1 Introduction

In this document, you will find the proofs to all lesson/worksheet problems.

The first section will be dedicated to Lecture Problems while the second will be on Worksheet Problems.

2 Lecture Problems

2.1 Direct Proof

Problem 2.1.1. *Prove that the product of two odd integers is odd.*

Proof. Let $u = 2u_1 + 1, v = 2v_1 + 1$ be two odd integers. Then, $uv = (2u_1 + 1)(2v_1 + 1) = 4u_1v_1 + 2u_1 + 2v_1 + 1 = 2(2u_1v_1 + u_1 + v_1) + 1 = 2m + 1$, which is odd. $\blacksquare$

2.2 Proof By Contrapositive

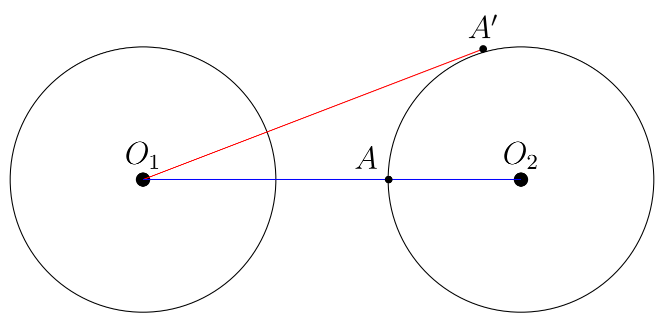
Problem 2.2.1. *Prove that the product of two odd integers is odd.*

Proof. Let $2m = uv$ be an even integer. Then, WLOG, $u = 2k$, $v = \dfrac{m}{k}$ (or vice versa), where $k|m$ (k divides m) Hence, either u or v must be even. If the product being not odd ($\neg q$) implies u or v being even ($\neg p$), then the two integers being odd (p) implies that the product is odd (q).

2.3 Proof by Contradiction

Problem 2.3.1. *Prove that the point on circle ω_1 that is closest to the center of a different circle ω_2 lies on the line joining the centers of ω_1 and ω_2*

Proof. Assume that the length of $\overline{A'O_2}$ for a point A not on the line O_1O_2 is less than $\overline{AO_2}$ for $A \in \overleftrightarrow{O_1O_2}$



Then, $\overline{A'O_2} < \overline{AO_2}$. However, as A' is closer to O_2's x-coordinate than A, the triangle $\overset{\triangle}{O_1AA'}$ is an obscene triangle with $\angle O_1AA' > 90°$ Hence, $\overline{A'O_2} > \overline{AO_2}$. Contradiction $\Rightarrow\!\Leftarrow$.

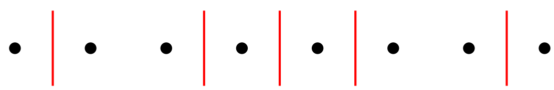
2.4 Proof By Bijection

Problem 2.4.1. *A composition of an integer n is a set of positive integers $a = (a_1, ..., a_k)$ s.t.*

$$\sum_{i=1}^{k} a_i = n$$

Prove that the number of compositions of n is 2^{n-1}

Proof. We'll use the stars & bars counting method. Assume that there is a ball for each 1 in n, equaling a total of n balls. For each composition, you start from in between the first and the second ball either put a bar there or you don't. The number of balls in between the kth and $k+1$th bars is equal to a_k. Number of bars $+1$ determines the length of the composition. For example, the diagram below shows a composition of 8, where the length of the composition is $5 + 1 = 6$ and the composition is $a = (1, 2, 1, 1, 2, 1)$.



There are $n-1$ slots and two choices (no bar/bar); hence, the number of distinct compositions is $2 \times 2 \times ... \times 2 = 2^{n-1}$

2.5 Proof By Exhaustion

Problem 2.5.1. *Find $n \in \mathbb{Z}^+$ s.t. $m = n^2 + n + 3$ is a perfect square.*

Proof. See that $n^2 < m = n^2 + n + 3 < (n+1)^2 = n^2 + 2n + 1$ for $n \geq 3$. As no perfect square exists in between two consecutive perfect squares, m can't be a perfect square for $n > 2$. Then, the only cases left are $n = 1, 2$. It is easy to see that $n = 1$ doesn't yield a perfect square for m while $n = 2$ gives $m = 4 + 2 + 3 = 9 = 3^2$. Thus, the only solution is $n = 2$ ■.

2.6 Choosing Your Battles

The detailed proof outlines were given in the slides. Completing the steps are left as an exercise to the reader.

2.7 Induction

Problem 2.7.1. *Prove that:*

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

Proof. Let's check the base case $n = 0$:

$$2^0 = 1 = 2^1 - 1$$

Assume that the formula is true for $n = N$. Take $n = N + 1$.

$$\sum_{k=0}^{N+1} 2^k = 2^{N+1} + \sum_{k=0}^{N} 2^k = 2^{N+1} + 2^{N+1} - 1 = 2^{N+2} - 1 \quad \blacksquare$$

Problem 2.7.2. *Prove that:*

$$F_n = \frac{(a^n - b^n)}{\sqrt{5}}, \quad a = (1+\sqrt{5})/2, b = (1-\sqrt{5})/2$$

where F_n is the nth Fibonacci number for $n \geq 1$.

Proof. Let's check the base case $n = 1$:

$$F_1 = 1, F_1 = \frac{(a-b)}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

Assume that the formula is true for $n = N$. Take $n = N + 1$. There is no easy way to relate F_N and F_{N+1} by themselves. However, if we just add another assumption valid within the induction frame, we can reach our answer. Assume that the formula is also true for $n = N - 1$. Notice that a and b are the solutions to the quadratic equation $x^2 - x - 1 = 0$. Hence, $a + 1 = a^2$ and $b + 1 = b^2$. Then;

$$F_{N+1} = F_N + F_{N-1} = \frac{1}{\sqrt{5}}(a^{n-1}(a+1) - b^{n-1}(b+1)) = \frac{a^{n-1}a^2 - b^{n-1}b^2}{\sqrt{5}} = \frac{a^{n+1} - b^{n+1}}{\sqrt{5}}$$

$\blacksquare$

2.8 The Pigeonhole Principle

Problem 2.8.1. *In a party, there are 6 people who either know or don't know each other. Prove that there is a group of three in which either no one knows another, or all know each other.*

Proof. Visualize the problem as a graph: Assume that there is no trio satisfying the condi-

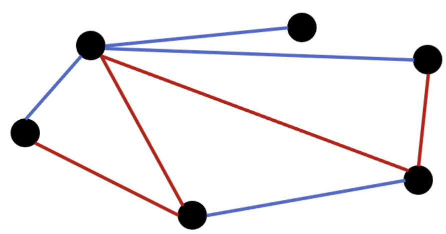


Figure 1: Nodes are people, blue lines connect friends, red lines connect non-friends

tions. Then, take a person. There must be 5 edges with the person as a node. As there are two color options and 5 edges, at least 3 edges must have the same color. WLOG, choose this color as blue (friends). If the other nodes of the three edges have at least one blue edge in between them, a trio of friendship is found. If they have none, then there must be a red triangle corresponding a trio in which nobody knows each other. Contradiction, there must be at least one trio satsifying at least one condition. $\Rightarrow\!\Leftarrow$

2.9 Extremal Principle

Problem 2.9.1. *Prove that $\sqrt{p}$ is irrational for a prime p*

Proof. Assume otherwise. Then, $\sqrt{p} = \frac{a}{b}$, where $\frac{a}{b}$ is in its **lowest terms** (a, b are coprime). Thus, $p = \frac{a^2}{b^2}$. Then, $p|a$. Hence, $p^2|a^2$. Let $a = pa_1$. Then, $1 = \frac{pa^2}{b^2} \Rightarrow \frac{b^2}{a_1^2} = p$. By the same argument, $p|b \Rightarrow b = pb_1$. Hence, $\frac{a}{b} = \frac{a_1}{b_1}; \quad a_1 < a, b_1 < b$ However, we assumed that $\frac{a}{b}$ was in its lowest terms. Contradiciton, $\sqrt{p}$ is irrational $\Rightarrow\!\Leftarrow$.

2.10 Invariance Principle

Consider the following board:

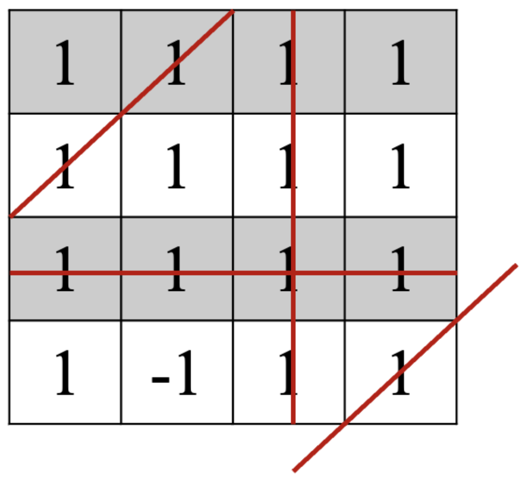


Figure 2: Initial State of the Board

You're only allowed to act on rows, columns, and major/minor diagonals. In each step, you can multiply each number by -1 in the row/column/diagonal you've chosen. Can you make the board have all 1s?

Proof. The key observation is that you can only act on an even number of boxes on the outermost non-corner squares (there are 8 of them) in each move. Thus, the sign of their product never changes. Their initial product is $1^8 \cdot (-1) = -1$; hence, their product will always be equal to -1. Then, there must exist at least one -1 after each move. The board can not have all ones.

3 Worksheet Problems

3.1 Warm Up: Direct Proofs + Modular Arithmetic

Problem 3.1.1. *Prove that the product of two integers that have a remainder of* 1 *when divided by* 3 *also has a remainder of* 1*.*

Proof. Let $m = 3m' + 1, n = 3n' + 1$. Then $mn = 9m'n' + 3m' + 3n' + 1 = 3(3m'n' + m' + n') + 1$ ■

Problem 3.1.2. *Prove that the product of two integers that have a remainder of* 1 *and* $n-1$ *when divided by* n *has a remainder of* $n-1$*.*

Proof. Let $a = na' + 1, b = nb' + n - 1$. Then, $ab = (na' + 1)(n(b' + 1) - 1) = n^2a'b' + n(b' + 1) - na' - 1 = n(na'b' + b' - a' + 1) - 1$ ■

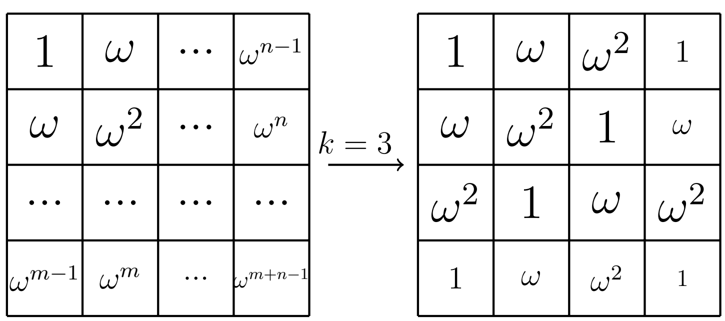
Problem 3.1.3. *Prove that the remainder of the product of two integers when divided by* n *is the product of the remainders of the two integers divided by* n*. Alternatively, if* $m = ab$*,* $a \equiv a_1 \pmod{n}$*,* $b \equiv b_1 \pmod{n}$*;* $m \equiv a_1b_1 \pmod{n}$

Proof. Let $a = na' + a_1, b = nb' + b_1$. Then, $ab = n^2a'b' + nb'a_1 + na'b_1 + a_1b_1 = n(na'b' + b'a_1 + a'b_1) + a_1b_1$ ■

3.2 What are Complex Numbers Doing Here? (Tiling)

Problem 3.2.1. *A* $m \times n$ *chess board can be covered with* $1 \times k$ *tiles (with rotation). Prove that either* m *or* n *is divisible by* k *(*$k|n$ *or* $k|m$*).*

Proof. Let $\omega = e^{2i\pi/k}$. Assign ω^{x+y} to the square (x, y) (starting from $(0, 0)$).



See that each tile covers exactly one times each root of unity. Thus, summing up the numbers each tile covers, we get

$$S = \omega^a + ... + \omega^{a+k-1} = 1 + \omega + ... + \omega^{k-1} = \frac{1 - \omega^k}{1 - \omega} = 0$$

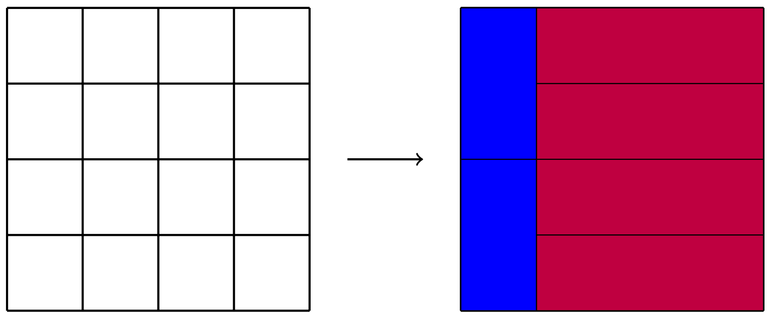
Thus, if we are able to cover the board with tiles, the total sum of the board should also be 0. Then,

$$1 + \omega + ... + \omega^{n-1} + \omega(1 + ... + \omega^{n-1}) + ... + \omega^{m-1}(1 + ... + \omega^{n-1}) = (1 + ... + \omega^{m-1})(1 + ... + \omega^{n-1})$$

$$= \frac{1-\omega^m}{1-\omega}\frac{1-\omega^n}{1-\omega} = 0$$

Thus, either $\omega^n = 1$ or $\omega^m = 1$. WLOG, let $\omega^n = 1$. Then, $e^{2i\pi n/k} = 1 \Leftrightarrow k|n$. Thus, either $k|m$ or $k|n$. ■

Problem 3.2.2. *A given chessboard can be tiled using* $1 \times m$ *horizontal tiles and* $n \times 1$ *vertical strips. prove that it can be tiled using only one of these strips.*



Proof. Let $\zeta = e^{2i\pi/m}, \xi = e^{2i\pi/n}$. Assign $\zeta^x\xi^y$ to square (x, y). Then, a horizontal tile starting from (x, y) would cover the numbers whose sum is $\xi^y(\zeta^x + ...\zeta^{x+m-1}) = \zeta^x\xi^y\frac{1-\zeta^m}{1-\zeta} = 0$. Similarly, the sum of the numbers that the vertical strips cover is also 0. Then, the sum of the numbers on the board must be equal to 0.

1	ζ	ζ^2	ζ^3
ξ	$\zeta\xi$	$\zeta^2\xi$	$\zeta^3\xi$
1	ζ	ζ^2	ζ^3
ξ	$\zeta\xi$	$\zeta^2\xi$	$\zeta^3\xi$

The sum of the numbers on the board are (if the dimensions of the board is $n_1 \times m_1$:

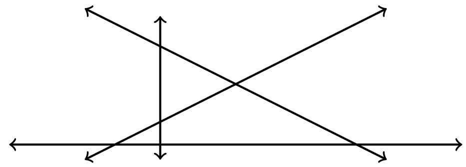
$$1+\zeta+...+\zeta^{m-1}+\xi(1+...+\zeta^{m-1})+...+\xi^{n-1}(1+...+\zeta^{m-1}) = (1+...+\xi^{n-1})(1+...+\zeta^{n-1})$$

$$= \frac{1-\zeta^{m_1}}{1-\zeta}\frac{1-\xi^{n_1}}{1-\xi} = 0$$

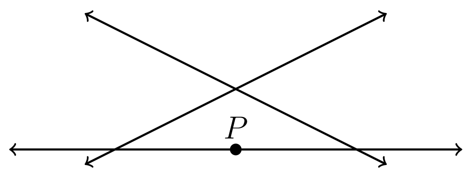
Then, either $m|m_1, n|n_1$. Hence, you can tile the board with either one of the tiles.

3.3 Inductive Proofs: Lines + A Helpful Inequality

Problem 3.3.1. *If n lines are drawn in a plane, no two lines are parallel and no three coincide, how many regions do they separate the plane into?*



Proof. We'll use induction. First, we must gain some intuition on the answer. See that the first line divides the plane into 2 regions. The second divides the first region once and the second region once, resulting in 4 regions. Intuitively, originate the lines from a random point without a line. For example, in the figure below, originate the horizontal line from point P.



See that the region where the line originates is seperated into two, and with each intersection, the line divides another region into two. Thus, if the line intersects with k other lines, $k+1$ regions are added. Then, the nth line intersects with $n-1$ other lines and thus adds n more regions. Denote the region count for n lines with $S(n)$. $S(1) = 2$ and

$$S(n) = S(n-1)+n = ... = S(1)+2+3+...+n = 1+1+2+3+...+n = \frac{(n)(n+1)}{2}+1 = \frac{n^2+n+2}{2}$$

Now, let's prove this intuitive formula formally with induction.

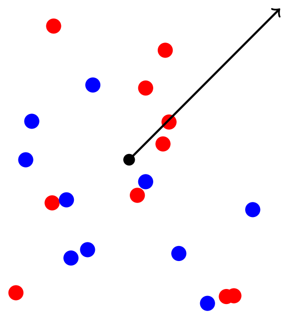
Base Case ($n = 1$): $S(1) = 2 = \frac{1+1+2}{2}$

Inductive Step: Assume that $S(n) = \frac{n^2+n+2}{2}$. To find $S(n+1)$, place the $n+1$th line. Take a point P such that no intersection point exists to one side of the point. The line divides the region in which P lies by 2. Then, with each intersection with a line, the line enters another region which it again divides into two. Because no three lines coincide and no two lines are parallel, there are exactly n intersection points. Hence, $S(n+1) = S(n)+n+1 = \frac{n^2+n+2}{2}+n+1 = \frac{n^2+3n+4}{2} = \frac{(n+1)(n+2)+2}{2}$ ∎

Induction may seem redundant here, because we dived deep in the intuition portion. Regularly, you wouldn't write the intuition; rather, you would write just the conjecture and the induction.

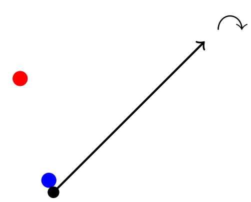
Problem 3.3.2. *$2n$ dots are placed around the outside of the circle. n of them are colored red and the remaining n are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the correct point.*

Proof. Let's draw a diagram:



Think of the arrow as a sweeper that sweeps the dots, and when the number of blues exceed red, it restarts. We'll prove the assertion via induction.

Base Case: For $n = 1$, if you start after the blue dot and before the red dot (cw), you'll get a successful trip.



Inductive Step: Assume that you can have a successful trip for $n = k$. For $n = k + 1$, take adjacent points (successively swept in a clockwise rotation) that are red and blue clockwise. If you can't take such points, then the reds and the blues are clustered and the solution is trivial. After you disregard these points, it is always possible to have a successful trip as you have $2k$ dots. Also see that because you either start before the red dot/after the blue dot, during clockwise rotation, the red dot will be swept first. Thus, the condition of number of red dots passed being at least the number of blue dots passed isn't violated during the sweeping of the two points. Hence, it is possible to have a successful trip for $n = k + 1$ ■

Problem 3.3.3. *Prove the AM-GM Inequality:* $\dfrac{x_1 + ... + x_n}{n} \geq \sqrt[n]{x_1 ... x_n}$

Proof. This solution is due to Wikipedia Contributors. We'll obtain the intuition backwards. We want to prove $\dfrac{x_1 + ... + x_n}{n} \geq \sqrt[n]{x_1 ... x_n}$. If we manipulate this equation:

$$\frac{x_1 + ... + x_n}{\sqrt[n]{x_1 ... x_n}} = \frac{x_1}{\sqrt[n]{x_1 ... x_n}} + ... + \frac{x_n}{\sqrt[n]{x_1 ... x_n}} \geq n$$

Define by $a_k = \dfrac{x_k}{\sqrt[n]{x_1 ... x_n}}$. Notice that $a_1 a_2 ... a_n = 1$. Then, if we prove that for numbers satisfying $a_1 ... a_n = 1$, $a_1 + ... + a_n \geq n$, our proof is completed.

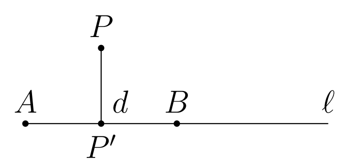
Let's prove by induction. For the base case, let's prove a more general statement for $n = 2$. For numbers $a_1 < 1, a_2 > 1$, $a_1 + a_2 > a_1 a_2 + 1$. This is easy to see as if we multiply both sides of $a_2 > 1$ by $1 - a_1$, we get $a_2 - a_1 a_2 > 1 - a_1 \Rightarrow a_2 + a_1 > a_1 a_2 + 1$. See that the greater sign becomes an equals sign when $a_1 = a_2 = 1$.

Inductive Step: Assume that the inequality holds for all $n \leq N$. Take $N + 1$ numbers satisfying $a_1 ... a_N a_{N+1} = 1$. It is easy to see that equality holds for all numbers being 1. If at least one number isn't one, then there must WLOG exist i such that $a_i > 1$. Hence, there must also exist j such that $a_j < 1$. WLOG, let $i = N, j = N + 1$. Write the condition as $a_1 ... a_N a_{N+1} = (a_1 ... a_{N-2})(a_N a_{N+1}) = 1$. Then, from our induction hypothesis, $a_1 + ... + a_{N-2} + a_N a_{N+1} > n - 1$ Now, using the base case, $a_1 + ... + a_{N-2} + a_{N-1} + a_N > a_1 + ... + a_{N-2} + a_N a_{N+1} + 1 > n$ ■

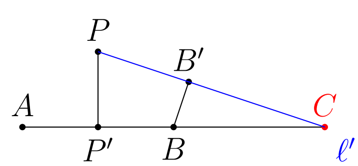
3.4 Extremal Principle: Sylvester-Gallai + Some Diophantine Equations

Problem 3.4.1. *(Sylvester-Gallai Theorem) Prove that if there are finite amount of points in a plane, all connected by lines, that there is either a line that passes through only two of them, or they are collinear.*

Proof. Pair each line with the point closest to it. Label these pairs with the distance of the point to the line. Take the pair with the **least** label. Let the line be ℓ, the points that ℓ connect be A, B and the closest point P. Then,



d is the distance PP' which is less than any other line-point pair. Assume that ℓ passes through three points, the thid being C. If this wasn't true, the condition in the question would be satisfied. Then, there must also exist a line ℓ' passing through P and C. (If C is in between A and B, you can switch the points used and arrive at the same proof.)



Then, the distance $|BB'|$ would be less than $|PP'|$, which contradicts our initial hypothesis. Thus, there must either exist a line that passes through only two of them, or all points are collinear. ■

Problem 3.4.2. *Prove that the equation $x^2 + y^2 = 3z^2$ has no solution. For which numbers instead of* 3 *can you use an analogous proof?*

Proof. First, see that the square of any number when divided by 3 (mod 3) gives either 0 or 1. Then, for x^2+y^2 to be divisible by 3, both remainders must be 0 $(0+1=1, 1+0=0, 1+1=2, 0+0=0)$. Then, $3|x, 3|y$. Assume that (x_0, y_0, z_0) is the solution with smallest $z = z_0$. Let $x_0 = 3x_1, y_0 = 3y_1$. Then, the equation becomes $9x_1^2 + 9y_1^2 = 3z^2$. Dividing both sides by 3, we get $3x_1^2 + 3y_1^2 = z_0^2$. Then, $3|z_0^2 \Rightarrow 3|z_0$. Let $z_0 = 3z_1$. Then, $x_1^2 + y_1^2 = 3z_1^2$. Hence, (x_1, y_1, z_1) is also a solution and $z_1 < z_0$. Thus, our initial hypothesis was contradicted: The equation has no solution. ■

For the second part, the answer is primes of the form $p = 4k + 3$. Proof is left as an exercise to the reader. (Hint: Use Euler's criterion for quadratic residues)

Problem 3.4.3. *Prove that the equation $x^4 + y^4 = w^2$ has no solution. For which case does this account in Fermat's Last Theorem?*

Proof. Assume that (x, y, w) is the solution with minimal w. It is clear that (x^2, y^2, w) is a Pythagorean triple which can not be simplified more (primitive). Then, Then, w is odd. WLOG, assume x is odd and y is even. Because the numbers form a Pythagorean triple, there exist relatively prime u, v (exactly one is odd) s.t. $u > v$ and:

$$x^2 = u^2 - v^2, \quad y^2 = 2uv, \quad w = u^2 + v^2$$

See that a number squared gives either 1 or 0 as a remainder when divided by 4. If x is odd, then u is odd and v is even. $y^2 = 2uv = (2v)u$, and $2v, u$ must be perfect squares. Then, $u = a^2, v = 2b^2$ and we get $x^2 = a^4 - 4b^4 \Rightarrow x^2 + (2b^2)^2 = a^2$. Thus, we get another Pythagorean triple. Then, we may write:

$$x = c^2 - d^2, \quad 2b^2 = 2cd, \quad a^2 = c^2 + d^2$$

$b^2 = cd$; thus, $c = m^2, d = n^2$. Then, $m^4 + n^4 = a^2$, and $a < w$. Hence, our initial hypothesis is contradicted: There is no solution ■

3.5 The Pigeonhole Principle: Erdös-Szekeres + A Tournament

Problem 3.5.1. *(Erdös-Szekeres Theorem, Hard) Prove that there exists either an monotone increasing or a monotone decreasing sequence of length $n+1$ in a sequence of distinct integers of length $n^2 + 1$*

Proof. Let the sequence be $a_1, a_2, ..., a_{n^2+1}$. Label each a_i with an ordered pair consisting of the sizes of the largest monotone increasing sequence (c_i) and the largest monotone decreasing (d_i) sequence ending with a_i. Assume otherwise. Then, all $1 \leq c_i, d_i \leq n$. Then, there are $n \cdot n = n^2$ different possible labels. There are $n^2 + 1$ numbers; hence, two labels must be the same. Let these be $(c_i, d_i), (c_j, d_j)$, the labels of a_i, a_j. Then, $c_i = c_j = c, d_i = d_j = d$. If $a_i > a_j$, $d_i < d_j$ because a_j can be added to the end of the decreasing sequence ending with a_i. Similarly, if $a_i < a_j$, $c_i < c_j$. These yield contradictions. Thus, there exists either an monotone increasing or a monotone decreasing sequence of length $n + 1$ in a sequence of distinct integers of length $n^2 + 1$. ■

Problem 3.5.2. *There is a tournament in which 10 players participate. Each match-up happens once; a draw is 0 points, a win is worth +1 point and a lose is -1 points. If more than 70% of these games ended in a draw prove that there must be two players with the same total of points.*

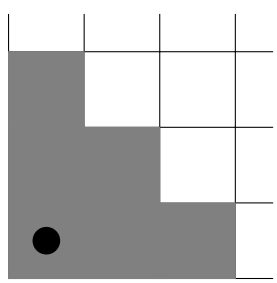
Proof. In a tournament with n participants, there are $\dfrac{n(n-1)}{2}$. In our tournament, then, there are 45 games. %70 of 45 is 31.5; hence, at least 32 games have ended in a draw. Then, there are at most 13 games in which points were given out. Assume that all the points are different. Then, at least 9 players have positive or negative scores. Hence, at least 5 people have a score of same sign. WLOG let this sign be positive. Then, these players have collected at least $1+2+3+4+5=15$ points. However, this means that at least 15 games did not end in a draw, which yields a contradiction. ■

3.6 My Personal Favorite: The Invariance Principle

Problem 3.6.1. *Consider all lattice squares* (x, y) *with* x, y *nonnegative integers. Assign to each its lower left corner as a label. We shade the squares* $(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)$*. There is a chip on each of the six squares. Can the shaded area be emptied with finite repetition of the following move:*

If squares $(x+1, y)$ *and* $(x, y+1)$ *are empty, you can take the chip from* (x, y) *and put two chips on* $(x+1, y)$ *and* $(x, y+1)$*.*

Proof. Assign the number 2^{-x-y-1} to each lattice point (x, y) on the grid.



$\frac{1}{8}$	$\frac{1}{16}$		
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	

Recall that with each move, the chip on (x, y) disappears and two chips on $(x+1, y)$ and $(x, y+1)$ are placed. Denote the sum of values assigned to squares with chips by S. Initially $S=1/2$. With each move, $S \rightarrow S-2^{-x-y-1}+2^{-x-y-2}+2^{-x-y-2}=S-2^{-x-y-1}+2 \cdot 2^{-x-y-2}=S-2^{-x-y-1}+2^{-x-y-1}=S$. Then, the sum is invariant under the allowed move. Hence, the total sum is always 1/2.

Now, let's look at the total value sum of the board. By the geometric series, $1/2^{k}+1/2^{k+1}+\ldots=1/2^{k-1}$. Then:

$\frac{1}{8}$	$\frac{1}{16}$		
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	
1	1/2	1/4	···

Therefore, the total sum on the board is $1 + 1/2 + 1/4 + ... = 2$. See that if the gray area is empty, the maximum value the chip-filled squares can sum up to is $2 - 1/2 - (1/4 + 1/4) - (1/8 + 1/8 + 1/8) = 5/8$. Now, notice that only one chip may exist at the leftmost column and the bottommost row. Thus, we must the attainable value drops by at least $2 \cdot (1/32 + 1/64 + ...) = 2/16 = 1/8$. The attainable value is at most $5/8 - 1/8 = 4/8 = 1/2$. However, we are performing finite amount of moves; hence, we must never be able to attain an infinite sum, meaning that $S < 1/2$ when the gray area is emptied. However, we've found that $S = 1/2$ is invariant. This yields a contradiction: the gray area can not be emptied. ■

Problem 3.6.2. *The integers from* 1 *to* 70 *are written on a board. In each move, two numbers* a *and* b *are erased from the board and the number* $|a - b|$ *is written on it. Can we be left only with the number* 24 *on the board after a finite amount of moves?*

Proof. Let's prove something stronger. Numbers from 1 to $4n + 2$ are written on a board ($70 = 4 \cdot 17 + 2$). The same move applies. We'll show that no even number $2m$ can be yielded at the end of finite amount of moves ($24 = 2 \cdot 12$). See that replacing a, b with $|a - b|$ doesn't change the parity of the sum. To see this, set WLOG $a \geq b$ and denote the sum by S. Then, $S \to S - a - b + a - b = S - 2b$. If S is even, S stays even, and vice versa. Now, let's compute S. $S = \dfrac{(4n + 2)(4n + 3)}{2} = (2n + 1)(4n + 3)$ which is odd. If only one number $2m$ is left on the board, $S = 2m$. However, S was odd $\Rightarrow\!\Leftarrow$.

Problem 3.6.3. *The integers from* 1 *to* 70 *are written on a board. In each move, two numbers* a *and* b *are erased from the board and the numbers* $4a + b$ *and* $3b$ *are written on it. Can we get the sum of the numbers on the board to 100032 after a finite amount of moves?*

Proof. Similarly, denote the sum on the board with S. After a move $S \to S - a - b + 4a + b + 3b = S + 3a + 3b$. Then, the remainder of S divided by 3 is invariant under this move. $S = 35 \cdot 71 = 3 \cdot 828 + 1 \equiv 1 \pmod 3$ and $100032 \equiv 0 \pmod 3$. Hence, we can not get to 100032 after a finite amount of moves ■.

4 Acknowledgements

I would like to thank Azer Kerimov and Murat Yoğurtçu for the ideas for problems 3.6.1 and 3.5.2, respectively.

References

[1] 5 Fermat's Method of Descent. www.math.uci.edu/ ndonalds/math180b/5descent.pdf.

[2] Cool Induction Problems. circles.math.ucla.edu/circles/lib/data/Handout-142-159.pdf. Accessed 30 Oct. 2023.

[3] Engel, Arthur. Problem-Solving Strategies. Springer, 1998.

[4] Wikipedia Contributors. "Sylvester–Gallai Theorem." Wikipedia, Wikimedia Foundation, 26 Oct. 2023, en.wikipedia.org/wiki/Sylvester%E2%80%93Gallai_theorem. Accessed 30 Oct. 2023.

[5] Zhao, Tiling, and Yufei Zhao. MOP 2007 Blue Group Tiling: Coloring and Weights. 2007, yufeizhao.com/olympiad/tiling.pdf. Accessed 30 Oct. 2023.