Proof of the MacMahon Formula for Plane Partitions

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1 Introduction

The MacMahon Formula for Plane Partitions is the well-known and celebrated generating function for plane partitions. The statement is as follows:

$$\mathbf{PL}(q) = \prod_{i=1}^{\infty} \frac{1}{(1-q^i)^i}$$

There exists different proofs for this formula; however, many of them assume a myriad of prerequisite knowledge. In this paper, it is aimed to give the proof with the least amount of pre-requisite advanced mathematical knowledge. Basic knowledge of algebra, combinatorics, and determinants will suffice. Some theorems on symmetric functions won't be proven in this paper, but an interested reader can refer to [2] for more information. This proof will synthesize papers/books [1][2][3][4].

1.1 Outline of the Proof

Because the proof will contain seemingly unmotivated lemmas and theorems, it is useful to outline the proof here. This outline is synthesized from [4][3] with an emphasis on accessibility of a non-expert reader.

1. The **Ferrers Diagram** representation for integer partitions will be introduced. This will be helpful in developing a representation for plane partitions.

2. Filling in the Ferrers Diagrams with numbers, we will obtain a representation, **Young Tableaux**, for plane partitions that are strictly decreasing in columns (column-strict).

3. The generating function for Reverse Semistandard Young Tableaux, the **Schur function**, will be defined.

4. The generating function for plane partitions bounded in an arbitrary box will be related to the Schur function of rectangular column-strict plane partitions.

5. Through an alternative definition of the Schur function, the **Bialternant Formula**, a formula that is related to the shape of the plane partition for the Schur function will be developed.

6. Incorporating this formula into the relation between plane partitions bounded in a box and rectangular column-strict plane partitions, a formula that doesn't explicitly refer to the Schur function will be found for the generating function.

7. Letting the dimensions of the box go to infinity, the MacMahon formula for plane partitions will be derived.

1.2 Integer Partitions

Define an **integer partition** of a positive integer n as the set $P = \{p_1, p_2, \ldots, p_k\}$ where $\sum_{i=1}^{k} p_i = n$. The number k is called the **number of parts** of partition P and p_i is the ith **part** of partition P.
Integer partitions can be visualized in various ways, with one being **Ferrers Diagrams**. In Ferrers Diagrams, unit squares are arranged into rows and columns such that each row is one part of the partition. For example, the Ferrers Diagrams of two different partitions of 9 can be found below:

Figure 1: Partitions $P_1 = \{5, 3, 1\}$ and $P_2 = \{4, 3, 2\}$ of 9

Although a closed form formula isn't known for the partition function $p(n)$ that gives the number of distinct partitions for a given integer n, a generating function for $p(n)$ is well-known. The proof will not be given here, but the generation function is as follows:

$$P(q) = \sum_{n=0}^{\infty} p(n)n = \prod_{j=1}^{\infty} \frac{1}{1 - q^j}$$

1.3 Plane Partitions

A **Plane Partition** is the 3-D analogue of an integer partition. A plane partition of n is a 2-D array of numbers $\pi_{i,j}$ such that $|\pi| \equiv \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \pi_{i,j} = n$. One of the visualizations of a plane partition would be by stacking cubes on top of each other. For example, a plane partition of 15 would be represented as follows:

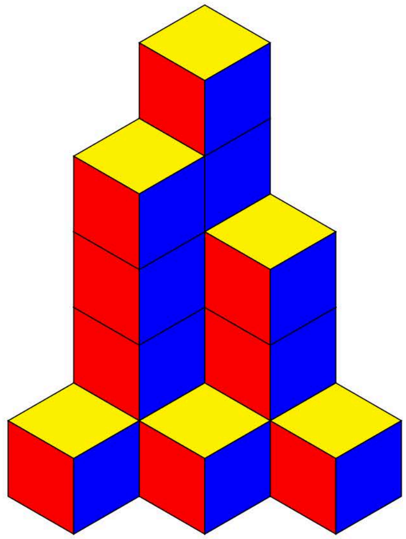
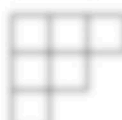


Figure 2: A Plane Partition of 15

Here, $\pi_{1,1} = 5, \pi_{1,2} = 3, \pi_{1,3} = 1, \pi_{2,1} = 4,\ \pi_{2,2} = 1, \pi_{3,1} = 1$, and $\pi_{i,j} = 0$ for $(i,j) \neq (1,1), (1,2), (1,3), (2,1), (2,2), (3,1)$.
There are some additional constraints in the definition of a plane partition. The parts in both a row and a column should be weakly decreasing, i.e. $\pi_{i,j} \geq \pi_{i,j+1}\ \pi_{i,j} \geq \pi_{i+1,j}$.
Under these constraints, an alternative representation for plane partitions that uses Ferrers Diagrams can be proposed. Define the shape of a plane partition $sh(\pi)$ by the ordered set of integers $(\pi_1, ..., \pi_k)$ where $\pi_i = \#\{\pi_{i,j} | j > 1 \text{ and } \pi_{i,j} > 0\}$ or simply, the length of row i in the cube representation. Draw the Ferrers Diagram representing $sh(\pi)$, viewing the set of integers as an integer partition. Then, fill each box with the respective height of cubes, or fill box at (i,j) with $\pi_{i,j}$.
Let's look at an example. The shape of the partition in Figure 2, $sh(\pi) = (3, 2, 1)$. Then, the following Ferrers diagram should be drawn:



Now, we should fill in the Ferrers diagram. From top-left row-wise, the diagram should be filled in with 5,3,1,4,1,1, in this order. Now, the diagram becomes:

5	3	1
4	1	
1		

If the diagram is filled with strictly decreasing numbers down a column, such a diagram is called a **reverse semistandard Young tableau** (RSSYT). The difference between weak