Economics 121, Fall 2016
Exam 1 Solutions

1 Question 1

1.1 Setting up the problem

$$\underset{x_1,x_2}{max}\{x_1^2x_2^3\} s.t. p_1x_1 + p_2x_2 = I$$

$$\mathscr{L} = x_1^2x_2^3 + \lambda(x_1p_1 + x_2p_2 - I)$$

First order conditions:

$$\frac{\partial \mathscr{L}}{\partial x_1} = 2x_1x_2^3 + \lambda p_1 = 0$$

$$\frac{\partial \mathscr{L}}{\partial x_2} = 3x_1^2x_2^2 + \lambda p_2 = 0$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = x_1p_1 + x_2p_2 - I = 0$$

1.2 Demand

Dividing the first FOC by the second FOC:

$$\frac{2x_1x_2^3}{3x_1^2x_2^2} = \frac{-\lambda p_1}{-\lambda p_2} \Rightarrow \frac{2x_2}{3x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{3x_1p_1}{2p_2}$$

Plugging in x_2 in the budget constraint

$$x_1p_1 + \frac{3x_1p_1}{2p_2}p_2 = I \Rightarrow 2.5x_1p_1 = I \Rightarrow x_1 = \frac{2I}{5p_1}$$

$$x_2 = \frac{3\frac{2I}{5p_1}p_1}{2p_2} = \frac{3I}{5p_2}$$

$$(x_1^*, x_2^*) = (\frac{2I}{5p_1}, \frac{3I}{5p_2})$$

1.3 Tax

The new budget constraint:

$$p_1x_1 + (1+r)p_2x_2 = I + rp_2x_2^*$$

Verifying that the bundle (x_1^*, x_2^*) is still affordable

$$p_1x_1^* + (1+r)p_2x_2^* \overset{?}{\leq} I + rp_2x_2^*$$

$$p_1x_1^* + p_2x_2^* + rp_2x_2^* \overset{?}{\leq} I + rp_2x_2^*$$

$$p_1x_1^* + p_2x_2^* = I$$

We know that this equation always holds due to the budget constraint (we made sure the budget constraint holds for (x_1^*, x_2^*) when solving question [1.2]).

*Alternative solution***:**

We can plug in the value for (x_1^*, x_2^*)

$$p_1\frac{2I}{5p_1} + (1+r)p_2\frac{3I}{5p_2} \overset{?}{<} I + r\frac{3I}{5p_2}p_2$$

$$\frac{2I}{5} + (1+r)\frac{3I}{5} \overset{?}{<} I + r\frac{3I}{5}$$

$$I + r\frac{3I}{5} = I + r\frac{3I}{5}$$

Solving the consumer's new optimization problem

Notice that the problem is identical to the original problem but with a different income and with a different price. We can call the new income I^t and the new price p_2^t

$p_2^t = (1+r)p_2$

$I^t = I + rp_2x_2^*$

$p_1^t = p_1$ (price 1 did not change)

Now we can plug in these income and price in our demand functions from question [1.2]

$$(x_1^t, x_2^t) = (\frac{2I^t}{5p_1^t}, \frac{3I^t}{5p_2^t}) = (\frac{2(I + rp_2x_2^*)}{5p_1}, \frac{3(I + rp_2x_2^*)}{5(1+r)p_2})$$

Alternative solution:

$$\mathscr{L} = x_1^2x_2^3 + \lambda[x_1p_1 + x_2p_2(1+r) - I - rp_2x_2^*]$$

FOC:

$$\frac{\partial\mathscr{L}}{\partial x_1} = 2x_1x_2^3 + \lambda p_1 = 0$$

$$\frac{\partial\mathscr{L}}{\partial x_1} = 3x_1^2x_2^2 + \lambda p_2(1+r) = 0$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = x_1 p_1 + x_2 p_2 (1+r) - I - r p_2 x_2^* = 0$$

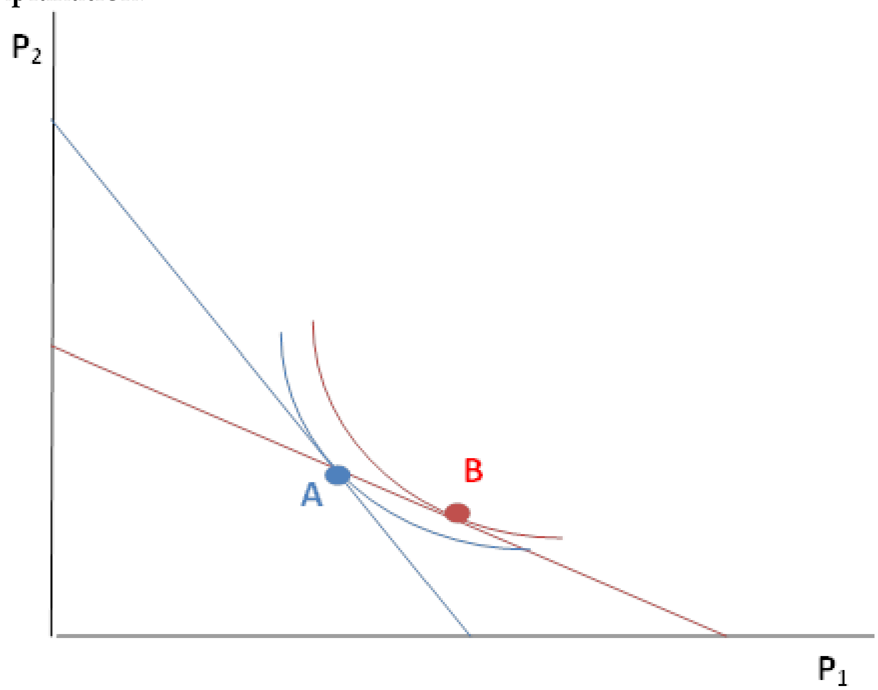
$$\frac{2x_2}{3x_1} = \frac{p_1}{p_2(1+r)} \Rightarrow x_2 = \frac{3x_1 p_1}{2p_2(1+r)}$$

Plugging in x_2 in the budget constraint

$$x_1 p_1 + \frac{3x_1 p_1}{2p_2(1+r)} p_2(1+r) = I + r p_2 x_2^* \Rightarrow x_1^t = \frac{2(I + r p_2 x_2^*)}{5p_1}$$

$$x_2^t = \frac{3\frac{2(I+rp_2x_2^*)}{5p_1} p_1}{2p_2(1+r)} = \frac{3I(I + r p_2 x_2^*)}{5p_2(1+r)}$$

Explanation:



The consumer's original bundle is point A where the original budget line is tangent to the indifferent curve. After the changes in prices and income the consumer moves to the red budget line. Notice that the original bundle is still affordable (this is what we checked in the previous part) and therefore the budget lines have to intersect at point A. Also notice that the new budget line is less steep than the previous budget line, since the second good is now more expansive. Finally, notice that there are points in the new budget lines (such as point B) that are "northeast" to the original indifference curve, and therefore necessarily increase utility. Point B increase consumption of good 1 and decreases consumption of good 2.

1.4 Comparing optimal consumption bundles

$I = 60$, $p_1 = 4$, $p_2 = 6$, $r = 1$

$$(x_1^*, x_2^*) = (\frac{2I}{5p_1}, \frac{3I}{5p_2}) = (6, 6)$$

$$(x_1^t, x_2^t) = (\frac{2(I + rp_2x_2^*)}{5p_1}, \frac{3(I + rp_2x_2^*)}{5(1+r)p_2}) = (\frac{192}{20}, \frac{288}{60}) = (\frac{48}{5}, \frac{24}{5})$$

As the values above show, after the tax the consumer has increased the consumption of good 1 and decreased the consumption of good 2 (if we plug in the value to the utility function it is also possible to show that the utility of the consumer increased, but that is not required).

One's first thought might be that if the government both taxes and subsidizes the consumer, then the person's behavior might remain unchanged. A good first observation is then that the excise tax makes good 2 relatively more expensive. We would ordinarily expect this to prompt a decrease in good 2 (we must be careful, lest good 2 is a Giffen good, though we know these are rare) and to have an ambiguous effect on good 1 (depending on whether they are substitutes or complements). Noting the change in relative prices is thus a first step, but something more is needed. The next point is to observe that the two taxes are calculated so that the original consumption bundle is still affordable. This ensures that the new budget makes available some bundles that were previously unaffordable, and that the person prefers. But given the increase in the price of good 2 and the affordability of the original bundle, the only such new bundles involve less good 2 and more good 1.

Intuitively, the tax was an excise tax, imposed on one good as a function of its consumption, while the subsidy was a lump sum subsidy that is not affected by the consumer's behavior. An excise tax makes one good relatively more expensive than the other. Even if the consumer receives the amount of the tax that the consumer would have paid in if the persoins' behavior remaing unchanged back in the form of a subsidy, the effect on prices remain and encourages the consumer to substitute units of good 2 with units of good 1.

2 Question Two

2.1. (16 points)

The Lagrangian is:

$$\mathcal{L} = x_1 - x_1^2 + x_2 + \lambda\,(p_1x_1 + x_2 - 10)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 - 2x_1 + \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 + \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = p_1x_1 + x_2 - 10 = 0$$

To find the demand function we rearrange the first two FOCs as:

$$1 - 2x_1 = -\lambda p_1$$

$$1 = -\lambda$$

We then divide these to get:

$$\frac{1-2x_1}{1} = \frac{-\lambda p_1}{\lambda}$$

$$1 - 2x_1 = p_1$$

Then we rearrange this to get:

$$x_1(p_1) = \frac{1-p_1}{2}$$

Note that unlike most problems we have studied this does not depend on income.
The elasticity of demand for good 1 iis:

$$e(p_1, p_2, I) = -\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1}$$

Computing the derivative

$$\frac{\partial x_1}{\partial p_1} = -\frac{1}{2}$$

So our elasticity of demand is:

$$e(p_1) = -\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1} = \frac{p_1}{2x_1}$$

Subsituting in our equation for x_1 we get:

$$e(p_1) = \frac{p_1}{2\frac{1-p_1}{2}} = \frac{p_1}{1-p_1}$$

As we noted in class, an alternative approach to elasticity is to look at the derivative of expenditure - $\frac{\partial}{\partial p_1}p_1 x_1$. This tells you something about elasticity, and in particular, it will tell you whether demand is elastic or inelastic, but it does not give you the elasticity. If you are asked simply whether demand is elastic or inelastic, looking at the derivative of expenditure is fine, but if asked to find the elasticity, one must do more. Notice also that once we get to the stage x_1 - i.e.$e(p_1) = \frac{p_1}{2x_1}$, we are not yet done, because x_1 depends upon p_1. We want to find elasticity entirely in terms of p_1, p_2, and I.

The good will be elastic if $e > 1$, inelastic if $e < 1$, and unit elastic if $e = 1$.
If $e > 1$

$$\frac{p_1}{1-p_1} > 1$$

$$p_1 > 1 - p_1$$

$$2p_1 > 1$$

$$p_1 > \frac{1}{2}$$

Likewise if $e < 1$.

$$\frac{p_1}{1 - p_1} < 1$$

$$p_1 < 1 - p_1$$

$$2p_1 < 1$$

$$p_1 < \frac{1}{2}$$

and if $e = 1$,

$$\frac{p_1}{1 - p_1} = 1$$

$$p_1 = 1 - p_1$$

$$2p_1 = 1$$

$$p_1 = \frac{1}{2}$$

Thus for $0 < p_1 < \frac{1}{2}$ demand for good 1 will be inelastic, for $p_1 = \frac{1}{2}$ it will be unit elastic, and for $p > \frac{1}{2}$ it will be elastic.

2.2. (10 points)

Remember that the elasticity is given by:

$$e(p_1) = -\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1}$$

Computing the derivative of our new demand function $x_1 = \frac{I - \bar{x}_2}{p_1}$ we get:

$$\frac{\partial x_1}{\partial p_1} = -\frac{I - \bar{x}_2}{p_1^2}$$

So we get:

$$e(p_1) = -\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1} = \frac{p_1}{x_1}\frac{I - \bar{x}_2}{p_1^2}$$

and substituting in our equation for x_1 (don't forget to use the new demand function here) we get:

$$e(p_1) = \frac{p_1}{\frac{I - \bar{x}_2}{p_1}}\frac{I - \bar{x}_2}{p_1^2} = \frac{(I - \bar{x}_2)p_1^2}{(I - \bar{x}_2)p_1^2} = 1$$

So our elasticity is now 1 (unit elastic) for any choice of prices p_1. Alternately one could have noted that unit elasticity means that expenditure on a good doesn't change as the prices do. Our expenditure on good 1 will always be everything we have left over after purchasing good two - so it shouldn't be affected by the price for good 1.

This is different from our elasticity we found in part 2.1 - because there the elasticity depended on price. The long-run demand may be more elastic or less elastic than the short-run demand, depending on price.

2.3 (6 points)

A good first step in answering this question is to note that, from our results we can tell that demand is not necessarily more elastic in the long-run. In our example depending on prices demand could be either more elastic, less elastic, or exactly the same in the long-run relative to the short-run.

A good next step is to think about why we expect demand to be more elastic in the long run. The idea is that it is easier to adjust one's consumption in the long run than in the short run. For example, if the price of oil increases, in the long run one might buy an electric car, move closer to work, install solar panels, insulate one's house, and so on. All of these will allow a person to consume less oil, but all take time, and are more easily accomplished in the loing run than in the short run. Hence, we expect quantity to be more responsive to price in the long run, and so for demand to be more elastic in the long run.

Now let's compare this intuition to the work we have just done. In the example, we were thinking about an increase in the price of oil, and were arguing that in the long run it is easier to adjust one's consumption of oil in the long run than in the short run. In [2.2], however, we were talking about a change in the price of good 1, but then comparing a short run situation in which it was difficult to adjust the quantity of good 2 with a long run situation in which the quantity of good 2 could be freely adjusted. Instead, we might have approximated the considerations of the previous paragraph by assuming that in the short run, it is impossible to adjust the quantity of good 1, and that good 1 can be freely adjusted in the long run. If we had done this, we would indeed have found that demand is more elastic in the long run than in the short run.

Finally, it would be good to put these findings together. Will demand be more elastic in the long run? If the salient difference between the long run is that it is easier to adjust one's consumption of the good in question, then generally yes, demand will be more elastic in the long run. This is the usual thinking. If the salient difference is that it is easier to adjust the quantities of other goods, then things are less clear.

3 Question 3

3.1

Intuitively, if both goods were to have zero income effects, then an increase in income would lead to zero consumption change. This means that the consumer is not fully spending her income, which cannot be utility maximizing.

We can check this intuition in the following way.

First, we write the budget constraint. This is written in more detail in the exam. They key is to recognize that the x_1 and x_2 appearing here are not arbitrary, but are given by the demand functions.

$$p_1 x_1 + p_2 x_2 = I$$

Then we differentiate it with respect to I.

$$p_1 \frac{dx_1}{dI} + p_2 \frac{dx_2}{dI} = 1$$

We can now verify that it cannot be the case that both goods have zero income effects. Both goods having zero income effects would mean that $\frac{dx_1}{dI} = 0 = \frac{dx_2}{dI}$, which cannot satisfy the equation above. Nonetheless, it may be possible that one good has a zero income effect; the other good now has to have a positive income effect to satisfy this equation.

3.2

To find the compensated demand function for this good, we set up the expenditure minimization problem.

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2$$
$$\text{s.t.}$$
$$x_1 - x_1^2 + x_2 = \bar{U}$$

Then we set up the corresponding Lagrangian problem.

$$L(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \lambda(x_1 - x_1^2 + x_2 - \bar{U})$$

Next, we derive the first order conditions.

$$\frac{dL}{dx_1} = p_1 + \lambda(1 - 2x_1) = 0$$
$$\frac{dL}{dx_2} = p_2 + \lambda = 0$$
$$\frac{dL}{d\lambda} = x_1 - x_1^2 + x_2 - \bar{U} = 0$$

Dividing $\frac{dL}{dx_1}$ by $\frac{dL}{dx_2}$, we get $x_1^c = \frac{p_2 - p_1}{2p_2}$, which is equal to the ordinary demand function that we derived in [2.1].