Correct answers are ticked in green.

MTH

GS-2014 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 8, 2013**
Duration : Two hours (2 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine-readable. Read the instructions given on the reverse of the answer sheet before you start filling it up. Use only HB pencils to fill-in the answer sheet.

2. There are thirty (30) multiple choice questions divided into two parts. Part I consists of 20 questions and Part II consists of 10 questions. Bachelors students who have applied only for the Integrated Ph.D. program at TIFR CAM, Bangalore will only be evaluated on Part I. All other students (including Bachelors students applying to the Ph.D. programs at both, TIFR, Mumbai and Bangalore) will be evaluated on both Parts I and II.

3. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. **Each corect answer will get 1 mark. There is no negative marking for wrong answers. A question not answered will not get you any mark.** Do not mark more than one circle for any question : this will be treated as a wrong answer.

4. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

8. Notation and Conventions used in this test are given on page 1 of the question paper.

NOTATION AND CONVENTIONS

$\mathbb{N}$:= Set of natural numbers = $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$:= Set of integers
$\mathbb{Q}$:= Set of rational numbers
$\mathbb{R}$:= Set of real numbers
$\mathbb{C}$:= Set of complex numbers
$\mathbb{R}^*$:= Set of non-zero real numbers
$\mathbb{C}^*$:= Set of non-zero complex numbers
$\mathbb{R}^n$:= n-dimensional vector space over $\mathbb{R}$

$(a, b) := \{x \in \mathbb{R} | a < x < b\}$
$[a, b) := \{x \in \mathbb{R} | a \leq x < b\}$
$[a, b] := \{x \in \mathbb{R} | a \leq x \leq b\}$

A sequence is always indexed by the set of natural numbers.
The cyclic group with n elements is denoted by $\mathbb{Z}/n$.
Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology.
For any set S, the cardinality of the set is denoted by $|S|$.

Part I

1. Let A, B, C be three subsets of $\mathbb{R}$. The negation of the following statement
For every $\epsilon > 1$, there exists $a \in A$ and $b \in B$ such that for all $c \in C$, $|a - c| < \epsilon$ and $|b - c| > \epsilon$

   is

   A. there exists $\epsilon \leq 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a - c| \geq \epsilon$ or $|b - c| \leq \epsilon$
   B. there exists $\epsilon \leq 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a - c| \geq \epsilon$ and $|b - c| \leq \epsilon$
   C. there exists $\epsilon > 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a - c| \geq \epsilon$ and $|b - c| \leq \epsilon$
   D. there exists $\epsilon > 1$, such that for all $a \in A$ and $b \in B$ there exists $c \in C$ such that $|a - c| \geq \epsilon$ or $|b - c| \leq \epsilon$. ✓

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function, then:

   A. f has to be uniformly continuous
   B. there exists an $x \in \mathbb{R}$ such that $f(x) = x$ ✓
   C. f cannot be increasing
   D. $\lim_{x \to \infty} f(x)$ exists.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lim_{x \to +\infty} f'(x) = 1$, then

   A. f is bounded
   B. f is increasing
   C. f is unbounded ✓
   D. f' is bounded.

4. Let f be the real valued function on $[0, \infty)$ defined by

$$f(x) = \begin{cases} x^{\frac{2}{3}} \log x \text{ for } x > 0 \\ 0 \text{ if } x = 0. \end{cases}$$

Then

A. f is discontinuous at $x = 0$
B. f is continuous on $[0, \infty)$, but not uniformly continuous on $[0, \infty)$
C. f is uniformly continuous on $[0, \infty)$
D. f is not uniformly continuous on $[0, \infty)$, but uniformly continuous on $(0, \infty)$.

5. Let $a_n = (n+1)^{100} e^{-\sqrt{n}}$ for $n \geq 1$. Then the sequence $(a_n)_n$ is

A. unbounded
B. bounded but does not converge
C. bounded and converges to 1
D. bounded and converges to 0.

6. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Which of the following statements is always true?

A. $\int_0^1 f^2(x)\,dx = \left(\int_0^1 f(x)\,dx\right)^2$
B. $\int_0^1 f^2(x)\,dx \leq \left(\int_0^1 |f(x)|\,dx\right)^2$
C. $\int_0^1 f^2(x)\,dx \geq \left(\int_0^1 |f(x)|\,dx\right)^2$
D. $\int_0^1 f^2(x)\,dx \lneq \left(\int_0^1 f(x)\,dx\right)^2$.

7. Let $f_n(x)$, for $n \geq 1$, be a sequence of continuous nonnegative functions on $[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0.$$

Which of the following statements is always correct?

A. $f_n \to 0$ uniformly on $[0,1]$
B. f_n may not converge uniformly but converges to 0 point-wise
C. f_n will converge point-wise and the limit may be non-zero
D. f_n is not guaranteed to have a point-wise limit.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x)-f(y)| \geq \frac{1}{2}|x-y|$, for all $x, y \in \mathbb{R}$. Then

A. f is both one-to-one and onto
B. f is one-to-one but may not be onto
C. f is onto but may not be one-to-one
D. f is neither one-to-one nor onto.

9. Let $A(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, where $\theta \in (0, 2\pi)$. Mark the correct statement below.

A. $A(\theta)$ has eigenvectors in $\mathbb{R}^2$ for all $\theta \in (0, 2\pi)$
B. $A(\theta)$ does not have an eigenvector in $\mathbb{R}^2$, for any $\theta \in (0, 2\pi)$
C. $A(\theta)$ has eigenvectors in $\mathbb{R}^2$, for exactly one value of $\theta \in (0, 2\pi)$
D. $A(\theta)$ has eigenvectors in $\mathbb{R}^2$, for exactly 2 values of $\theta \in (0, 2\pi)$

10. Let $\mathcal{C} \subset \mathbb{Z} \times \mathbb{Z}$ be the set of integer pairs (a, b) for which the three complex roots r_1, r_2 and r_3 of the polynomial $p(x) = x^3 - 2x^2 + ax - b$ satisfy $r_1^3 + r_2^3 + r_3^3 = 0$. Then the cardinality of $\mathcal{C}$ is

A. $|\mathcal{C}| = \infty$
B. $|\mathcal{C}| = 0$ ✓
C. $|\mathcal{C}| = 1$
D. $1 < |\mathcal{C}| < \infty$.

11. Let A be an $n \times n$ matrix with real entries such that $A^k = 0$ (0-matrix), for some $k \in \mathbb{N}$. Then

A. A has to be the 0 matrix
B. trace(A) could be non-zero
C. A is diagonalizable
D. 0 is the only eigenvalue of A. ✓

12. There exists a map $f : \mathbb{Z} \to \mathbb{Q}$ such that f

A. is bijective and increasing
B. is onto and decreasing
C. is bijective and satisfies $f(n) \geq 0$ if $n \leq 0$ ✓
D. has uncountable image.

13. Let S be the set of all tuples (x, y) with x, y non-negative real numbers satisfying $x + y = 2n$, for a fixed $n \in \mathbb{N}$. Then the supremum value of

$$x^2y^2(x^2 + y^2)$$

on the set S is

A. $3n^6$
B. $2n^6$ ✓
C. $4n^6$
D. n^6.

14. Let G be a group and let H and K be two subgroups of G. If both H and K have 12 elements, which of the following numbers cannot be the cardinality of the set $HK = \{hk : h \in H, k \in K\}$?

A. 72
B. 60
C. 48
D. 36.

15. How many proper subgroups does the group $\mathbb{Z} \oplus \mathbb{Z}$ have?

A. 1
B. 2
C. 3
D. infinitely many.

16. X is a metric space. Y is a closed subset of X such that the distance between any two points in Y is at most 1. Then

A. Y is compact
B. any continuous function from $Y \to \mathbb{R}$ is bounded
C. Y is not an open subset of X
D. none of the above.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let S be a non-empty proper subset of $\mathbb{R}$. Which one of the following statements is always true? (Here $\bar{A}$ denotes the closure of A and A^o denotes the interior of A.)

A. $f(S)^o \subseteq f(S^o)$
B. $f(\bar{S}) \subseteq \overline{f(S)}$
C. $f(\bar{S}) \supseteq \overline{f(S)}$
D. $f(S)^o \supseteq f(S^o)$.

18. What is the last digit of 97^{2013}?

A. 1
B. 3
C. 7 ✓
D. 9.

19. For $n \in \mathbb{N}$, we define

$$s_n = 1^3 + 2^3 + 3^3 + \cdots + n^3.$$

Which of the following holds for all $n \in \mathbb{N}$?

A. s_n is an odd integer
B. $s_n = n^2(n+1)^2/4$ ✓
C. $s_n = n(n+1)(2n+1)/6$
D. none of the above.

20. Let C denote the cube $[-1,1]^3 \subset \mathbb{R}^3$. How many rotations are there in $\mathbb{R}^3$ which take C to itself?

A. 6
B. 12
C. 18
D. 24. ✓

Part II

21. Let $f : [0,1] \to [0,\infty)$ be continuous. Suppose

$$\int_0^x f(t)\,\mathrm{d}t \geq f(x), \text{ for all } x \in [0,1].$$

Then

A. no such function exists
B. there are infinitely many such functions
C. there is only one such function ✓
D. there are exactly two such functions.

22. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous map such that $f(x) = 0$ for only finitely many values of x. Which of the following is true?

A. either $f(x) \leq 0$ for all x, or, $f(x) \geq 0$ for all x ✓
B. the map f is onto
C. the map f is one-to-one
D. none of the above.

23. Let S_n be the symmetric group of n letters. There exists an onto group homomorphism

A. from S_5 to S_4
B. from S_4 to S_2 ✓
C. from S_5 to $\mathbb{Z}/5$
D. from S_4 to $\mathbb{Z}/4$.

24. Let H_1, H_2 be two distinct subgroups of a finite group G, each of order 2. Let H be the smallest subgroup containing H_1 and H_2. Then the order of H is

A. always 2
B. always 4
C. always 8
D. none of the above. ✓

25. Which of the following groups are isomorphic?

A. $\mathbb{R}$ and $\mathbb{C}$ ✓
B. $\mathbb{R}^*$ and $\mathbb{C}^*$
C. $S_3 \times \mathbb{Z}/4$ and S_4
D. $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$.

26. The number of irreducible polynomials of the form $x^2 + ax + b$, with a, b in the field $\mathbb{F}_7$ of 7 elements is:

A. 7
B. 21 ✓
C. 35
D. 49.

27. X is a topological space of infinite cardinality which is homeomorphic to $X \times X$. Then

A. X is not connected
B. X is not compact
C. X is not homemorphic to a subset of $\mathbb{R}$
D. none of the above. ✓

28. Let X be a non-empty topological space such that every function $f : X \to \mathbb{R}$ is continuous. Then

A. X has the discrete topology ✓
B. X has the indiscrete topology
C. X is compact
D. X is not connected.

29. Let $f : X \to Y$ be a continuous map between metric spaces. Then $f(X)$ is a complete subset of Y if

A. the space X is compact ✓
B. the space Y is compact
C. the space X is complete
D. the space Y is complete.

30. How many maps $\phi : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ are there, with the property that $\phi(ab) = \phi(a) + \phi(b)$, for all $a, b \in \mathbb{N} \cup \{0\}$?

A. none
B. finitely many ✓
C. countably many
D. uncountably many.