Solving the Black-Scholes Equation: An Application of the Fourier Transform

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Abstract

A. The aim of this paper is to outline and discuss a non-traditional method of solution to the Black-Scholes equation. The focus will be on transforming the equation into the form of the Heat Equation, and then applying the use of the Fourier transform to solve. In an effort to provide a concise mathematical analysis, economic theorems will be avoided providing a mathematically rigorous solution to the Black-Scholes equation. Secondarily, the goal of this paper is to discuss the historical significance of the Black-Scholes, and how the implementation of this solution leads to varying real world consequences.

I. Introduction

The Black-Scholes Equation is one of the most important equations in finance, as it allows for accurate pricing of derivatives. In financial math, solutions to the equation are by definition: "Tradeable Derivatives." (1) The equation's importance has not gone unnoticed, Myron Scholes was awarded the 1997 Nobel Prize in Economic Sciences (2) for his work in developing the equation. Unfortunately Fischer Black passed away in 1995 (3), and the prize is not awarded posthumously. Warren Buffet, a well known financier, even goes as far as to refer to the equation as "Holy Writ in Finance" (4) in a 2009 letter to Berkshire Hathaway shareholders. The equation is stated as follows:

$$\frac{\partial F}{\partial T} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial^2 F}{\partial S^2} = rF \tag{1}$$

The equation was also a primary cause for the financial collapse of 2008, as argued by Ian Stewart, emeritus professor of mathematics at the University of Warwick: "The financial sector called it the Midas Formula and saw it as a recipe for making everything turn to gold. But the markets forgot how the story of King Midas ended." (5) In short, the Black-Scholes was revered to such a point that the underlying assumptions were ignored, and derivatives were sold at much higher prices as their true value.

II. Background

The Black-Scholes equation is a partial differential equation composed of two constants r and σ, and two variables $F(t, S)$ and $S(x, t)$. "r" represents the risk-free interest rate, assumed to be constant in time. "σ" represents the volatility of the underlying asset, for most derivatives the stock S, and is also assumed to be constant. $S(x, t)$ represents the value of the underlying asset S at time t and is assumed to follow geometric 2D Brownian motion. $F(t, S)$ represents the value of the trade-able derivative F, dependent on the values of S and the time t. (6)

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The derivation of the equation bases upon the idea of a risk-less portfolio. A portfolio is simply a combination of holdings. Holdings are anything with value: stocks, derivatives, land, etcetera. A risk-less portfolio consists of buying a derivative of stock S, denoted as F, and shorting λ units of stock S, yielding a portfolio (1)

$$P = F - \lambda S \tag{2}$$

The partial derivatives of the above equation, and the boundary conditions that describe a risk-less investment can then be used to derive the Black-Scholes equation. The derivation bases upon many financial boundary conditions, and will not be done, in this paper. A derivation can be found in "An Introduction to Stochastic Calculus with Applications to Finance." (1) The equation can be derived and expressed in many forms, for this paper we will use the same form as stated in the "Introduction."

$$\frac{\partial F}{\partial T} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial^2 F}{\partial S^2} = rF \tag{3}$$

Importantly, a keen, or even not so keen, observer may note that the pricing of a derivative is based upon the notion of there not only existing a risk-less interest rate, but also a risk-less investment. Interestingly, a market that is valued at an equivalent value of "12 trillion ... 20% of the world's Gross Domestic Product."(7) is priced on this notion.

The equation can be used to price any tradeable derivative, however the solutions for certain derivatives are more mathematically accessible than others. For this reason, and to illustrate the usefulness of the Fourier Transform, the paper will solely look at the solution for a common derivative, the European Call option.

An European Call option is a derivative that gives it's owner the option to buy a stock at the strike price K, at the maturity date T. Since it is an option, the buyer can also choose not to purchase the stock at maturity. (1) Mathematically speaking the value of the option is as follows:

$$V_T = \max(S_T - K, 0) \tag{4}$$

The following figure, demonstrates graphically the value of an European call at it's maturity date T (8)

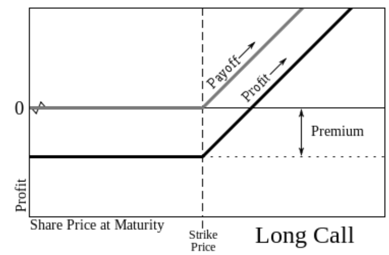

Figure 1: *Profits on a call option*

III. Method of Solution

I. Reduction to a Diffusion Equation

The first step will be to reduce the Black-Scholes Equation, stated as a partial differential equation, into the form of a Diffusion Equation. Specifically into the form of the Heat Equation. Once the equation is reduced to this form, it can readily be solved, yielding a mathematically rigorous solution to the Black-Scholes Equation. The process is as follows

$$\frac{\partial F}{\partial T} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial^2 F}{\partial S^2} = rF \tag{5}$$

Using the following substitutions

$$S = Ke^x \tag{6}$$

$$F(t, S) = Kv(x, \tau) \tag{7}$$

$$\tau = (T - t)\frac{\sigma^2}{2} \tag{8}$$

The partial differential equation can then be reduced into a form with constant coefficients

$$\frac{\delta v}{\delta \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\delta v}{\delta x} - \frac{2r}{\sigma^2}v \tag{9}$$

In order to reduce the equation to the form of a diffusion equation, the constants in front of the dv/dx and v terms must be set to zero, using this assumption, note that the solution to Eq. (9) takes the form

$$v(x, \tau) = f(\tau)g(x)h(x, \tau) \tag{10}$$

By substitution of this solution Eq. (10) into Eq.(9) and by setting the constants associated with the dv/dx and v terms to 0, g(x) and f(τ) can be solved for and the solution can be expressed as a function of h(x,τ), which satisfies the heat equation.

$$F(t,S) = Ke^{-(a^2/4+a+1)\tau}e^{-(a/2)x}h(x,\tau) \quad (11)$$

$$a = \frac{2r}{\sigma^2} - 1 \quad (12)$$

II. Solving the Heat Equation

Now an expression for h(x,τ) is all thats left to be found and we will have found the solution to the equation, in terms of the known quantities. However, we know h(x,τ) is the solution to the heat equation, so that is our starting point

$$\frac{\partial h}{\partial \tau} = \frac{\partial^2 h}{\partial x^2} \quad (13)$$

This equation could then be solved by separation of variables, or more elegantly by the use of the Fourier transform, and its inverse.

$$\mathcal{F}(f(x))(k) = \tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx} f(x)dx \quad (14)$$

$$\mathcal{F}^{-1}(\tilde{f}(k))(x) = f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx} \tilde{f}(k)dk \quad (15)$$

What we really need is the Fourier transform of the second derivative of h(x,τ) with respect to x. We can find this by using the equation for the Fourier transform and integration by parts, or by using the general formula for the Fourier transform of $f^n(x)$

$$\mathcal{F}(f^n(x))(k) = (ik)^n \mathcal{F}(f(x))(k) \quad (16)$$

Therefore

$$\frac{\partial^2 \tilde{h}}{\partial x^2} = -k^2 \tilde{h} \quad (17)$$

The answer to this equation is found by integration with respect to τ

$$\int \frac{\partial \tilde{h}}{\partial \tau}\partial\tau = \int -k^2 \tilde{h}\partial\tau \quad (18)$$

$$\tilde{h}(k,\tau) = \tilde{h}(k,0)e^{-k^2\tau} \quad (19)$$

Now the inverse Fourier transform needs to be applied to find our original goal of an expression for h(x,τ). To accomplish this $\tilde{h}(k,\tau)$ needs to be reduced to a product of $\tilde{h}_1$ and $\tilde{h}_2$

$$\tilde{h}(k,\tau) = \tilde{h}_1\tilde{h}_2 \quad (20)$$

$$\tilde{h}_1 = \tilde{h}(k,0) \quad (21)$$

$$\tilde{h}_2 = e^{-k^2\tau} \quad (22)$$

The inverse transforms are then found by equation 15

$$\mathcal{F}^{-1}(\tilde{h}_1) = h(x,0) = \tilde{h}_1 \quad (23)$$

$$\mathcal{F}^{-1}(\tilde{h}_2) = \frac{1}{\sqrt{2\tau}}e^{-x^2/(4\tau)} = \tilde{h}_2 \quad (24)$$

Then by the convolution theorem

$$(f * g)(x,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x-y,t)g(y,t)dy \quad (25)$$

$$(h_2 * h_1)(x,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} h_2(x-y,t)h_1(y,t) \quad (26)$$

Plugging in the values for h_1 and h_2 found in equations 23 and 24, into equation 26 yields the solution to h(x,τ)

$$h(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} \exp\left[-\frac{(x-y)^2}{4\tau}\right]h(y,0)dy \quad (27)$$

y is a change of variable, that results from the inverse Fourier transform of h_2

$$y = \sqrt{\tau}\,[k - ix/(2\tau)] \quad (28)$$

Plugging this expression for h(x,τ) back into the expression for $F(t,S)$, equation 11, yields the general solution to the Black-Scholes Equation. (6)

$$F(t,S) = Ke^{-(a^2/4+a+1)\tau}e^{-(a/2)x} * \frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} \exp\left[-\frac{(x-y)^2}{4\tau}\right]h(y,0)dy \quad (29)$$

Hence, the solution for the Black-Scholes equation has been found, in terms of all the known variables, and constants: S, K, T, t, r, and σ. So now any trade-able derivative can be valued. It's just a matter of tedious computation.

IV. Valuation of the European Call

Since the solution consists of three variables, we could graph the solution as a plane in the $F(t,S)$, S-K, τ coordinate system. This plane would represent every possible value of the derivative. Noting that S, K, τ, and $F(t,S)$ all have to be positive to have real world significance.

As an example, we will take the simple trade-able derivative of an European call with a Strike Price of 1, volatility of 0, and a riskless interest rate of 0.1. Using these values the equation simplifies quite succinctly

$$F(S,\tau) = \begin{Bmatrix} S - e^{-.1\tau},\ if\ S - e^{-.1\tau} > 0 \\ 0,\ if\ S - e^{-.1\tau} \leq 0 \end{Bmatrix} \quad (30)$$

Graphically the value $F(S,\tau)$ can be seen for different values of the stock price and the time left until maturity, τ

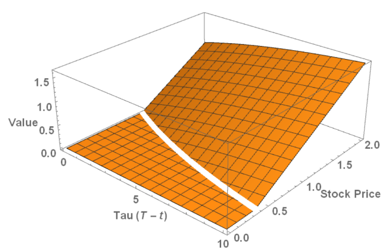

Figure 2: *Value of an European Call*

However this is the simplified example, For the more general case the Black-Scholes needs to be rewritten into a more manageable form that can be used to evaluate an European Call, with realistic volatility. This process will set the stage for a real world example presented in section five, "A Real World Example."

By inspection, the form of our general solution can be seen to be rather cumbersome. Calculating the integral every time you want to know what the price will be the next day is not very efficient. Fortunately the integral is a Gaussian integral with the following solution. Φ represents the Cumulative Distribution Function of the Normal Distribution, also known as the CDF.

$$\frac{1}{\sqrt{4\pi\tau}} \int_0^\infty \exp[-\frac{(x-y)^2}{4\tau} + Cy]dy = e^{C(x+C\tau)}\Phi(\frac{x+2\tau C}{\sqrt{2C}}) \quad (31)$$

If we use the formula for the Gaussian integral, Equation 31, along with our previous solution form, Equation 29, the resulting solution is now more manageable. We will also introduce a substitution of d_1 and d_2 in order to write the solution in a more concise and explicit manner

$$F(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \quad (32)$$

$$d_1 = \frac{\ln(S/K) + (T-t)(r - \sigma^2/2)}{\sigma\sqrt{T-t}} \quad (33)$$

$$d_2 = \frac{\ln(S/K) + (T-t)(r + \sigma^2/2)}{\sigma\sqrt{T-t}} \quad (34)$$

V. A Real World Example

For this example we are going to use the stock ETR:ARL, a bank in the Netherlands. The value of ETR:ARL was 25.39$ (10) on February 25th, 2016. On the same day, interest rate on a 1-year U.S. Treasury Bond, was 0.53% (9). This will be used as the risk-free interest rate as is commonly done in finance. The volatility (σ) will be approximated to be the standard deviation of opening prices for the past 6 months. This data was obtained from Yahoo! Finance (10) and the standard deviation was calculated via Eq. 35, below, yielding a volatility of 2.10

$$\sigma \approx SD = \sqrt{\frac{\sum(x_i - \bar{x})^2}{N-1}} \quad (35)$$

By using Eq. 32, Eq. 33, and Eq. 34 we can create the value equation and plot for the call option of ETR:ARL, for an arbitrary strike price of K=25.

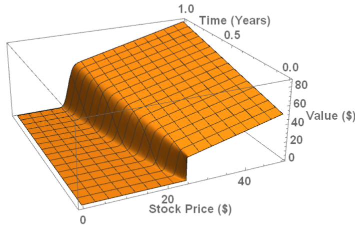

Figure 3: *Value of the call option on ETR:ARL, K=25*

We can now come up with an option price for any time period, stock price, and strike price. We will continue using the values for volatility, risk-less interest rate, and strike price that we used in creating the value plot. We will also decide to sell the option at opening on February 26th, 2016 Let's make the end date on our option the Thursday before the third Friday of March. A strange "standard" in finance. Tau in this case is therefore 14, the number of trading days in between our start and end date. The ticker price of ETR:ARL on February 25th 2016 at closing was $27.24. Plugging in these values into Equation 36 yields the value of our option: $29.66, and lastly by subtracting the price of the underlying stock we find that we should price our option at $2.92. The price of our option can also be represented graphically for every strike price.

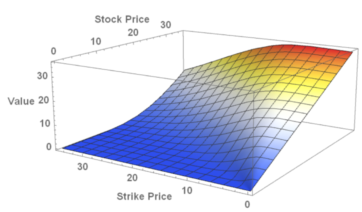

Figure 4: *Value of the call option on ETR:ARL*

Eurex Exchange, one of the world's leading derivatives exchanges, prices the option we calculated at $2.79.(11) So we overpriced the derivative by 4%. This deviation could be due to either Eurex Exchange using a more sophisticated pricing model, or more likely that they use a different method for estimating volatility.

VI. Discussion

Overall, it has been shown that the Fourier Transform is a powerful method of solution to the Black-Scholes equation. The method of solution via the Fourier transform provides a complete solution method that is normally not explored in financial math. For example in both *Stochastic Processes and Advanced Mathematical Finance* and *An Introduction to Stochastic Calculus with Applications to Finance* the solution to the heat equation is simply a given.

From a mathematician's standpoint, much of the underlying math, and therefore intuition regarding the solution is lost by omitting this step. The heat equation portion of the equation is the time evolution component, arguably the most important part. To simply accept the solution as Equation 27, a not very intuitive result, is a travesty towards the power of complex analysis.

After the solution for a general derivative was obtained via the Fourier Transform, the valuation of the European call was performed. First, a simple example was given where volatility was omitted. This example allowed the intuition to be seen as to how the Black-Scholes can be expressed in terms of the Cumulative Density Function of the Standard Normal Distribution.

This simplified form allowed the real world example of buying an option on ETR:ARL to be solved. The figures in this section should have generally shown how each variable interacts with the value of the derivative. Hopefully a broader understanding of each variables effect can be gleamed from this example.

VII. Conclusion

In conclusion, I hope that the reader has learned the mathematical process required to solve the Black-Scholes Equation and acquired a sense for how a real world derivative can be priced and graphed. Lastly, I want to touch back on the historical significance the misinterpretation of this equation has had historically. If someone told you that they could invest your money for absolutely no risk at a fixed rate, you hopefully wouldn't believe them. The Black-Scholes equation requires this assumption. If you happen to be the CEO of a large investment bank and a ratings agency tells you that a mortgage backed security has a 100% chance of returns at a fixed rate. Think about how that could possibly be, before you plug in the variables into a Black-Scholes equation, and pay for derivatives worth nothing.

> Accept reality and dare not question it
>
> Walt Whitman (12)

VIII. Citations

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