Planck-Scale Rotating Black Holes in $f(R) = R + aR^2$ Gravity as Dark Matter Candidates

Abstract

We examine metric $f(R)$ gravity with $f(R) = R + aR^2$ in the high-curvature regime, with emphasis on microscopic, rapidly rotating black holes. A Hartle–Thorne slow-rotation expansion is developed through second order in the rotation parameter. The resulting equations exhibit curvature-dependent suppression of rotational deformations and of frame-dragging gradients when $f'(R_0) \gg 1$. This provides a controlled setting in which evaporation arguments based on General Relativity can fail near Planckian curvature, permitting long-lived remnants that may contribute to dark matter.

1 Gravitational Framework

We consider metric $f(R)$ gravity with action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, f(R) + S_{\text{matter}}, \qquad f(R) = R + aR^2, \quad a > 0. \tag{1}$$

The field equations are

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu)\, f'(R) = 8\pi G T_{\mu\nu}, \tag{2}$$

with $f'(R) = 1 + 2aR$.

2 Static Background and High-Curvature Scale

Let $(\mathcal{M}, g^{(0)})$ be a static, spherically symmetric background with metric

$$ds_0^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{3}$$

and Ricci scalar $R_0(r)$. In the regime $R_0 \sim a^{-1}$ one has

$$f'(R_0) = 1 + 2aR_0 \gg 1, \tag{4}$$

so the characteristic length scale of curvature corrections is $\ell_a := \sqrt{a}$ and a microscopic horizon may satisfy $r_H \sim \ell_a$.

3 Hartle–Thorne Slow-Rotation Expansion Through Second Order

3.1 Ansatz and bookkeeping

Introduce a dimensionless rotation parameter $\epsilon \ll 1$ and expand the stationary, axisymmetric metric as

$$\begin{aligned} ds^2 = & -e^{2\Phi(r)}\left[1+2\epsilon^2\, h(r,\theta)\right]dt^2 + e^{2\Lambda(r)}\left[1+2\epsilon^2\, m(r,\theta)\right]dr^2 \\ & + r^2\left[1+2\epsilon^2\, k(r,\theta)\right]\Big(d\theta^2+\sin^2\theta\, d\varphi^2\Big) - 2\epsilon\,\omega(r)\, r^2\sin^2\theta\, dt\, d\varphi + \mathcal{O}(\epsilon^3), \end{aligned} \tag{5}$$

where ω is first order and the functions h, m, k are second order.

A standard Legendre decomposition is adopted:

$$h(r,\theta) = h_0(r)+h_2(r)P_2(\cos\theta), \quad m(r,\theta) = m_0(r)+m_2(r)P_2(\cos\theta), \quad k(r,\theta) = k_2(r)P_2(\cos\theta), \tag{6}$$

with $P_2(x) = \frac{1}{2}(3x^2-1)$. The scalar curvature is similarly expanded:

$$R(r,\theta) = R_0(r) + \epsilon^2\Big(R_{20}(r) + R_{22}(r)P_2(\cos\theta)\Big) + \mathcal{O}(\epsilon^3), \tag{7}$$

so that

$$f'(R) = f'(R_0) + \epsilon^2 f''(R_0)\Big(R_{20} + R_{22}P_2(\cos\theta)\Big) + \mathcal{O}(\epsilon^3), \qquad f''(R_0) = 2a. \tag{8}$$

3.2 First-order frame dragging equation

To first order in ϵ, the only new field equation is the (t,φ) component of (2), which reduces to

$$\frac{d}{dr}\left[r^4 f'(R_0)\, e^{-(\Phi+\Lambda)}\,\frac{d\omega}{dr}\right] = 0. \tag{9}$$

Hence

$$\frac{d\omega}{dr} = \frac{C}{r^4 f'(R_0)}\, e^{\Phi+\Lambda}, \tag{10}$$

where C is an integration constant determined by asymptotic behavior (or matching to an exterior region). In particular, when $f'(R_0) \gg 1$ one obtains the curvature-suppressed gradient

$$\omega'(r) = \mathcal{O}\left(\frac{1}{f'(R_0)}\right). \tag{11}$$

3.3 Second-order sector: structure of the equations

At order ϵ^2, the field equations (2) yield coupled equations for

$$h_0, m_0 \quad \text{(monopole sector)}, \qquad h_2, m_2, k_2 \quad \text{(quadrupole sector)},$$

together with the curvature perturbations R_{20}, R_{22}. The system splits into $\ell = 0$ and $\ell = 2$ blocks because the only angular dependence in (5) is via $P_2(\cos\theta)$.

Monopole block ($\ell = 0$). The $\ell = 0$ equations may be written schematically as a linear system

$$\mathcal{L}_0 \begin{pmatrix} h_0 \\ m_0 \\ R_{20} \end{pmatrix} = \begin{pmatrix} S_0^{(tt)} \\ S_0^{(rr)} \\ S_0^{(R)} \end{pmatrix}, \tag{12}$$

where $\mathcal{L}_0$ is a matrix of second-order radial differential operators depending on the background (Φ, Λ, R_0) and on $f'(R_0)$ and $f''(R_0)$, and where the sources are quadratic in the first-order rotation:

$$S_0^{(\cdot)}(r) = \alpha_0^{(\cdot)}(r)\,\omega'(r)^2 + \beta_0^{(\cdot)}(r)\,\omega(r)^2. \tag{13}$$

Using (11), the dominant scaling in the high-curvature regime is

$$S_0^{(\cdot)} = \mathcal{O}\left(\frac{1}{f'(R_0)^2}\right), \tag{14}$$

provided ω remains bounded and ω' controls the rotational stress terms.

Quadrupole block ($\ell = 2$). Similarly, the $\ell = 2$ sector is

$$\mathcal{L}_2 \begin{pmatrix} h_2 \\ m_2 \\ k_2 \\ R_{22} \end{pmatrix} = \begin{pmatrix} S_2^{(tt)} \\ S_2^{(rr)} \\ S_2^{(\theta\theta)} \\ S_2^{(R)} \end{pmatrix}, \tag{15}$$

with sources again quadratic in rotation:

$$S_2^{(\cdot)}(r) = \alpha_2^{(\cdot)}(r)\,\omega'(r)^2 + \beta_2^{(\cdot)}(r)\,\omega(r)^2 + \gamma_2^{(\cdot)}(r)\,\omega(r)\omega'(r). \tag{16}$$

The same curvature-suppression mechanism yields

$$S_2^{(\cdot)} = \mathcal{O}\left(\frac{1}{f'(R_0)^2}\right) \qquad \text{(in the regime } f'(R_0) \gg 1\text{)}. \tag{17}$$

3.4 A representative second-order equation

To make the second-order content explicit, we record a representative equation of Hartle–Thorne type, generalized by $f'(R_0)$, for the quadrupole deformation combination

$$\Xi(r) := h_2(r) + m_2(r).$$

A typical linear combination of the (θ, θ) and (r, r) components yields

$$\Xi'' + \left(\frac{2}{r} + \Phi' - \Lambda'\right) \Xi' - \frac{6}{r^2} e^{2\Lambda}\, \Xi = \mathcal{Q}(r)\, \omega'(r)^2 + \mathcal{H}(r)\, f''(R_0)\, R_{22} + \mathcal{O}\big(\omega\omega'\big), \tag{18}$$

where $\mathcal{Q}(r)$ and $\mathcal{H}(r)$ are background-dependent functions determined by $(\Phi, \Lambda, R_0, f'(R_0))$. By (10) one has $\omega'^2 = \mathcal{O}(f'(R_0)^{-2})$. Moreover, the scalar perturbation R_{22} satisfies an effective massive radial equation sourced by the same rotation terms:

$$\left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \Phi' - \Lambda'\right) \frac{d}{dr} - e^{2\Lambda} \left(\frac{6}{r^2} + m_\phi^2\right)\right] R_{22} = \mathcal{S}_R(r)\, \omega'(r)^2 + \mathcal{O}\big(\omega\omega'\big), \tag{19}$$

with $m_\phi^2 = (6a)^{-1}$ and some background function $\mathcal{S}_R(r)$.

Equations (18)–(19) exhibit explicitly that the second-order rotational deformations and curvature multipoles are driven by ω'^2, and hence are suppressed when $f'(R_0) \gg 1$.

4 Surface Gravity and Evaporation (Formal Statement)

Let $\chi^\mu = \partial_t^\mu + \Omega_H \partial_\varphi^\mu$ be the horizon generator for the stationary axisymmetric geometry. The surface gravity is

$$\kappa^2 = -\frac{1}{2} (\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)\Big|_{r=r_H}. \tag{20}$$

In the slow-rotation regime one has the expansion

$$\kappa = \kappa_0 + \epsilon^2 \kappa_2 + \mathcal{O}(\epsilon^3), \tag{21}$$

where κ_0 is determined by the static background (3) and κ_2 depends linearly on $(h_0, m_0, h_2, m_2, k_2)$ evaluated at r_H. If the high-curvature regime produces $f'(R_0) \gg 1$ near r_H and if κ_0 is regulated by the R^2 corrections, then the semiclassical temperature $T_H = \kappa/(2\pi)$ is not forced to diverge as $M \to M_{\rm Pl}$, and remnant formation becomes consistent with the modified near-horizon field equations.

5 Cosmological Interpretation

Assume that a nonzero fraction of the early-universe energy density collapses into microscopic horizons with $r_H \sim \sqrt{a}$ and with angular momentum in the slow-rotation regime of the above expansion (or near it). If evaporation is suppressed and leaves long-lived remnants with $M \sim M_{\rm Pl}$, then their macroscopic behavior is that of pressureless matter:

$$p_{\rm BH} \approx 0, \qquad \rho_{\rm BH}(t) \propto a(t)^{-3}. \tag{22}$$

Such relics can therefore contribute to the cold dark matter budget.

6 Conclusion

A Hartle–Thorne slow-rotation expansion through second order in ϵ for $f(R) = R + aR^2$ gravity yields: (i) a first-order frame-dragging equation with curvature-suppressed gradient proportional to $1/f'(R_0)$, and (ii) second-order monopole and quadrupole deformation equations sourced by ω'^2 and curvature multipoles. In the high-curvature regime $f'(R_0) \gg 1$, rotational deformations are correspondingly suppressed, providing a controlled setting in which Planck-scale rotating black holes can be long-lived and serve as dark matter candidates.