1 Question 1

1.1 The Lagrange function is:

$$L(x_1, x_2, \lambda) = x_1 - \frac{1}{2}x_1^2 + x_2 - \frac{1}{2}x_2^2 + \lambda[I - p_1x_1 - p_2x_2]$$

Thus, the first order conditions (FOCs) become:

$$\begin{aligned}
&L_1 = 1 - x_1 - \lambda p_1 = 0 \\
&L_2 = 1 - x_2 - \lambda p_2 = 0 \\
&I - p_1x_1 - p_2x_2 = 0
\end{aligned}$$

1.2 The second order conditions (SOCs) are:

$$\begin{vmatrix} 0 & -p_1 \\ -p_1 & -1 \end{vmatrix} = -p_1^2 < 0$$

$$\begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & -1 & 0 \\ -p_2 & 0 & -1 \end{vmatrix} = p_1^2 + p_2^2 > 0$$

Notice that this solution expands the determinants, but you were not asked to do so, you needed only to state the inequalities involving the determinants.

1.3 To find the demand for good 1, lets work with the first two FOCs. From these, we get:

$$\frac{1 - x_1}{1 - x_2} = \frac{p_1}{p_2} \tag{1.1}$$

We can then solve (1.1) for x_2 and replace in L_3 to obtain (after some simple rearrangements) the demand for good 1:

$$x_1(p_1, p_2, I) = \frac{p_1I - p_2p_1 + p_2^2}{p_1^2 + p_2^2}$$

It is then easy to see that good 1 is normal:

$$\frac{dx_1(p_1, p_2, I)}{dI} = \frac{p_1}{p_1^2 + p_2^2} > 0 \tag{1.2}$$

1.4 This part of this question is quite difficult. A common answer here was that the utility function failed to be quaisconcave. This is an understandable impression, since the squared terms make the utility function look similar to a function ($U(x_1, x_2) = x_1^2 + x_2^2$) that we have examined in a problem set, and that is not quasiconcave. Here, however, the squared terms enter with a negative rather than a positive sign. As it happens, this utility function is quaisconcave, though that is not easy to show.

The difficulty with this utility function is that it is not increasing everywhere. The function is increasing in x_1 and x_2 for relatively small values of these variables, but not for larger values. We can see this by noticing that the marginal utilities are not positive everywhere. That is, for good $i \in \{1, 2\}$ we have:

$$\frac{dU}{dx_i} = 1 - x_i. \tag{1.3}$$

This marginal utility is positive, and hence the utility function is increasing, as long s $x_1 < 1$ (with a similar result for x_2). Notice that we have said nothing about the units in which x_1 and x_2 are measured, so a value of 1 may be a tiny or an enormous quantity of the good in question.

Does it make sense to work with a utility function that is not increasing everywhere? This depends on income and prices. If I is not too large and p_1 and p_2 not too mall, then when we solve the first order conditions, we will find a solution on the increasing part of the utility function. More generally, if I is not too large and p_1 and p_2 not too mall, then the utility function will be increasing throughout the feasible set. In this case, it is irrelevant that the utility function is decreasing outside the feasible set, and we can proceed as usual. Hence, the FOCs will identify a maximum whenever the income is not "too high" with respect to the prices. Indeed, as long as the solution (i.e, $x_1(p_1, p_2, I)$ and $x_2(p_1, p_2, I)$) to this problem does not involve negative marginal utilities, the FOCs will pin down a maximum. Intuitively, whenever we wrote and solve the FOCs we are assuming that the budget constraint holds with equality. As long as we are sill on the increasing part of the utility function this will be true. To see this a bit more formally, notice that if we evaluate (1.3) at the optimal demand of, say, good 1, we get that:

$$\frac{dU}{dx_1} < 0 \Leftrightarrow I > p_1 + p_2 \tag{1.4}$$

What do we make of second-order conditions in this case? The key here is to notice that they are independent of $x_1(p_1, p_2, I)$ and $x_2(p_1, p_2, I)$. They depend only on prices, and even here in such a way that they will always be satisfied. Hence, if income is not "too high" (in the sense of (1.4)), we can solve the first order conditions, confirm that the second order conditions hold, and be confident that we have found a maximum. Now suppose that (1.4) is violated. In this case we can solve the first-order conditions and again confirm that the second-order conditions hold, but we have not found a maximum. What has gone wrong? When we solves a similar problem (on the problem set) with the utility function $U(x_1, x_2) = x_1^2 + x_2^2$, the second-=order conditions showed us that we have am minimum. But here we do not have a minimum, because the second-order conditions for a maximum hold. Instead, what we have here is a local maximum, but not a global maximum. In particular, when we form the Lagrangian, we are assuming that the budget constraint holds with equality. When the utility function is everywhere increasing, this is automatic, and poses no problems. In this cases, with a utility function is not everywhere increasing, the budget constraint may not hold with equality. In this case, the first-order conditions fine a local maximum, meaning that of all the points on the budget line, they pick out the one with the highest utility (and hence the second-order conditions hold). But if (1.4) fails, there is an even higher utility available from some point inside the budget line.

The type of utility function examined in this problem is convenient and is often used, because it is easy to work with. But if one uses it, one should make sure that tne environment is such that (1.4) holds.

2 Question 2

2.1. The agent's utility maximization problem is:

$$Max_{c,h} \;\; U(c,h)$$

$$s.t. \;\; c = w(T - h - kc) + I$$

The Lagrangian is

$$\mathcal{L} = U(c,h) + \lambda(c(1+wk) - wT + wh - I)$$

First-order conditions are given by

$$\frac{d\mathcal{L}}{dc} = \frac{dU}{dc} + \lambda(1+wk) = 0$$

$$\frac{d\mathcal{L}}{dh} = \frac{dU}{dh} + \lambda w = 0$$

$$\frac{d\mathcal{L}}{d\lambda} = c(1+wk) - wT + wh - I = 0$$

From the first two first-order conditions, we get:

$$\frac{\frac{dU}{dc}}{\frac{dU}{dh}} = \frac{1+wk}{w}$$

The left side is the marginal rate of substitution and the right side is the effective price of consumption relative to price of leisure. In particular, the important point here is to notice that the price of consumption is effectively $1+wk$. For an example of how this might be important, one explanation for increasing obesity rates in the US is that the price of food has fallen. Much of this price decrease has taken place not in the form of lower prices at the supermarket or restaurant, but reductions in preparation time as a result of new food handling and preparation techniques and new (primarily fast food) restaurants.

2.2. We have with the following relationship between ordinary and compensated demand functions.

$$h^c\left(w,\overline{U}\right) = h\left(w, E\left(w,\overline{U}\right)\right)$$

Differentiating both sides w.r.t w, we get

$$\frac{dh^c}{dw} = \frac{dh}{dw} + \frac{dh}{dE}\frac{dE}{dw}$$

$$\Longrightarrow \frac{dh^c}{dw} = \frac{dh}{dw} + \frac{dh}{dI}\left(ck + h - T\right)$$

The term in the brackets above is derived as follows. Firstly, the expenditure function is given by

$$E\left(w, \overline{U}\right) = c\left(1 + wk\right) + wh - wT$$

Differentiating both sides w.r.t w, we get

$$\frac{dE}{dw} = ck + h - T + \left(1 + wk\right)\frac{dc}{dw} + w\frac{dh}{dw}$$

Now consider the expenditure minimization problem (EMP) for this case.

$$Min_{c,h} \;\; c\left(1 + wk\right) + wh - wT$$

$$s.t. \;\; U\left(c, h\right) = \overline{U}$$

The Lagrangian is

$$\mathcal{L} = c\left(1 + wk\right) + wh - wT + \mu\left(U\left(c, h\right) - \overline{U}\right)$$

Note: we denote the multiplier by μ since we already used λ in 2.1. First-order conditions are given by

$$\frac{d\mathcal{L}}{dc} = 1 + wk + \mu\frac{dU}{dc} = 0$$

$$\frac{d\mathcal{L}}{dh} = w + \mu\frac{dU}{dh} = 0$$

$$\frac{d\mathcal{L}}{d\mu} = U\left(c, h\right) - \overline{U} = 0$$

Using the first two FOCs in the equation for $\frac{dE}{dw}$ that we derived above , we get

$$\frac{dE}{dw} = ck + h - T - \mu\frac{dU}{dc}\frac{dc}{dw} - \mu\frac{dU}{dh}\frac{dh}{dw}$$

which simplifies to

$$\frac{dE}{dw} = ck + h - T - \mu\frac{dU}{dw} = ck + h - T$$

where $\frac{dU}{dw} = 0$ follows by differentiating the constraint $U\left(c, h\right) = \overline{U}$ w.r.t to w.

Finally, we can rewrite the equation that contains the demand derivatives to obtain the Slutzky equation in the usual form

$$\frac{dh}{dw} = \frac{dh^c}{dw} + \left(T - ck - h\right)\frac{dh}{dI}$$

2.3. This question asked you to use your answer to question 2.1 to comment on when leisure would be a Giffen or not a Giffen good. Many people simply assumed that the reference should have been to question [2.2]. A few people asked whether it should be to [2.1] or [2.2]. Suspect that many others were simply interpreting it to be [2.2], the standard response was "you can use either one."

So, let us start with [2.2]. This gives you the Slutzky equation

$$\frac{dh}{dw} = \frac{dh^c}{dw} + (T - ck - h)\,\frac{dh}{dI}.$$

From this, we can see that leisure will be a Giffen good (i.e. $\frac{dh}{dw} > 0$) when $\frac{dh}{dI}$ is positive and large enough so that it can offset the substitution effect ($\frac{dh^c}{dw}$), which is always negative. This is a point that we have made in class. Why stress this? It is common to talk about Giffen goods in the ordinary consumption context, where a Giffen good must be a strongly inferior good. This is sufficiently common that one can find assertions that that all Giffen goods must be inferior. Here is a context where a Giffen good need not be inferior, a context that is important from a policy point of view (since much of our current economic discussion and policy is concerned with labor market participation), and a context in which goods may well be Giffen.

Next, how would we use 2.1 to answer this question? The idea here was that 2.1 directed your attention to the fact that the price of consumption was 1+kw. Hence, changes in w affect not only the price of leisure, but also the price of consumption.

To see what difference this makes, first consider what many texts offer as the standard Slutzky equation (written for good 2):

$$\frac{dx_2}{dp_2} = \frac{dx_1^c}{dp_2} - x_2\frac{dx_2}{dI}.$$

Here, the price of good 2 affects only the price of good 2, and we get a Slutzky equation in which it takes a strongly inferior good to give a positively sloped demand curve (i.e., a Giffen good). Now consider the standard labor/leisure case, where the Slutzky equation is

$$\frac{dh}{dw} = \frac{dh^c}{dw} + (T - h)\,\frac{dh}{dI}$$

How the price w of good 2 (leisure) affects both the price of leisure and income. In terms of the Slutzky equation, other than changes in notation, the changes is that $-x_2$ (which is negative) has been replaced by $(T - h)$, which is positive, so that a positive sloped demand function now requires a strongly positive income effect. Finally, look at the current setting, where the Slutzky equation is

$$\frac{dh}{dw} = \frac{dh^c}{dw} + (T - ck - h)\,\frac{dh}{dI}.$$

Here, the price w of good 2 (leisure) affects the price of good 2, affects income, and also affects the price of good 1. The effect on the Slutsky equation is that $T - h$ is now replaced by $T - ck - h$, which is still positive, but is likely to be smaller. This means that it takes an even larger income effect to give a positively sloped demand, and makes a Giffen good less likely.

3 Question 3

3.1 Question 3.1

Under proportional income tax scheme at rate t in both periods, agent's budget constraint becomes:

$$c + \frac{c}{1+r} + \frac{B}{1+r} = (1-t)I + \frac{(1-t)I}{1+r}. \tag{3.1}$$

Under excise tax on consumption at rate τ in both periods, agent's budget constraint becomes:

$$(1+\tau)c + \frac{(1+\tau)c}{1+r} + \frac{B}{1+r} = I + \frac{I}{1+r}. \tag{3.2}$$

3.2 Question 3.2

The key this question is to note that, other than differences in notation, the argument matches the case considered in class.

The same tax revenue assumption gives us the following relationship:

$$\tau c^{\tau} + \frac{\tau c^{\tau}}{1+r} = tI + \frac{tI}{1+r}, \tag{3.3}$$

where c^{τ} is the optimal consumption bundle under excise tax scheme.

First, show optimal bundle under excise tax scheme, (c^{τ}, B^{τ}) satisfies budget constraint of one under income tax. Plug (c^{τ}, B^{τ}) in income tax budget constraint:

$$c^{\tau} + \frac{c^{\tau}}{1+r} + \frac{B^{\tau}}{1+r} = (1-t)I + \frac{(1-t)I}{1+r}, \tag{3.4}$$

which holds if and only if,

$$c^{\tau} + \frac{c^{\tau}}{1+r} + \frac{B^{\tau}}{1+r} = I + \frac{I}{1+r} - (tI + \frac{tI}{1+r}), \tag{3.5}$$

which holds if and only if (by plugging in same tax revenue relationship),

$$c^{\tau} + \frac{c^{\tau}}{1+r} + \frac{B^{\tau}}{1+r} = I + \frac{I}{1+r} - (\tau c^{\tau} + \frac{\tau c^{\tau}}{1+r}), \tag{3.6}$$

which is after rearrangement,

$$(1+\tau)c^{\tau} + \frac{(1+\tau)c^{\tau}}{1+r} + \frac{B^{\tau}}{1+r} = I + \frac{I}{1+r}. \tag{3.7}$$

The last relationship holds by definition of budget constraint under excise tax. For utility comparison, a diagram is helpful. Notice that we cannot run this argument the other way—it is not the case that the optimal bundle under the income tax must lie on the excise tax budget line, and this is the root of the asymmetry between the taxes.

Note that after income tax relative prices do not change; however with excise tax it does as effective price of consumption is $(1+\tau)$, instead of 1.

The indifference curve in the figure represents the utility maximization of agent under excise tax scheme, hence this indifference curve is tangent to *excise* tax budget constraint, which is flatter than income tax budget curve. This tangency has to occur at the intersection of two budget curves, as we proved the excise tax bundle should satisfy income tax budget constraint (and it satisfies excise tax budget constraint by definition). Because of this, there will be some part of income tax budget constraint lying above the indifference curve. Since utility is increasing in (c, B), the agent can find bundles giving strictly higher utility and feasible (on or below budget constraint of the income tax). As agent is utility maximizer, she will find higher utility level under income tax scheme, hence income tax is better if optimal level of bequest B is positive.

3.3 Question 3.3

First, notice that if the agents in our model do not derive utility from bequests, ie. $u(c, B) = u(c)$ and $B = 0$, the the agent will spend all of her income on consumption:

$$c^{\tau} = \frac{I}{1+\tau}, c^{t} = (1-t)I \tag{3.8}$$

Using the same tax revenue relationship and plugging in c^{τ} value we get:

$$t = \frac{\tau}{1+\tau}. \tag{3.9}$$

But then budget constraint under two schemes become identical, ie. consumption level. We can see this by plugging previous equation into eg. income tax consumption:

$$c^{t} = (1 - \frac{\tau}{1+\tau})I = \frac{I}{1+\tau} = c^{\tau}. \tag{3.10}$$

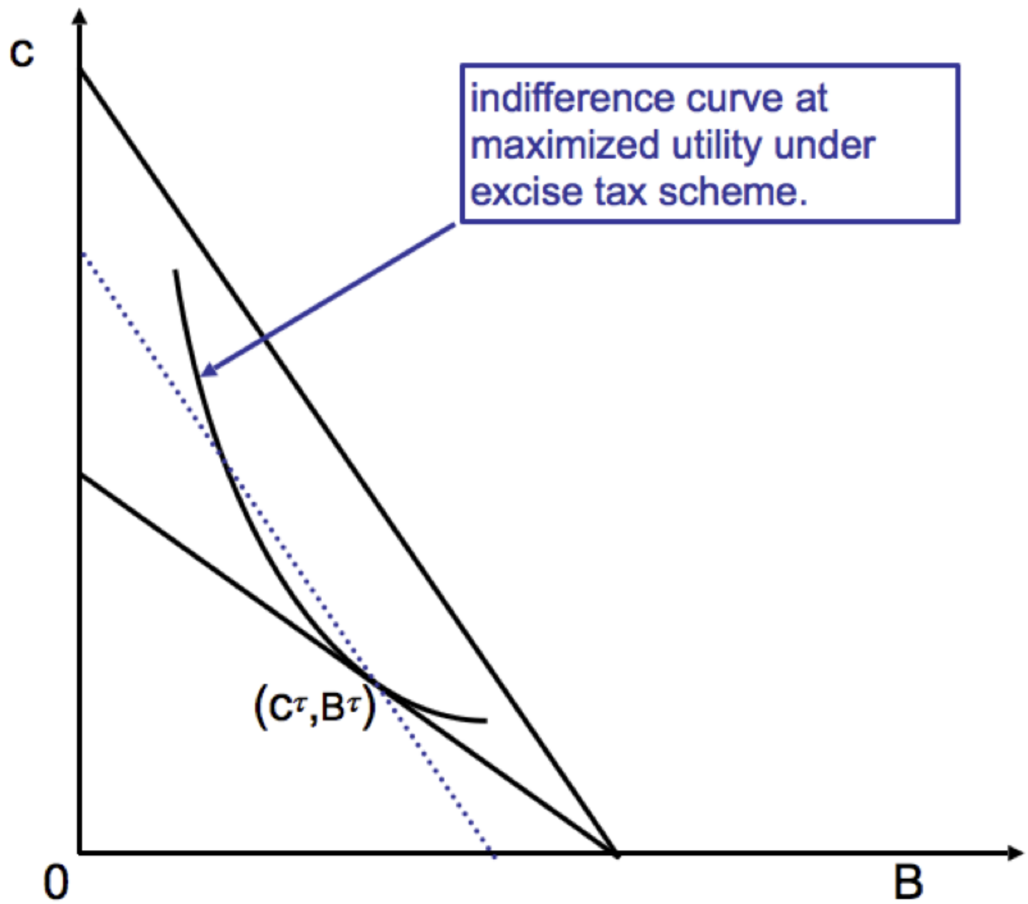
In this case, our model suggests that the two tax schemes are equivalent. One could then make a cost for the advantages of either of them, based on considerations outside the model, perhaps arising out of a view that one is easier to collect than another, of that one more readily gives rise to fraud, and so on.

Another setting under which the two taxes would be equivalent can be obtained by considering and economy of overlapping generations, as in the problem set. For example, consider an agent altruistically cares about her future generations. This would mean she cares about bequests because it increases consumption of her offspring, ie $u = \sum_{i=1}^{\infty} u(c_i)$ (assuming separable/additive structure across generations). Assume each generation i live for two periods (i,i+1), and leaves bequest B_i at $i+1$. B_i can be used by generation $i+2$, who lives for $(i+2, i+3)$. Then intuitively, what would happen is parent's bequest enters as income for next generation, which goes to consumption of offspring. So basically, the thing parents compare boils down to their own consumption versus next i's generation consumption. Then if we write the marginal rate of substitution and equate to price ratio of consumption, we have the following relation under both income tax and excise tax scheme:

$$\frac{u'(c_0)}{u'(c_i)} = \frac{1}{\frac{1}{(1+r)^i}} = (1+r)^i. \tag{3.11}$$

($(1+\tau)$ cancels out under excise tax). Then it remains to show budget constraints remain same under both schemes. For this, intuitively we can say parent's bequest enters as expenditure (LHS of budget constraint) and enters as income for offspring budget constraint, so in Lagrange problem, bequests "cancel out". Hence, we can treat budget constraints in Lagrangian as if bequests are not there (ie $B = 0$). Then we expect from previous discussion for $B = 0$, income or excise tax leaves agents indifferent. Once again, there are many considerations outside the model that might prompt one to prefer one tax or the other.

The most obvious consideration left outside the model is that we have assumed that there are no obstacles to the ability of people to borrow and save, so that they can freely move their income across time to achieve any consumption bundle consistent with their budget constraint. In practice, many people cannot easily carry their consumptions across periods. In this case it depends on what kind of imperfection we have, and depending on the case, we may have income or excise tax better.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