Simulation-Free Stochastic Analysis: An Introduction

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SUMMARY. *Monte-Carlo simulations have long been a standard tool for the stochastic analysis of investment programs. Yet, simulation-free estimates of various random variables have also been well-known in finance and beyond. This introductory paper presents several examples of stochastic analysis without Monte-Carlo simulations.*

Numerous institutional and individual investors endeavor to fund their financial commitments by investing in risky assets. For decades, Monte-Carlo simulations have been a standard tool for the stochastic analysis of investment portfolios, financial commitments, and their relations. This tool is popular among investment consultants, asset managers, plan sponsors, and others.

Yet Monte-Carlo simulations possess certain deficiencies. Most importantly, simulation results are illustrations rather than conclusive quantitative demonstrations. This tool may require an extensive and expensive computational infrastructure. Simulation results may be hard to replicate and materially depend on sample sizes, software/hardware platforms and other technicalities.

Most funding programs (e.g., DB, DC, and 529 plans) routinely employ simulations for their analysis. Many appear unaware that reliable estimates of some of the key variables can be obtained without simulations. Moreover, many appear oblivious of the fact that simulation-free stochastic analysis has long been universally recognized.

The goal of this introductory paper is to present several examples of simulation-free stochastic analysis and provide proper context for more advanced analysis in future publications. These examples demonstrate that simulation-free stochastic analysis has been

- well-known for a long time,
- well-known in finance,
- well-known in actuarial science.

Tosses of a Symmetrical Coin

Let us consider the following question: what is the probability that the number of heads in 1,000,000 tosses of a symmetrical coin is greater than 499,000 but less than or equal to 501,000? To answer this question, there is no need to employ Monte-Carlo simulations or perform 1,000,000 tosses of a symmetrical coin. The answer is 95.449968%; anyone with a basic understanding of the binomial distribution and MS Excel can get this result.

Obviously, neither MS Excel nor Monte-Carlo simulations were available in the 18th century. Yet Abraham de Moivre could have estimated the answer to this question in the 1730s. The estimate generated by the normal approximation to the sums of binomial random variables that de Moivre developed is 95.449974%. This is simulation-free stochastic analysis in the early 18th century.

Modern Portfolio Theory

Modern Portfolio Theory (MPT) is a classic framework for the stochastic analysis of portfolio returns. MPT utilizes the language of probability theory and the basic measurements of random variables (e.g. means, variances, correlations). The analysis of portfolio means and variances is instrumental to generating efficient portfolios – the key concept of MPT.

Yet there is no need for Monte-Carlo simulations in MPT. Its stochastic analysis of portfolio returns is perfectly simulation-free for a simple reason. The required measurements of portfolio return – the arithmetic mean and the variance – can be calculated directly from the capital market assumptions. Specifically, the arithmetic mean of portfolio return is calculated as follows:

$$M = R^T w$$

where w is the vector of portfolio weights, R is the vector of arithmetic mean returns for the underlying asset classes.

The variance of portfolio return is calculated as follows:

$$V = w^T C w$$

where w is the vector of portfolio weights and C is the covariance matrix. These "simulation-free" formulas are the foundation of MPT developed in the 1950s.

Stochastic Present Values

The concept of present value is at the heart of finance in general and actuarial science in particular. While deterministic present values are much more common than their stochastic counterparts, stochastic present values (SPV) are recognized as well. The renowned Black-Scholes formula is arguably the most prominent example of simulation-free stochastic analysis of SPVs. Also, there is a substantial body of research on stochastic annuities and their present values that does not employ simulations.

Conventional funding problems can be translated into the standard language of probability theory using the concept of SPV. Robust estimates of SPVs are instrumental to generating optimal asset allocation, contribution, and spending strategies for funding problems.

Here are a couple of examples of simulation-free analysis of SPVs. The first example deals with the distributions of SPVs of perpetuities. If portfolio returns are lognormal, then the SPV of a perpetuity has inverse Gamma distribution. This and similar results make possible simulation-free estimates of various measurements of shortfall (e.g., shortfall probability).[1]

The second example deals with the moments of the distributions of SPV. Let us consider a series of n payments of \$1, $\{r_k\}$ be a series of independent identically distributed portfolio returns, and $R_k = 1 + r_k$ be return factors. The corresponding SPV is defined as follows:

$$SPV = \sum_{k=1}^{n} \frac{1}{R_1 \ldots R_k}$$

The stochastic properties of this SPV are directly related to the funding objective. Specifically, this series of payments is funded if and only if the corresponding

[1] Dufresne (1990), Milevsky (1997), Milevsky (2005), Milevsky (2006), Dufresne (2007).

SPV is no greater than existing asset value A_0.[2] The probability of funding is equal to $\Pr\left(SPV \le A_0\right)$.

Let us also define the following values:

$$m_1 = E\left(\frac{1}{R_k}\right)$$

$$m_2 = E\left(\frac{1}{R_k^2}\right)$$

$$S_1 = \sum_{k=1}^{n} m_1^k$$

$$S_2 = \sum_{k=1}^{n} m_2^k$$

The mean of SPV is calculated as follows:

$$E\left(SPV\right) = S_1$$

The variance of SPV is calculated as follows:[3]

$$Var\left(SPV\right) = \frac{m_2 + m_1}{m_2 - m_1} S_2 - \frac{2m_2}{m_2 - m_1} S_1 - S_1^{\,2}$$

These formulas assume stationary series of payments and portfolios. Similar methodologies generate estimates of the key measurements of SPVs for any series of payments, contributions, and glide paths. These estimates can be used for the development of optimal glide paths as well as contribution and spending strategies for a broad variety of funding problems. No simulations are needed.

Overall, simulation-free stochastic analysis has a long history and is well-established in finance. Simulation-free methodologies possess significant advantages and should be expected to receive increasing attention in the future.

[2] See Mindlin (2009).
[3] See Kellison (2009). See also Mindlin (2009) for an expanded version of this result.

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Important Information