi}(K)\}$$

and,

Thus,

$$\{v_x\} = \frac{\partial}{\partial x}\left[([I]-[H])\{\overline{\phi}(K)\}\right]$$

$$\{v_y\} = \frac{\partial}{\partial y}\left[([I]-[H])\{\overline{\phi}(K)\}\right]$$

The uncertainty in the $\{v_x\}$ and $\{v_y\}$ matrices is introduced through the presence of the matrix [H]. However, the two velocity matrices are almost deterministic as [H] is small. Nevertheless, even for minor a soil whose heterogeneity is only mildly random, the seepage path of the flow is random.

Conclusion

It can be concluded that the presence of a very high degree of random heterogeneity in the hydraulic conductivity field of a soil stratum appreciably alters the velocity distribution as compared to a perfectly homogeneous soil in the aquifer. The equations developed are very complex. If the heterogeneity of soil is mildly random, the velocity matrices are almost deterministic. Localised changes in point velocity do occur but these are unlikely to have a major impact on, for example, the discharge into a pumping well.

References

Bhattacharya, A.K., 2005. "Groundwater flow in a weakly-random hydraulic conductivity field" *Electronic Journal of Geotechnical Engineering, Volume 10*, 2005 – BundleE, ejge paper 2005-0582.

Bhattacharya, A.K. and Kumar, D., 2012. "Effect of very mild random tremors on saturated sub-surface flow" Proceedings, *International Symposium on Engineering under Uncertainty: Safety Assessment and Management*, 4-6 January, 2012, Bengal Engineering and Science University, Shibpur, Howrah. Paper No. CNP002.

Bhattacharya, A.K. and Choudhury, S., 2001. "Uncertainty Analysis of Groundwater Flow during Earthquakes." *Proceedings, All India Seminar on Disaster Management and Civil Engineering Solutions,* 7 - 8 December, Calcutta, India.

Bhattacharya, A.K., Chandanam, A., Thakur, A. and Meena, R., 2016. "A multi-noded finite element analysis of groundwater flow beneath a sheetpile" Proceedings, *Sixth International Groundwater Conference*, 11-13 February, 2016, Chennai, India. Paper ID TS5 – 32. Pages V – 301-308.

Choudhury, S., 2002. "Evaluation of Seepage under Sheet-pile using the Finite Element Method with Six-Noded Triangular Elements" *ME Thesis submitted to Bengal Engineering College (Deemed University)*, Howrah, India.

Maji, S., Mondal, S., Bhattacharya, A.K., Manna, M.C. and Choudhury, S., 2002. "Finite Element Analysis of Flow under a Sheetpile" Proceedings, *International Conference on Water Resource Management in Arid Regions*, 23 – 27, March 2002, Kuwait.

Manna, M.C., Bhattacharya, A.K., Choudhury, S. and Maji, S., 2003. "Groundwater flow beneath a sheetpile analysed using six-noded triangular finite elements" *Journal of the Civil Engineering Division, The Institution of Engineers (India)*, Vol. 84, August 2003. Pages 121 – 129.

Mondal, S., 2001. "Finite Element Analysis of Seepage Flow under Dam" *ME Thesis submitted to Bengal Engineering College (Deemed University)*, Howrah, India.