GS2019 - Mathematics Question Paper

Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \dots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
- $M_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices with the Euclidean metric, and I denotes the identity matrix in $M_n(\mathbb{R})$.
- All rings are associative, with a multiplicative identity.
- For any prime number p, $\mathbb{F}_p$ denotes the finite field with p elements.
- If A and B are sets, then $A - B$ refers to $\{x \in A \mid x \notin B\}$.
- For a ring R, $R[x]$ denotes the polynomial ring in one variable over R, and $R[x, y]$ denotes the polynomial ring in two variables over R.

PART A

Answer the following multiple choice questions.

1. The following sum of numbers (expressed in decimal notation)

$$1 + 11 + 111 + \cdots + \underbrace{11\ldots1}_{n}$$

is equal to

✓ (a) $(10^{n+1} - 10 - 9n)/81$
(b) $(10^{n+1} - 10 + 9n)/81$
(c) $(10^{n+1} - 10 - n)/81$
(d) $(10^{n+1} - 10 + n)/81$

2. For $n \geq 1$, the sequence $\{x_n\}_{n=1}^{\infty}$, where:

$$x_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}$$

is

✓ (a) decreasing
(b) increasing
(c) constant
(d) oscillating

3. Define a function:

$$f(x) = \begin{cases} x + x^2\cos(\frac{\pi}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Consider the following statements:

(i) $f'(0)$ exists and is equal to 1
(ii) f is not increasing in any neighborhood of 0
(iii) $f'(0)$ does not exist
(iv) f is increasing on $\mathbb{R}$.

How many of the above statements is/are true?

(a) 0
(b) 1
✓ (c) 2
(d) 3

4. Consider differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with the property that for all $a, b \in \mathbb{R}$ we have:

$$f(b) - f(a) = (b-a)f'\left(\frac{a+b}{2}\right).$$

Then which one of the following sentences is true?

(a) Every such f is a polynomial of degree less than or equal to 2

(b) There exists such a function f which is a polynomial of degree bigger than 2

(c) There exists such a function f which is not a polynomial

(d) Every such f satisfies the condition $f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$ for all $a, b \in \mathbb{R}$

5. Let V be an n-dimensional vector space and let $T : V \to V$ be a linear transformation such that

$$\text{Rank } T \leq \text{Rank } T^3.$$

Then which one of the following statements is necessarily true?

(a) Null space(T) = Range(T)

(b) Null space$(T) \cap$ Range$(T) = \{0\}$

(c) There exists a nonzero subspace W of V such that Null space$(T) \cap$ Range$(T) = W$

(d) Null space$(T) \subseteq$ Range(T)

6. The limit

$$\lim_{n\to\infty} n^2 \int_0^1 \frac{1}{(1+x^2)^n} dx$$

is equal to

(a) 1

(b) 0

(c) $+\infty$

(d) $1/2$

7. Let A be an $n \times n$ matrix with rank k. Consider the following statements:

(i) If A has real entries, then AA^t necessarily has rank k

(ii) If A has complex entries, then AA^t necessarily has rank k.

Then

(a) (i) and (ii) are true

(b) (i) and (ii) are false

(c) (i) is true and (ii) is false

(d) (i) is false and (ii) is true

8. Consider the following two statements:

(E) Continuous functions on $[1, 2]$ can be approximated uniformly by a sequence of even polynomials (i.e., polynomials $p(x) \in \mathbb{R}[x]$ such that $p(-x) = p(x)$).

(O) Continuous functions on $[1, 2]$ can be approximated uniformly by a sequence of odd polynomials (i.e., polynomials $p(x) \in \mathbb{R}[x]$ such that $p(-x) = -p(x)$).

Choose the correct option below.

(a) (E) and (O) are both false
✓ (b) (E) and (O) are both true
(c) (E) is true but (O) is false
(d) (E) is false but (O) is true

9. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{\sin(x^3)}{x}$. Then f is

✓ (a) bounded and uniformly continuous
(b) bounded but not uniformly continuous
(c) not bounded but uniformly continuous
(d) not bounded and not uniformly continuous

10. Let
$$S = \{x \in \mathbb{R} \mid x = \text{Trace}(A) \text{ for some } A \in M_4(\mathbb{R}) \text{ such that } A^2 = A\}.$$
Then which of the following describes S?

(a) $S = \{0, 2, 4\}$
(b) $S = \{0, 1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4\}$
✓ (c) $S = \{0, 1, 2, 3, 4\}$
(d) $S = [0, 4]$

11. Let f be a continuous function on $[0, 1]$. Then the limit $\lim\limits_{n\to\infty} \displaystyle\int_0^1 nx^n f(x)\, dx$ is equal to

(a) $f(0)$
✓ (b) $f(1)$
(c) $\sup\limits_{x\in[0,1]} f(x)$
(d) The limit need not exist

12. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$, defined by
$$f_n(x) = \frac{1}{n} \exp(-n^2 x^2).$$
Then which one of the following statements is true?

(a) Both the sequences $\{f_n\}$ and $\{f_n'\}$ converge uniformly on $\mathbb{R}$
(b) Neither $\{f_n\}$ nor $\{f_n'\}$ converges uniformly on $\mathbb{R}$
(c) $\{f_n\}$ converges pointwise but not uniformly on any interval containing the origin

(d) $\{f_n'\}$ converges pointwise but not uniformly on any interval containing the origin

13. Let the sequence $\{x_n\}_{n=1}^{\infty}$ be defined by $x_1 = \sqrt{2}$ and $x_{n+1} = (\sqrt{2})^{x_n}$ for $n \geq 1$. Then which one of the following statements is true?

    (a) The sequence $\{x_n\}$ is monotonically increasing and $\lim_{n\to\infty} x_n = 2$

    (b) The sequence $\{x_n\}$ is neither monotonically increasing nor monotonically decreasing

    (c) $\lim_{n\to\infty} x_n$ does not exist

    (d) $\lim_{n\to\infty} x_n = \infty$

14. Consider functions $f : \mathbb{R} \to \mathbb{R}$ with the property that $|f(x) - f(y)| \leq 4321|x - y|$ for all real numbers x, y. Then which one of the following statements is true?

    (a) f is always differentiable

    (b) There exists at least one such f that is continuous and such that $\lim_{x\to\pm\infty} \dfrac{f(x)}{|x|} = \infty$

    (c) There exists at least one such f that is continuous, but is non-differentiable at exactly 2018 points, and satisfies $\lim_{x\to\pm\infty} \dfrac{f(x)}{|x|} = 2018$

    (d) It is not possible to find a sequence $\{x_n\}$ of real numbers such that $\lim_{n\to\infty} x_n = \infty$ and further satisfying $\lim_{n\to\infty} \left|\dfrac{f(x_n)}{x_n}\right| \leq 10000$

15. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$, defined by

$$f_n(x) = \frac{\sqrt{1 + (nx)^2}}{n}.$$

    Then which one of the following statements is true?

    (a) $\{f_n\}$ and $\{f_n'\}$ converge uniformly on $\mathbb{R}$

    (b) $\{f_n'\}$ converges uniformly on $\mathbb{R}$ but $\{f_n\}$ does not

    (c) $\{f_n\}$ converges uniformly on $\mathbb{R}$ but $\{f_n'\}$ does not

    (d) $\{f_n\}$ converges uniformly to a differentiable function on $\mathbb{R}$

16. The number of ring homomorphisms from $\mathbb{Z}[x, y]$ to $\mathbb{F}_2[x]/(x^3 + x^2 + x + 1)$ equals

    (a) 2^6

    (b) 2^{18}

    (c) 1

    (d) 2^9

17. Let $X \subset \mathbb{R}^2$ be the subset

$$X = \{(x, y) \mid x = 0, |y| \leq 1\} \cup \left\{(x, y) \mid 0 < x \leq 1, y = \sin \frac{1}{x}\right\}.$$

    Consider the following statements:

(i) X is compact

(ii) X is connected

(iii) X is path connected.

How many of the statements (i)-(iii) is/are true?

(a) 0

(b) 1

✔ (c) 2

(d) 3

18. Consider the different ways to colour the faces of a cube with six given colours, such that each face is given exactly one colour and all the six colours are used. Define two such colouring schemes to be equivalent if the resulting configurations can be obtained from one another by a rotation of the cube. Then the number of inequivalent colouring schemes is

(a) 15

(b) 24

✔ (c) 30

(d) 48

19. Let $C^\infty(0,1)$ stand for the set of all real-valued functions on $(0,1)$ that have derivatives of all orders. Then the map $C^\infty(0,1) \to C^\infty(0,1)$ given by

$$f \mapsto f + \frac{df}{dx}$$

is

(a) injective but not surjective

✔ (b) surjective but not injective

(c) neither injective nor surjective

(d) both injective and surjective

20. A stick of length 1 is broken into two pieces by cutting at a randomly chosen point. What is the expected length of the smaller piece?

(a) 1/8

✔ (b) 1/4

(c) $1/e$

(d) $1/\pi$

PART B

Answer whether the following statements are True or False.

F 1. There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) \subseteq \mathbb{R}-\mathbb{Q}$ and $f(\mathbb{R}-\mathbb{Q}) \subseteq \mathbb{Q}$.

T 2. If $A \in M_{10}(\mathbb{R})$ satisfies $A^2 + A + I = 0$, then A is invertible.

F 3. Let $X \subseteq \mathbb{Q}^2$. Suppose each continuous function $f : X \to \mathbb{R}^2$ is bounded. Then X is necessarily finite.

F 4. If A is a 2×2 complex matrix that is invertible and diagonalizable, and such that A and A^2 have the same characteristic polynomial, then A is the 2×2 identity matrix.

F 5. Suppose A, B, C are 3×3 real matrices with $\text{Rank } A = 2, \text{Rank } B = 1, \text{Rank } C = 2$. Then $\text{Rank } (ABC) = 1$.

F 6. For any $n \geq 2$, there exists an $n \times n$ real matrix A such that the set $\{A^p \mid p \geq 1\}$ spans the $\mathbb{R}$-vector space $M_n(\mathbb{R})$.

T 7. The matrices

$$\begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

are similar.

F 8. Consider the set $A \subset M_3(\mathbb{R})$ of 3×3 real matrices with characteristic polynomial $x^3 - 3x^2 + 2x - 1$. Then A is a compact subset of $M_3(\mathbb{R}) \cong \mathbb{R}^9$.

F 9. There exists an injective ring homomorphism from the product ring $\mathbb{R} \times \mathbb{R}$ into $C(\mathbb{R})$, where $C(\mathbb{R})$ denotes the ring of all continuous functions $\mathbb{R} \to \mathbb{R}$ under pointwise addition and multiplication.

T 10. $\mathbb{R}$ and $\mathbb{R} \oplus \mathbb{R}$ are isomorphic as vector spaces over $\mathbb{Q}$.

T 11. If 0 is a limit point of a set $A \subseteq (0, \infty)$, then the set of all $x \in (0, \infty)$ that can be expressed as a sum of (not necessarily distinct) elements of A is dense in $(0, \infty)$.

F 12. The only idempotents in the ring $\mathbb{Z}_{51}$ (i.e., $\mathbb{Z}/51\mathbb{Z}$) are 0 and 1. (An idempotent is an element x such that $x^2 = x$).

T 13. Let A be a commutative ring with 1, and let $a, b, c \in A$. Suppose there exist $x, y, z \in A$ such that $ax + by + cz = 1$. Then there exist $x', y', z' \in A$ such that $a^{50}x' + b^{20}y' + c^{15}z' = 1$.

F 14. The ring $\mathbb{R}[x]/(x^5 + x - 3)$ is an integral domain.

F 15. Given any group G of order 12, and any n that divides 12, there exists a subgroup H of G of order n.

T 16. Let H, N be subgroups of a finite group G, with N a normal subgroup of G. If the orders of G/N and H are relatively prime, then H is necessarily contained in N.

F 17. If every proper subgroup of an infinite group G is cyclic, then G is cyclic.

T 18. Each solution of the differential equation

$$y'' + e^x y = 0$$

remains bounded as $x \to \infty$.

F 19. There exists a uniformly continuous function $f : (0, \infty) \to (0, \infty)$ such that

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converges.

F 20. Let $v : \mathbb{R} \to \mathbb{R}^2$ be C^∞ (i.e., has derivatives of all orders). Then there exists $t_0 \in (0, 1)$ such that $v(1) - v(0)$ is a scalar multiple of $\frac{dv}{dt}\big|_{t=t_0}$.