NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 19, 2013

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions that follow.

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 11 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded for questions involving more than one answer only if **all** the correct answers are given. **There will be no partial credit**.
- Calculators are **not allowed**.

Notation

- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers.
- $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) denotes the n-dimensional Euclidean space over $\mathbb{R}$ (respectively, over $\mathbb{C}$), and is assumed to be endowed with its 'usual' topology. $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will denote the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and is identified with $\mathbb{R}^{n^2}$ (respectively, $\mathbb{C}^{n^2}$) when considered as a topological space.
- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
- The space of continuous real-valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup-norm' metric. The space of continuously differentiable real-valued functions on $[a, b]$ is denoted by $\mathcal{C}^1[a, b]$.
- The derivative of a function f is denoted by f' and the second derivative by f''.
- The transpose of a vector $x \in \mathbb{R}^n$ (respectively, an $n \times n$ matrix A) will be denoted by x^T (respectively, A^T).
- The symbol I will denote the identity matrix of appropriate order.
- The determinant of a square matrix A will be denoted by $\det(A)$ and its trace by $\mathrm{tr}(A)$.
- $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) will denote the group of invertible $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) with the group operation being matrix multiplication.
- Unless specified otherwise, all logarithms are to the base e.

Section 1: Algebra

1.1 Find the number of elements of order two in the symmetric group S_4 of all permutations of the four symbols $\{1, 2, 3, 4\}$.

1.2 Let G be the group of all invertible 2×2 upper triangular matrices (under matrix multiplication). Pick out the normal subgroups of G from the following:
a. $H = \{A \in G \ : \ a_{12} = 0\}$;
b. $H = \{A \in G \ : \ a_{11} = 1\}$;
c. $H = \{A \in G \ : \ a_{11} = a_{22}\}$,
where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}.$$

1.3 Let $G = GL_n(\mathbb{R})$ and let H be the (normal) subgroup of all matrices with positive determinant. Identify the quotient group G/H.

1.4 Which of the following rings are integral domains?
a. $\mathbb{R}[x]$, the ring of all polynomials in one variable with real coefficients.
b. $\mathbb{M}_n(\mathbb{R})$.
c. The ring of complex analytic functions defined on the unit disc of the complex plane (with pointwise addition and multiplication as the ring operations).

1.5 Find the condition on the real numbers a, b and c such that the following system of equations has a solution:

$$\begin{array}{ccccccc} 2x & + & y & + & 3z & = & a \\ x & + & & & z & = & b \\ & & y & + & z & = & c. \end{array}$$

1.6 Let $\mathcal{P}_n$ denote the the vector space of all polynomials in one variable with real coefficients and of degree less than, or equal to, n, equipped with the standard basis $\{1, x, x^2, \cdots, x^n\}$. Define $T : \mathcal{P}_2 \to \mathcal{P}_3$ by

$$T(p)(x) = \int_0^x p(t)\ dt + p'(x) + p(2).$$

Write down the matrix of this transformation with respect to the standard bases of $\mathcal{P}_2$ and $\mathcal{P}_3$.

1.7 Determine the dimension of the kernel of the linear transformation T defined in Question 1.6 above.

1.8 A symmetric matrix in $\mathbb{M}_n(\mathbb{R})$ is said to be *non-negative definite* if $x^T Ax \geq 0$ for all (column) vectors $x \in \mathbb{R}^n$. Which of the following statements are true?
a. If a real symmetric $n \times n$ matrix is non-negative definite, then all of its eigenvalues are non-negative.
b. If a real symmetric $n \times n$ matrix has all its eigenvalues non-negative, then it is non-negative definite.
c. If $A \in \mathbb{M}_n(\mathbb{R})$, then AA^T is non-negative definite.

1.9 Only one of the following matrices is non-negative definite. Find it.

a.

$$\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}.$$

b.

$$\begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}.$$

c.

$$\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}.$$

1.10 Let B be the real symmetric non-negative definite 2×2 matrix such that $B^2 = A$ where A is the non-negative definite matrix in Question 1.9 above. Write down the characteristic polynomial of B.

Section 2: Analysis

2.1 Evaluate:

$$\lim_{n\to\infty} \sin\left(\left(2n\pi + \frac{1}{2n\pi}\right)\sin\left(2n\pi + \frac{1}{2n\pi}\right)\right).$$

2.2 Evaluate:

$$\lim_{n\to\infty} \frac{1}{n}[(n+1)(n+2)\cdots(n+n)]^{\frac{1}{n}}.$$

2.3 Which of the following series are convergent?

a.

$$\sum_{n=1}^{\infty} \frac{\frac{1}{2}+(-1)^n}{n}.$$

b.

$$\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1}-\sqrt{n}).$$

c.

$$\sum_{n=1}^{\infty} \frac{\sin(n^{\frac{3}{2}})}{n^{\frac{3}{2}}}.$$

2.4 Which of the following functions are uniformly continuous?
a. $f(x) = x\sin\frac{1}{x}$ on $]0,1[$.
b. $f(x) = \sin^2 x$ on $]0,\infty[$.
c. $f(x) = \sin(x\sin x)$ on $]0,\infty[$.

2.5 Find the points where the following function is differentiable:

$$f(x) \;=\; \begin{cases} \tan^{-1} x, & \text{if } |x| \leq 1, \\ \frac{\pi x}{4|x|} + \frac{|x|-1}{2}, & \text{if } |x| > 1. \end{cases}$$

2.6 Which of the following sequences/series of functions are uniformly convergent on $[0,1]$?
a. $f_n(x) = (\cos(\pi n!x))^{2n}$.
b.

$$\sum_{m=1}^{\infty} \frac{\cos(m^6 x)}{m^3}.$$

c. $f_n(x) = n^2 x(1-x^2)^n$.

2.7 Let $f \in \mathcal{C}^1[0,1]$. For a partition

$$(\mathcal{P}) : 0 = x_0 < x_1 < x_2 < \cdots < x_n = 1,$$

define

$$S(\mathcal{P}) \;=\; \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

Compute the supremum of $S(\mathcal{P})$ taken over all possible partitions $\mathcal{P}$.

2.8 Write down the Taylor series expansion about the origin in the region $\{|x| < 1\}$ for the function

$$f(x) \;=\; x\tan^{-1}(x) - \frac{1}{2}\log(1+x^2).$$

2.9 Write down all possible values of i^{-2i}.

2.10 What is the image of the set $\{z \in \mathbb{C} \ : \ z = x + iy, x \geq 0, y \geq 0\}$ under the mapping $z \mapsto z^2$.

Section 3: Topology

3.1 Let (X,d) be a metric space. For subsets A and B of X, define

$$d(A,B) \;=\; \inf\{d(a,b) \;:\; a \in A, b \in B\}.$$

Which of the following statements are true?
a. If $\overline{A} \cap \overline{B} = \emptyset$, then $d(A,B) > 0$.
b. If $d(A,B) > 0$, then there exist open sets U and V such that $A \subset U, B \subset V, U \cap V = \emptyset$.
c. $d(A,B) = 0$ if, and only if, there exists a sequence of points $\{x_n\}$ in A converging to a point in B.

3.2 Let X be a set and let (Y,τ) be a topological space. Let $g : X \to Y$ be a given map. Define

$$\tau' \;=\; \{U \subset X \;:\; U = g^{-1}(V) \text{ for some } V \in \tau\}.$$

Which of the following statements are true?
a. τ' defines a topology on X.
b. τ' defines a topology on X only if g is onto.
c. Let g be onto. Define the equivalence relation $x \sim y$ if, and only if, $g(x) = g(y)$. Then the quotient space of X with respect to this relation, with the topology inherited from τ', is homeomorphic to (Y,τ).

3.3 Find pairs of homeomorphic sets from the following:
$A = \{(x,y) \in \mathbb{R}^2 \;:\; xy = 0\}$;
$B = \{(x,y) \in \mathbb{R}^2 \;:\; x + y \geq 0, xy = 0\}$;
$C = \{(x,y) \in \mathbb{R}^2 \;:\; xy = 1\}$;
$D = \{(x,y) \in \mathbb{R}^2 \;:\; x + y \geq 0, xy = 1\}$.

3.4 Let (X,τ) be a topological space. A map $f : X \to \mathbb{R}$ is said to be *lower semi-continuous* if for every $\alpha \in \mathbb{R}$, the set $f^{-1}(]-\infty,\alpha])$ is closed in X. It is said to be *upper semi-continuous* if, for every $\alpha \in \mathbb{R}$, the set $f^{-1}([\alpha,\infty[)$ is closed in X. Which of the following statements are true?
a. If $\{f_n\}$ is a sequence of lower semi-continuous real valued functions on X, then $f = \sup_n f_n$ is also lower semi-continuous.
b. Every continuous real valued function on X is lower semi-continuous.
c. If a real valued function is both upper and lower semi-continuous, then it is continuous.

3.5 Let

$$S \;=\; \{A \in \mathbb{M}_n(\mathbb{R}) \;:\; \mathrm{tr}(A) = 0\}.$$

Which of the following statements are true?
a. S is nowhere dense in $\mathbb{M}_n(\mathbb{R})$.
b. S is connected in $\mathbb{M}_n(\mathbb{R})$.
c. S is compact in $\mathbb{M}_n(\mathbb{R})$.

3.6 Let S be the set of all symmetric non-negative definite matrices (see Question 1.8) in $\mathbb{M}_n(\mathbb{R})$. Which of the following statements are true?
a. S is closed in $\mathbb{M}_n(\mathbb{R})$.
b. S is connected in $\mathbb{M}_n(\mathbb{R})$.
c. S is compact in $\mathbb{M}_n(\mathbb{R})$.

3.7 Which of the following sets are compact in $\mathbb{M}_n(\mathbb{R})$?
a. The set of all upper triangular matrices all of whose eigenvalues satisfy $|\lambda| \leq 2$.
b. The set of all real symmetric matrices all of whose eigenvalues satisfy $|\lambda| \leq 2$.
c. The set of all diagonalizable matrices all of whose eigenvalues satisfy $|\lambda| \leq 2$.

3.8 Let X be the set of all real sequences. Consider the subset

$$S = \left\{ x = (x_n) \in X \ : \ \begin{array}{l} x_n \in \mathbb{Q} \ \text{ for all } n, \\ x_n = 0, \ \text{except for a finite number of } \ n \end{array} \right\}.$$

Which of the following statements are true?
a. S is dense in ℓ_1, the space of absolutely summable sequences, provided with the metric

$$d_1(x,y) = \sum_{n=1}^{\infty} |x_n - y_n|.$$

b. S is dense in ℓ_2, the space of square summable sequences, provided with the metric

$$d_2(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2 \right)^{\frac{1}{2}}.$$

c. S is dense in ℓ_∞, the space of bounded sequences, provided with the metric

$$d_\infty(x,y) = \sup_n \{|x_n - y_n|\}.$$

3.9 Which of the following statements are true?
a. There exists a continuous function $f : \{(x,y) \in \mathbb{R}^2 \ : \ 2x^2 + 3y^2 = 1\} \to \mathbb{R}$ which is one-one.
b. There exists a continuous function $f :]-1,1[\ \to \ ]-1,1]$ which is one-one and onto.
c. There exists a continuous function $f : \{(x,y) \in \mathbb{R}^2 \ : \ y^2 = 4x\} \to \mathbb{R}$ which is one-one.

3.10 Which of the following statements are true?
a. Let $f :]0,\infty[\to]0,\infty[$ be such that

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y|$$

for all x and y. Then f has a fixed point.
b. Let $f : [-1,1] \to [-1,1]$ be continuous. Then f has a fixed point.
c. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$. Then there exists a point $x_0 \in \mathbb{R}$ such that

$$f(x_0) = f\left(x_0 + \frac{T}{2}\right).$$

Section 4: Applied Mathematics

4.1 Find all the solutions $(\lambda, u), u \not\equiv 0$, of the problem:

$$\begin{aligned} u'' + \lambda u &= 0, \text{ in }]0,1[, \\ u(0) = & \; 0 = u'(1). \end{aligned}$$

4.2 Find the constant c such that the following problem has a solution:

$$\begin{aligned} -u'' &= c \text{ in }]a,b[, \\ u'(a) = -1 \quad &, \quad u'(b) = 1. \end{aligned}$$

4.3 Evaluate:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(3x^2 + 2\sqrt{2}xy + 3y^2)} \, dxdy.$$

4.4 Find the stationary function $y = y(x)$ of the integral

$$\int_0^4 [xy' - (y')^2] \, dx$$

satisfying the conditions $y(0) = 0$ and $y(4) = 3$.

4.5 Let $L(y)$ denote the Laplace transform of a function $y = y(x)$. If y and y' are bounded, express $L(y'')$ in terms of $L(y), y$ and y'.

4.6 Find the singular points of the differential equation

$$x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$$

and state whether they are regular singular points or irregular singular points.

4.7 Let (λ_1, y_1) and (λ_2, y_2) be two solutions of the problem

$$\begin{aligned} (p(x)y'(x))' + \lambda q(x)y(x) &= 0 \text{ in }]a,b[, \\ y(a) = & \; 0 = y(b) \end{aligned}$$

where p and q are positive and continuous functions on $]a,b[$. If $\lambda_1 \neq \lambda_2$, evaluate

$$\int_a^b q(x)y_1(x)y_2(x) \, dx.$$

4.8 Solve:

$$xy'' - y' = 3x^2.$$

4.9 Let $f \in \mathcal{C}[a,b]$. Write down Simpson's rule to approximate

$$\int_a^b f(x) \, dx$$

using the points $x = a, x = (a+b)/2$ and $x = b$.

4.10 What is the highest value of n such that Simpson's rule (see Question 4.9 above) gives the exact value of the integral of f on $[a,b]$ when f is a polynomial of degree less than, or equal to, n?

Section 5: Miscellaneous

5.1 Let $m > n$. In how many ways can we seat m men and n women in a row for a photograph if no two women are to be seated adjacent to each other?

5.2 Let $n \in \mathbb{N}$ be fixed. For $r \leq n$, let C_r denote the usual binomial coefficient $\binom{n}{r}$ which gives the number of ways of choosing r objects from a given set of n objects. Evaluate:

$$C_0 + 4C_1 + 7C_2 + \cdots + (3n+1)C_n.$$

5.3 Let
$A =$ the set of all sequences of real numbers,
$B =$ the set of all sequences of positive real numbers,
$C = \mathcal{C}[0,1]$ and $D = \mathbb{R}$.
Which of the following statements are true?
a. All the four sets have the same cardinality.
b. A and B have the same cardinality.
c. A, B and D have the same cardinality, which is different from that of C.

5.4 For a positive integer n, define

$$\Lambda(n) \;=\; \begin{cases} \log p, & \text{if } n = p^r,\ p \text{ a prime and } r \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Given a positive integer N, evaluate:

$$\sum_{d|N} \Lambda(d)$$

where the sum ranges over all divisors d of N.

5.5 Let a, b and c be real numbers. Evaluate:

$$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}.$$

5.6 Write down the equation (with leading coefficient equal to unity) whose roots are the squares of the roots of the equation

$$x^3 - 6x^2 + 10x - 3 \;=\; 0.$$

5.7 Let $A = (0,1)$ and $B = (1,1)$ in the plane $\mathbb{R}^2$. Determine the length of the shortest path from A to B consisting of the line segments AP, PQ and QB, where P varies on the x-axis between the points $(0,0)$ and $(1,0)$ and Q varies on the line $\{y = 3\}$ between the points $(0,3)$ and $(1,3)$.

5.8 Let $x_0 = a, x_1 = b$. If

$$x_{n+2} \;=\; \frac{1}{3}(x_n + 2x_{n+1}),\ \ n \geq 0,$$

find $\lim_{n\to\infty} x_n$.

5.9 Which of the following statements are true?
a. If a, b and c are the sides of a triangle, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2} \geq \frac{1}{2}.$$

b. If a, b and c are the sides of a triangle, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2} \leq 1.$$

c. Both statements above are true for all triples (a, b, c) of strictly positive real numbers.

5.10 Let $f \in \mathcal{C}[a, b]$. Assume that $\min_{x\in[a,b]} f(x) = m > 0$ and let $M = \max_{x\in[a,b]} f(x)$. Which of the following inequalities are true?
a.

$$\frac{1}{M}\int_a^b f(x)\, dx + m\int_a^b \frac{1}{f(x)}\, dx \geq 2\sqrt{\frac{m}{M}}(b-a).$$

b.

$$\int_a^b f(x)\, dx \int_a^b \frac{1}{f(x)}\, dx \geq (b-a)^2.$$

c.

$$\int_a^b f(x)\, dx \int_a^b \frac{1}{f(x)}\, dx \leq (b-a)^2.$$