INVESTIGATING PROBABILITY IN CYLINDRICAL DICE

Terpsiadou Myrto [1]

ABSTRACT

When we toss a coin, we expect to get either heads or tails. The contingency of the coin landing on its side is practically impossible. But what if we could differentiate a coin's geometrical characteristics to make it act as a die with 3 surfaces, i.e., there is an 1/3 chance of falling on each of its sides? In this paper, we explored the geometrical characteristics that a cylinder should have to act as a fair three-sided die. We proposed two theoretical models which present different solutions -the arc model and the surface area model- and formed hypotheses for different contingencies. Finally, we presented the results in the form of relative frequencies; The arc model hypotheses were verified while the surface area model hypotheses were rejected. We then concluded that the condition for a cylinder to act as a fair die is for the Height to Radius ratio to be 1.155.

STATED RESEARCH QUESTION

Coins and dice are used daily to "make decisions" in games of chance. A coin has an 1/2 chance of falling on each of its two bases, while a die has an 1/6 chance of falling on each of its sides. The coin, however, being cylindrical, has a third side, the cylindrical surface, on which it is practically impossible to land due to its morphology. Is it possible, then, to change the geometrical characteristics of a cylindrical coin, so that it acts as a fair die?

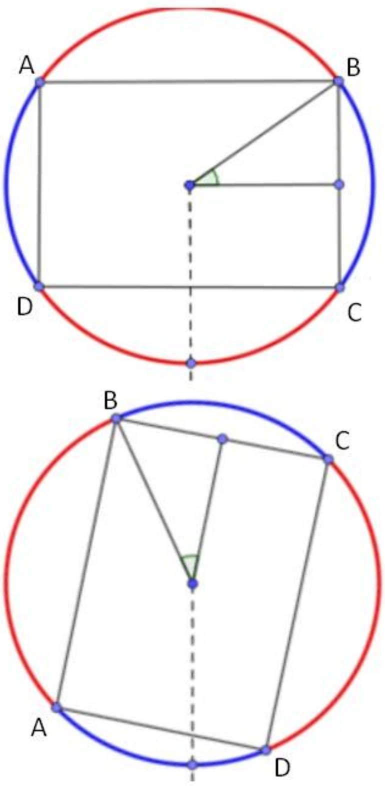
To study the possible changes in geometrical characteristics, firstly, we must determine the physical characteristics of the cylinder. These include a smooth surface, a consistent density - which remains constant in all samples but also the same at all points of each sample. (Controlled parameters) Moreover, since this experiment is a random experiment, factors such as human ability and environmental conditions are excluded -assuming that they are not extreme-. The independent parameter is the height to radius ratio of the cylinder and the dependent parameter is the relative frequency of the fall on each side of the cylinder in each experiment.

BACKGROUND RESEARCH

Theoretical model 1- the arc model:

Initially, we studied how the relationship between the radius and the height of the cylinder is linked to the probability of it landing on each surface.

When the die hits the floor, it lands on the side where the center of gravity is closer to the floor. If the vertical line connecting the centre of mass to the earth intersects the arcs AB or CD, then the die will land on one of its bases. If it intersects the arcs AC or BD, then it will land on the cylindrical surface. In this way, we can calculate the probability of falling on the cylindrical surface, or on the two bases, for which the possibilities are equal. (Figure 1) Based on this model we make hypotheses Ia, IIa and IIIa



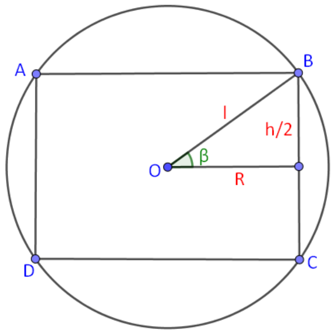


Figure 2: *The angle β*

Key for figure 2: η= $\frac{h}{R}$ και $l = \sqrt{R^2 + \left(\frac{h}{2}\right)^2}$

$$\boxed{\begin{array}{l} arcAB = arcCD = arcAD + arcBC = \\ 2arcBC = 120^\circ \; and \; \frac{h}{2} = \frac{l}{2} \Leftrightarrow h = l \\ so \; triangle \; BOC \; is \; equilateral \end{array}}$$

Figure 1: *The vertical of the weight dissecting arcs CD and AC*

Theoretical model 2- The surface area model

In a common die, the areas of the 6 sides are equal. In a coin, the area of the cylindrical surface is significantly smaller than the base areas. In proportion to the dice, we assume that the probability is proportional to the surface area. So, theoretically, we need to create a cylinder where the area of the cylindrical surface is equal to the area of one of the bases. Based on this model we make hypotheses Ib, IIb and IIIb

HYPOTHESES

Hypothesis Ia) Cylinders where β = 30° will have an equal chance of landing on each of the 3 sides.
Hypothesis IIa) Cylinders where β>30° will have a greater chance of landing on their cylindrical surface.
Hypothesis IIIa) Cylinders where β<30° will have a greater chance of landing on one of their bases.

Hypothesis Ib) When H=R:

$$\left.\begin{array}{l} E_A = E_B = \pi R^2 \\ E_C = 2\pi R h = 2\pi R^2 \end{array}\right. \Longrightarrow \left.\begin{array}{c} 2E_A = 2E_B = E_C \\ E_A + E_B = E_C \end{array}\right\} \begin{array}{l} \text{A and B} = \frac{1}{4} \text{ probability} \\ \text{C} = \frac{1}{2} \text{ probability} \end{array}$$

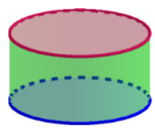


Figure 3: *Cylinder where H=2R*

Hypothesis IIb) When H=2R:

$$\left.\begin{array}{l} E_A = E_B = \pi R^2 \\ E_K = 2\pi R h = 4\pi R^2 \end{array}\right. \Longrightarrow \left.\begin{array}{c} 4E_A = 4E_B = E_K \\ 2(E_A + E_B) = E_K \end{array}\right\} \begin{array}{l} \text{A and B} = \frac{1}{6} \text{ probability} \\ \text{C} = \frac{4}{6} \text{ probability} \end{array}$$

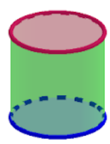


Figure 4: *Cylinder where H=R*

Hypothesis IIIb) When H=R/2:

$E_A=E_B=\pi R^2$ $\Longrightarrow$ $E_A=E_B=E_C$ } A, B and C=$\frac{1}{3}$ probability

$E_C=2\pi Rh=\pi R^2$

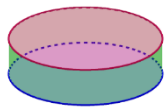


Figure 5: *Cylinder where H=R/2*

VARIABLES AND CONSTANTS

As stated above, in this experiment, the human factor is completely disregarded. Furthermore, parameters such as the height of the toss are also irrelevant. The surface on which the cylinder landed remained constant in every experiment and was non-tilted and smooth.
The dependent variable is the relative frequency of each side of the cylinder in every experiment. The independent variable is the height to radius ratio in each cylinder.

MATERIALS

For this experiment, 10 cylinders with different geometrical characteristics which are shown in Table 1 were 3d printed, all from the same material. Then, a “1” and a “2” were marked on each base to tell them apart.

	Radius	**Height**	**Type**
1	2	4	h=2R, β>30°
2	2	2	h=R, β≈ 30°
3	2	1	h=R/2, β<30°
4	1	2	h=2R, β>30°
5	1	1	h=R, β≈ 30°
6	1	0.5	h=R/2, β<30°
7	0.5	1	h=2R, β>30°
8	0.5	0.5	h=R, β≈ 30°
9	0.5	0.25	h=R/2, β<30°
10	2.31	2	h/R=1.155, β=30°

Table 1: *The 10 cylinders used*

Figure 6: *left: H=2R, middle: H=R, right: H=R/2*

PROCEDURE

4 experiments were conducted to test the hypotheses and 10 cylinders were used (Table 1). Each cylinder was thrown and the side on which it landed was noted. After 500 trials with each cylinder, the relative frequencies were calculated (Table 2) and were also made into pie graphs.

In the first experiment, we tested the impact of the size of the cylinder in probability, when the ratio of the measurements remains the same. For this experiment, we used 3 groups of 3 cylinders each that had the same Height to Radius ratio in different sizes. In the table, each group is marked with the same colour.

The first part of the second experiment tested hypothesis Ia where β=30° and H/R=1.155. For this experiment cylinder 10 in the table was used. The second part tested hypothesis Ib with the cylinders where H/R=1 and β≈ 30°. Here, cylinders 2, 5, 8 were used, as shown in the figure on the right.

Experiment 3 tested Hypotheses IIa and IIb using cylinders where H=2R and β>30°. In this experiment, cylinders 1, 4, 7 were used, as presented in the figure on the right.

In experiment 4 we tested Hypotheses IIIa and IIIb where H=R/2 and β<30°. Here, cylinders 3, 6, 9 were used, as shown in the third picture on the right.

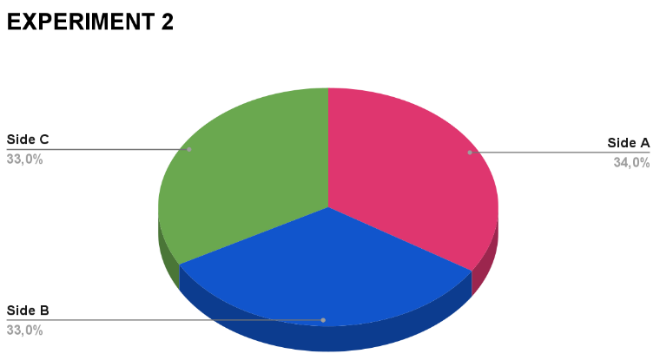
RESULTS

	Radius	Height	Type	Relative Frequencies
1	2	4	h=2R, β>30°	A=9%, B=10%, Γ=81%
2	2	2	h=R, β≈ 30°	A=34%, B=33%, Γ=33%
3	2	1	h=R/2, β<30°	A=46%, B=47%, Γ=7%
4	1	2	h=2R, β>30°	A=10%, B=10%, Γ=80%
5	1	1	h=R, β≈ 30°	A=33%, B=33%, Γ=34%
6	1	0.5	h=R/2, β<30°	A=47%, B=46%, Γ=7%
7	0.5	1	h=2R, β>30°	A=10%, B=11%, Γ=79%
8	0.5	0.5	h=R, β≈ 30°	A=34%, B=33%, Γ=33%
9	0.5	0.25	h=R/2, β<30°	A=47%, B=47%, Γ=6%
10	2.31	2	h/R=1.155, β=30°	A=33%, B=34%, Γ=33%

Table 2: *The relative frequencies of each side in each experiment*

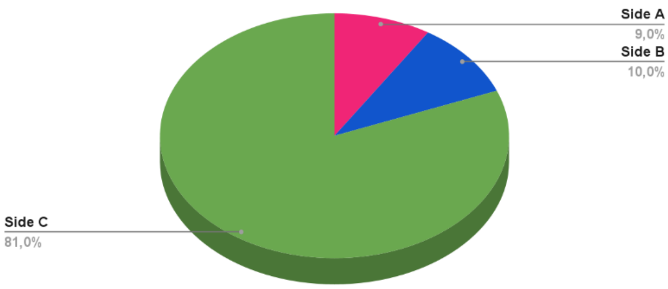
Experiment 1: The results showed that the size does not make a difference in probability. (All the dice with the same ratio presented similar results)

Experiment 2: In both parts, the relative frequencies for each side were equal, which means that dice with height equal to their radius are fair, having the same probability to land on either one of their three sides.



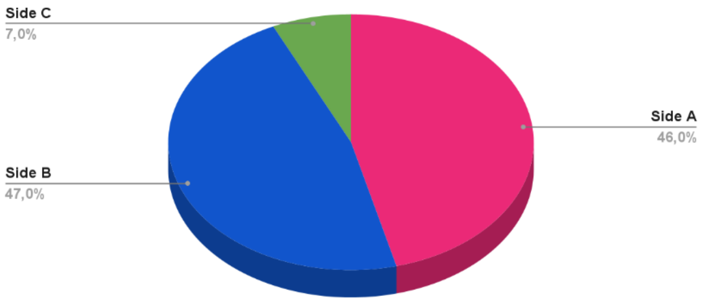
Experiment 3: The relative frequencies for the cylindrical side were significantly more than the ones for the bases, which means that cylinders with a height double their radius (and β>30°) will be far more likely to fall on their cylindrical side, rather than one of the bases.

EXPERIMENT 3



Experiment 4: The relative frequencies for the bases were significantly more than the ones for the cylindrical side, which means that dice with a radius double their height (and β<30°) will be far more likely to fall on each one of their bases, rather than the cylindrical side.

EXPERIMENT 4



DISCUSSION

In experiment 1 we showed that if the dimensions of two cylinders are proportional to each other, the size of the cylinder does not differentiate the results. In experiment 2 we showed that the probability that the cylinder will fall on each of its sides is equal when β= 30°, so H/R=1.155 (Figure 5)
Experiment 3 results showed that the probability of the cylinder falling on a base is much less than the probability of falling on the cylindrical surface when its height is twice the radius and when b>30° (Figure 6). Lastly, experiment 4 showed that the probability of the cylinder falling on a base is much greater than the probability of falling on the cylindrical surface when its height is half the radius and when b<30° (Figure 7).
The final answer to the stated research question is that for a cylinder to be able to act as a die, i.e., to have an equal chance of falling on each of its sides, it must have an angle β of 30° or a Height to Radius ratio equal to 1.155 (Figure 2).
Some sources of error in this experiment could be the 3d printer's accuracy and the number of trials.
In general, this study shows interest not only in the mathematical world of probability but also in everyday life applications that include the use of dice or games of luck.

CONCLUSIONS

Theoretically, the fair die is the one where H/R=1.155, that confirms the arc model. Experimentally, there were two «successful» dice that had a similar morphology. That is, in the first the ratio η is H/R=1 while in the second, H/R=1.155. Similarly, in the first b≈27° while in the second β = 30°, so it can be observed that there is room for a slight deviation (at least 3°) of the geometric characteristics of the cylinder that does not experimentally affect the probability of falling on each side in a non-negligible extend. The deviation in the results will probably be noticed in an experiment with a larger number of trials.

Another topic for further research could be the golden ratio application on these dice to investigate the results. Another possible extend of this research is to precisely determine the maximum deviation in the H/R ratio without non-negligible effects on probability.

REFERENCES

Clark, Matthew P.A., and Brian D. Westerberg. "How Random Is the Toss of a Coin?" *CMAJ*, CMAJ, 8 Dec. 2009, https://www.cmaj.ca/content/181/12/E306.

Meyer, Paul L. *Introductory Probability and Statistical Applications*. Addison-Wesley, 1986.

Murray, Daniel B., and Scott W. Teare. "Probability of a Tossed Coin Landing on Edge." *Physical Review E*, American Physical Society, 1 Oct. 1993, https://pre.aps.org/abstract/PRE/v48/i4/p2547_1.

Teaching Bayesian Model Comparison with the ... - Iypt.ru. https://iypt.ru/wp-content/uploads/2021/09/Teaching-Bayesian-Model-Comparison-With-the-Three-.pdf.

Yong, Ee Hou, and Lakshminarayanan Mahadevan. "Probability, Geometry, and Dynamics in the Toss of a Thick ..." *ResearchGate*, American Journal of Physics, 2011, https://www.researchgate.net/publication/258487755_Probability_geometry_and_dynamics_in_the_toss_of_a_thick_coin.

Zohair. "NO." *Theory at ITU*, 2 Mar. 2018, https://itu18.wordpress.com/2018/03/02/lecture-6-randomized-rounding/.

Mahadevan, L., and Ee Hou Yong. "Probability, Physics, and the Coin Toss." *Physics Today*, American Institute of PhysicsAIP, 1 July 2011, https://physicstoday.scitation.org/doi/full/10.1063/PT.3.1178.