Principles of
Optimal Glide Path Design

Dimitry Mindlin, ASA, MAAA, PhD
President

CDI Advisors LLC
dmindlin@cdiadvisors.com

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The importance of asset allocation for institutional and individual investors is broadly recognized. Typically, DC plan participants are responsible for their asset allocation decisions, and most of them are not well-equipped to make optimal decisions. Several types of financial products endeavor to assist DC plan participants to make better decisions, but, in most cases, their approaches to asset allocation are inadequate. Unfortunately, the scientific foundation of optimal asset allocation for DC plans is still in its infancy.

This paper takes initial steps to correct this situation. It presents a simple, yet flexible and powerful framework for the design of optimal long-term investment strategy (a.k.a. glide path). The paper presents several commonsense principles and demonstrates how these principles lead to specific methodologies of optimal glide path design.

The paper argues that the process of optimal glide path design should have the following key aspects. First, an optimal glide path should represent a Nash equilibrium solution and be designed via the process of "backward induction" – from the future to the present. Second, the optimization objective should be based on the minimization of the asset value required to fund a specific standard of living in retirement. The paper demonstrates that optimal portfolios generated by this framework maximize the investor's standard of living in retirement.

This author is optimistic that this glide path optimization framework will eventually become the mainstream tool for the process of funding financial commitments.

Asset Allocation's Déjà Vu

DC plans are rapidly becoming a dominant component of the retirement system. Unlike DB plan participants that have little responsibility over the plan's asset allocation and contribution decisions, DC plan participants select their portfolios, determine their contributions, and enjoy (or regret) the results of their investment strategies. So, the very first question a typical DC investor faces is: What is the optimal portfolio today?

Does a typical DC plan investor have sufficient knowledge to select an optimal portfolio? Does the investor have appropriate tools to do so? In general, the answer to both questions is "no." As a result, portfolio selections of many DC plan participants are sub-optimal. Yet, it is universally recognized that asset allocation is one of the key factors affecting the investment performance.

The investor's challenges do not stop here. The investor would be well-advised to design the entire *glide path* – a multi-period asset allocation strategy that consists of the current portfolio and a series of portfolios the investor expects to utilize in the future.

There are at least two major reasons to envision the entire glide path at the present. First, the investor needs to know how much to contribute to the investment program today. Expected future portfolio selections affect contribution rates at the present. For example, the investor may expect her portfolios to get more conservative as the investor gets closer to retirement. Portfolios that are more conservative have lower expected returns and may require higher contributions today. Knowing only today's portfolio is insufficient if the objective is to create a prudent long-term investment strategy.

Second, portfolio selections should be interconnected. A portfolio selection in a particular year affects portfolio selections in all years. Ideally, every portfolio selection should be the best response to all other portfolio selections. In particular, today's portfolio should be the best response to every expected portfolio selection over the investor's entire lifetime. As demonstrated in later sections, the investor should make all future portfolio selections first, and only then answer the question the investor faced in the first place: What is the optimal portfolio today?

Target Date Funds and Their Glide Paths

As a reflection of the challenges retirement investors face, financial products that provide portfolio design at the present and automatically adjust optimal portfolios in the future have been getting increasingly popular. In particular, target date funds (TDF), both conventional and custom, have experienced steady inflows of money in recent years, partially assisted by the fact that TDFs are a qualified default investment alternative (QDIA) for auto-enrollment. This trend is widely expected to continue.

One of the most important features of a (custom) TDF is the fund's glide path. The structure of the glide path is one of the key factors affecting the fund's long-term performance. The quality of the glide path design is one of the main characteristics of the fund's ability to serve the best interests of its investors. Overall, glide path design is vitally important for the success of an investment program.

Yet, the scientific foundation of optimal glide path design is still in its infancy. The glide paths of many (if not most) TDF lack theoretical substance. Even a seemingly straightforward question – why should an investor expect her portfolio to change as the investor ages? – had been unresolved for about four decades.[1] All in all, there is a need for a scientifically rigorous quantitative approach to optimal glide path design.

Criteria for a Better Framework

This paper presents a theoretical framework for glide path design for an investor with financial commitments to fund. A financial *commitment*, or just commitment for brevity, is defined as a future cash flow. The investor's goal is to fund the commitment, i.e. to ensure that the money is readily available every time a payment is due. The pursuit of this goal is called the *funding problem*. Given the multi-year nature of the commitments, the investor needs an optimal glide path that serves the investor's best interests.

One of the most important applications of optimal glide path design is retirement investing. A retirement investor wishes to achieve a certain standard of living in retirement. To support a specific standard of living in retirement, there must be an income stream generated by the investor's assets. This stream is the investor's commitment, and the investor's goal is to fund this commitment. As a measurement of the standard of living in retirement, it is common to employ the concept of *replacement ratio* defined as the percentage of the investor's last pre-retirement income available in retirement. In this paper, it is assumed that the investor's commitment is a post-retirement income stream specified by a particular replacement ratio.

The paper presents several basic principles of the development of optimal glide paths. The primary challenge in this development is the identification of the appropriate objective for the optimization of glide paths. Consequently, these principles mainly deal with the investor's objectives. The paper demonstrates how these principles lead to specific methodologies of optimal glide path design.[2]

It should be noted that a different set of principles may lead to a different optimization methodology. To assess the appropriateness of a particular set of principles, there is a need for *criteria of usefulness* to determine whether a particular methodology serves the investor's best interests.

This paper offers the following criteria. It is reasonable to assume that a retirement investor endeavors to maximize her standard of living in retirement (replacement ratio). Therefore, a sensible way to assess the value of a particular methodology is to inquire whether the methodology maximizes the investor's replacement ratio.

This criteria, however, has a problem, even though the criteria is reasonable as a rule. The investor's replacement ratio is uncertain, so it cannot be maximized as a whole. The right way to approach this problem is to optimize appropriate *risk measurements* of replacement ratio. Therefore, in order to satisfy the criteria of usefulness, a particular methodology should demonstrate that it optimizes certain risk measurements of replacement ratio.

The methodology presented in this paper satisfies this criteria. As demonstrated in subsequent sections, the portfolio selection methodology presented in this paper indeed maximizes the investor's replacement ratio.

Overall, this author is optimistic that the approach to optimal glide path design presented in this paper will be helpful to investors with financial commitments.

Principles and Objectives

The section presents several principles that serve as a basis for the methodology of optimal glide path design.

Throughout this paper, it is assumed that an investor's primary goal is to fund a financial commitment. The definition of commitment covers a broad range of future cash flows. Here are a few examples of the types of financial commitments investors may have.

- A commitment may be perfectly known (e.g. to accumulate $1M at retirement).
- A commitment may be known but uncertain (e.g. to have the replacement ratio of 50%).[3]

- A commitment may be unknown and uncertain (e.g. to have the highest replacement ratio).

The investor makes contributions to the investment program and reallocates the program's assets on a regular basis. In this paper, it is assumed that these contributions have been pre-determined (e.g. as a series of percentages of income). As discussed in Mindlin [2010], this assumption represents a version of the optimization of the *retirement triangle* "Commitment-Cost-Risk" suitable for DC plan participants. When the "Cost" component of the retirement triangle is pre-determined, then the objectives "minimize risk given commitment" and "maximize commitment given risk" lead to the same set of efficient portfolios.[4]

As discussed above, the investor wishes to have an optimal glide path that serves the investor's best interests. The principles presented below deal with *the objective of glide path optimization*, which is called just "the objective" for brevity. It is imperative to recognize that different objectives may generate different optimal glide paths.

<u>Principle 1</u>. *The objective is expressed in terms of the investor's financial commitment.*

This principle essentially instructs to "keep your eye on the ball." The investor's primary goal should be at the heart of the objective. For example, the objective of "*maximizing the probability of achieving the replacement ratio of 50%*" complies with Principle 1. As another example, the objective of "*maximizing the probability of achieving a positive return in the last working year*" does not comply with Principle 1. This objective ignores the investor's primary goal and the multitude of risks the investor faces in retirement.

<u>Principle 2</u>. *The objective is quantitative.*

This principle states that the objective should represent a *numerical* measure of success of the investment program. The principle also implies that the optimal glide path is the one that produces the best measure of success. For example, "*maximizing the probability of achieving the replacement ratio of 50%*" is an objective function that complies with Principle 2. As another example, the objective of "*having the most comfortable lifestyle in retirement*" does not comply with Principle 2, as the term "comfortable lifestyle" is not quantitative.

<u>Principle 3</u>. *The objective incorporates the investor's risk tolerance as related to the process of funding the investor's financial commitment.*

A certain level of risk is unavoidable in a typical investor's investment program. A vast majority of investors, including retirement investors, endeavor to fund their commitments by virtue of investing in risky assets. Therefore, it is possible that the investor's existing assets and future expected contributions would be insufficient to fund the commitment. This possibility should be reflected in the objective.

For example, the objective of "*maximizing the probability of achieving the replacement ratio of 50%*" complies with Principle 3. As another example, the objective of "*achieving the replacement ratio of 50%*" does not comply with Principle 3. While it is not unreasonable for the investor to target the replacement ratio of 50% as a general goal, this objective ignores the risks inherent in the investment program and does not provide sufficient information for glide path optimization.

It should be noted that the investor's risk tolerance can be expressed in many ways. In the abovementioned example, the risk tolerance is expressed as the *probability* to achieve a certain replacement ratio. Other risk measurements include, but are not limited to, the shortfall size, shortfall volatility, the risk-adjusted expected cost, etc.[5]

Principle 4. ***Every portfolio selection in a glide path is based on the investor's objective.***

Principle 4 states that the investor never ignores the commitment and the risk of its funding. As a result, the design of the entire glide path is driven by the commitment and the investor's risk tolerance, which may evolve but never disappears.

Principle 5. ***The investor and the investor's expectations are rational.***

This principle is sometimes expressed in game theory as "a rational player is a *habitual payoff maximizer*." Principle 5 means that the investor expects to optimize the objective whenever the investor makes a decision. In other words, at any decision point, the investor will not select a portfolio that generates a sub-optimal value of the objective function when a different portfolio would generate a better value.

Principle 6. ***The investor revisits her asset allocation on a regular basis.***

Principle 6 is just a reflection of broadly recognized practices most investors follow.

Principle 7. ***The investor pays the lowest amount possible for a given item.***

As far as the funding problem is concerned, Principle 7 conveys that the investor prefers to have the lowest asset value required to fund a pre-specified commitment, subject to the investor's risk tolerance. This principle also implies that investors generally prefer to contribute less to achieve a pre-specified goal. Doing so simply leaves more money in the investor's possession at the present. While ultimate success or failure of the investment program depends, among other things, on the level of contributions to fund the program, it should be recognized that the resources available to fund financial commitments are usually limited.

For example, let us assume that investment strategy *A* requires $1.0M to have an 80% chance of achieving a pre-specified replacement ratio and investment strategy *B* requires $1.1M to have an 80% chance of achieving *the same* pre-specified replacement ratio. Principle 6 tells us that the investor prefers strategy *A* to strategy *B*, as strategy *A* requires less money to deliver the same replacement ratio at lower "cost" and has the same probability of success of 80%.

In subsequent sections, Principles 1–7 are utilized to develop a robust methodology of optimal glide path design.

From Decision Theory to Game Theory

The principles presented in the prior section deal with various aspects of the investor's objectives. Assuming, for a moment, that a proper quantitative objective has been established, it would seemingly make sense for the investor to select the glide path that optimizes the objective at the present. Indeed, on the surface, it appears that the optimization of the right objective today should lead to the right glide path without further ado. In fact, the maximization of the investor's expected utility is a common approach to solving various economic problems.

This line of thinking, however, would be an unwise oversimplification of the problem of optimal glide path design. A glide path is an inherently multi-period object, and its design should incorporate multiple decision points. Applying the reasoning of decision theory to this problem and performing the optimization of the objective *only once* would imply that the investor's risk tolerance exists today and disappears in the future.

More specifically, why can't the investor optimize the portfolios all at once today, given the right objective? Because by doing so the investor would effectively ignore future portfolio selections and the evolution of the investor's risk tolerance. The assumption of selecting the entire glide path based solely on today's risk tolerance and employing this glide path "no matter what" is plainly unreasonable. In terms of the previous section, this approach violates Principles 4 and 6.

The problem of optimal glide path design should involve not only the optimization of the objective at the present, but also rational expectations of future optimal portfolio selections. This problem involves multiple decisions makers with a common goal (to fund the commitment), their actions and preferences. These characteristics of the problem suggest that *game theory* (rather than decision theory) would provide the right framework for finding optimal solutions.[6]

The next section takes initial steps in this direction.

Glide Path Design and Nash Equilibria

This section explores optimal glide path design from the perspective of *game theory*.

According to Principle 1, the investor's commitment should be the front and center of the optimization methodology. According to Principles 2, 3, and 4, for every portfolio selection, the investor has a quantifiable objective that is related to the investor's commitment and incorporates the investor's risk tolerance. According to Principle 5, for every portfolio selection, the optimal portfolio is the one that optimizes this objective.

Principle 6 implies that the investor rebalances her portfolio regularly. For simplicity, let us assume that the investor rebalances her portfolio at the beginning of each year. As discussed above, the investor seeks an optimal glide path that serves the investor's best interests. Thus, the investor is expected to select a number of portfolios that would form an optimal glide path.

It is constructive to consider that different investors – the investor herself at the present and her aging "clones" in all subsequent years – make these portfolio selections. The investor and her aging "clones" are called *the investor's cohort* throughout this paper.[7] Every member of the investor's cohort (called just "investor" for simplicity) has risk tolerance and an objective that incorporates this risk tolerance. Every investor selects the optimal portfolio that optimizes the objective.

All investors share the same goal – to fund the commitment. The investors' decisions are interrelated, as every portfolio affects the outcome of the funding process. Therefore, the investors should seek a mutually beneficial strategy.

Game theory provides an effective framework for this situation. The investor's cohort represents *a set of players*. Portfolios selections represent a player's *set of actions*. Every player has an objective function that determines *preferences* for the player's actions. A set of players, their actions and preferences define *a strategic game*.

A mutually beneficial strategy calls for every player's action to be the best response to other players' actions. Such a strategy represents one of the central notions of game theory – the concept of *Nash Equilibrium* (NE). Equivalently, a set of actions represents a NE strategy if no player can benefit by unilaterally changing her action. The challenge is to find a Nash equilibrium strategy for the "game" of optimal glide path design.

Backward Induction

As a strategic game, the funding problem has the following distinctive properties.

1. Every player has preceding and succeeding players.
2. Every player's action depends on the actions of all succeeding players. Every player's objective incorporates the portfolio selections of all succeeding players. Indeed, when a member of the investor's cohort selects a portfolio, the objective depends on the future portfolio selections (succeeding players) only, as the preceding players are in the past at the time of this selection.
3. The players – the members of the investor's cohort – know everything about each other. After all, there is only one real player – the investor – and all other players are the investor's imaginary "clones." Therefore, players know everything about each other. According to Principle 5, the investor's expectations are rational, which means that all players are rational. Moreover, all players know that all players are rational, and all players know that all players know that all players are rational, and so on *ad infinitum*. In other words, the rationality of all players is common knowledge.[8]

These properties suggest a natural "order of operations" in the selection of portfolios that constitute an optimal glide path. Optimal portfolios should be selected *backwards* – from the future to the present – as follows.

1. The last player (the one in the last year of the mortality table) selects her optimal portfolio.

2. Taking this portfolio as given, the second to last player selects her optimal portfolio.
3. The process goes on until the first player (the real investor) selects her optimal portfolio. The first player's selection incorporates the portfolios selected by preceding players (i.e. all imaginary members of the investor's cohort). This selection completes the glide path.

This process is commonly called *backward induction*. R. Aumann, the 2005 Nobel Prize laureate in economics (jointly with T. Schelling), describes backward induction as "the oldest idea in game theory" and asserts that "nothing seems simpler or more natural." In fact, a well-known result in game theory (proved by R. Aumann) states that, under certain conditions, *common knowledge of rationality necessarily implies backward induction*.[9] It also can be shown that a glide path generated by the process of backward induction is a NE strategy. This glide path is called a *NE glide path* in this paper.

How is a NE glide path related to the investor's best interests? Does a NE glide path maximize the investor's replacement ratio? The next section lays the foundation for the answers to these questions.

Required Assets and Risk Measurements

So far, the actual makeup of the investor's objectives has played no role in this discussion. In order to design a NE glide path that maximizes the investor's replacement ratio, let us deal with the specifics of the investor's objectives.

It is important to note that an objective has two key components. First, an objective must have an *objective function* – a quantitative risk measurement of the funding process. Second, an objective has an *optimization direction* – minimization or maximization. For example, the objective of maximizing the probability of achieving the replacement ratio of 50% has two components: the objective function (the probability of achieving the replacement ratio of 50%) and the optimization direction (maximization).

To identify the right objectives for the backward induction optimization process, let us utilize the principle of paying the lowest amount possible for a given item (Principle 7). Applying this principle to the funding problem, the investor endeavors to have the lowest asset value required today to fund a pre-specified commitment. Therefore, the objective function is defined as the asset value required today to fund a pre-specified commitment. The investor endeavors to minimize this function.

However, this endeavor has the same problem as the desire to maximize the replacement ratio straight up, as discussed in a prior section. The problem is this objective function does not incorporate the investor's risk tolerance. The asset value required today to fund a pre-specified commitment is uncertain, or, in other words, it is a random variable. This asset value is a *stochastic present value* of all cash flows in the funding problem (including pre-retirement contributions and post-retirement spending). This stochastic present value is called *Required Assets* (*RA*).[10]

A random variable (e.g. replacement ratio, portfolio return or required assets) cannot be optimized as a whole like a conventional function, but it can be optimized at a certain level of risk. In other words, one has to identify a relevant *risk measurement* of the random variable to be used in the optimization methodology. For example, while the investor cannot minimize the required assets as a whole, the investor may want to minimize the required assets *to have a P% chance* to fund the commitment. It can be shown that this objective is equivalent to minimizing the *P*-th percentile of *RA*.[11] The risk measurement in this example is the *P*-th percentile of *RA*, and the objective is to minimize this risk measurement.

With the objective of required assets minimization, the backward induction procedure presents an intuitively clear model of the actual multi-year process of optimal portfolio selection. For every future year, the backward induction procedure contains a step that represents this year. Indeed, having reached a particular future year, the investor is expected to minimize the required assets (subject to the investor's risk tolerance) regardless of the accumulated asset value. This expectation is built into the optimal portfolio selections in all preceding years.

To summarize, a sensible objective should consist of a risk measurement of the required assets (the objective function) and an optimization direction (minimization or maximization). Each player's payoff can be defined as the difference between some arbitrary asset value and the required assets (subject to the investor's risk tolerance). The investor subsequently may want to produce a NE glide path that that maximizes the payoff in for every member of the investor's cohort (via backward induction). One of the most important conclusions of this paper is that, under certain conditions, *the minimization of required assets utilized in the design of a NE glide path necessitates the maximization of replacement ratio*.

Asset Allocation's Déjà Vu All Over Again

Let us get back to the fundamental question a typical DC investor faces: What is the optimal portfolio today? As discussed in prior sections, this portfolio is the *first* portfolio in a NE glide path and, ironically, the result of the *last* step in the backward induction process.

Today's optimal portfolio represents arguably the most important practical result of the process of optimal glide path design. This portfolio plays a special role for several reasons. First, this portfolio is the only *actual* portfolio that is implemented at the present, as the other portfolios in the glide path are just *expected* to be implemented in the future.

Second, the performance of this portfolio represents the performance of the whole investment program (target date fund, managed account, etc.) today. As the investor (and/or her advisors) monitors the investment program and the service providers that manage it, this portfolio is the front and center of attention.

Third, the year in which this portfolio is employed – the first year – is the only year for which the investor knows the beginning-of-year asset value. Since assets are typically invested in risky financial instruments, the beginning-of-year asset values in all subsequent years are uncertain.

The last observation places an important "*boundary condition*" on the process of optimal glide path design. As the backward induction procedure unfolds, every investor minimizes her objective function – a risk measurement of the required assets. In particular, the first investor minimizes her objective function, generates an optimal portfolio and the lowest value of the objective function.

If this lowest required asset value is greater (less) than the investor's existing asset value, then, to offset the asset shortfall (surplus), the replacement ratio should be decreased (increased). This is the primary reason that the minimization of required assets leads to the maximization of replacement ratio.

It is important to keep in mind that the existing asset value and expected future contributions (plus their investment earnings) are the only sources of funding for the investor. Since expected future contributions (and the investor's risk tolerance in all years) are given, then *the lowest required asset value generated by the first investor must be equal to the existing asset value*. This "boundary condition" requires an adjustment of the replacement ratio to ensure that there are no spurious assets in the glide path design methodology. After all, money neither materializes out of nowhere nor disappears without a trace.

Having recognized the importance of the first portfolio in a glide path, let us get back to the question whether the required assets minimizing portfolio is in the investor's best interests. In particular, does this portfolio maximize the replacement ratio? The answer to this question is *yes, it does*. Given the investor's risk tolerance, expected future contributions and the "boundary condition," the first portfolio in a required assets minimizing NE glide path maximizes the replacement ratio. A formal proof of this statement is presented in the Appendix.

To recap this discussion, here is a list of the key elements needed for of the design of optimal glide path.

1. The commitment data. In case of investing for retirement, this data includes (possibly, average) current age, retirement age, compensation, existing asset value, expected contributions.
2. An objective function that represents a risk measurement of the required assets, specified for each year.
3. A risk tolerance parameter for the objective function, specified for each year. This series of risk tolerance parameters is called the investor's *riskpath*.

In other words, in order to design an optimal glide path, we need some basic information about the investor (e.g. a participant or a group of participants of a DC plan), the type of risk tolerance and numeric measurements of risk tolerance. This information completely determines a specific NE glide path that maximizes the replacement ratio. This observation is of particular importance if the objective is to create a comprehensive suite of custom target date funds.

The next section contains a few examples of optimal glide paths generated by this methodology.

Examples of Optimal Glide Paths

The examples in this section are based on the version of the methodology presented in prior sections that is analogous to mean-variance optimization – a classic approach to one-period portfolio optimization.

As discussed above, in order to specify an optimal glide path, one needs an objective function and a risk tolerance parameter for every year. In this section, the objective function in every year k is defined as "risk-adjusted expected cost" C_k as $C_k = E_k + t_k \cdot S_k$, where E_k is mean of required assets RA_k, S_k is standard deviation of RA_k, RA_k, is the required assets (the stochastic present value) in year k, and $t_k \geq 0$ is a risk aversion parameter. E_k and $t_k \cdot S_k$ can be interpreted as the investor's "reward" and "penalty" for risk taking. The investor's objective is to minimize the "risk-adjusted expected cost".[12]

The most straightforward way to specify the evolution of the investor's risk tolerance is to assume that the riskpath is constant. *Exhibit 1* presents the optimal glide path generated for a 35 year old with constant risk aversion parameter $t = 1$ (i.e. $t = 1$ in all years).

Exhibit 1

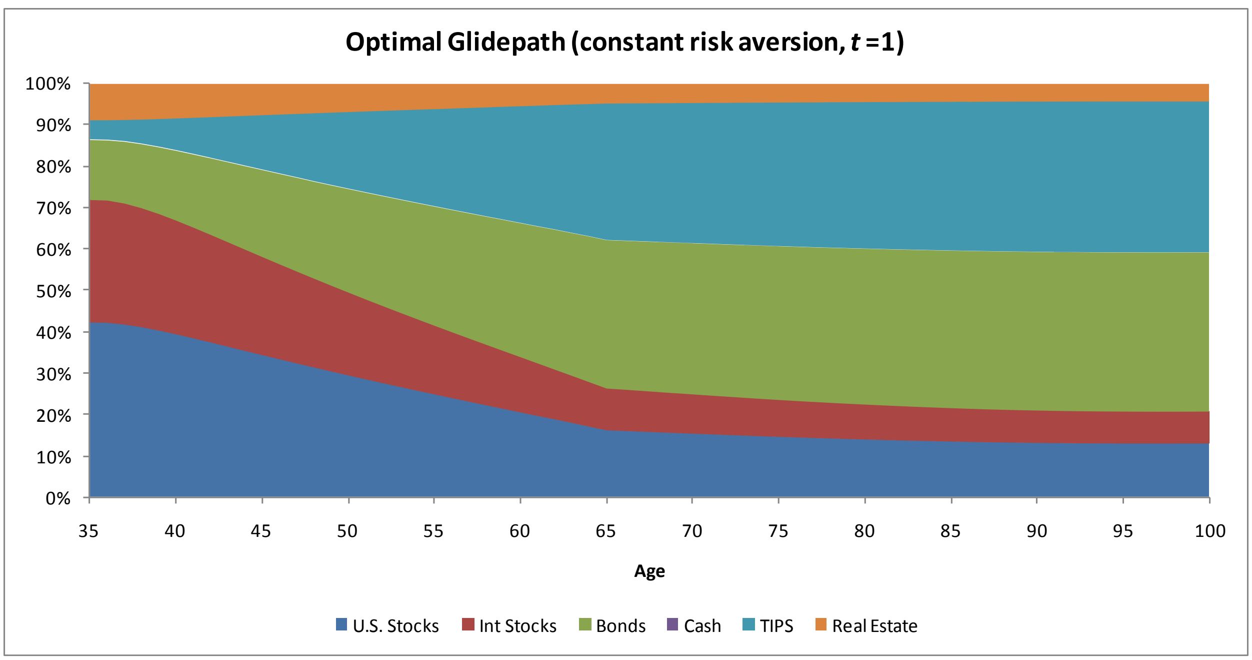
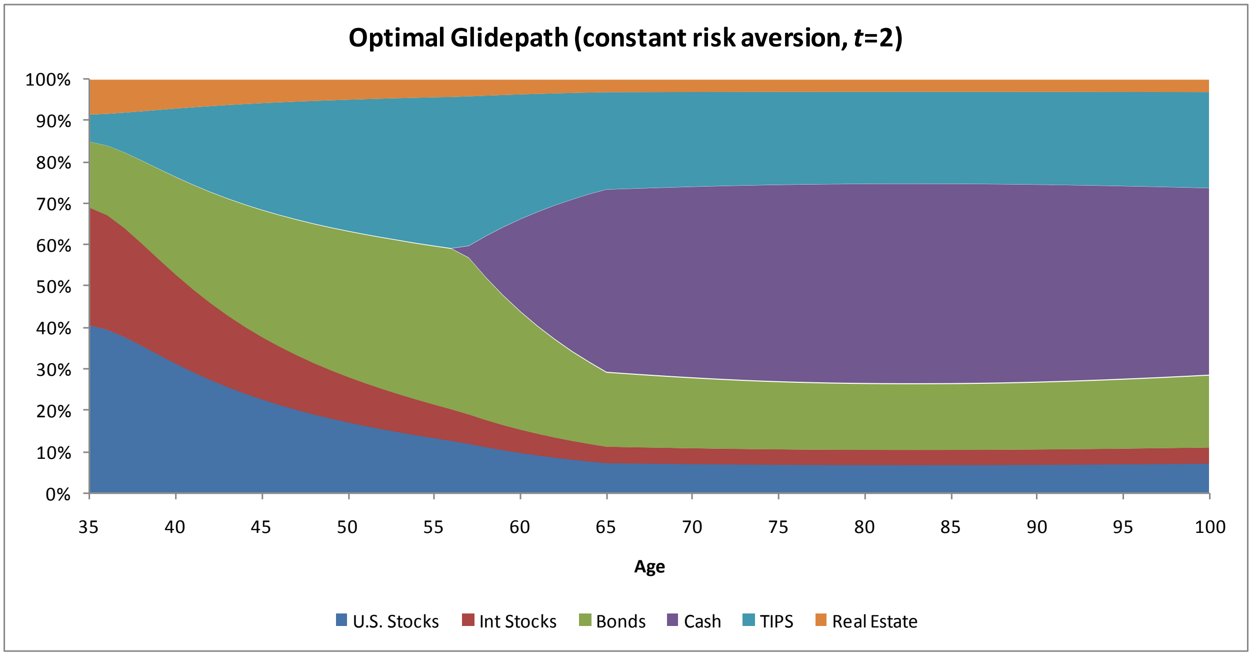


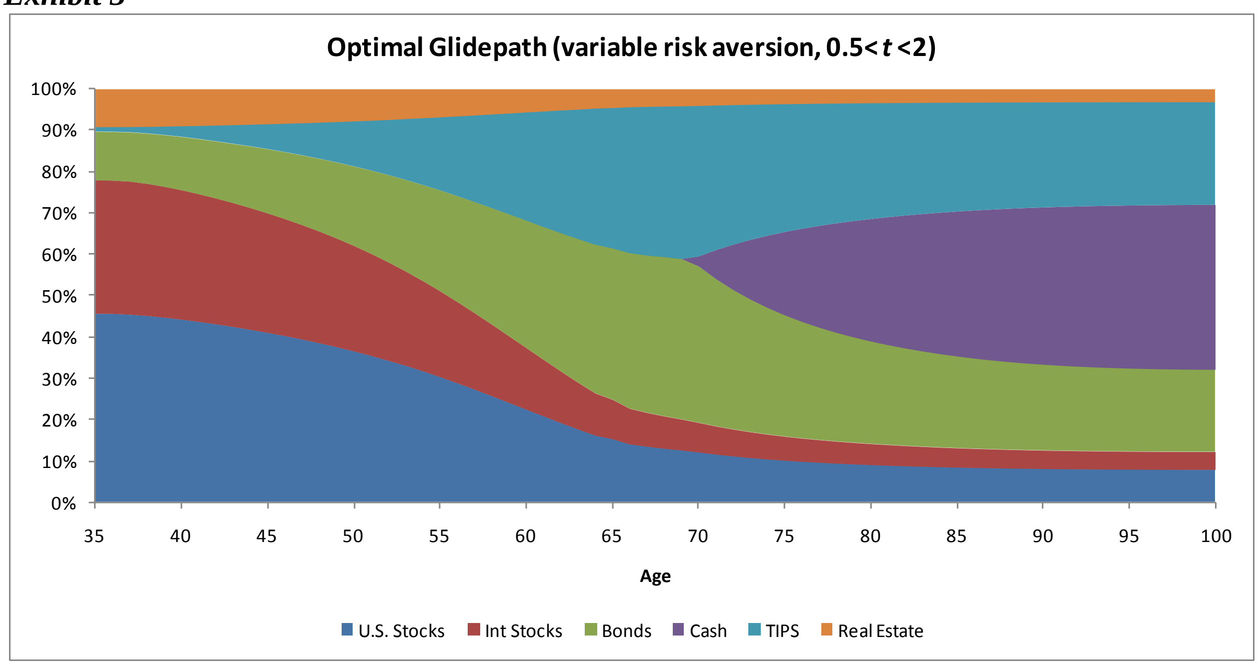
Exhibit 2 presents the optimal glide path generated for the constant risk aversion parameter $t = 2$.

Exhibit 2



It should be noted that the assumption of constant risk aversion is a choice, not a necessity. The framework presented in this paper allows wide latitude in the determination of objective functions and riskpaths. For example, *Exhibit 3* presents the optimal glide path generated for a riskpath that varies the risk aversion parameter *t* from 0.5 to 2, which may be considered as a reflection of the conventional wisdom – more risk for the young, less risk for the old.

Exhibit 3



Conclusion

Retirement investors face numerous challenges, and the selection of optimal portfolios is certainly one of them. There is a clear need for a disciplined scientifically rigorous approach that would bring clarity and powerful analytical tools to the process of optimal glide path design for retirement investors.

This author believes that the framework presented in this paper establishes a solid foundation for the development of optimal portfolios that serve the investors' best interests. The paper obviously just scratches the surface, and many fascinating problems remain unresolved. Yet, this author is optimistic that retirement investors and their service providers – target date funds, managed accounts vendors, consultants and advisors – will find this framework valuable from both scientific and practical perspectives.

FOR LICENSING INFORMATION AND MORE DETAILS ABOUT THE OPTIMAL GLIDE PATH METHODOLOGIES PRESENTED IN THIS PAPER, PLEASE CONTACT DIMITRY MINDLIN, *DMINDLIN@CDIADVISORS.COM*.

APPENDIX: Replacement Ratio Maximization

This appendix contains a formal proof that *the minimization of required assets necessitates the maximization of replacement ratio*.

Let us assume that a retirement investor (or a group of investors) has determined the stream of contributions, objective functions, and the riskpath. Let us also assume that the process of backward induction is complete for all years except the first one, and the glide path portfolios are in place for all years except the first one. As discussed in prior sections, the last step in this process is to find the optimal portfolio today via minimizing the objective function (a risk measurement of required assets) in the first year and finding the replacement ratio that satisfies "the boundary condition" (i.e. matches the existing asset value).

The question is, would the minimization of required assets lead to the highest replacement ratio? The following proposition shows that the answer to this question is "yes."

Proposition. Let $F(R,X)$ be the objective function (a risk measurement of the asset value required to fund a commitment) in the first year, where R is a replacement ratio and X is a portfolio. Let X_1 be the portfolio that minimizes $F(R,X)$ and satisfies the boundary condition for replacement ratio R_1 and the existing asset value A:

$$F(R_1, X_1) = A \qquad (1)$$

Then R_1 is the highest replacement ratio that satisfies the boundary condition.

Proof. Let X_2 be any portfolio that satisfies "the boundary condition" for replacement ratio R_2:

$$F(R_2, X_2) = A \qquad (2)$$

From (1) and (2), we have

$$F(R_1, X_1) = F(R_2, X_2) \qquad (3)$$

Since X_1 is optimal, then

$$F(R_1, X_1) \le F(R_1, X_2) \qquad (4)$$

Substituting (4) to (3), we have

$$F(R_2, X_2) \le F(R_1, X_2) \qquad (5)$$

Since $F(R, X)$ is a measurement of the asset value required to fund a commitment, then a higher replacement ratio requires higher asset value at the present. Therefore, $F(R, X)$ is an increasing function of replacement ratio R for any portfolio X. Consequently, from (5) we have

$$R_1 \ge R_2 \qquad (6)$$

Inequality (6) shows replacement ratio R_1 is the highest. Q.E.D.

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Endnotes

[1] In 1969, P. Samuelson and R. Merton demonstrated that a rational investor should maintain the same portfolio regardless of the investor's age, see Samuelson [1969], Merton [1969] and Merton [1971]. This result runs counter to conventional practices, and P. Samuelson called it "a failure," see the transcript of the P. Samuelson's interview conducted for the History of Finance project (sponsored by the American Finance Association) in 2005, http://www.afajof.org/association/historyfinance.asp. Since then, numerous authors have endeavored to find a sensible portfolio selection framework that would justify the rationality of evolving portfolios. It has been demonstrated that the presence of "human capital" and the mean-regressing properties of portfolio returns, under certain conditions, may substantiate the evolution of rational investors' portfolios. Finally, Mindlin [2009a] demonstrated that a Nash equilibrium long-term investment strategy generally consists of evolving portfolios.
[2] For a more detailed discussion of optimization objectives, see Mindlin [2010].
[3] This commitment is uncertain because there are inflation and longevity risks, among other risks.
[4] See Mindlin [2010] for more details.
[5] For the definition of "the risk-adjusted expected cost", see Mindlin [2010].
[6] This observation is of great importance for the justification of the rationality of evolving portfolios, see Mindlin [2009a].
[7] The concept of the investor's cohort was introduced in Mindlin [2009].
[8] Common knowledge is a term from *epistemic modal logic*. See, for example, Aumann [1995] or Gintis [2009], chapter 4.
[9] See Aumann [1995].
[10] For more details about stochastic present values, see Mindlin [2009b] and Mindlin [2010].
[11] See Mindlin [2009b].
[12] See Mindlin [2010] for more details about Commitment Driven Investing and its analogy to Modern Portfolio Theory.

Important Information