Commitment Driven Investing and Time Diversification

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ABSTRACT

The classic works of P. Samuelson and R. Merton in the 1960-70s demonstrated that a rational investor (a rational maximizer of expected utility) should maintain *the same* portfolio regardless of the investor's age. This conclusion stood out against conventional practices that advocated the opposite views – the investor's portfolio should be rationally expected *to evolve* as the investor ages.

It is well-known that P. Samuelson was dissatisfied with this result. Over the next four decades, P. Samuelson and numerous researchers have endeavored to find a sensible portfolio selection framework that would justify the rationality of evolving portfolios (glide paths). The results of these efforts have been unsatisfactory so far, largely due to the presence of certain overly restrictive assumptions.

This paper provides a new portfolio selection framework and a theoretical justification for the conventional practices. This framework differs from the classic approach in its primary objective (to fund a specific financial commitment vs. to maximize expected utility) and in the recognition that multiple portfolio selections should form a Nash equilibrium solution to the problem of funding the commitment. *The paper demonstrates that the missing component in the Samuelson-Merton framework was the concept of Nash equilibrium.*

"*Thus, how we set up the goals … matters a lot. … Which goal makes the most sense to you?*"

Fischer Black, *The Plan Sponsor's Goal*, 1995

The Time Diversification Problem

The time-diversification problem occupies a special place among unresolved problems in finance. It is hard to name another area of finance that has been discussed so much, produced scientifically rigorous recommendations of perfect clarity, and seen so much money invested *against* these recommendations.

"Folk wisdom and casual introspection" recommend that investors should have more risk in their portfolios when young, shifting their wealth to safer portfolios as the investors age. This is a common investment practice recommended by numerous financial counselors, as well as the foundation of life-cycle investing that has gained considerable popularity in recent years. However, Samuelson [1969] and Merton [1969]-[1971] demonstrated that, under certain assumptions, "*you do the same thing when you have one year to go … as you could do if you had 100,000 years to go*."[1]

This "defiance of folk wisdom and casual introspection" satisfied neither Samuelson nor numerous financial economists and practitioners.[2] So began the quest to find a sensible framework that would demonstrate that a rational investor should do "something different over time." Numerous authors have attempted to prove that rational investors should alter their portfolios over time. Several factors that may justify the rationality of evolving portfolios have been introduced in academic literature. Most notably, the presence of "human capital" and the mean-regressing properties of portfolio returns, under certain conditions, may substantiate the evolution of rational investors' portfolios. Nevertheless, the time-diversification problem has never been declared solved, only that "*some very considerable progress has been made*."[3]

Practitioners have no illusions about the state of affairs in the science of life-cycle investing. A recent research report from *Vanguard* provides a perfectly clear assessment of this area.

> "*Some of the finest investment minds have participated on both sides, without providing a conclusion. … The debate over time diversification has been long-running and remains unresolved*."[4]

A recent research report from *Ibbotson Associates* concurs.

> "*Little rigorous work has been done to answer how and why the equity-bond glide path should evolve throughout an investor's lifetime, and even less work has been done to answer how and why intra-stock and intra-bond splits should evolve over time*."[5]

Thus, we have a decades-old quest to justify the rationality of evolving portfolios that has involved "some of the finest investment minds." We also acknowledge that "little rigorous work has been done" to answer some of the most important questions of life-cycle investing. How could these statements coexist without contradiction?

To answer this question, let us take a closer look at some classic results. According to Samuelson [1992], there is a "*mandate that rational investors should be myopic, ignoring the length of their time horizon*." This mandate, however, is valid only "*for those who maximize expected utility*."[6] Lots of rigorous work has been done to answer how rational utility maximizers should invest their assets. But what about those who do not wish to "maximize expected utility" and pursue other goals instead?[7]

This paper identifies a different goal that compels a rational investor to do "something different over time." This paper's position is *the conventional approaches to the time diversification problem are unconvincing not because they produce the wrong answers, but because they answer the wrong questions*.

The framework this paper presents – *Commitment Driven Investing* (CDI) – explains the evolution of a rational investor's portfolio and its "time diversification" properties. The key features that distinguish CDI from other models are the explicit presence of a financial commitment in the funding problem and the recognition that multiple asset allocation decisions a rational investor makes over the investor's time horizon should form a Nash equilibrium solution to the problem of funding the commitment.

Different Aspects of the Time Diversification Problem

The term "time diversification" has several interconnected aspects. The most recognized meaning of the term signifies the belief that time "diversifies away" the riskiness of equities. Do equities possess time diversification properties? This seemingly straightforward question has proven to be highly controversial. While good arguments have been presented on both sides of the debate, a conclusive answer to this question has eluded the debate's participants.

The emerging consensus appears to be that a conclusive answer to this question may not even exist. Some risk measurements of equities increase with time; some other risk measurements of equities decrease with time. This resolution appears to be as good as it gets for this aspect of the problem.

This resolution, however, is unsatisfactory as a general solution because the most recognized meaning of the term "time diversification" is not necessarily the most important one. To a practitioner, a much more relevant aspect of the time diversification problem is the problem of optimal equity allocation and the evolution of this allocation over time (this aspect of the problem is sometimes called "time-series diversification"). Should a rational investor expect to reduce her equity exposure as she ages? This seemingly straightforward question has confounded academic researchers and practitioners alike.

Yet another meaning of the term "time diversification" is related to *different* investors at different ages, as opposed to *one* investor and the evolution of her portfolio over time (this aspect of the problem is sometimes called "cross-sectional diversification"). Should a younger person hold more equities than an older one? This seemingly straightforward question has proven to be as controversial as the abovementioned "seemingly straightforward" questions.

Overall, the problem of time diversification has long been the subject of raging debates that show no signs of abating.

It is important to clarify the place of this paper in this debate. If one believes in the "more time – less risk" property of equities, then one is understandably in favor of both time-series and cross-sectional time diversifications. After all, if "more time" means "less risk," then younger people, who possess more time by definition, should hold more equities. On the other hand, if one supports time-series and cross-sectional diversifications, it does not necessarily mean that one believes in the "more time – less risk" property of equities. The solutions this paper presents possess both time-series and cross-sectional diversification properties. At the same time, this paper advances no arguments regarding long-term or short-term riskiness of equities.

Commitments, Goals and Nash Equilibria

The approach this paper introduces is based on the idea that investors do not invest in a vacuum – they contribute their resources and take a multitude of risks primarily *to fund their financial commitments*. A financial commitment is defined as a series of payments in the future.[8] As Peter Bernstein put it, "*... the purpose of accumulating wealth is to spend money at some future point for some purpose.* "[9] The challenge is to bring this "money at some future point for some purpose" – a financial commitment – directly into portfolio optimization.

A commitment is the reason a particular investment program exists in the first place. The objective of funding the investor's commitment is the driving force behind the asset allocation and contribution policies as well as the guiding light for risk taking. To highlight the role the concept of commitment plays in the funding problem, this framework is called *Commitment Driven Investing* (CDI).

It should be clear from the paper's epigraph that this author subscribes to the notion that the assumption regarding the investor's goals is of paramount importance. The very existence of a specific commitment makes possible an explicit articulation of the investor's goals regarding the cost of funding and the risk of failing to fund the commitment. When a multi-period commitment is explicitly defined, the investor is at liberty to put the expected evolution of the investor's preferences into effect.

Let us consider, for example, an investor who has made a commitment to fund a certain cash flow over the next N years. The investor's challenge is to specify her risk preferences and the evolution of these preferences, design the initial portfolio and the evolution of this portfolio, and, finally, determine how much money to contribute.

Assuming that the investor rebalances her portfolio at the beginning of each year, the investor is expected to make N portfolio selections. It is useful to think that different investors (rather than the same investor at different points in time) make these portfolio selections. Effectively, there are N investors – the investor herself at the present and her aging "clones" in all subsequent years. Every investor selects a portfolio in her particular year. All investors have the same goal – to fund the commitment. *The presence of the commitment and the existence of the common goal compel the investors to seek a mutually beneficial strategy*. The investor and her aging "clones" are called *the investor's cohort* throughout this paper.

The decisions of the members of the investor's cohort are interrelated. Problems regarding the interactions of a group of decision makers are one of the subjects of *game theory*. The investor's cohort represents *a set of players*. Various portfolios available to a player represent the player's *set of actions*. Given a level of risk tolerance, a player may prefer a portfolio that minimizes cost for the player's level of risk tolerance. Therefore, each player has certain *preferences* for the player's actions (the lower cost, the better). A set of players, their actions and preferences define *a strategic game*.[10]

Every player must deal with other players and respond to their actions. Therefore, the goal of the investor's cohort is to develop a strategy according to which every player's action is the best response to other players' actions. Such a strategy represents one of the central notions of game theory – the concept of *Nash equilibrium*. (A set of actions represents a Nash equilibrium if no player can benefit by unilaterally changing her action.)[11]

The next section discusses this example in more detail and, in particular, specifies a series of portfolio selections that represents a Nash equilibrium for the funding problem in the example.

Getting back to the time diversification debate, the key question is whether the portfolios that represent a Nash equilibrium should be the same. As a rule, the answer is "no." In particular, ***the investor's cohort, even though subject to constant risk tolerance and facing random-walk securities returns, would not rationally invest the same fraction in equities at all ages***.

It is informative to compare these statements to the following statement from Samuelson [1989].

> "*Folk wisdom recommends that we should be more risk tolerant when young, reducing as we approach retirement the fraction of wealth put into risky equities and increasing our safe-cash exposure. However, Samuelson [1969] found that* ***a rational maximizer of expected utility****,* ***even though subject to constant relative risk aversion and facing random-walk securities returns, would rationally invest the same fraction in equities at all ages***." (Emphasis is mine – DM).

Obviously, the presence of "time diversification" in CDI and the absence of "time diversification" in Samuelson's framework are in conflict. To grasp the nature of this conflict, it should be noted that Samuelson's "rational maximizer of expected utility" and "a rational Nash equilibrium constructor" in CDI have entirely different goals. The differences between the goals of "maximizing expected utility over a lifetime" and "constructing a Nash equilibrium solution" may truly "matter a lot."

Example: $1 in *N* Years

Let us assume that the investor has made a commitment to fund just one payment of $1 in *N* years. We assume that the goal to fund the commitment is the investor's *primary objective* and the failure to fund the commitment is the investor's *primary risk*.

It should be emphasized that the primary risk is defined as an event the investor wishes to avoid (the failure to fund the commitment). In order to manage this risk, the risk should be measured. The measurements of this risk include, but are not limited to, the probability of the "event-to-avoid" and the size and volatility of shortfall if the "event-to-avoid" happens. This paper focuses primarily on the probability of the "event-to-avoid" due to its importance and simplicity. Consequently, the risk tolerance is expressed as the probability that a particular asset value at the present is sufficient to fund the commitment. Other measurements, however, are also important for a comprehensive risk analysis.

We also make the following assumptions.

- The investor contributes a certain asset value at the present and makes no other contributions.
- The investor wishes to minimize this asset value (the cost of funding the commitment) given the investor's risk tolerance.
- The distributions of portfolio returns are lognormal.[12]
- Portfolio returns in different years are independent.
- There are no transaction costs.
- There is no risk-free asset. In other words, a Treasury zero-coupon bond that pays $1 at the end of *N* years is not obtainable.

For a given series of stochastic portfolio returns $R_1, \ldots, R_N$, let us define random variable *RA* as the stochastic present value of $1 in *N* years.

$$RA = \frac{1}{(1+R_1)\ldots(1+R_N)} \tag{1}$$

Definition (1) provides asset value *RA* required at the present to fund $1 in *N* years (this is the reason behind the abbreviation *RA* – "*Required Assets*").

The investor's risk tolerance and the cost of funding the commitment can be expressed in terms of stochastic present value *RA*. For a given asset value $*A* and a series of portfolios returns $R_1, \ldots, R_N$, probability *p* that $*A* is sufficient to fund $1 in *N* years is equal to

$$p = \Pr\left(A(1+R_1)\ldots(1+R_N) \geq 1\right) = \Pr\left(RA \leq A\right) \tag{2}$$

It clear from (2) that if the investor wishes to have probability p to fund the commitment, then asset value A is equal to the pth percentile of RA. Therefore, the problem of cost minimization is equivalent to the problem of finding the lowest pth percentile of the corresponding RA. (Proposition 1 in the Appendix contains a formula to calculate percentiles of RA).

It should be emphasized that the procedure of finding the lowest pth percentile of RA utilizes risk tolerance at the present and ignores future risk tolerances. In this case, it can be shown that, given the investor's risk tolerance at the present, a series of portfolios that minimize the cost of funding the commitment does not possess time-series diversification properties. In other words, if the investor is allowed to select portfolios in all years and use her risk tolerance only at the present, then *the optimal strategy contains the same portfolio in all years* (see Proposition 2 in the Appendix). In terms of the investor's cohort, this result essentially tells us that *if a player could disregard the succeeding players and their risk preferences, the player's optimal strategy would be to select the same portfolio every year*.

Exhibit 1 contains numerical examples for the problem of funding \$1 in one year (N=1), two years (N=2), and three years (N=3) under the assumption that the investor's risk tolerance (70%) exists only at the present. The optimal strategy for the two year problem is 58% equities and 42% cash in both years; the optimal strategy for the three year problem is 100% of equities in all three years. The optimal portfolios, however, do depend on number of years N and risk tolerance p. Thus, time-series diversification is absent, but cross-sectional diversification is present.

Exhibit 1.

The Funding Problem \$1 in *N* Years - Risk Tolerance 70% at the Present Only

	One Year Problem (N = 1)			Two Year Problem (N = 2)			Three Year Problem (N = 3)		
Year	Risk Tolerance	Equities	Cash	Risk Tolerance	Equities	Cash	Risk Tolerance	Equities	Cash
1	70%	10%	90%	70%	58%	42%	70%	100%	0%
2				-	58%	42%	-	100%	0%
3							-	100%	0%

Capital Market Assumptions

	Equities	Cash
Geometric Return	8%	3.25%
St Dev of Return	13%	1%
Correlation between Equities and Cash		0

Now, let us assume that risk tolerance preferences do not disappear. In light of this assumption, let us revisit the funding problem paying full attention to the investor's cohort. As a "strategic game," this problem has the following distinctive properties. First, every player has preceding and succeeding players. Second, every player's action depends on the actions of all succeeding players. These properties suggest a natural "order of operations" in the quest to find optimal portfolios – from year N back to the present.

Let us consider the *N*th (last) player in the investor's cohort and her risk tolerance. The last player wishes to have probability p_N to fund \$1 in *one* year and, at the same time, minimize the cost of funding the commitment. Her *RA* is defined as

$$RA_N = \frac{1}{1+R_N} \tag{3}$$

As discussed before, the last player's goal is to minimize the p_N th percentile of RA_N in (3). Let us assume that portfolio R_N^* delivers the lowest p_N th percentile of RA_N.

Next, let us consider the (*N*-1)th player and her risk tolerance. This player wishes to have probability p_{N-1} to fund \$1 in *two* years and, at the same time, minimize the cost of funding the commitment. For any portfolios R_{N-1} and R_N, the corresponding *RA* is equal to

$$RA_{N-1} = \frac{1}{\left(1+R_{N-1}\right)\left(1+R_N\right)} \tag{4}$$

However, the (*N*-1)th player is not at liberty to use *any* portfolio R_N – the selection of portfolio R_N is the *N*th player's prerogative. The (*N*-1)th player expects the *N*th player to find her optimal portfolio R_N^* and subsequently incorporates this portfolio into the (*N*-1)th player's decision-making process. Therefore, RA_{N-1} in (4) becomes

$$RA_{N-1} = \frac{1}{\left(1+R_{N-1}\right)\left(1+R_N^*\right)} \tag{5}$$

The (*N*-1)th player's action, therefore, is to minimize the p_{N-1}th percentile of RA_{N-1} in (5) given portfolio R_N^* and get her optimal portfolio R_{N-1}^*.

Similarly, the *k*th player uses the optimal portfolios $R_{k+1}^*,\ldots,R_N^*$ selected by her succeeding players to define RA_k as

$$RA_k = \frac{1}{\left(1+R_k\right)\left(1+R_{k+1}^*\right)\ldots\left(1+R_N^*\right)} \tag{6}$$

and subsequently minimize the p_kth percentile of RA_k in (6) given portfolios $R_{k+1}^*,\ldots,R_N^*$. This process continues until all optimal portfolios $R_1^*,\ldots,R_N^*$ have been selected. The lowest p_1th percentile of RA_1 is the desired asset value to be invested at the present. It can be shown that *portfolios* $R_1^*,\ldots,R_N^*$ *represent the Nash equilibrium solution* to the funding problem.

Exhibit 2 contains numerical examples for the problem of funding \$1 in one year (*N*=1), two years (*N*=2), and three years (*N*=3) under the assumption that the investor's risk tolerance is constant (70%) in every year.

Exhibit 2.

The Funding Problem \$1 in *N* Years - Risk Tolerance 70% in Every Year

Year	One Year Problem (N = 1) Risk Tolerance	Equities	Cash	Two Year Problem (N = 2) Risk Tolerance	Equities	Cash	Three Year Problem (N = 3) Risk Tolerance	Equities	Cash
1	70%	10%	90%	70%	18%	82%	70%	29%	71%
2				70%	10%	90%	70%	18%	82%
3							70%	10%	90%

Capital Market Assumptions

	Equities	Cash
Geometric Return	8%	3.25%
St Dev of Return	13%	1%
Correlation between Equities and Cash		0

The time-series diversification properties of the optimal strategies for the two year and three year problems in Exhibit 2 are clear – the optimal portfolios there obviously shift from equities to cash over time. Note that the year 2 portfolio in the optimal strategy for the two-year problem and the year 3 portfolio in the optimal strategy for the three-year problem are the same as the optimal portfolio for the one-year problem. Similarly, the year 1 portfolio in the optimal strategy for the two-year problem and the year 2 portfolio in the optimal strategy for the three-year problem are the same.

Apart from the differences in the time diversification properties, the results presented in Exhibits 1 and 2 are strikingly different in conventional terms as well. The optimal portfolios in Exhibit 2 are much more conservative in the "asset-only" space than the ones in Exhibit 1. For example, the all-equity glide path in the three-year funding problem that ignores future risk preferences loses its over-aggressiveness as soon as these future risk preferences are introduced. As we see, the incorporation of future risk tolerance preferences may have profound impact on optimal portfolios, even when these preferences remain constant.

Attributes of CDI

This paper does not attempt to present CDI in its entirety, as the main subject of the paper is time diversification. Some of the most valuable features of CDI – e.g. multi-stream commitments, human capital, annuities, and matching bond portfolios – are outside of the scope of this paper.[13] In fact, in order to illuminate the issue of time diversification in CDI, the author has endeavored to present the simplest possible version of CDI in this paper. A much more realistic example of a Nash equilibrium glide path for a DC plan participant is presented in Exhibit 3 in the Appendix.[14]

However, it is important to highlight a few attributes of CDI that distinguish this approach from other approaches. This section presents a concise discussion of these attributes.

The first attribute is related to the articulation of the investor's goals. The investor's goals must be stated clearly. Unfortunately, a clear articulation of the investor's goals is not a common attribute of most publications on the subject of asset allocation. In CDI, the investor's goal, in its simplest form, is dual: given a financial commitment, to *minimize risk at a given level of cost*, and *minimize cost at a given level of risk*. One of the key results in CDI is these objectives are in perfect harmony with each other – they lead to the same set of efficient portfolios called the *cost-risk efficient frontier*.

The second attribute is related to the issue "what to manage and when to manage it." As we all know, the amount of literature devoted to the analysis of investment risks is immense. A large portion of this literature deals with investment portfolios and their returns. A number of risk measurements of portfolio returns are well-known (e.g. standard deviation, shortfall probability, semi-variance, value-at-risk). Also well-known are various methodologies that optimize portfolios based on these measurements. It is generally recognized that all risk measurements have their pros and cons, and there is no single risk measurement superior to all others.[15]

Yet, all *portfolio return*-based methodologies utilize one obscured assumption. It is assumed that the asset value is known at the present and the goal is to manage the unknown future value. For example, think of the classic mean-variance optimization (MVO). Given one dollar at the *present*, MVO minimizes the variability of the *future* value of this dollar given the mean of the *future* value. In general, *the object of return-based optimization (RBO) is in the future.*

For an investor with a financial commitment to fund, the situation is exactly the opposite. As was discussed in the previous section, if the investor has a commitment to fund \$1 in *N* years, then the desired future value (\$1) is perfectly known. On the other hand, the present value – the amount to invest at the present – is not known and should be optimized. In the simplest form of the mean-variance version of CDI, given one dollar in the *future*, CDI minimizes the variability of the *present* value of this dollar given the mean of the *present* value.[16] In a certain sense, CDI is RBO in reverse, as RBO's "known present – optimized future" principle becomes CDI's "known future – optimized present" principle. Hence, *the object of the commitment-based optimization is at the present.*

The third attribute is related to the main object of our analysis. By definition, the main object of all return-based optimization methodologies is portfolio return and its measurements. Portfolio return is a one-period stochastic object that "knows nothing" about the investor or her financial commitments. While there have been attempts to fix this problem (most notably, the replacement of portfolio return by surplus return in surplus optimization), the results of these attempts have not been broadly accepted in DB or DC universes.

The main object of analysis in CDI is *RA* ("Required Assets") – a stochastic present value of the investor's financial commitment. *RA* "knows something" about the investor and her commitment; in fact, it "knows" a lot about the commitment. For example, for a DC plan investor, *RA* "knows something", among other things, about the investor's age, salary, deferral rate, and time until retirement. For a DB plan, *RA* "knows something," among other things, about the age/service characteristics of the plan, its demographic assumptions and the benefit package. Most importantly, *RA* "knows something" about the investor's portfolio and its evolution in the future. Thus, *RA* contains vital information about the key components of the funding problem.

As a technical tool, *RA* possesses a number of valuable qualities; some of them were on display in the previous section. *RA* is a multi-period stochastic object that can accommodate different rebalancing rules, volatile commitment streams, annuities, matching bond portfolio, and much more. It helps translate the problems of funding financial commitments into the standard language of probability theory and utilize this theory's analytical tools to tackle those problems.[17]

Overall, the analysis of stochastic present values is valuable to a variety of funding problems (DC and DB plans, foundations, endowments, etc.) This author is cautiously optimistic that this analysis will eventually become a mainstream technique for institutional and individual investors.

Conclusion

The time diversification debate has been going on for a long time and is likely to continue. According to some models, *a rational investor, in pursuit of her goals*, "*would invest the same fraction in equities at all ages*." This paper presents a different framework – Commitment Driven Investing – that looks at the problem of time diversification form a different angle. In this framework, *a cohort of rational investors, in pursuit of their goals, would not generally invest the same fraction in equities at all ages*. Thus, the investors' goals are at the heart of the time diversification conundrum.

In addition to setting the right goals, investors must ask the right questions. Here is a small sample of questions that investors may want to ponder.

- What is the purpose of accumulating wealth?
- When does risk taking happen – at the present, at some point in the future, or all the time between now and some point in the future?
- What do we know – the future value or the present value?
- What do we need to know – the future value or the present value?
- Do risk preferences ever disappear?

The identification of the right goal is one of the most important steps in the process of finding optimal solutions, as Fischer Black taught us in Black [1995]. Maximizing expected utility gives one set of strategies, and maximizing the likelihood of funding one's financial commitment gives another. We are finally beginning to grasp the importance of F. Black's final message to us.[18]

"*Which goal makes the most sense to you?*"

APPENDIX.

For a given series of lognormal portfolio returns $R_1, \ldots, R_N$, let us define random variable *RA* as

$$RA = \frac{1}{(1+R_1)\ldots(1+R_N)}$$

If $E(R_k) = m_k$, $Var(R_k) = s_k^2$ for all $k \leq N$, let us also define C_1 and C_2 as

$$C_1 = \prod_{k=1}^{N} (1+m_k)$$

$$C_2 = \prod_{k=1}^{N} \left(1 + \frac{s_k^2}{(1+m_k)^2}\right)$$

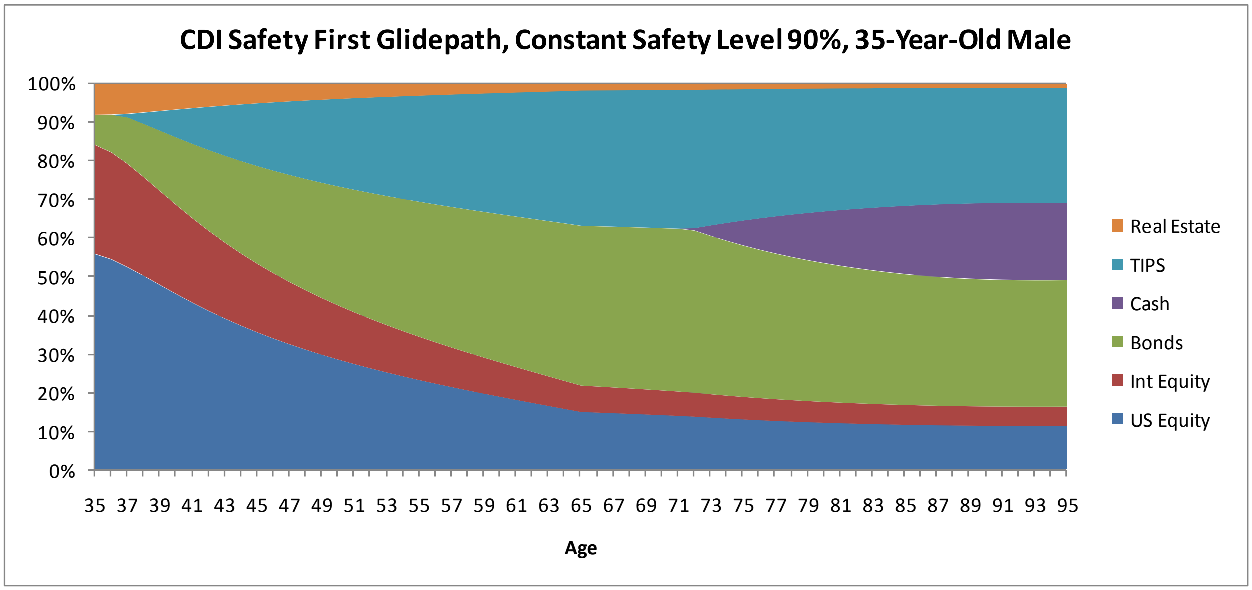
Proposition 1. Given lognormal portfolio returns $R_1, \ldots, R_N$, the *p*th percentile of *RA* is equal to

$$Q(p) = \frac{\sqrt{C_2}}{C_1} e^{\Phi^{-1}(p)\sqrt{\ln C_2}}$$

where $\Phi^{-1}(p)$ is the *p*th percentile of the standard normal distribution.

Proposition 2. For a given probability *p*, a series of portfolios that minimizes the *p*th percentile of *RA* (see Proposition 1) contains identical portfolios.

Exhibit 3.



Source: *CDI Advisors LLC*

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ENDNOTES

[1] From the interview conducted for the History of Finance project sponsored by the American Finance Association (called the Interview throughout this paper), page 2. The Interview is quoted according to the transcript posted at http://www.afajof.org/association/historyfinance.asp.

[2] As a reflection of this dissatisfaction, Samuelson stated in the Interview (p.3), "*… I felt that the first pass at the problem was a failure. I hadn't proved that you should (do) something different over time*." Samuelson's desire to justify the rationality of a non-myopic view of life-cycle investing is reflected in the titles of some of his papers. Twenty years after Samuelson [1969], Samuelson wrote a paper named "A Case *At Last* for Age-Phased Reduction in Equity" – note the priceless "*At Last*" in the title (see Samuelson [1989]). Another "*At Last*" appeared in the title

of Samuelson [1992] – "*At Last*, a Rational Case for Long-Horizon Risk Tolerance and for Asset-Allocation Timing?" Yet another "*At Last*" appeared in the title of section 7 in Samuelson [1991] – "Intuition *At Last* Vindicated." Moreover, at the conclusion of Samuelson [1991], Samuelson declared, "*My three-decade search for confirmation of enhanced risk tolerance among long horizon investors has … achieved success*." In retrospect, however, the arguments presented in Samuelson [1989] and Samuelson [1991] did not look terribly compelling.

3 Quoted from the Interview, page 3.

4 See Bennyhoff [2008].

5 See Idzorek [2008].

6 The mandate also requires investors to possess "constant relative-risk aversion", see Samuelson [1992].

7 While the goal of "maximizing expected utility" is popular in academic literature, this popularity is far from universal. More than half a century ago, Roy [1952] stated the following. "*In calling in a utility function to our aid, an appearance of generality is achieved at the cost of a loss of practical significance and applicability in our results. A man who seeks advice about his actions will not be grateful for the suggestion that he maximise expected utility."* Also, Rabin and Thaler [2001] compellingly argue that "… *it is time for economists to recognize that expected utility is an ex-hypothesis, so that we can concentrate our energies on the important task of developing better descriptive models of choice under uncertainty*."

8 These payments may be of uncertain timing, magnitude and likelihood.

9 See Bernstein [2003].

10 See, for example, Osborne [2004].

11 For example, see Osborne [2004], chapter 2.

12 Portfolio return *R* is called lognormal if $1+R=e^{X}$, where *X* is a normal random variable.

13 See Mindlin [2009a] for more details.

14 Exhibit 3 contains the Nash equilibrium glide path (CDI Safety-First unconstrained optimization, 90% safety level for all ages) for a 35-year-old male, who retires in 30 years, has salary $50,000, account balance $50,000, contributes 6% of salary and gets 3% employer match.

15 See, for example, Levy [2006].

16 Similar to return based methodologies, CDI has different versions, e.g. mean-variance and Safety-First.

17 For the basics of stochastic present values, see Kellison [2009], chapter 12 and Milevsky [2006], chapter 9.

18 Sadly, F. Black passed away soon after the publication of Black [1995]. For a more detailed analysis of Black [1995], see Mindlin [2009b] .

Important Information