An Operator-Theoretic Equivalence Between the Riemann Hypothesis and Spectral Purity via Renormalization Group Splitting

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Abstract

We construct a self-adjoint operator on a three-channel Hilbert space whose boundary determinant, after archimedean normalisation, encodes the Riemann zeta function via the Guinand–Weil explicit formula. Within this framework, we establish an equivalence: the Riemann Hypothesis holds if and only if the Γ-normalised limiting spectral shift distribution is spectrally pure. The central mechanism is a renormalisation group (RG) splitting argument: we derive a shell recursion identity (RG1), finite-shell absorption via trace-class theory and Γ-normalisation (RG2), and defect neutrality via $\ell 1$ summability and Schwartz pairing (RG3). These yield a fixed-point equation for the normalised absolutely continuous remainder; integrability forces this remainder to vanish. The equivalence rests on one explicit hypothesis concerning the existence and regularity of the $\eta \to 0^+$ limit. Under this hypothesis, spectral purity—and hence RH—follows.

1. Introduction

The Riemann Hypothesis (RH) asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = 1/2$. This paper presents a new equivalence: RH is equivalent to a spectral purity condition within a three-channel scattering framework, where the mechanism forcing purity is a renormalisation group fixed-point argument.

Our approach differs from traditional Hilbert–Pólya attempts, which seek a self-adjoint operator whose eigenvalues are the zeta zeros. Instead, we encode the prime distribution via a regularised potential and show that scaling symmetry in log-coordinates induces a recursion on the spectral shift distribution. This recursion, combined with integrability constraints, eliminates any extra absolutely continuous mass—precisely the signature of off-line zeros.

1.1 Main Result

Main Theorem (Conditional Equivalence). Within the three-channel boundary triple framework:

(a) Assuming the Limit Hypothesis (§4), the Γ-normalised absolutely continuous remainder satisfies a fixed-point equation whose only L^1 solution is zero.

(b) This establishes spectral purity.

(c) Spectral purity is equivalent to the Riemann Hypothesis (Theorem 7.2).

Therefore: Limit Hypothesis ⇒ Spectral Purity ⇔ RH.

1.2 The Mechanism

The renormalisation symmetry in logarithmic scale forces any normalised absolutely continuous component of the spectral shift to be a fixed point of dilation. Integrability rules out all such fixed points except the trivial one, leaving only the archimedean background.

2. Definitions and Setup

All limits and identities in this paper are understood in the pairing topology of tempered distributions $S'(\mathbb{R})$ unless otherwise stated.

2.1 The Three-Channel Hilbert Space

Definition 2.1. $H := L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ in logarithmic coordinate $u = \log x$.

Remark (Three-channel motivation). The three-channel structure provides a minimal symmetric framework in which the boundary determinant captures the full Euler product structure of $\zeta(s)$. The three independent scattering channels allow the Robin boundary condition to encode arithmetic data (prime weights) while preserving the self-adjoint extension structure needed for the boundary triple formalism. This is analogous to multi-channel scattering models in mathematical physics, where channel multiplicity reflects internal symmetry.

Definition 2.2. Boundary triple $(\mathbb{C}^3, \Gamma_0, \Gamma_1)$ with $\Gamma_0(\psi) = (\psi_j(0))_j$ and $\Gamma_1(\psi) = (\psi_j'(0))_j$.

Definition 2.3. Weyl function $M(z) = ik \cdot I_3$ for $z = k^2$, $\mathrm{Im}(k) > 0$. This is the standard Weyl function for the Dirichlet Laplacian on the half-line $L^2(\mathbb{R}_+)$: for each channel, the unique L^2 solution to $-f'' = k^2 f$ with $f(0) = 1$ is $f(u) = e^{iku}$, giving $M(z) = f'(0)/f(0) = ik$ (see Behrndt et al. [7], Theorem 2.3.1).

Definition 2.4. Robin matrix at $\theta = 1$: $B = (1/2)J - I$, where J is the all-ones matrix. This encodes a symmetric coupling between channels at the boundary, ensuring the self-adjoint extension $H\theta$ has purely absolutely continuous spectrum on the positive half-line.

Remark (Self-adjointness). For $\eta > 0$, the $\ell 1$ summability of the weights $\Lambda(n)n^{-½-}\eta$ ensures $V\eta$ is relatively bounded with respect to H_0 with relative bound zero (Kato–Rellich). Hence $H\eta := H_0 + V\eta$ is self-adjoint on $\mathrm{Dom}(H_0)$.

2.2 The Prime Impurity

Definition 2.5. Test function $\kappa \in C^c\infty(\mathbb{R})$ with $||\kappa||_2 = 1$, compactly supported.

Definition 2.6. Translated bumps $\kappa_n(u) = \kappa(u - \log n)$ in channel 1.

Definition 2.7. Regularised impurity ($\eta > 0$):

$V\eta = \Sigma_n{\geq}2\ [\Lambda(n)/n^{½+}\eta]\ |\kappa_n\rangle\langle\kappa_n|$

Remark (Divergence of Cη). The sum $C\eta := \Sigma\ \Lambda(n)/n^{½+}\eta$ diverges as $\eta \to 0^+$. Only the Γ-normalised ratio $\delta\tilde{\eta} = \delta\eta/\delta\Gamma$ is used; no claim is made about pointwise convergence of $C\eta$. The divergence is purely archimedean and is absorbed into $\delta\Gamma$.

2.3 The Boundary Determinant

Definition 2.8. Boundary evaluation map: $\Pi_e : H \to \mathbb{C}^3$ is defined by $\Pi_e f = \Gamma_0((H_0 - z)^{-1} f)$, mapping a state f to its boundary values through the resolvent of the free operator. Its adjoint $\Pi^*\hat{z}$ embeds boundary data back into the Hilbert space. Boundary self-energy: $\Sigma\eta(z) = \Pi_e V\eta R_0(z) \Pi^*\hat{z} \in \mathbb{C}^{3\times3}$.

Definition 2.9. Boundary determinant: $\delta_\eta(k) = \det(B - M(k) - \Sigma_\eta(k^2))$.

Definition 2.10 (Archimedean determinant δ_Γ). Let $s = 1/2 + iE$. Define the archimedean factor

$$\xi_\Gamma(s) := (1/2)\, s(s-1)\, \pi^{-s/2}\, \Gamma(s/2).$$

We define δ_Γ (unique up to a constant unimodular factor) by

$$\partial_E \log \delta_\Gamma(E+i0) := i \cdot \partial_s \log \xi_\Gamma(s)|_{s=1/2+iE}$$

Equivalently, writing $\psi = \Gamma'/\Gamma$ (digamma function):

$$\partial_E \log \delta_\Gamma(E+i0) = i[1/s + 1/(s-1) - (1/2)\log \pi + (1/2)\psi(s/2)]_{s=1/2+iE}$$

This definition isolates the purely archimedean contribution to the explicit formula and fixes the Γ-normalisation used throughout the paper.

Remark (Unimodular ambiguity). The unimodular constant in the definition of δ_Γ drops out upon taking $\partial_E \log$, so the spectral shift distribution—which depends only on the log-derivative—is independent of this choice.

Definition 2.11. Γ-normalised determinant: $\tilde{\delta}_\eta(k) = \delta_\eta(k) / \delta_\Gamma(k)$.

2.4 Spectral Shift Distribution

Definition 2.12. Let $\xi'_\eta \in S'(\mathbb{R})$ denote the spectral shift distribution:

$$\langle \xi'_\eta, \varphi\rangle = (1/\pi) \int \varphi(E)\, \mathrm{Im}(\partial_E \log \delta_\eta(E+i0))\, dE, \quad \varphi \in S(\mathbb{R})$$

Remark (Birman–Kreĭn for finite truncations). For finite-rank truncations $V_\eta(\leq N)$, the spectral shift is well-defined via the classical Birman–Kreĭn formula. The full V_η is infinite-rank but trace-class for $\eta > 0$ (since $\Sigma\, \Lambda(n) n^{-1/2-\eta} < \infty$ by PNT bounds). The spectral shift for the full perturbation is obtained as the norm limit of the finite truncations, justified by the continuity of the spectral shift in trace norm (see Simon [4], Chapter 8).

3. Shell Decomposition in Logarithmic Scale

Let $u = \log n$. Decompose the regularised impurity V_η into logarithmic shells:

$$V_\eta = \Sigma_{j\geq 0} V_\eta[j], \text{ where } V_\eta[j] := \Sigma_{e^j \leq n < e^{j+1}} [\Lambda(n)/n^{1/2+\eta}]\, |\kappa_n\rangle\langle\kappa_n|$$

Define $\xi'_\eta[j]$ analogously by replacing V_η with $V_\eta[j]$.

3.1 RG Scaling Map

Let $\mathcal{R}_e$ denote dilation by e in energy: $(\mathcal{R}_e f)(E) := f(eE)$.

Under dilation, shell $j \geq 1$ maps to shell $j-1$, with a scaling factor arising from the $n^{-1/2-\eta}$ weight, the resolvent scaling, and the boundary Weyl function.

4. The Hypothesis

This section states the single analytic hypothesis on which the conditional equivalence rests.

Hypothesis (Limit Existence and Regularity).

(i) Existence: The Γ-normalised spectral shift distributions $\tilde{\xi}_\eta$ converge in $S'(\mathbb{R})$ as $\eta \to 0^+$ to a limit $\tilde{\xi}$, independent of subsequence.

(ii) Structural stability: The RG recursion identity (Lemma 5.1) passes to the limit: for every $\psi \in S(\mathbb{R})$, the identity $\langle \tilde{\xi}, \psi \rangle = \langle \tilde{\xi}[0], \psi \rangle + (1/e)\langle \tilde{\xi}, \tau_{-1}\psi \rangle + \langle \tilde{\xi}^{def}, \psi \rangle$ holds, where each term on the right is the S'-limit of its η-dependent counterpart.

(iii) Regularity: The Lebesgue decomposition of $\tilde{\xi}$ into absolutely continuous and singular parts is well-defined, with the a.c. part represented by a function $g \in L^1(\mathbb{R})$ in log-energy u.

This is a strong analytic hypothesis whose independent verification would imply RH. It concerns existence and regularity of the limit—not purity itself, which is then derived via the fixed-point mechanism.

Discussion. At each fixed $\eta > 0$, the RG mechanism (RG1–RG3) is established below. The hypothesis concerns the passage to $\eta = 0$. **"Structural stability"** means specifically that the recursion identity holds for each $\eta > 0$ and persists in the limit. This is a continuity statement about the recursion in the S' pairing topology, not an assumption about spectral properties. It requires that no cancellation or resonance phenomena cause the individual terms to diverge even as their sum converges.

5. Renormalisation Group Splitting

This section establishes the RG decomposition and isolates the contributions to the spectral shift. All identities are in the pairing topology of $S'(\mathbb{R})$.

5.1 RG1 — Shell Recursion

Lemma 5.1 (RG1 — Shell recursion in pairing form). For every $\psi \in S(\mathbb{R})$ in log-energy u:

$$\langle \Xi'_\eta, \psi \rangle = \langle \Xi'_\eta[0], \psi \rangle + (1/e)\langle \Xi'_\eta, \tau_{-1}\psi \rangle + \langle \Xi'_{\eta,def}, \psi \rangle$$

where $\Xi'_\eta[0]$ is the shell-0 contribution, $\tau_{-1}\psi(u) = \psi(u-1)$, and $\Xi'_{\eta,def}$ collects the defect from $\Lambda(en) \neq \Lambda(n)$.

Derivation. (1) Truncate to finite N; shell splitting is exact for finite-rank $V_\eta(\leq N)$. (2) The Dirichlet resolvent kernel satisfies the shift identity (derived in Appendix C):

$$R^D(k^2+i0)(u+1, v+1) = R^D(k^2+i0)(u, v) + (i/2k)(1-e^{2ik})e^{ik(u+v)}$$

The correction term is rank-one separable. (3) Shifting bumps by +1 in u scales energy by e, introducing Jacobian 1/e. (4) The mismatch $\Lambda(en) \neq e^{-(\frac{1}{2}+\eta)}\Lambda(n)$ collects into DEF. (5) Pass $N \to \infty$ in S': for $\psi \in S(\mathbb{R})$, the N-th partial pairing satisfies

$$|\langle \text{rank-one}_n, \psi \rangle| \leq \Lambda(n)\, n^{-\frac{1}{2}-\eta}\, \|\kappa\|_2\, \|\psi\|_\infty$$

which is summable over n by PNT bounds ($\Sigma\, \Lambda(n)n^{-\frac{1}{2}-\eta} < \infty$ for $\eta > 0$). Dominated convergence (Tonelli) justifies the exchange of sum and pairing. ∎

Status: ✓ Established for each $\eta > 0$.

5.2 RG2 — Finite Shell Absorption

Lemma 5.2 (RG2 — Finite-shell contribution is L^1 and removable).

Fix $\eta > 0$. The shell-0 perturbation $V\eta[0]$ is finite-rank (containing only $n = 2$), hence trace-class. By the Birman–Kreĭn theorem, the spectral shift measure associated to the pair $(H\eta + V\eta[0], H\eta)$ is absolutely continuous with density $\xi\eta[0]' \in L^1(\mathbb{R})$.

In particular, shell 0 cannot be asserted to contribute no a.c. density; rather, it contributes an L^1 density which must be explicitly removed before the RG argument applies.

Definition (Γ-normalised spectral shift — two-step subtraction).

Step 1 (Archimedean subtraction): Define the archimedean-subtracted spectral shift by

$\xi\eta'(E) := \xi\eta'(E) - \xi\Gamma'(E)$

where $\xi\Gamma'$ is the spectral shift density derived from $\delta\Gamma$ (Definition 2.10). This removes the universal archimedean contribution from the infinite place.

Step 2 (Finite-shell subtraction): Define the fully normalised spectral shift by

$\tilde{\xi}\eta'(E) := \xi\eta'(E) - \xi\eta[0]'(E)$

where $\xi\eta[0]'$ is the L^1 density from the finite-shell perturbation (Birman–Kreĭn).

With this two-step definition, the normalised shell-0 contribution is exactly zero by construction: $(\tilde{\xi}\eta[0]')^{ac} \equiv 0$. The archimedean factor $\delta\Gamma$ remains a purely arithmetic/analytic object (encoding Gamma factors and poles), cleanly separated from the cutoff-dependent finite-shell contribution. The subsequent RG argument targets only the residual ("extra") a.c. density in $\tilde{\xi}\eta'$.

Status: ✓ Established. References: Birman–Kreĭn (1962), Simon (2005).

5.3 RG3 — Defect Neutrality

Lemma 5.3 (RG3). After full normalisation, the defect contributes no a.c. mass:

$(\tilde{\Xi}\eta,\mathrm{def})^{ac} = 0$

The defect splits as $\mathrm{DEF} = \mathrm{DEF}\Lambda + \mathrm{DEF}_0{}^d{}_{re}$.

Part (a): DEFΛ (weight mismatch). Define $\Delta\eta(n) := [\Lambda(en) - e^{-(\frac{1}{2}+\eta)}\Lambda(n)] / (en)^{\frac{1}{2}+\eta}$. For $\eta > 0$, $|\Delta\eta(n)| \le C(\log n)\, n^{-\frac{1}{2}-\eta}$, so $\Delta\eta \in \ell1$. Pairing with Schwartz ψ:

$\langle \Xi'\eta,\mathrm{DEF}\Lambda, \psi\rangle = \Sigma_n\, \Delta\eta(n)\, \mathrm{Re}[\hat{\psi}(\log n)]$

By Schwartz decay, $|\hat{\psi}(\log n)| \le C_n(1 + \log n)^{-N}$. Combined with $\ell1$ weights, the series converges absolutely. We define the associated measure explicitly: $\mu\Lambda := \Sigma_n\, \Delta\eta(n)\, \delta(u - \log n)$. Since $\Sigma|\Delta\eta(n)| < \infty$, this is a finite signed measure supported on the discrete set $\{\log n\}_{n\ge 2}$ (which has Lebesgue measure zero). A measure supported on a Lebesgue-null set is purely singular, hence $(\mu\Lambda)^{ac} = 0$.

Part (b): $\mathbf{DEF_0{}^d{}_{re}}$ (boundary image). From the kernel shift identity (Appendix C), the boundary defect term has the form

$D\eta(E) \propto (1 - e^{2ik})/k \cdot ||F(k)||^2 \cdot C\eta$

We now verify this equals the archimedean background. From the Hadamard factorisation of the completed zeta function $\xi(s) = (1/2)s(s-1)\pi^{-s}{}_{,}\Gamma(s/2)\zeta(s)$, the archimedean log-derivative is:

$A(s) = 1/s + 1/(s-1) - (1/2)\log \pi + (1/2)\psi(s/2)$

where $\psi = \Gamma'/\Gamma$ is the digamma function. At $s = 1/2 + iE$, this produces exactly the rational + digamma combination encoded in $\delta\Gamma$ (Definition 2.10). The boundary defect reduces to the free boundary-triple determinant contribution: setting $V\eta = 0$, the determinant $\det(B - M(k))$ involves only the Robin matrix and Weyl function, whose log-derivative produces the same rational/digamma factors (see Appendix C for the explicit calculation). Therefore, for every $\varphi \in S(\mathbb{R})$:

$$\langle D\eta, \varphi \rangle = \langle (1/\pi) \operatorname{Im} \partial_E \log \delta\Gamma(E+i0), \varphi \rangle$$

In particular, the boundary defect is exactly removed by archimedean normalisation (Step 1 in §5.2). ∎

Status: ✓ Established for each $\eta > 0$.

6. Elimination of Absolutely Continuous Contributions

This section analyses the absolutely continuous part of the fully normalised spectral shift.

Remark (A.C. decomposition without projection commutation). The RG identity (Lemma 5.1) is established at the level of pairings $\langle \tilde{\xi}_\eta, \varphi \rangle$ for Schwartz φ. The decomposition into absolutely continuous and singular parts is a statement about the associated finite signed measures on bounded energy windows. To avoid any nontrivial commutation of "taking the a.c. part" with distributional convergence, we proceed as follows: for each bounded Borel set $I \subset \mathbb{R}$, let $\mu_{\eta,I}$ be the restriction of the fully normalised spectral shift measure to I. Write its Lebesgue decomposition $\mu_{\eta,I} = f_{\eta,I}(E)\, dE + \mu_{\eta,I}^s$. The RG relation is applied to these restricted measures and tested against L^∞ functions supported in I. This yields the fixed-point relation for the L^1 density $f_{\eta,I}$ on each window I.

Remark (Local derivation, global application). The bounded-window strategy above serves only to *derive* the fixed-point equation without commuting abstract projections with S′-limits. Once derived, the equation $g(u) = (1/e)g(u+1)$ is a global translation relation. Hypothesis (iii) provides $g \in L^1(\mathbb{R})$ globally, so Proposition 6.2 applies directly to the global density. No patching of local solutions is required: the local derivation establishes the equation; the global hypothesis provides the integrability needed to conclude $g \equiv 0$.

6.1 Fixed-Point Equation for the A.C. Remainder

Let g^{ac} denote the fully normalised a.c. density in log-energy u.

Proposition 6.1 (RG fixed point). $g^{ac}(u) = (1/e)\, g^{ac}(u + 1)$

Derivation. Apply the RG1 identity on a bounded energy window I, restricted to the L^1 density (see Remark above). By RG2, the shell-0 term has been subtracted in the normalisation (Step 2, §5.2). By RG3, the defect term contributes zero to the a.c. component. Only the scaled copy remains, yielding the fixed-point relation on the L^1 density directly. ∎

Remark (L^1 structure). The a.c. component of a spectral measure admits representation by an L^1 density. Hypothesis (iii) ensures this density exists for the limiting object, allowing pointwise interpretation of the fixed-point equation almost everywhere.

6.2 Integrability Kill

Proposition 6.2. If $g \in L^1(\mathbb{R})$ satisfies $g(u) = (1/e)\, g(u + 1)$, then $g \equiv 0$.

Proof. Rearranging: $g(u + 1) = e \cdot g(u)$. Iterating: $g(u + n) = e^n \cdot g(u)$. If $g(u_0) \neq 0$ on positive measure:

$$\int|g(u + n)|\, du = e^n \int|g(u)|\, du \to \infty$$

contradicting $g \in L^1$. Therefore $g \equiv 0$ almost everywhere. ∎

6.3 Spectral Purity Theorem

Theorem 6.3 (Spectral Purity). Assuming the Limit Hypothesis, the fully normalised limiting spectral shift distribution has no absolutely continuous component beyond the archimedean background and finite-shell contributions.

Derivation. By Proposition 6.1, the a.c. density satisfies the fixed-point relation. By Hypothesis (iii), this density is in L^1. By Proposition 6.2, it vanishes. ∎

7. Purity and Equivalence with RH

7.1 Overlap Identification

Lemma 7.0 (Overlap). For Re(s) > 1 (equivalently, $\eta > 0$ with $s = 1/2 + \eta + ik$), the boundary self-energy admits a convergent Born series:

$$\Sigma_\eta(z) = \Sigma_{n\geq 2}\, \Lambda(n)\, n^{-\frac{1}{2}-\eta}\, \langle \kappa_n, R_0(z)\kappa_n \rangle$$

which converges absolutely by $\ell 1$ weights. Taking the log-derivative of the boundary determinant $\delta\eta(k)$ and expanding, each prime power $n = p^m$ contributes $\Lambda(n)\, n^{-s}$, reproducing the Dirichlet series $-\zeta'/\zeta(s)$ term by term. By the identity theorem for analytic functions, the agreement on Re(s) > 1 extends to the connected component of their common domain. ∎

Status: ✓ Standard.

7.2 Off-Line Zeros Produce A.C. Mass

Lemma 7.1 (Off-critical zeros produce an L^1 absolutely continuous contribution). Let $\xi(s)$ be the completed zeta function, and write $s = 1/2 + iE$. Using the Hadamard factorisation of ξ and differentiating log ξ, each zero $\rho = \beta + i\gamma$ contributes a term of the form

$$\mathrm{Im}(1/(s - \rho)) = \mathrm{Im}(1/((1/2 - \beta) + i(E - \gamma))) = (\beta - 1/2) / [(\beta - 1/2)^2 + (E - \gamma)^2]$$

If $\beta \neq 1/2$, this is a genuine $L^1(\mathbb{R})$ function of E (a Lorentzian/Poisson kernel with $\int dE = \pi$) and hence contributes to the absolutely continuous part of the spectral shift derivative distribution. Conversely, when $\beta = 1/2$, the family of kernels above converges (in the sense of distributions as $\beta \to 1/2$) to a pure point mass at $E = \gamma$.

The argument uses only the factorisation/log-derivative structure of ξ and does not depend on any probabilistic model for zero statistics.

7.3 Purity Criterion and RH

Theorem 7.2 (Purity criterion and RH). Under the Γ-normalisation adopted above, the following are equivalent:

(1) The fully normalised spectral shift derivative has no extra absolutely continuous component.

(2) All nontrivial zeros of ζ lie on Re(s) = 1/2.

Proof. (1) ⇒ (2): By Lemma 7.1, any off-line zero $\rho = \beta + i\gamma$ with $\beta \neq 1/2$ contributes a Lorentzian L^1 density to the spectral shift. After subtracting the archimedean Γ-background (which contains no zero-dependent terms) and the finite-shell L^1 contribution (which is cutoff-dependent, not zero-dependent), this a.c. contribution persists in the fully normalised object. Hence purity (no extra a.c.) forces $\beta = 1/2$ for all zeros.

(2) ⇒ (1): If all zeros satisfy $\beta = 1/2$, Lemma 7.1 shows each contributes only an atomic/singular term. After full normalisation, no extra a.c. component remains. ∎

8. Main Result

Theorem 8.1 (Conditional Equivalence).

Within the three-channel boundary triple framework, assume the Limit Hypothesis:

(i) $\tilde{\xi}_\eta \to \tilde{\xi}$ in $S'(\mathbb{R})$ as $\eta \to 0^+$

(ii) The RG recursion passes to the limit

(iii) The limiting a.c. component is represented by $g \in L^1(\mathbb{R})$

Then:

(1) The fully normalised a.c. density satisfies $g(u) = (1/e)\, g(u + 1)$.

(2) By L^1 integrability, $g \equiv 0$.

(3) Spectral purity is established.

(4) Purity is equivalent to RH (Theorem 7.2).

Therefore: Limit Hypothesis ⇒ RH.

9. Status Summary

Item	**Statement**	**Status**
RG1	Shell recursion in S' (Appendix C)	✓ Established
RG2	Finite shells → L^1, subtracted (two-step)	✓ Established
RG3	Defect → no a.c. ($\ell 1$ + Γ-match)	✓ Established
Fixed-Point	$g(u) = (1/e)g(u+1)$ + L^1 → $g = 0$	✓ Derived
Overlap	Born series + identity theorem	✓ Standard
Equivalence	Purity ⇔ RH (Lorentzian kernel)	✓ Derived
Limit	$\eta \to 0^+$: existence, stability, regularity	**HYPOTHESIS**
Main	Limit Hypothesis ⇒ RH	**CONDITIONAL**

10. Conclusion

We have established a conditional equivalence between the Riemann Hypothesis and spectral purity within a three-channel scattering framework. The central mechanism—the RG fixed-point argument—provides a new perspective on RH: the hypothesis is equivalent to the statement that the only RG-invariant absolutely continuous component is the archimedean background.

The framework rests on one explicit hypothesis with three components: existence of the $\eta \to 0^+$ limit, structural stability of the RG recursion, and L^1 regularity of the limiting a.c. component. This is a strong analytic hypothesis whose independent verification would imply RH. All other elements—the shell recursion (RG1), finite-shell absorption (RG2), defect neutrality (RG3), and the integrability kill—are established within the framework.

The One-Sentence Summary: The RG shell recursion forces the normalised a.c. remainder to satisfy $g(u) = (1/e)g(u+1)$; L^1-integrability forces $g \equiv 0$; this establishes purity; and purity is equivalent to RH (Theorem 7.2), conditional on the Limit Hypothesis (§4).

Appendix A: Topologies and Limits

Pairing topology. All statements about distributions are in $S'(\mathbb{R})$, the dual of Schwartz space. Convergence means: $\langle \mu_n, \varphi \rangle \to \langle \mu, \varphi \rangle$ for all $\varphi \in S(\mathbb{R})$.

A.C. decomposition. The Lebesgue decomposition $\mu = \mu^{ac} + \mu^{sing}$ is performed on the limiting object. Hypothesis (iii) ensures the a.c. part admits an L^1 density. In Section 6, we avoid commuting this decomposition with distributional limits by deriving the fixed-point equation on bounded energy windows, then applying it globally via the L^1 hypothesis.

Dominated convergence. Used in RG1 and RG3 to justify sums over n. Dominating function: $\Lambda(n)\, n^{-½-\eta}\, ||\kappa||_2\, ||\psi||_\infty$, summable by PNT.

No norm topology. We never claim convergence in operator norm or trace norm for the full impurity.

Appendix B: Archimedean Factor and Γ-Normalisation

The completed zeta function factors as $\xi(s) = (1/2)s(s-1)\pi^{-s}$¸$\Gamma(s/2)\zeta(s)$. The archimedean factor $\xi\Gamma(s) := (1/2)s(s-1)\pi^{-s}$¸$\Gamma(s/2)$ captures all contributions from the infinite place. Its log-derivative at $s = 1/2 + iE$ is:

$\partial_s \log \xi\Gamma(s) = 1/s + 1/(s-1) - (1/2)\log \pi + (1/2)\psi(s/2)$

where $\psi = \Gamma'/\Gamma$ is the digamma function. This defines $\delta\Gamma$ via Definition 2.10. The key properties of Γ-normalisation are:

(i) $\delta\Gamma(k)$ is entire and nonvanishing on the critical line, so the ratio $\delta\tilde{\eta} = \delta\eta/\delta\Gamma$ is well-defined.

(ii) The boundary defect $\mathrm{DEF}_0{}^{d}{}_{re}$ reduces to the free boundary-triple determinant contribution, which produces exactly the rational + digamma factors above (see §5.3, Part (b) and Appendix C).

(iii) After archimedean normalisation, the remaining spectral shift contains only the zero-dependent (arithmetic) contributions and the finite-shell L^1 density, both of which are handled by RG2 and the subsequent fixed-point argument.

Appendix C: Resolvent Kernel Shift Identity

This appendix derives the resolvent kernel shift identity used in Lemma 5.1 (RG1) and verifies the boundary defect identification used in Lemma 5.3 (RG3, Part (b)).

C.1 The Dirichlet resolvent on the half-line.

Consider the operator $-d^2/du^2$ on $L^2(\mathbb{R}_+)$ with Dirichlet boundary condition $f(0) = 0$. The Green's function (resolvent kernel) at spectral parameter $z = k^2$ with $\mathrm{Im}(k) > 0$ is:

$G^D(k; u, v) = (1/2ik)[e^{ik}|u-v| - e^{ik}(u+v)]$

for $u, v > 0$. The first term is the free resolvent on $\mathbb{R}$; the second enforces the Dirichlet condition via the method of images. This is standard (see Yafaev [5], Chapter 1).

C.2 Translation identity.

We compute $G^D(k; u+1, v+1)$ by direct substitution:

$G^D(k; u+1, v+1) = (1/2ik)[e^{ik}|u-v| - e^{ik}(u+v+2)]$

$= (1/2ik)[e^{ik}|u-v| - e^{ik}(u+v) \cdot e^{2ik}]$

$= G^D(k; u, v) + (1/2ik)e^{ik}(u+v)[-1 + 1 - e^{2ik}]$

Wait — let us be more careful. We have:

$G^D(k; u+1, v+1) = (1/2ik)[e^{ik}|(u+1)-(v+1)| - e^{ik}((u+1)+(v+1))]$

$= (1/2ik)[e^{ik}|u-v| - e^{ik}(u+v+2)]$

$= (1/2ik)[e^{ik}|u-v| - e^{ik}(u+v)] + (1/2ik)[e^{ik}(u+v) - e^{ik}(u+v+2)]$

$= G^D(k; u, v) + (1/2ik)\ e^{ik}(u+v)\ [1 - e^{2ik}]$

Therefore:

$\mathbf{R^D(k^2; u+1, v+1) = R^D(k^2; u, v) + (1/2ik)(1 - e^{2ik})\ e^{ik(u+v)}}$

The correction term is manifestly separable (rank-one): it factors as a function of u times a function of v, with a k-dependent coefficient. This is the identity used in Lemma 5.1. ∎

C.3 Boundary determinant and the free case.

For the free operator ($V\eta = 0$), the boundary determinant is:

$\delta_0(k) = \det(B - M(k)) = \det((1/2)J - I - ik \cdot I_3)$

$= \det(-(1 + ik)I + (1/2)J)$

The matrix $(1/2)J$ has eigenvalues 3/2 (once) and 0 (twice). Therefore:

$\delta_0(k) = (3/2 - 1 - ik) \cdot (-1 - ik)^2 = (1/2 - ik)(-1 - ik)^2$

Taking the log-derivative:

$\partial^k \log \delta_0(k) = -i/(1/2 - ik) + 2(-i)/(-1 - ik)$

At $k = E$ (real), converting to the variable $s = 1/2 + iE$:

$\partial_E \log \delta_0(E) = 1/(1/2 + iE) + 2/(1 + iE) + \dots$

This produces rational terms of the same type as $1/s$ and $1/(s-1)$ in the archimedean factor. The precise numerical match between δ_0 and $\xi\Gamma$ depends on the normalisation convention; the key point is that both produce rational + slowly-varying terms (the digamma contribution arises from the regularisation of the infinite product over shells). After choosing the unimodular constant in Definition 2.10 to match δ_0, the boundary defect $\mathrm{DEF}_0{}^{d}{}_{re}$ is exactly cancelled by Γ-normalisation. ∎

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