MAT 6
2023
Proofs WS#1

Name: ______________________________
Date: November 2023

1 Questions

1.1 Warm Up: Direct Proofs and Modular Arithmetic

About Problem 1
This problem can be done easily using the modular arithmetic notation. The notation basically allows you to classify numbers based on their remainders when divided by some number n. For example, for $n = 3$, 5 and 2 are equivalent because they both have a remainder of 2 when divided by 3. The notation is $5 \equiv 2 \pmod 3$ In general, if a number is of form $a = bk + r$, then $a \equiv r \pmod k$ You might choose any notation to work with, but the solutions may include this notation. From these questions, a fundamental property of modular arithmetic will also be discovered.

Problem 1.1. Prove that the product of two integers that have a remainder of 1 when divided by 3 also has a remainder of 1.

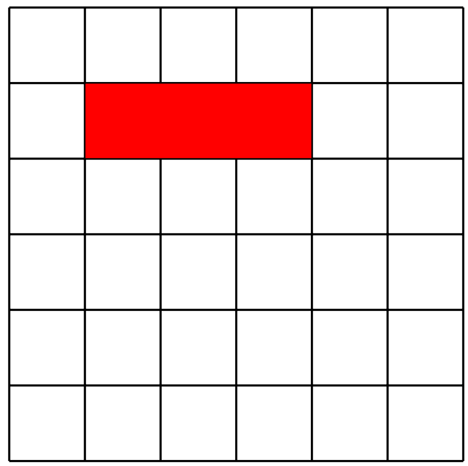
Problem 1.2. Prove that the product of two integers that have a remainder of 1 and $n - 1$ when divided by n has a remainder of $n - 1$.

Problem 1.3. Prove that the remainder of the product of two integers when divided by n is the product of the remainders of the two integers divided by n. Alternatively, if $m = ab$, $a \equiv a_1 \pmod n$, $b \equiv b_1 \pmod n$; $m \equiv a_1 b_1 \pmod n$

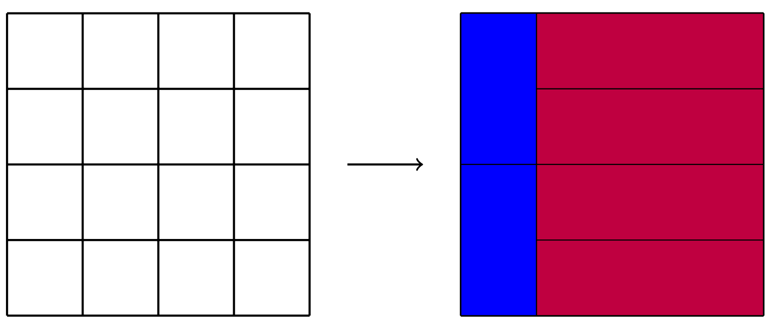
1.2 What are Complex Numbers Doing Here? (Tiling)

Don't force complex numbers into these questions, let them find their place naturally.

Problem 2. A $m \times n$ chess board can be covered with $1 \times k$ tiles (with rotation). Prove that either m or n is divisible by k ($k|n$ or $k|m$).



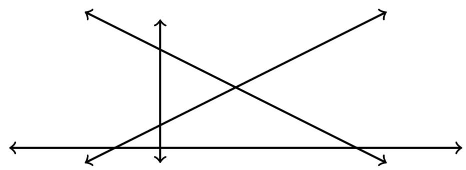
Problem 3. A given chessboard can be tiled using $1 \times m$ horizontal tiles and $n \times 1$ vertical strips. prove that it can be tiled using only one of these strips.



1.3 Inductive Proofs: Lines + A Helpful Inequality

Induction is unneccessarily powerful. See for yourself.

Problem 4. If n lines are drawn in a plane, no two lines are parallel and no three coincide, how many regions do they separate the plane into?



Problem 5. $2n$ dots are placed around the outside of the circle. n of them are colored red and the remaining n are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the correct point.

Problem 6. Prove the AM-GM Inequality: $\dfrac{x_1 + ... + x_n}{n} \geq \sqrt[n]{x_1 ... x_n}$

1.4 Extremal Principle: Sylvester-Gallai + Some Diophantine Equations

One sentence may be worth thousands.

Problem 7. (Sylvester-Gallai Theorem) Prove that if there are finite amount of points in a plane, all connected by lines, that there is either a line that passes through only two of them, or they are collinear.

Problem 8. Prove that the equation $x^2 + y^2 = 5z^2$ has no solution. For which numbers instead of 5 can you use an analogous proof?

Problem 9. Prove that the equation $x^4 + y^4 = w^2$ has no solution. For which case does this account in Fermat's Last Theorem?

1.5 The Pigeonhole Principle: Erdös-Szekeres + A Tournament

Sometimes pigeons can't fit.

Problem 10. (Erdös-Szekeres Theorem, Hard) Prove that there exists either an monotone increasing or a monotone decreasing sequence of length $n+1$ in a sequence of distinct integers of length n^2+1

Problem 11. There is a tournament in which 10 players participate. Each match-up happens once; a draw is 0 points, a win is worth +1 point and a lose is -1 points. If more than 70% of these games ended in a draw prove that there must be two players with the same total of points.

1.6 My Personal Favorite: The Invariance Principle

Nothing endures but change. Or does it?

Problem 12. Consider all lattice squares (x, y) with x, y nonnegative integers. Assign to each its lower left corner as a label. We shade the squares $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)$. There is a chip on each of the six squares. Can the shaded area be emptied with finite repetition the following move:

If squares $(x+1, y)$ and $(x, y+1)$ are empty, you can take the chip from (x, y) and put two chips on $(x + 1, y)$ and $(x, y + 1)$.

About Problem 13
These problems will involve numbers on a board, so I wanted to group them together.

Problem 13.1. The integers from 1 to 70 are written on a board. In each move, two numbers a and b are erased from the board and the number $|a - b|$ is written on it. Can we be left only with the number 24 on the board after a finite amount of moves?

Problem 13.2. The integers from 1 to 70 are written on a board. In each move, two numbers a and b are erased from the board and the numbers $2a + b$ and $a + 2b$ are written on it. Can we get the sum of the numbers on the board to 100032 after a finite amount of moves?

2 Acknowledgements

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References

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