Present Values, Investment Returns and Discount Rates

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The concept of *present value* lies at the heart of finance in general and actuarial science in particular. The importance of the concept is universally recognized. Present values of various cash flows are extensively utilized in the pricing of financial instruments, funding of financial commitments, financial reporting, and other areas.

A typical funding problem involves a *financial commitment* (defined as a series of future payments) to be funded. A financial commitment is funded if all payments are made when they are due. *A present value of a financial commitment is defined as the asset value required at the present to fund the commitment.*

Traditionally, the calculation of a present value utilizes *a discount rate* – a deterministic return assumption that represents investment returns. If the investment return and the commitment are certain, then the discount rate is equal to the investment return and the present value is equal to the sum of all payments discounted by the compounded discount rates. The asset value that is equal to this present value and invested in the portfolio that generates the investment return will fund the commitment with certainty.

In practice, however, perfectly certain future financial commitments and investment returns rarely exist. While the calculation of the present value is straightforward when returns and commitments are certain, uncertainties in the commitments and returns make the calculation of the present value anything but straightforward. When investment returns are uncertain, a single discount rate cannot encompass the entire spectrum of investment returns, hence the selection of a discount rate is a challenge. In general, the asset value required to fund an uncertain financial commitment via investing in risky assets – the present value of the commitment – is uncertain (stochastic).[1]

While the analysis of present values is vital to the process of funding financial commitments, uncertain (stochastic) present values are outside of the scope of this paper. This paper assumes that *a present value is certain (deterministic)* – a present value is assumed to be a number, not a random variable in this paper. The desire to have a deterministic present value requires a set of assumptions that "assume away" all the uncertainties in the funding problem.

In particular, it is generally necessary to assume that all future payments are perfectly known at the present. The next step is to select a proper measurement of investment returns that serves as the discount rate for present value calculations. This step – the selection of the discount rate – is the main subject of this paper.

One of the main messages of this paper is *the selection of the discount rate depends on the objective of the calculation*. Different objectives may necessitate different discount rates. The paper defines investment returns and specifies their relationships with present and future values. The key measurements of investment returns are defined in the context of return series and, after a concise discussion of capital market assumptions, in the context of return distributions. The paper concludes with several examples of investment objectives and the discount rates associated with these objectives.

1. Investment Returns

This section discusses one of the most important concepts in finance – investment returns.

Let us define the investment return for a portfolio of assets with known asset values at the beginning and the end of a time period. If *PV* is the asset value invested in portfolio *P* at the beginning of a time period, and *FV* is the value of the portfolio at the end of the period, then the portfolio return R_P for the period is defined as

$$R_P = \frac{FV - PV}{PV} \tag{1.1}$$

Thus, given the beginning and ending values, portfolio return is defined (retrospectively) as the ratio of the investment gain over the beginning value. Definition (1.1) establishes a relationship between portfolio return R_P and asset values *PV* and *FV*.

Simple transformations of definition (1.1) produce the following formula:

$$FV = PV\left(1 + R_P\right) \tag{1.2}$$

Formula (1.2) allows a forward-looking (prospective) calculation of the end-of-period asset value *FV*. The formula is usually used when the asset value at the present *PV* and portfolio return R_P are known (this explains the notation: *PV* stands for “Present Value”; *FV* stands for “Future Value”).

While definition (1.1) and formula (1.2) are mathematically equivalent, they utilize portfolio return R_P in fundamentally different ways. The return in definition (1.1) is certain, as it is used retrospectively as a measurement of portfolio performance. In contrast, the return in formula (1.2) is used prospectively to calculate the future value of the portfolio, and it may or may not be certain.

When a portfolio contains risky assets, the portfolio return is uncertain by definition. Most institutional and individual investors endeavor to fund their financial commitments by virtue of investing in risky assets. The distribution of uncertain portfolio return is usually analyzed using a set of forward-looking capital market assumptions that include expected returns, risks, and correlations between various asset classes. Later sections discuss capital market assumptions in more detail.

Given the present value and portfolio return, formula (1.2) calculates the future value. However, many investors with future financial commitments to fund (e.g. retirement plans) face a different challenge. Future values – the commitments – are usually given, and the challenge is to calculate present values. A simple transformation of formula (1.2) produces the following formula:

$$PV = \frac{FV}{1+R_P} \tag{1.3}$$

Formula (1.3) represents the concept of *discounting procedure*. Given a portfolio, formula (1.3) produces the asset value *PV* required to be invested in this portfolio at the present in order to accumulate future value *FV*. It must be emphasized that return R_P in (1.3) is generated by the actual portfolio *P*, as *there is no discounting without investing*. Any discounting procedure assumes that the assets are actually invested in a portfolio that generates the returns used in the procedure.

Formulas (1.2) and (1.3) are mathematically equivalent, and they utilize portfolio return in similar ways. Depending on the purpose of a calculation in (1.2) or (1.3), one may utilize either a particular measurement of return (e.g. the expected return or median return) or the full range of returns.[2] The desirable properties of the future value in (1.2) or present value in (1.3) would determine the right choice of the return assumption.

Future and present values are, in a certain sense, inverse of each other. It is informative to look at the analogy between future and present values *in the context of a funding problem*, which would explicitly involve a future financial commitment to fund. Think of an investor that has $*P* at the present and has made a commitment to accumulate $*F* at the end of the period by means of investing in a portfolio that generates investment return *R*.

Similar to (1.2), the future value of $*P* is equal to

$$FV = P(1+R) \tag{1.4}$$

Similar to (1.3), the present value of $*F* is equal to

$$PV = \frac{F}{1+R} \tag{1.5}$$

The shortfall event is defined as failing to accumulate $*F* at the end of the period:

$$FV < F \tag{1.6}$$

The shortfall event can also be defined *equivalently* in terms of the present value as $*P* being insufficient to accumulate $*F* at the end of the period:

$$P < PV \tag{1.7}$$

In particular, the shortfall probability can be expressed in terms of future *and* present values:

$$\textit{Shortfall Probability} = \Pr(FV < F) = \Pr(PV > P) \tag{1.8}$$

If the shortfall event happens, then *the shortfall size* can also be measured in terms of future and present values. The future shortfall $F-FV$ is the additional amount the investor will be required to contribute at the end of the period to fulfill the commitment. The present shortfall $PV-P$ is the additional amount the investor is required to contribute at the present to fulfill the commitment.

Clearly, there is a fundamental connection between future and present values. However, this connection goes only so far, as there are issues of great theoretical and practical importance that distinguish future and present values. As demonstrated in a later section, similar conditions imposed on future and present values lead to different discount rates.

Uncertain *future values* generated by the uncertainties of investment returns (and commitments) play no part in financial reporting. In contrast, various actuarial and accounting reports require calculations of *present values*, and these present values must be deterministic (under current accounting standards, at least). Therefore, there is a need for a deterministic discounting procedure.

Conventional calculations of deterministic present values usually utilize a single measurement of investment returns that serves as *the discount rate*. Since there are numerous measurements of investment returns, the challenge is to select the most appropriate measurement for a particular calculation. To clarify these issues, subsequent sections discuss various measurements of investment returns.

2. Measurements of Investment Returns: Return Series

This section discusses the key measurements of series of returns and relationships between these measurements. Given a series of returns $r_1,\ldots,r_n$, it is desirable to have a measurement of the series – a single rate of return – that, in a certain sense, would reflect the properties of the series. The right measurement always depends on the objective of the measurement. The most popular measurement of a series of returns $r_1,\ldots,r_n$ is its *arithmetic average A* defined as the average value of the series:

$$A=\frac{1}{n}\sum_{k=1}^{n}r_k \tag{2.1}$$

As any other measurement, the arithmetic average has its pros and cons. While the arithmetic average is an unbiased estimate of the return, the probability of achieving this value may be unsatisfactory. As a predictor of future returns, the arithmetic average may be too optimistic.

Another significant shortcoming of the arithmetic return is it does not "connect" the starting and ending asset values. The starting asset value multiplied by the compounded arithmetic return factor (1 + *A*) is normally greater than the ending asset value.[3] Therefore, the arithmetic average is inappropriate if the objective is to "connect" the starting and ending asset values. The objective that leads to the arithmetic average as the right choice of discount rate is presented in Section 5.

Clearly, it would be desirable to “connect” the starting and ending asset values – to find a single rate of return that, given a series of returns and a starting asset value, generates the same future value as the series. This observation suggests the following important objective.

Objective 1: To "connect" the starting and ending asset values.

The concept of *geometric average* is specifically designed to achieve this objective. If A_0 and A_n are the starting and ending asset values correspondingly, then, by definition,

$$A_0(1+r_1)\dots(1+r_n)=A_n \tag{2.2}$$

The geometric average G is defined as the single rate of return that generates the same future value as the series of returns. Namely, the starting asset value multiplied by the compounded return factor $(1+G)^n$ is equal to the ending asset value:

$$A_0(1+G)^n=A_n \tag{2.3}$$

Combining (2.2) and (2.3), we get the standard definition of the geometric average G:

$$G=-1+\prod_{k=1}^{n}(1+r_k)^{1/n} \tag{2.4}$$

Let us re-write formulas (1.2) and (1.3) in terms of present and future values. If A_n is a future payment and $r_1,\dots,r_n$ are the investment returns, then the present value of A_n is equal to the payment discounted by the geometric average:

$$A_0=\frac{A_n}{(1+r_1)\dots(1+r_n)}=\frac{A_n}{(1+G)^n} \tag{2.5}$$

Thus, the geometric average connects the starting and ending asset values (and the arithmetic average does not). Therefore, *if the primary objective of discount rate selection is to connects the starting and ending asset values, then the geometric average should be used for the present value calculations*.

To present certain relationships between arithmetic and geometric averages, let us define variance V as follows: [4]

$$V=\frac{1}{n}\sum_{k=1}^{n}(r_k-A)^2 \tag{2.6}$$

If V = 0, then all returns in the series are the same, and the arithmetic average is equal to the geometric average. Otherwise (if V > 0), the arithmetic average is greater than the geometric average ($A > G$).[5]

There are several approximate relationships between arithmetic average A, geometric average G, and variance V. These relationships include the following relationships that are denoted (R1) – (R4) in this paper.

$$G \approx A - V/2 \tag{R1}$$

$$(1+G)^2 \approx (1+A)^2 - V \tag{R2}$$

$$1+G \approx (1+A)\exp\left(-\frac{1}{2}V(1+A)^{-2}\right) \tag{R3}$$

$$1+G \approx (1+A)\left(1+V(1+A)^{-2}\right)^{-1/2} \tag{R4}$$

These relationships produce different results, and some of them work better than the others in different situations. Relationship (R1) is the simplest, popularized in many publications, but usually sub-optimal and tends to underestimate the geometric return.[6] Relationships (R2) – (R4) are slightly more complicated, but, in most cases, should be expected to produce better results than (R1).

The geometric average estimate generated by (R4) is always greater than the one generated by (R3), which in turn is always greater than the one generated by (R2).[7] Loosely speaking,

(R2) < (R3) < (R4)

In general, "inequality" (R1) < (R2) is not necessarily true, although it is true for most practical examples. If $A > V/4$, then the geometric average estimate generated by (R1) is less than the one generated by (R2).[8]

There is some evidence to suggest that, for historical data, relationship (R4) should be expected to produce better results than (R1) – (R3). See Mindlin [2010] for more details regarding the derivations of (R1) – (R4) and their properties.

Example 2.1. $n = 2, r_1 = -1\%, r_2 = 15\%$. Then arithmetic mean A, geometric mean G, and variance V are calculated as follows.

$$A = \frac{1}{2}(-1\% + 15\%) = 7.00\%$$

$$G = \sqrt{(1-1\%)(1+15\%)} - 1 = 6.70\%$$

$$V = \frac{1}{2}\sum_{k=1}^{2}(r_k - A)^2 = 0.64\%$$

Note that $(1+G)^2 = (1+A)^2 - V$, so formula (R2) is exact in this example.

Given \$1 at the present, future value *FV* is

$$FV = 1\cdot(1-1\%)(1+15\%) = 1.1385$$

If we apply arithmetic return *A* to \$1 at the present for two years, we get

$$(1+7\%)^2 = 1.1449$$

which is greater than future value *FV* = 1.1385.

If we apply geometric return *G* to \$1 at the present for two years, we get

$$(1+6.70\%)^2 = 1.1385$$

which is equal to future value *FV*, as expected.

Given \$1 in two years, present value *PV* is

$$PV = \frac{1}{(1-1\%)(1+15\%)} = 0.8783$$

If we discount \$1 in two years using geometric return *G*, we get

$$\frac{1}{(1+6.70\%)^2} = 0.8783$$

which is equal to present value *PV*, as expected.

If we discount \$1 in two years using arithmetic return *A*, we get

$$\frac{1}{\left(1+7.00\%\right)^2}=0.8734$$

which is less than present value PV = 0.8783.

3. Capital Market Assumptions and Portfolio Returns

This section introduces capital market assumptions for major asset classes and outlines basic steps for the estimation of portfolio returns.

It is assumed that the capital markets consist of n asset classes. The following notation is used throughout this section:

m_i mean (arithmetic) return;
s_i standard deviation of return;
c_{ij} correlation coefficient between asset classes i and j.

A portfolio is defined as a series of weights $\{w_i\}$, such that $\sum_{i=1}^{n} w_i = 1$. Each weight w_i represents the fraction of the portfolio invested in the asset class i.

Portfolio mean return A and variance V are calculated as follows:

$$A=\sum_{i=1}^{n} w_i m_i \qquad (3.1)$$

$$V=\sum_{i,j=1}^{n} w_i w_j s_i s_j c_{ij} \qquad (3.2)$$

Let us also define *return factor* as 1 + R. It is common to assume that the return factor has *lognormal* distribution (which means ln(1 + R) has *normal* distribution). Under this assumption, parameters μ and σ of the lognormal distribution are calculated as follows:

$$\sigma^2=\ln\left(1+V\left(1+A\right)^{-2}\right) \qquad (3.3)$$

Using σ calculated in (3.3), parameter μ of the lognormal distribution is calculated as follows:

$$\mu=\ln\left(1+A\right)-\frac{1}{2}\sigma^2 \qquad (3.4)$$

Given parameters μ and σ, the P^{th} percentile of the return distribution is equal to the following:

$$R_P = \exp\left(\mu + \sigma\Phi^{-1}(P)\right) - 1 \qquad (3.5)$$

where Φ is the standard normal distribution. In particular, if P = 50%, then $\Phi^{-1}(P) = 0$. Therefore, the median of the return distribution under the lognormal return factor assumption is calculated as follows.

$$R_{0.5} = \exp(\mu) - 1 \qquad (3.6)$$

Example 3.1. Let us consider two uncorrelated asset classes with mean returns 8.00% and 6.00% and standard deviations 20.00% and 10.00% correspondingly. If a portfolio has 35% of the first class and 65% of the second class, its mean and variance are calculated as follows.

$$A = 8.00\% \cdot 35\% + 6.00\% \cdot 65\% = 6.70\%$$

$$V = (20.00\% \cdot 35\%)^2 + (10.00\% \cdot 65\%)^2 = 0.9125\%$$

It is interesting to note that the standard deviation of the portfolio is 9.55% ($= \sqrt{0.9125\%}$), which is lower than the standard deviations of the underlying asset classes (20.00% and 10.00%). Assuming that the return factor of this portfolio has lognormal distribution, the parameters of this distribution are

$$\sigma = \sqrt{\ln\left(1 + \frac{0.9125\%}{(1+6.70\%)^2}\right)} = 0.0893$$

$$\mu = \ln(1+6.70\%) - \frac{0.0893^2}{2} = 0.0609$$

From (3.5), the median return for this portfolio is

$$R_{0.5} = \exp\left(0.0609 + 0.0893 \cdot \Phi^{-1}(0.5)\right) - 1 = 6.27\%$$

From (3.5), the 45th percentile for this portfolio is

$$R_{0.45} = \exp\left(0.0609 + 0.0893 \cdot \Phi^{-1}(0.45)\right) - 1 = 5.09\%$$

4. Measurements of Investment Returns: Return Distributions

The previous section presented the relationships between the arithmetic and geometric averages defined for a series of returns. This section develops similar results when *return distribution R is given*.

In this case, the *arithmetic* average (mean) return A is defined as the expected value of R:

$$A = E(R) \tag{4.1}$$

The *geometric* average (mean) return G is defined as follows:

$$G = \exp\left(E\left(\ln(1+R)\right)\right) - 1 \tag{4.2}$$

These arithmetic and geometric average returns are the limits of the arithmetic and geometric averages of appropriately selected series of independent identically distributed returns. Specifically, let $\{r_k\}$ be a series of independent returns that has the same distribution as R. Let us define arithmetic averages A_n and geometric averages G_n for $r_1, \ldots, r_n$:

$$A_n = \frac{1}{n}\sum_{k=1}^{n} r_k \tag{4.3}$$

$$G_n = -1 + \prod_{k=1}^{n} (1+r_k)^{1/n} \tag{4.4}$$

According to the Law of Large Numbers (LLN), A_n converge to E. Also, from (4.4) we have

$$\ln(1+G_n) = \frac{1}{n}\sum_{k=1}^{n} \ln(1+r_k) \tag{4.5}$$

Again, according to the LLN, $\frac{1}{n}\sum_{k=1}^{n} \ln(1+r_k)$ converge to the expected value $E\left(\ln(1+R)\right)$. From (4.5), $\ln(1+G_n)$ converges to $E\left(\ln(1+R)\right)$ as well. Consequently, G_n converge to $\exp\left(E\left(\ln(1+R)\right)\right) - 1$, which, according to (4.2), is equal to G.

To recap, A_n converges to A and G_n converges to G when n tends to infinity. As discussed above, the approximations (R1) – (R4) are true for A_n and G_n, where V_n is defined as in (2.6):

$$V_n = \frac{1}{n}\sum_{k=1}^{n} (r_k - A_n)^2 \tag{4.6}$$

Since V_n converge to the variance of returns V when n tends to infinity, the approximations (R1) – (R4) are true for A and G as well. As was discussed before, if the primary objective of discount rate selection is to connects the starting and ending asset values, then the geometric mean is a reasonable choice for the discount rate.

This conclusion, however, is valid over relatively long time horizons only. Over shorter time horizons, the geometric average of series $\{r_k\}$ has non-trivial volatility and cannot be considered approximately constant. More importantly, the investor may have objectives other than connecting the starting and ending asset values. All in all, additional conditions of stochastic nature may be required to select a reasonable discount rate. Such conditions are discussed in the next section.

For large *n*, the Central Limit Theorem (CLT) can be used to analyze the stochastic properties of the geometric average. According to the CLT applied to $\frac{1}{n}\sum_{k=1}^{n}\ln(1+r_k)$, the geometric average return factor $1+G_n$ defined as

$$1+G_n=\prod_{k=1}^{n}(1+r_k)^{1/n}=\exp\left(\frac{1}{n}\sum_{k=1}^{n}\ln(1+r_k)\right)$$

is approximately lognormally distributed. If the mean and standard deviation of $\ln(1+r_k)$ are μ and σ correspondingly, then the parameters of the geometric average return factor are μ and $\frac{\sigma}{\sqrt{n}}$.

Assuming that the return factor has *lognormal* distribution, it can be shown that relationship (R4) is exact:[9]

$$1+G=(1+A)\left(1+V(1+A)^{-2}\right)^{-1/2} \tag{4.7}$$

An important property of lognormal return factors is *the geometric mean return is equal to the median return*. Indeed, if μ and σ are the parameters of the lognormal distribution, then $\ln(1+R)$ is normal and

$$G=\exp\left(E\left(\ln(1+R)\right)\right)-1=\exp(\mu)-1 \tag{4.8}$$

which is the median of the return distribution according to (3.6).

Thus, if a discount rate were chosen at random (not that this is a great idea), then there would be a 50% chance for the discount rate to be greater than the geometric mean and a 50% chance to be less than the geometric mean. Similarly, if a present value were calculated using randomly selected discount rate, then there would be a 50% chance that a present value is greater than the present value calculated using the geometric mean.[10]

Given arithmetic mean *A* and variance *V*, formula (4.7) produces geometric return *G*. If there is a need to calculate the arithmetic mean when the geometric mean and the variance are given, then the arithmetic mean is calculated as follows:

$$1+A=(1+G)\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4V}{(1+G)^2}}} \tag{4.9}$$

<u>*Example 4.1*</u>. This example is a continuation of *Example 3.1*. In this example, $A=6.70\%$ and $V=0.9125\%$. According to (4.7),

$$G=\frac{1+0.067}{\sqrt{1+\frac{0.009125}{(1+0.067)^2}}}-1=6.27\%$$

which is equal to the median return calculated in *Example 3.1*. Note that the geometric returns for the individual asset classes are 6.19% and 5.53%. It is noteworthy that the geometric return for the portfolio that has 35% of the first class and 65% of the second class is 6.27%, which higher than the geometric returns of the individual classes.

Let us take a look at the stochastic properties of the geometric average for this portfolio. Under the lognormal return factor assumption, the parameters of the return distribution are $\mu=0.0609$ and $\sigma=0.0893$ (see *Example 3.1*). If *n* = 10, then the geometric average return factor $1+G_n$ is approximately lognormally distributed with parameters $\mu=0.0609$ and $\frac{\sigma}{\sqrt{n}}=0.283$. The mean, median and standard deviation are 6.32%, 6.27% and 3.00% correspondingly. Note significant decreases of the mean and standard deviation of the geometric average compared to the original return distribution (6.32% vs. 6.70% and 3.00% vs. 9.55%), while the median remains the same.

5. Examples of Discount Rate Selection

As was discussed in the previous section, the investor may have objectives other than connecting the starting and ending asset values. This section discusses and presents three additional examples of such objectives that lead to the selection of discount rates.

Let us consider a simple modification of the funding problem discussed earlier in the paper. Think of an investor that has made a commitment to accumulate $*F* at the end of the period by means of investing in a portfolio that generates (uncertain) investment return *R*. To fund the commitment, the investor wants to make a contribution that is the subject to certain conditions.

For convenience, let us recall *Objective 1* introduced in Section 2:

<u>Objective 1</u>: *To "connect" the starting and ending asset values.*

As was demonstrated in Section 2, the right discount rate for this objective is the geometric return.

<u>Objective 2</u>: *To have a "safety cushion".*

Let us assume that the investor's objective is have more than a 50% chance that investment returns are greater than the discount rate (the "safety cushion"). For example, if it is required to have a P% chance that the investment return is greater than the discount rate, then the discount rate that delivers this safety level is *the (100 - P)th percentile of the return distribution*.

<u>Objective 3</u>: *No expected gains/losses* ***in the future****.*

Let us assume that the investor's objective is to have neither expected gains nor losses *at the end of the period*. If C_f is the investor's contribution at the present, then this objective implies that the commitment is the mean of the (uncertain) future value of C_f :

$$0 = E(FV) = E\left(C_f\left(1+R\right)-F\right) \tag{5.1}$$

Equation (5.1) gives the following formula for contribution C_f (subscript f in C_f indicates that the objective is "no expected gains or losses *in the future*"):

$$C_f = \frac{F}{1+E(R)} \tag{5.2}$$

Formula (5.2) shows that the objective "no expected gains or losses in the future" leads to contribution C_f calculated as the present value of the commitment using the arithmetic mean return. Hence, the right discount rate d_f for this objective is the arithmetic mean return:

$$d_f = E(R) \tag{5.3}$$

As discussed in a prior section, there is a certain symmetry and fundamental connection between future and present values. In light of this discussion, the following objective is a natural counterpart to *Objective 3*.

<u>Objective 4</u>: *No expected gains/losses* ***at the present****.*

At first, this objective looks somewhat peculiar. Everything is supposed to be known at the present, so what kind of gains/losses can exist now? But remember that that the asset value required to fund the commitment – the present value of the commitment – is uncertain at the present. Therefore, the objective "today's contribution is the mean of the present value of the commitment" is as meaningful as the objective "the commitment is the mean of the future value of today's contribution" discussed in *Objective 3*.

If C_p is the contribution the investor makes at the present, then the objective "no expected gains/losses at the present" implies the following equation.

$$E(PV) = E\left(\frac{F}{1+R} - C_P\right) = 0 \tag{5.4}$$

Equation (5.4) gives the following formula for contribution C_p (subscript p in C_p indicates that the objective is "no expected gains or losses *at the present*"):

$$C_P = F \cdot E\left(\frac{1}{1+R}\right) \tag{5.5}$$

Formula (5.5) shows that the objective "no expected gains or losses *at the present*" leads to contribution C_P that is equal to the present value of the commitment using discount rate d_p:

$$C_p = \frac{F}{1+d_p} \tag{5.6}$$

where d_p is calculated from (5.5) and (5.6) as

$$d_p = \frac{1}{E\left(\frac{1}{1+R}\right)} - 1 \tag{5.7}$$

Note that Jensen inequality entails

$$E\left(\frac{1}{1+R}\right) > \frac{1}{1+E(R)} \tag{5.8}$$

Therefore, $d_p < d_f$.

Under the lognormal return factor assumption, we can tell more about discount rate d_p. Defining ρ_R as

$$\rho_R = 1 + \frac{V}{\left(1+E(R)\right)^2} \tag{5.9}$$

where V is the variance of return R, it can be shown that the expected value of the reciprocal return factor is

$$E\left(\frac{1}{1+R}\right)=\frac{\rho_R}{1+E(R)} \tag{5.10}$$

Combining (5.7) and (5.10), we get

$$d_p=\frac{1+E(R)}{\rho_R}-1 \tag{5.11}$$

Furthermore, under the lognormal return factor assumption, there is an interesting relationship between the geometric mean return G and discount rates d_p and d_f generated by *Objective 3* and *Objective 4*:

$$1+G=\sqrt{(1+d_p)(1+d_f)} \tag{5.12}$$

Thus, the geometric mean return G is the "geometric mid-point" between the discount rates generated by the objectives of no expected gains/losses in the future and at the present.

<u>*Example 5.1*</u>. This example is a continuation of *Example 3.1* and *Example 4.1*. As in these examples, $A=6.70\%$ and $V=0.9125\%$. Then $\rho_R=1.0080$ and

$$d_f=6.70\%$$
$$d_p=5.85\%$$

The 45th percentile of the return distribution is $R_{0.45}=5.09\%$ (see *Example 3.1*).

Conclusion

The selection of a discount rate is one of the most important assumptions for the calculations of present values. This paper presents the basic properties of the key measurements of investment returns and the discount rates associated with these measurements.

The paper shows that *the selection of the discount rate depends on the objective of the calculation*. The paper demonstrates the selection of discount rates for the following four objectives.

<u>*Objective 1*</u>*: To "connect" the starting and ending asset values*. The correct discount rate for this objective is *the geometric mean return*.

<u>*Objective 2*</u>*: To have a certain "safety cushion"*. The correct discount rate for this objective is *the (100 - P)th percentile of the return distribution* if it is required to have a *P*% chance that the investment return is greater than the discount rate.

<u>*Objective 3*</u>*: No expected gains/losses in the future*. The correct discount rate for this objective is *the arithmetic mean return*.

<u>*Objective 4*</u>*: No expected gains/losses at the present*. The correct discount rate for this objective is given in formula (5.7).

It is worth reminding that the main purpose of a discount rate is to calculate a deterministic present value. Yet, present values associated with vital funding problems are inherently stochastic. As a result, the presence of a discount rate assumption has significant pros and cons. The primary advantage of a discount rate is the simplicity of calculations. The main disadvantage is a discount rate based deterministic present value cannot adequately describe the present value of an uncertain financial commitment funded via investing in risky assets. This author believes that the direct analysis of present values and their stochastic properties is the most appropriate approach to the process of funding financial commitments, but this subject is outside of the scope of this paper.

This author hopes that the paper would be useful to practitioners specializing in the area of funding financial commitments.

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Endnotes

[1] There are exceptions, e.g. an inflation-adjusted cash flow with a matching TIPS portfolio.
[2] See Mindlin [2009] for more details.
[3] That is as long as the returns in the series are not the same.
[4] For the purposes of this paper, the concerns that the sample variance as defined in (2.6) is not an unbiased estimate are set aside.
[5] This fact is a corollary of the *Jencen's inequality*.
[6] For example, see Bodie [1999], p. 751, Jordan [2008], p. 25, Pinto [2010], p. 49., Siegel [2008], p. 22., DeFusco [2007], p 128, 155.
[7] That is, obviously, as long as the returns in the series are not the same and $V > 0$.
[8] Mindlin [2010] contains a simple example for which (R1) > (R2).
[9] See Mindlin [2010] for more details.
[10] The presence of discount rate is critical for these observations. In general, the median of the present value distribution calculated using the full range of returns (and without discount rates) is *not* equal to the present value calculated using the geometric mean (except when the cash flow contains just one payment). In other words, the median of present value is not the same as the present value at the median return. See Mindlin [2009] for more details regarding stochastic present values.

Important Information