What is the Hilbert Hotel and What Does Infinity Even Mean?

By Renzo Honorato

Photo by runnyrem on Unsplash

Odds are you have probably heard something about the fact that some infinities are bigger than others. And maybe you've even heard that whole story of the Hilbert Hotel (to those who haven't, don't worry, I'll introduce the argument later in this article), but haven't you ever wondered how such a counterintuitive fact is true? I mean, infinity is... Well, infinite, so how come something infinite is bigger than another thing that is also infinite? That question has haunted my being for a long time until I could finally grasp what that claim means, and by the end of this article, I hope to make that clearer to you too.

First of all, I am going to introduce the problem to make sure we are all on the same page here. If you're new to the nonsense world of infinite stuff, I highly recommend Veritasium's video on the topic on YouTube called "How An Infinite Hotel Ran Out Of Room".

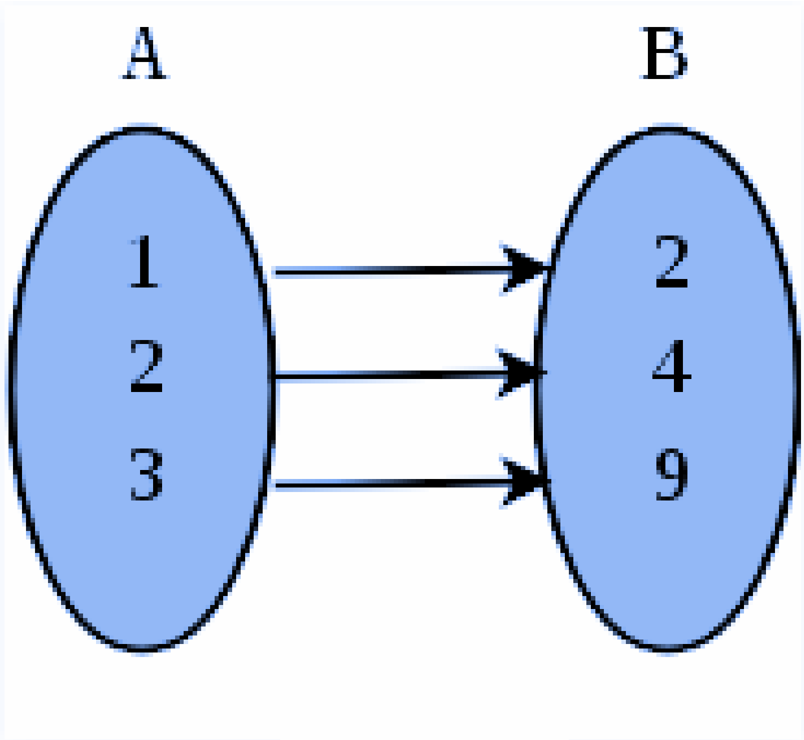
Let’s Talk About The Hilbert Hotel

Without further ado, the problem goes like this: Imagine you're on a road trip and you stop by a hotel to spend the night, called the Hilbert Hotel (named after David Hilbert, who first proposed this), and as soon as you enter the hotel the receptionist tells you that the hotel has an infinite number of rooms and in every one of them there is already someone, and you can see that the hotel is an infinite corridor with doors numbered from one up to infinity, from right to left. But that is not a problem—you tell him—there is a way to get me a room, just ask everyone to move to the room on their left, so the person in room #1 goes to room #2, the person in room #2 goes to room #3 and so on.

Something fishy is going on here. How come you can just make up a free room out of thin air?? But what we just did was to say that infinity plus one is still infinity. So there is always more room! Now I would like to ask you to stop and ponder about what that means, we have proved that you can always add a natural number (the numbers we use to count, like 1,2,3...) to infinity and it remains the same (the proof is analogous to numbers larger than 1, I invite you to try it), and I don't mean that infinity is so big that adding one to it is insignificant, I mean that infinity plus one **IS** infinity itself, it doesn't change at all. So before you arrived there was an infinite number of rooms and an equally infinite number of guests in the hotel, so every room was occupied, but, still, there was a way to get a room for you :0, and that is counterintuitive because infinity is weird, but if you think about it is completely logical that adding 1 to something infinite doesn’t change it, because infinity isn’t a really large number, it is nothing like a number, indeed, **infinity is the collection of all numbers**. Isn't that amazing?

Alright, so infinity plus some natural number is still infinity, but the Hilbert Hotel still has a lot to teach us about infinities. This time imagine you brought your whole infinite family to travel with you and when you arrive at the Hilbert Hotel the receptionist tells you that, again, they're full, but the mathematician in your heart tells him: well, actually, you can make room for all of my family, just ask every guest to move to the room that is two times the number of their room, that way person on room #1 goes to #2, the person on room #2 goes #4, #3 goes #6 and so on, by the time everyone has moved, all the infinitely many odd-numbered rooms will be free for all of my family.

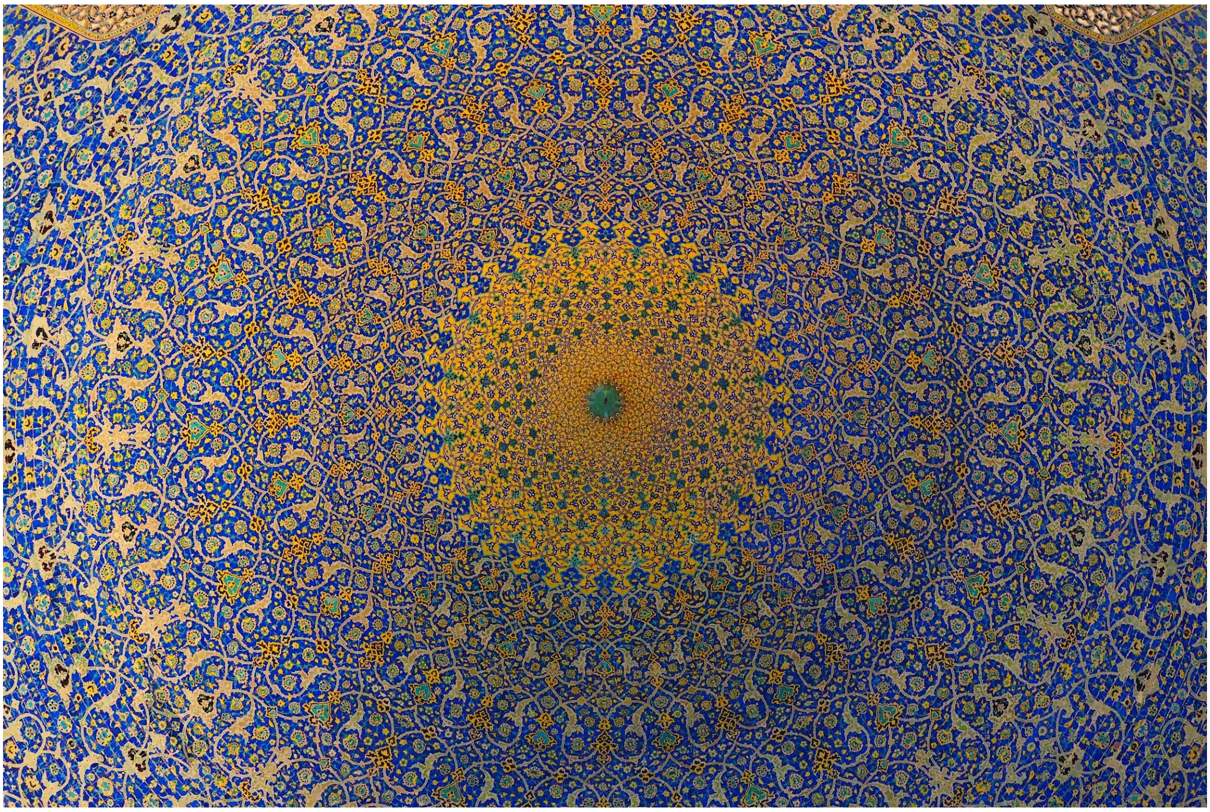
Read that again. **There are as many odd numbers as there are natural numbers** (and the same is true for even numbers). Yes, I mean it, try it yourself, if you assign every number a corresponding odd number up to infinity you'll see that they're the same (no one has gone to infinity to confirm it, but if any of you gets there, please contact me! Email at the end of the article ;)). That argument of assigning each element in a set to another element in another set is the definition of two sets being equal. And here comes the cool thing: you can do that to any set of multiples of an integer. For example, there are as many multiples of 3 as there are integers, as you can assign every integer a multiple of 3. In the study of sets it is common to call that a **bijection**, but don't carried away by the terminology, here is an example of a bijection:



This is a bijection, so you can affirm that there are as many elements on A as there are on B.

And this is the foundation of the arguments we've made so far! You can make a bijection from even numbers to odd numbers, so the sets have the same number of elements if, and only if, you can assign every element of a set to a single element of the other set. (By the way, expanding the relation in the image we can see that there are as many perfect squares as there are natural numbers).

By now you should have some intuition about the weird properties of infinity, and if some of the arguments don't make any sense at all you don't worry, it's normal, infinity is abstract and you just need some time to digest it, so read the last paragraphs again and try seeing the video I linked above, it should be helpful, but if you're ready to continue, we still have a lot to talk about.



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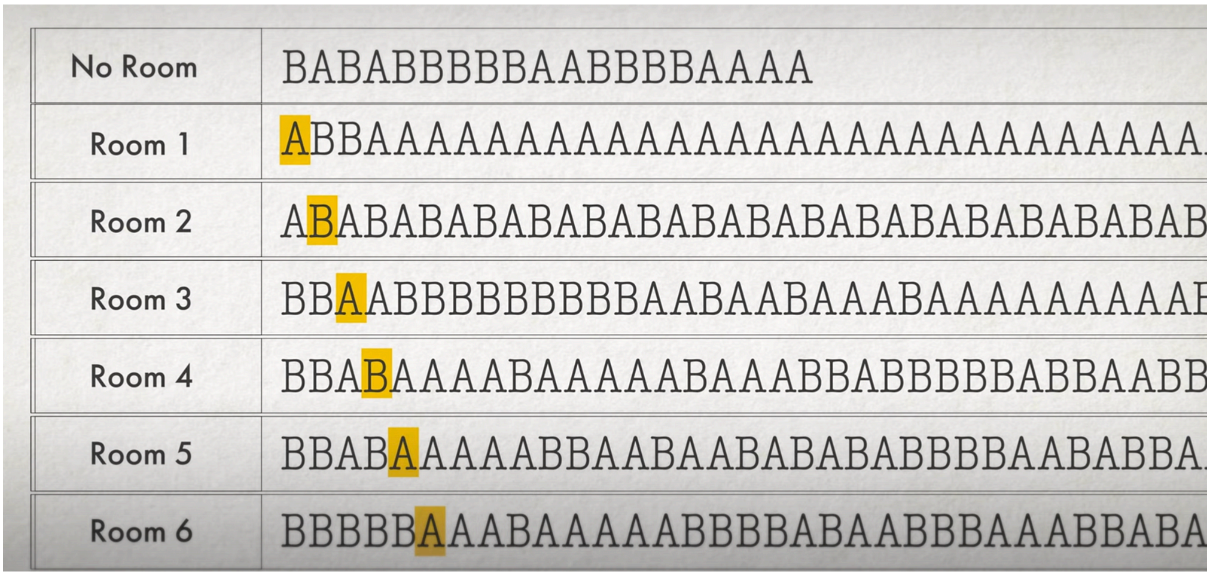
Different Kinds of Infinity

Does this image also make you feel small? Well, that is how infinity should make you, just by trying to represent something that looks kinda endless we can get a glimpse of what infinity looks like, and it's terrifying.

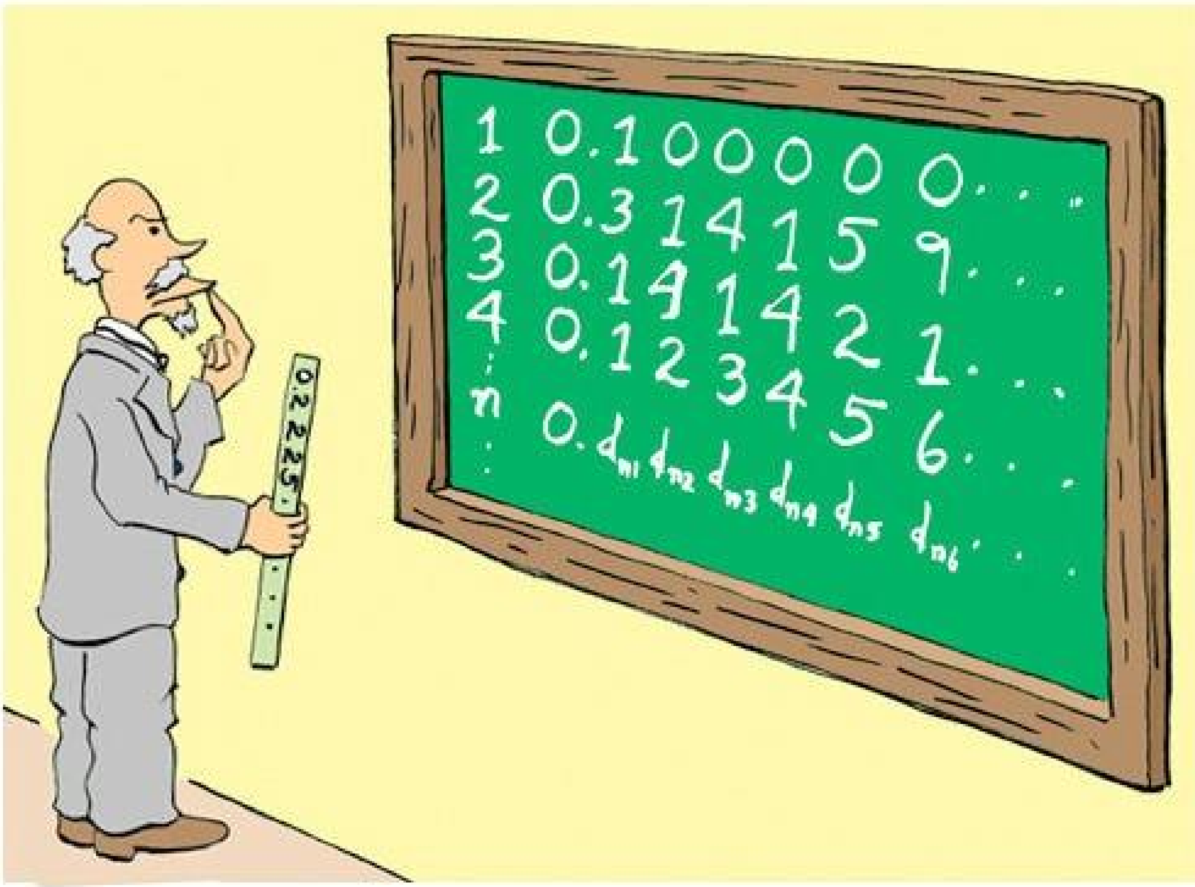
So far we've been talking about a very particular kind of infinity, the ones we can count. Of course, you would need an infinite amount of time to count it, but given unlimited time it would be possible to count all of the rooms in the Hilbert Hotel. Those are known, intuitively, as countable infinities, but there is another kind of infinity called an uncountable infinity, which we'll be talking about now.

Again, imagine you're travelling and you stop by the Hilbert Hotel, but this time something is different, this time you are travelling in a party bus, where everyone is standing and partying and each person is named using an infinite string of A's and B's. So you go to the reception and ask for infinite rooms for your infinite guests, but the receptionist says that this time there is no way that you can make room for your infinite friends. And he explains to you why: "I can try to make a list of everyone that will spend the night in the hotel, but when I'm done I can look at the diagonal of the names (look at the image below) and at the first position of the first name I'll flip the letter, A turns into B and B turns into A, then I'll do the same for the second position of the second name and so on until I have done that to every name and then I'll get back an infinite string of A's and B's that is different of every other in at least one position, so that person wasn't listed, and after I list it I can do that process again, meaning that there will always be someone missing out.



Credit to Veritasium for this awesome explanation on the diagonalization argument!

Sadly he is right, there will always be someone who wasn't assigned a room... And although you'll have to sleep in a bus tonight, this argument proves something very curious: **it proves that some infinities are bigger than others!** Because the infinity of the people in the bus is larger than the infinity of the rooms in the hotel you can't fit everyone in the bus in the hotel, but why is this infinity bigger? The answer is that it is uncountable (as opposed to the countable infinity of the rooms in the hotel), so there is no way to count all the people in your party bus, therefore, even given an unlimited amount of time, there would be always some missing people that you haven't accounted for, which would mean that there are more people in the bus than there are rooms in the hotel because you cannot assign a room to each person, as there will always be someone that has not been assigned a room. That argument the receptionist used is known as Cantor's diagonal argument, which was proposed by mathematician Georg Cantor to prove that you can list all natural numbers (that is what we did in the last session, with countable infinities) and you can make a list of all rational numbers, **but you cannot make a list of all real numbers between 0 and 1** (which is what we were trying to do by assigning each person in the bus a room, but on the place of infinite A's and B's decimals are made of infinite digits from 0 up to 9, I encourage you to try using the diagonalization proof to convince yourself that there will always be some missing numbers in your infinite list). To use formal language: there is no way to establish a bijection from the natural numbers to the real numbers because some real numbers will be missing.



Cantor's diagonal argument proves that some infinite things are bigger than others

The way I like to think about is this: natural numbers are just dots lined up, but there is nothing in between two consecutive natural numbers (that is pretty intuitive, but you can prove that using the axiom of choice) but that is not true for real numbers because real numbers are a continuous line, not just points, so you can always zoom in more and more and find new numbers in between every two real numbers. Is it just me or is that awesome? Try saying that out loud: "There are more real numbers between 0 and 1 than there are natural numbers" and as you say that imagine zooming into the real number line, allow yourself to picture the infinitesimal beauty of the real numbers, doesn't matter how zoomed in you go, there will **always** be more numbers.

Think of the universe, all the stars, every single atom that makes up everything around us and beyond the limits of the Earth and the solar system. Our scale is practically infinite (that is also an interesting discussion, I'll probably be writing about that soon, so stay tuned!), but is nowhere near infinity. Take a second to truly, deeply, wrap your mind around the enormous size of the universe. And once you feel that uncomfortable cold and terrifying sensation deep down in your soul, that is the closest we can get to understanding infinity, and it goes much much beyond that.

And if that bothers you it means that I have achieved my goal with this article.