Stochastic Field Theory for Pricing Time Series

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Abstract

Various statistical processes with and without the presence of noise can be modeled as field theories in the continuum limit. We see that field theory may form an efficient alternative to the standard AI based approaches when discovering correlations in many realistic stochastic time series.

1 Introduction

The use of field theoretic methods in modeling options pricing has been well studied. There are many potential benefits of applying field theoretic methods to the equity market(see Kleinert). We make the latter rigorous in this paper, calculating the general forms of correlation functions in terms of parameters of the effective field theory. There are parameters that must be fitted to the data-a finite set of n couplings ('numbers') as opposed to functional estimation that is required in ML and neural networks. Clearly the amount of arbitrariness and the risk of over-fitting is exponentially lesser.

1.1 Connection with Neural Networks

The fashionable way to find correlations in time series is to use neural networks. Our approach might appear completely independent of it but there is a deep link between the two. In the literature, it has been often noted that in the continuum limit, neural networks can be approximated by a renormalization group flow.

The rest can be found here https://arxiv.org/pdf/1410.3831v1.pdf

1.2 Connection with ARCH, GARCH, ARMA

All classes of autoregressive models are highly specialized instances of the general field theory. In particular all the parameters $a_{i,j}$ are simply couplings that appear in the effective field theory Lagrangian some of which may be eliminated by simple symmetry checks. Others are fit from observation. A more detailed discussion appears in a later section.

2 Pricing as a Field Theory

2.1 Introduction

The effect of Gaussian noise in strongly coupled field theories has been well studied, both in and outside of a holographic context. An analytic solution can be obtained for this model. This simplified model is equivalent to the efficient market hypothesis; in practice, however, options models have been observed to be non-Gaussian (i.e, options markets have various inefficiencies). We expect the noise distribution to depend upon an assortment of macroeconomic variables. Such a noise dependence upon being integrated out yields modified interaction term(s) in the potential. We are in the regime of an interacting field theory, where a perturbation theory can be setup in terms of Feynman diagrams. The exploration of the non-perturbative regime is more non-trivial; however, we shall see that the presence of gravity duals can simplify this analysis considerably.

2.2 Linear Gaussian Case (Black Scholes)

Nakayama constructed super-symmetric path integrals for the Black Scholes Merton model. We begin with the Langevin Equation:
$\frac{\partial \phi(t)}{\partial t} = \frac{\partial V(\phi)}{\partial \phi} + \sigma \eta(t)$
Where $\eta(t)$ satisfies
$< \eta(t)\eta(0) >= \delta(t)$
The partition function corresponding to this is:
$Z = \int DXD\eta J(\phi)\delta(\frac{\partial \phi(t)}{\partial t} - \frac{\partial V(\phi)}{\partial \phi} - \sigma\eta(t))exp(-\int dt\frac{\eta^2}{2})$
Where $J(\phi)$ is the Jacobian associated with the imposition of the delta function constraint. It can be written as a path integral over Grassman fields:
$J(X) = \int D\psi D\bar{\psi} exp(= \frac{1}{\sigma^2}\int dt\bar{\psi}\partial_t\psi - \frac{\partial^2 V}{\partial X^2}\bar{\psi}\psi)$
Once the Gaussian noise is integrated out, we get:
$S_{bos} = \int dt\frac{1}{2\sigma}(\partial_t\phi - \partial_\phi W)^2$
One way to write the. field theory is without integrating out the delta function but instead writing it thus:
$\delta(...) = \int D\tilde{\phi}e^{\tilde{\phi}(...)}$
We get the following partition function:
$Z = \int D\phi D\tilde{\phi}D\eta exp(\tilde{\phi}(\frac{\partial \phi(t)}{\partial t} - \frac{\partial V(\phi)}{\partial \phi} - \sigma\eta(t)) + \frac{\eta^2}{2})$
Upon explicitly performing the integral over η we get:
$Z = \int D\phi D\tilde{\phi} exp(\tilde{\phi}(\frac{\partial \phi(t)}{\partial t} - \frac{\partial V(\phi)}{\partial \phi}) + \frac{\tilde{\phi}^2}{2\sigma})$
Note that the system has a non-trivial one point function, which can be interpreted as a background source field. In the Feynman diagram representation in the theory, we may have legs that connect to the boundary field denoted in Figure–. In the Black Scholes Merton model, the super-potential is linear $W(\phi) = \mu\phi$. This corresponds to the OU process, with Feynman rules given by 1(a), 1(c) and 1(d).

2.3 Nonlinear Extensions of BS

Noise can be formulated as a field theory, and for small enough noise, we have a path integral system that can be solved by perturbation theory or effective field theory. The latter can be useful in organizing correlation functions in terms

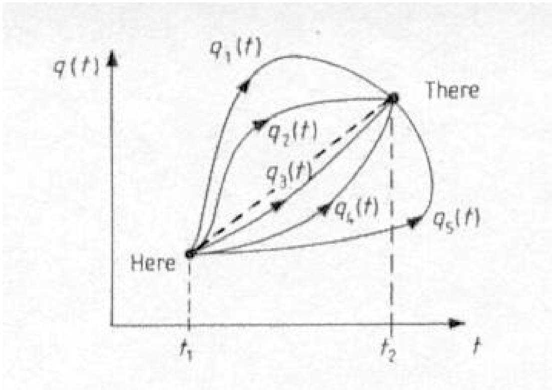

Figure 1: A schematic description of the path integral as a weighted sum over an infinity of possible trajectories of the price q(t) between two specified points

of Feynman diagrams. Tree level here encodes the linear regime, while loop expansions enter the non-linear regime. The effective action is:
$S_{bos} = \int dt \frac{1}{2\sigma^2}(\partial_t \phi - \partial_\phi V(\phi))^2$
The Black Scholes Merton model following the efficient market hypothesis corresponds to $W = \mu\phi$. Here we consider the most general distribution given thus:
$V(\phi) = \sum_{i=1}^{\infty} a_i \phi^i$
Where a_i are couplings that will in general change will scale $a_i(\Lambda, (M_j))$ (i.e, undergo a renormalization group flow) as well as possibly depend on macroeconomic variables. We can conduct a perturbative analysis in small coupling following the general treatment in the Appendix. First let us consider the Feynman rules. They are simply those given in Appendix 1 along with higher point vertices of the form corresponding to
$V_{m,n} = (\partial_t \phi(t))^m (\phi(t))^n$

3 Correlation Functions

The stochastic differential equation for the logarithms of prices reads:
$\cdot x = r_x + \eta(t)$
Where the noise variable $\eta(t)$ follows an as yet unspecified distribution. The constant drift is defined only up to a gauge choice- we choose the gauge where $< \eta(t) >= 0$. Now we can expand the exponential in a power series by Taylor expanding the Hamiltonian as follows:
$H(p) = ia_1 p + \frac{1}{2!} a_2 p^2 - i \frac{1}{3!} c_3 p^3 ...$
Where the drift has here simply been identified with the one point function:
$c_1 \rightarrow r_x$
Our stochastic differential equation now reads:
$\cdot (x)(t) = \eta(t)$
Now the probability of a path starting at (x_a, t_a) and ending at (x_b, t_b) is given by the following path integral:
$P(x_b, t_b | x_a, t_a) = \int D\eta \int_{x_{t_a}}^{x(t_b)=x_b} Dx exp(-\int_{t_a}^{t_b} dt H(\eta(t))) \delta(\cdot x - \eta)$
This expectation value of a quantity $f(x(t))$ at a particular point is given by

splitting the path integral into two parts:
$< f(x(t)) >= \int dxP(x_b, t_b|x, t)f(x)P(x, t|x_a, t_a)$
The expectation value is the 'one-point function'. This can be generalized to an n-point function defined by pinning the path integral at the n specified points:
$< f_1(x_1(t_1)...f_n(x_n(t_n))) >= \int \prod_i dx_i P(x_b, t_b|x_1, t_1)f(x_1)P(x_{t_2})f_2(x_2)...f(x_n)P(x_n, t_n|x_a, t_a)$
Initially we just focus on the correlation functions of the η variable. A correlation function can be expressed in terms of parameters of the position or momentum space Hamiltonian. We can go for one to the other thus:
$P(\eta(t)) = \int \frac{Dp}{2\pi} exp(\int_{t_a}^{t_b} dt(ip(t)\eta(t) - H(p(t))))$
For these kinds of actions, correlation functions can be constructed by simple functional differentiation of the following form:
$< \eta(t_1)...\eta(t_n) >= (-i)^n \int D\eta \int \frac{Dp}{2\pi}(\frac{\delta}{\delta(p(t_1))} \cdots \frac{\delta}{\delta(p(t_n))} e^{i\int_{t_a}^{t_b} p(t)\eta(t)})e^{-\int_{t_a}^{t_b} H(p(t))}$
We perform n partial integrations to get:
$< \eta(t_1)...\eta(t_n) >= (-i)^n \int D\eta \int \frac{Dp}{2\pi}(e^{i\int_{t_a}^{t_b} p(t)\eta(t)}) \frac{\delta}{\delta(p(t_1))} \cdots \frac{\delta}{\delta(p(t_n))} e^{-\int_{t_a}^{t_b} H(p(t))}$
$= i^n \frac{\delta}{\delta(p(t_1))} \cdots \frac{\delta}{\delta(p(t_n))} e^{-\int_{t_a}^{t_b} H(p(t))}$
We can thus get the lowest order correlation functions. The one point function vanishes by the definition of the variable η:
$< \eta(t_1) >= Z^{-1} \int D\eta\eta(t_1)exp(-\int_{t_a}^{t_b} dtH(\eta(t))) = 0$
For the rest we simply get down delta functions:
$< \eta(t_1)\eta(t_2) >= Z^{-1} \int D\eta\eta(t_1)\eta(t_2)exp(-\int_{t_a}^{t_b} dtH(\eta(t)))$
$= c_1^2 + c_2\delta(t_1 - t_2)$
$< \eta(t_1)\eta(t_2)\eta(t_3) >= Z^{-1} \int D\eta\eta(t_1)\eta(t_2)\eta(t_3)exp(-\int_{t_a}^{t_b} dtH(\eta(t)))$
$< \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) >= Z^{-1} \int D\eta\eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4)exp(-\int_{t_a}^{t_b} dtH(\eta(t)))$

3.1 Stochastic Volatility Models

It has been observed in market analysis for decades that 'volatility clusters'. Regimes of high volatility clump together, as do those of low volatility It is clear that the above formalism admits a straightforward generalization to models with time-varying volatility. This has been done in ...The simplest case is one in which the volatility is a Gaussian variable with mean η and standard deviation ζ. The stochastic differential equation is:
$\cdot\, x(t) = -\frac{v(t)}{2} + \sqrt{v(t)}\eta(t)$
Where the noise variable is:
$< \eta(t) >= 0, < \eta(t_1)\eta(t_2) >= \delta(t_1 - t_2)$
The variance follows the following equation:
$\cdot\, \sigma(t) = -(\sigma(t) - \sigma_0) + \epsilon\sqrt{\sigma(t)}\eta_\sigma(t)$
The parameter $\eta_\sigma(t)$ is the volatility of the volatility. the action is:
$S_{bos} = \int dt \frac{1}{2\sigma(t)}(\partial_t\phi - \partial_\phi V)^2 + \eta\sigma(t) + \zeta\sigma(t)^2$
Alternatively, using the auxiliary field formalism, the partition function can be written as:
$Z = \int D\phi D\tilde{\phi} D\sigma exp(\tilde{\phi}(\frac{\partial\phi(t)}{\partial t} - \frac{\partial V(\phi)}{\partial\phi} - \sigma\eta(t)) + \frac{\tilde{\phi}^2}{2\sigma(t)} + \frac{(\sigma(t)-\eta)^2}{2\zeta})$
This is a non-linear path integral over σ. It can only be evaluated order by order, or by treating σ as a background field. It is possible in principle to carry on making the volatility of the volatility ζ too stochastic, and so on.

3.2 Multi-scale Volatility

There is ample empirical evidence in the SP 500 that suggests that volatility has a mean reversion property at at least two time scales- one long and one short. In principle there may be more. In the canonical approach to stochastic volatility based on stochastic calculus, the consideration of multi-scale volatility is cumbersome even for two scales. The stochastic differential equation modeling it can be written thus:
$\cdot\, x = \mu + \sigma(t)\eta(t)$
The classical Black Scholes model corresponds to $\sigma(t)$ being a constant. In a volatility process with n time scales we will have:
$\sigma(t) = f(v_1, ..., v_n)$
The SDE obeyed by each of these is:
$\cdot\, v_i = (m_i - v_i) + \nu_i \eta(t)$
In field theory, any number of time scales can be introduced in the system. This can be by introducing an RG flow of σ by promoting $\sigma \to \sigma(k)$. Or by making σ a temporally and 'spatially' varying background field. If there are a finite number of fixed length scales we wish to introduce in the noise (i.e, volatility with a finite number of scales), we can simply expand the volatility in a Fourier basis:
$\sigma = \sum_{i=1}^{N} a_i sin(\omega_i t)$
Where the ω_is encode all the scales in the system that are fixed by observation or prior knowledge. The σ acts just as would time dependent couplings in a regular QFT. The integral will be explicitly performed over the terms in it.

4 Modeling Residual Correlations using Effective Field Theory

4.1 Motivation

A field theory is a general and useful way to encode correlations up to a very high degree of non-linearity.It has been used to describe extremely general statistical systems that are completely classical. Loop corrections in those systems account for classical-nonlinearity rather than quantum corrections. For classical statistical field theory, the gauge-gravity correspondence can be interpreted in these systems as a duality between a strongly non-linear system in d dimensions and a weakly non-linear (gravitational) system in $d + 1$ dimensions. Finance in particular deals with strongly nonlinear systems which is what makes trading or any kind of prediction extremely hard- the same reason strongly coupled quantum field theories are hard. Fortunately, physicists have come up with various solutions for this- one of the most useful being gravity duals. Indeed a gravity dual for linear Black Scholes was constructed by Nakayama, with a prescription of how to extends to non-linearity. We can expect a similar approach to work for equities. In fact the case of N equities where N is large is even simpler since large N methods may prove useful.

4.2 Connection to RG Flows

Here we will encode the nonlinear dependence of the factors on the price by considering the factors as slowly evolving background fields. The most rigorous known formalism to encode correlations between fluctuations across different timescales is the renormalization group. In the context of quantum theory, one observes that as one 'zooms in' or out of the time series, the theory that one uses to describe the phenomena changes. As we vary the length scale Λ at which the system is observed, certain parameters in the theory (couplings) change continuously.

4.3 Effective Field Theory of Noise

First we construct a field theory Lagrangian. There is no reason to expect locality or unitarity or any other notion of renormalizability a priori. For N equities the most general local Lagrangian reads:
$L = \int dt \sum_{i,j} M_{ij}\phi(t)^i \partial_t \phi(t)^j + \sigma_{ij}\phi^i(t)\phi^j(t) + ...\lambda_{ijk}(\alpha(t))\phi^i\phi^j\phi^k + ...$
Where all sums above run from 1 to N equities. We have ignored higher order corrections as we are solely interested in two and three point functions. Here the coupling tensors are to be fixed by observation. They will contain modes that vary more slowly with time- for instance, they may update quarterly. These will depend upon the fundamental factors (the α) and such dependence can be fixed using fitting algorithms or machine learning. If we are concerned with regimes where these do not change these may be numerically fixed from the data. In the large N limit we get: There is also no reason to assume any symmetry just from the Lagrangian. However it is time translation invariant. However, consider the fact that the Lagrangian is invariant under the transformations:
$\phi_i \to \alpha\phi, \lambda_{ijk} \to \alpha^{-3}\lambda_{ijk}, M_{ij} \to \alpha^{-2}M_{ij}$
This is precisely the generalised conformal symmetry introduced in ... Just by GCS, we find the

5 Multi-Investor Models and The Power of Large N

A problem involving multiple investors can be recast as a large N problem. In the limit of N very large, the behavior of a dynamical system is described in terms of mean field theory. At any finite value of N, the behavior of observables in a system is given by its mean field value plus contributions suppressed by orders of $1/N$. Physically speaking, there may be investors with differing degrees of experience and agendas who make the noise fluctuate on different scales. The simplest stochastic differential equation can be written as:
$\cdot\, x(t) = r_x + \sum_i^N \eta_i$
The sum runs over different groups of investors, each creating noise following a unique Levy distribution with noise falling off as:
$|x|^{-1-\lambda_i}$
Their individual probability distributions are:
$P_\lambda(\eta_\lambda) = exp(-\int_{t_a}^{t_b} dtI(ip(t)\eta_\lambda(t) - H_\lambda(\eta_\lambda(t)))$
With Hamiltonian:
$H_\lambda(p) = \frac{\sigma_\lambda^2 |p|^\lambda}{2}$

The overall probability of returning (x_b, t_b) given (x_a, t_a) is:
$g(x_b, t_b|x_a, t_a) = \prod_\lambda (\int D\eta_\lambda \int Dx exp(-\int_{t_a}^{t_b} H_\lambda(\eta_\lambda(t))))\delta(\cdot x(t) - \sum_i \eta_i)$
We rewrite the delta function in its Fourier representation as in previous sections:
$G(x_b, t_b|x_a, t_a) = \int \frac{Dp}{2\pi} \prod_\lambda (\int D\eta_\lambda) \int Dx e^{\int_{t_a}^{t_b} dt(p(t)\cdot x(t) - H_\lambda(\eta_\lambda(t))} e^{-i\sum_\lambda \int_{-\infty}^{\infty} dt p(t)\eta_\lambda(t)}$
Mixed correlation functions can be generated here by the usual functional differentiation:
$< \eta_1(t_1)...\eta_n(t_n) >= (-i)^n \int D\eta \int \frac{Dp}{2\pi} (\frac{\delta}{\delta(p(t_1))} \cdots \frac{\delta}{\delta(p(t_n))} e^{i\int_{t_a}^{t_b} p(t)\eta(t)}) e^{-\int_{t_a}^{t_b} H(p(t))}$

6 Symmetry Based Methods

There are two primary tools that have recently become popular in theoretical physics, that enable us to solve for the nonlinear behavior of a system by exploiting its symmetries. One is the Operator Product Expansion and the other involves the construction of 'duals' that live in a space of one dimension higher. The general way to construct a gravity dual is by considering the scaling symmetries of the system. First there are hints of a Lifschitz scaling $W \to \lambda W$ and $t \to \lambda^z t$ in certain markets. In markets that possess even a weakly broken Lifschitz symmetry, it is relatively simple to construct a gravity dual and calculate two or three point functions. In markets where this symmetry is absent, we have to work with less constraining symmetries such as GCS.

6.1 Lifschitz Gravity Dual

Gravity duals for Lifschitz theories are well known. Through the coset construction, one can prove that the unique metric that respects the given symmetry is the following:
$ds^2 = -\frac{L^{2z}}{r^{2z}} dt^2 + \frac{L^2}{r^2}(2dtdx + dx^2 + dr^2)$
Explicitly, we see that Lifschitz symmetries hold. The additional symmetries are translations, rotations, and Galilean boosts. We do not have an analogue of special conformal transformations for general z. We wish to express this metric as the solution of some action describing Einstein gravity coupled to matter fields. As shown in 'Lifschitz' the action can be written as that of a massive Maxwell field:
$S = \int d^6x\sqrt{-g}(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu})$
This can be solved for the vector field A and the metric $g_{\mu\nu}$.Indeed we will recover the asymptotically Lifschitz solution.

6.1.1 Philosophy

The general philosophy behind holography is that a nonlinear system with N strongly interacting degrees of freedom should have a weakly coupled (linear) dual in the large N limit. The dual called a 'gravity' dual merely satisfies the requirement that it is general coordinate invariant. In the abstract sense one does not have to link it to gravity or string theory. The market consists of N time interacting series where N is large- we also know that the time series have highly nonlinear correlations. Hence we start out assuming tha. Then it is reasonable to assume that $\delta\rho$ being an effective scalar field is dual to a scalar in the bulk. Thereafter we calculate all possible terms in the two, three and four point functions. Finally we compare our result with LSS.

6.2 Towards a Large-N Dual

We now consider free massive and massless scalar field on this background. calar field on this background. Often even when we wish to measure the correlation function of a part of the boundary stress tensor (sourced by the bulk metric) or a vector, we use a massive scalar field as a proxy. Therefore this assumption may be valid. From the boundary action perspective this is equivalent to introducing a non-normalizable 'source' coupling of the form
$S_{bound} = S_{bound} + \int \phi(\delta\rho)$
Such that functional differentiation with respect to ϕ gives us factors of ρ in the correlation function:
$\frac{\delta^n Z}{\delta\phi^n} =< (\delta\rho)_1 ...(\delta\rho)_n >$
We can then try all possible interaction vertices and both the massive and massless cases. This approach may be helpful in the case of LSS, where we know how the correlators at least at low k. The action is given by:
$I = \frac{1}{2}\int d^6\sqrt{g}(g^{mn}\partial_m\phi\partial_n\phi + M^2\phi^2)$
The field equations are expressed thus:
$u^{d+z}\partial_u(u^{-(d+z)}\partial_u\phi) + (u^{2z}\partial_t^2 + \partial^2)\phi - \frac{M^2}{u^2}\phi = 0$
Fourier transforming in time and spatial coordinates (but not u) we get:
$u^{D+z}\partial_\mu(u^{-(4+z)}\partial_u\phi) - (u^{2z}\omega^2 + k^2)\phi - \frac{M^2}{u^2}\phi - \frac{M^2}{u^2}\phi(\omega, k) = 0$
We can solve this equation separately in the massless case. Imposing regularity for $M = 0$, the following solution is obtained:
$\phi_{(0)}(\omega, k)e^{-\frac{1}{2}\omega x^2}U(\frac{k^2+\omega(1-d-z)}{4\omega}, \frac{1}{2}(1-d-z), \omega x^2)$
Where $\phi_{(0)}(\omega, k)$ is arbitrary and indeed determined by boundary conditions, while U is the hypergeometric Kummer function. For the massive case, we have the following:
$\phi = \phi_0(\omega, k)e^{-1/2\omega x^2}x^{2c}U(a, b, \omega x^2)$
Where a, b and c are:
$a = \frac{1}{2} - \frac{1}{4}\sqrt{(1+d+z)^2 + 4M^2} + \frac{k^2}{4\omega}$
$b = 1 - \frac{1}{2}\sqrt{(1+d+z)^2 + 4M^2L^2}$
$c = \frac{1}{4}(1+d+z) - \frac{1}{4}\sqrt{(1+d+z)^2 + 4M^2}$
We now carry out the asymptotic expansion near the boundary at $u = 0$ as is common in AdS/CFT. The scalar field near the boundary can be expressed in the following form:
$\phi = u^{\Delta_-}(\phi_0(t, x, u) + ...) + u^{\Delta_+}(\phi(t, x, u) + ...)$
Assuming non degenerate roots we can expand thus:
$\phi = u^{\Delta_-}(\phi_0(t, x) + ...) + u^{\Delta_+}(\tilde{\phi}_{(0)}(t, x))$
Where $\delta_\pm$ are the roots of:
$\Delta(\Delta - d - 1 - z) = M^2$
The exact form of the expansion u will depend upon the value of z. For instance for $0 < z < 1$ we will get the following:
$\phi(t, x, u) = \phi_0 + u^2\phi_{(2)}(t, x) + u^{2+2\eta} + ... + u^4\phi_4(t, x) + ...$
We will consider this case in this work, as they cover the cases of physical interest (n between -3 and 1). The equations of motion can be solved perturbatively for ϕ_n. All this will contribute to the one point function. We should be able to fix the renormalization conditions to send this to either ρ_0 or zero for the

purpose of calculating density correlators. We focus on the two point function. Expanding the solution in terms of the confluent hypergeometric function, we simply throw away the divergent terms as $u \to \epsilon$.The result for the two point function is:
$< O(\omega,k,z)O(\omega,k,z) >= -(3+z)\omega^{(3+z)/2}\frac{\Gamma((3+z)/2)}{\Gamma(\frac{1}{2}(3+z))}\frac{\Gamma(k^2/4\omega+\frac{5+z}{4})}{\Gamma(k^2/4-\frac{1}{4}(1+z))}$
Now we turn to the calculation of higher point functions that can come from a variety of bulk indices.

6.2.1 Bulk Vertices

We can consider the following three point interactions in the bulk that satisfy both Lorentz invariance and are relevant:
$L_{3,1} = \phi_1(x_1)\phi_2(x_2)\phi(x_3)$
$L_{3,2} = \partial_\mu\phi_1(x_1)\partial^\mu\phi_2(x_2)\phi(x_3)$
The corresponding three point functions:
$G_{3,1}(X,Y,Z) = \int \frac{da}{u^{d+3}}K_1(a;X)K_2(a;Y)K_3(a;Z)$
$G_{3,2}(X,Y,Z) = \int \frac{da}{u^{d+3}}\partial_\mu K_1(a;X)\partial^\mu K_2(a;Y)K_3(a;Z)$
Where the bulk-to-boundary K is given by, in the massless case:
$K = \phi_{(0)}(\omega,k)e^{-\frac{1}{2}\omega x^2}U(\frac{k^2+\omega(1-d-z)}{4\omega},\frac{1}{2}(1-d-z),\omega x^2)$
And in the massive case:
$K = \phi_0(\omega,k)e^{-1/2\omega x^2}x^{2c}U(a,b,\omega x^2)$
Where a, b and c are:
$a = \frac{1}{2} - \frac{1}{4}\sqrt{(1+d+z)^2+4M^2} + \frac{k^2}{4\omega}$
$b = 1 - \frac{1}{2}\sqrt{(1+d+z)^2+4M^2L^2}$
$c = \frac{1}{4}(1+d+z) - \frac{1}{4}\sqrt{(1+d+z)^2+4M^2}$
For z=2 the calculation using holography was done in a similar vein in 'Gravity duals'. Results match the functions derived in the previous section for the two point case. Therefore the two approaches are consistent Similarly for other ranges of z we can perform similar calculations. For four point functions we can have the ϕ^4 vertex or contribution from the three point vertices. We can write these down explicitly as follows:
$G_{4,1}(X,Y,Z,V) = \int \frac{da}{u^{d+3}}K_1(a;X)K_2(a;Y)K_3(a;Z)K_4(a;V)$
$G_{4,2}(X,Y,Z) = \int \frac{da}{u^{d+3}}K_1(a;X)K_2(a;Y)G(a;b)K_1'(b;Z)K_2'(b;z)$
Here G is a bulk-to-bulk propagator which can be calculated in a similar manner to the bulk-to-boundary propagator. All these diagrams are in Figures 10-12. The integrals could be evaluated explicitly but we are simply here interested in which of them if any, reproduce the LSS results at small k. The idea now is to use these functional forms to fit the parameters from data.

7 From Correlation Functions to Trading Strategy

Consider our observables. In a given time frame, we have the time series of a vector of N stocks along with corresponding α factors $\{p_t,\alpha_t\}_{t=-T}^{t=0}$. the prices update on a time scale τ_1 and the α factors on another time scale τ_2 with:
$\tau_1 << \tau_2$
A trading strategy is generally defined as maximizing a utility function:
$u(x) := E(\pi(x)) - (\gamma/2)V(\pi(x))$

We argue that correlation functions can be used to construct much more robust trading strategies than those constructed using absolute covariance matrices, etc. The following functions can be fit from observation:
$< \phi(0)\phi(t) >, < \phi(0)\phi(t_1)\phi(t_2) >, ...$
For $n = N + K$ we will construct all n-point functions. This is because they by definition account for the time translation invariance of the system, and their observed values can be We would ideally look for spikes in the three point function at a given configuration (P_i, t_i) with α factors updated. We can construct operators with specific spikes in their correlation functions. The space of such functions will correspond to the space of viable trading strategies if the spikes outweigh the transaction costs.

7.1 Pairs Trading

For markets in which pair trading strategies to hold, it is simple to construct a field theory. In such markets, there exist equities such that certain fixed linear combinations of them are Gaussian. Say we are concerned with M co-integrated groups each with n_M equities. Here we would have:
$\prod_{i=1}^{n_M} \int D\eta_i \delta(\sum_i^{m_N} \alpha_i \phi_i - \eta_i) exp(\frac{\eta_i^2}{\sigma^2})$
with the usual Fourier transform representation we get:
$\prod_{i=1}^{n_M} \int DpD\eta_i e^{p(\sum_i^{m_N} \alpha_i \phi_i - \eta_i)} exp(\frac{\eta_i^2}{\sigma^2})$
We now have to read off a series of correlation coefficients from the data.

7.2 Prices as 'collider' experiments

The aim of conducting experiments in a particle physics collider is to uncover the 'couplings' (or scale-dependent correlations) of different particles in the system. The experiment cannot be repeated. The market or any time series data is akin to the latter- every data point in the historical time series is a certain fixed measurement that can provide information on the correlation structure of the market.

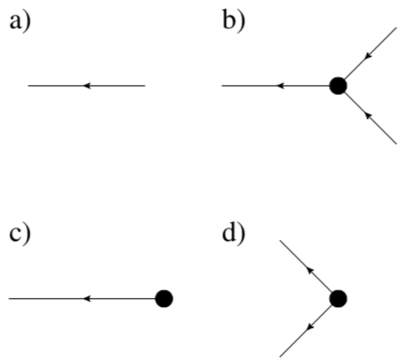

Figure 2: Feynman Diagrams for Generalized OU

8 Appendix

8.1 Perturbative Methods and Feynman Diagrams

Now let us compute correlation functions. The perturbation method essentially follows SDEs. It is an expansion in small noise. We write the functions integral thus:
$Z = \int D\phi exp(iS_{eff} + J\phi + \tilde{J}\tilde{\phi})$
Where the effective action can be split in the free and interacting parts:
$S = S_{free} + S_I$
Correlation functions can be calculated by functionally differentiating with respect to the currents:
$< \prod_{i=1}^{m} \phi(t_i) \prod_{j=1}^{n} \tilde{\phi}(t_j) >= \prod_{i=1}^{m} \frac{\delta}{\delta J(t_i)} \prod_{j=1}^{n} Z(\phi, \tilde{\phi}, J, \tilde{J})|_{J,\tilde{J}=0}$
We can see that the n-point correlation function is just the n-th moment of the price. We can get the cumulant from this by simply dividing the corresponding moment by Z. We can perturbatively expand the partition function around the free part:
$Z = \int D\phi e^{S_{free}}(1 + S_{int} + \int dtJ\phi) + \frac{1}{2!}(S_{int} + \int dtJ\phi + \tilde{J}\tilde{\phi}) + \frac{1}{3!}S_{int}^3)$
The interacting part of the above action can most generally be written as:
$S_{int} = \sum_{m,n=1}^{\infty} \frac{v_{m,n}}{m!n!}\phi^m\tilde{\phi}^n$
Where $v_{m,n}$ are interaction veritces. These will each correspond to a particular tree level Feynman diagram. Interaction between different moments can give rise to 'loops' in the same manner as in quantum field theory. The only different is that loop corrections account for quantum effects in the latter case, and non-linearities in the former. So far this discussion was extremely general. We specialize to markets in the next section.

8.2 Generalized Ornstein Unhelbeck Process

Let us consider an Ornstein Unhelbeck process with a trilinear vertex added on. It is given by the SDE:
$\dot{x}(t) + ax(t) + bx(t)^2 - \sqrt{D}\eta(t)$
The corresponding action is:

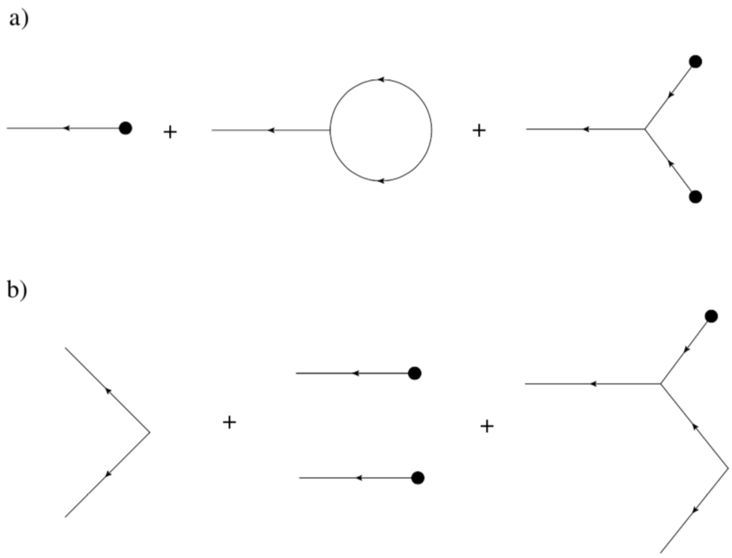

Figure 3: Feynman Rules for Generalized OU

$S(x,\tilde{x}) = \int(\tilde{x}(\dot{x}(t) + ax(t) + bx(t)^2 - y\delta(t-t_0)) - \frac{D}{2}\tilde{x(t)}^2 dt)$
The Feynman rules are the following: a) The inverse propagator is given thus:
$G(t-t') = (\frac{d}{dt} + a)^{-1}\delta(t-t')$
The one point function or mean is simply:
$< x(t) >= ye^{-a(t-t_0)}H(t-t_0)$
The Green's function is:
$< x(t_1 x(t_2)) >= e^{-a(t_1-t_2)}H(t_{1-t_2})$
Where H is the left discontinuous Heaviside step function. Higher point functions can be derived by simply functionally differentiating with respect to the currents. These follow the same diagrammatics as the corresponding QFT. The diagrams in FFigure 2 represent the tree level contributions to the mean or one point function and the variance or two point function.

8.3 From Noise to Action

While analyzing more general non-Gaussian noise distributions, it will often be useful to write the form of the noise in momentum space
$D = \int dp 2\pi e^{ipz} D$
Here we analyze how for a given noise profile, one can fix the coefficients of the potential. A form of noise that is considered sufficiently general to capture the behavior of stock and equity markets is the Levy distribution.
$L^{\lambda}_{\sigma^2(z)} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} L^{\lambda}_{\sigma^2(p)}$
Where
$L^{\lambda}_{\sigma^2(p)} = exp(-\sigma p^2)$
Non-gaussianities in noise can also be modeled using a Boltzmann distribution. In fact this suggests a notion of temperature in the noise. This may actb as an order parameter that characterizes phase transitions in the market. The Levy

Distribution is given by:
$L^{\lambda}_{\sigma^2}(z) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipz} L^{\lambda}_{\sigma^2}(p)$
Where
$L^{\lambda}_{\sigma^2}(p^2) = exp(-(\sigma^2 p^2)^{\lambda/2}/2)$
The Hamiltonian is read off from above:
$H(p) = \frac{1}{2}(\sigma^2 p^2)^{\frac{\lambda}{2}}$
The Gaussian distribution corresponds to the special case $\lambda = 2$. The large z or small momentum expansion yields:
$L^{\lambda}_{\sigma^2}(z) \approx \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipz}(1 - \frac{1}{2}(\sigma^2 p^2)^{\lambda/2})$
Upon performing the integral:
$L^{\lambda}_{\sigma^2}(z) \approx \frac{\lambda}{|z|^{1+\lambda}}$

8.4 Generalized Central Limit Theorem

The central limit theorem states that the convolution of infinitely many distributions of finite width approaches a Gaussian distribution or a Levy distribution. After a number t of convolutions of a distribution with Hamiltonian $H(p)$, we get:
$D(x,t) = \int \frac{dp}{2\pi} e^{ipz - tH(p)}$
For large t we simply use the saddle point approximation:
$tH'(p^*) = ix$
First let us assume that for small p, the Hamiltonian goes as p^2. We now expand around the saddle point to get:
$D(z,t) \approx e^{ip^* z - tH(p^*)} \int \frac{dp}{2\pi} e^{-(p-p^*)z - t\partial^2 H(p)(p-p^*)^2/2}$

$D(z,t) = \frac{e^{ip^* z - tH(p^*)}}{\sqrt{2\pi\sigma^2}} e^{-z^2/2t\sigma^2}$
$= \frac{e^{\sigma^2 p^{*2}/2 - tH(p^*)}}{\sqrt{2\pi\sigma^2}} e^{-(z - t\sigma^2 p^*)^2/2t\sigma^2}$
We get a Gaussian here. On the other hand if $H(p)$ starts off as $|p|^{\lambda}$ then the saddle point is governed by this term. We will get the Levy distribution up to a possible drift term:
$H(p) = -irp + H_{\lambda,\sigma,\beta}(p)$
Where $H_{\lambda,\sigma,\beta}$ is a Levy distribution with the following explicit form Empirically a lot of pricing time series fits this form. The specific parameters can be inferred from the data.

8.5 Gamma Distribution

The normalized Gamma distribution is:
$D_{\mu,\nu} = \frac{1}{\Gamma(\nu)} \mu^{\nu} z^{\nu-1} e^{-\mu z}$
This corresponds to the Hamiltonian:
$H = \nu log(1 - ip/\mu)$

8.6 Boltzmann Distribution

At extremely high frequencies, the returns in NASDAQ 100 and SP 500 the Boltzmann distribution provides an ideal fit:
$B(z) = \frac{1}{2T} e^{-|z|/T}$

In Fourier space this is expressed thus:
$B(p) = \int_{-\infty}^{\infty} dz e^{ipz} \frac{1}{2T} e^{-|z|/T} = \frac{1}{1+(Tp)^2} = e^{-H(p)}$
The Hamiltonian is:
$H(p) = log(1 + (Tp)^2)$
Apart from a set of extremely rare events, all events follow the Boltzmann law. The Boltzmann distribution allows us a measure of volatility to the market-the temperature T. This is observed to change very slowly with time, contingent on broader economic and political factors. Indeed the years corresponding to economic crises have precisely corresponded with those years where the market displayed an abnormally high temperature.

8.7 Student Distribution

The Student distribution is another one that has been proposed to account for heavy tails observed in finance:
$D_\delta(z) = N_\delta \frac{1}{\sqrt{2\pi\sigma^2}} e_\delta^{-z^2/2\sigma^2}$
Where e_δ^x is an approximation to an exponential distribution that corresponds to the latter in the limit $\delta \to 0$:
$e_\delta^x = (1 - \delta z)^{-1/\delta}$
The normalization is given by:
$N_\delta = \frac{\sqrt{\delta}\Gamma(1/\delta)}{\Gamma(1/\delta - 1/2)}$