CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
1st August 2021

Important information and instructions:

(1) Questions in Part A (Questions 1 – 10) will be used for screening. There will be a cut-off for Part A, which will not be more than 20 marks (out of 40).

(2) Each question in Part A has one or more correct answers. Enter your answers to these questions into the computer as instructed. Every question is worth 4 marks. A solution will receive credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(3) Your solutions to the questions in Part B (Questions 11– 20*) will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.)

(4) Answer 6 questions from Part B, on the pages assigned to them, with sufficient justification. Each question is worth 10 marks. Clearly indicate which six questions you would like us to mark in the six boxes on the front sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

(5) The scores in both the sections will be taken into account while making the final decision. In order to qualify for the PhD Mathematics interview, you must obtain at least 15 marks from among the starred questions 17*–20*.

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand, respectively, for the sets of non-negative integers, of integers, of rational numbers, of real numbers, of non-negative real numbers, of positive real numbers, and of complex numbers. For a field F and a positive integer n, $M_n(F)$ stands for the set of $n \times n$ matrices over F and $\mathrm{GL}(n, F)$ for the set of invertible $n \times n$ matrices over F. I_n denotes the $n \times n$ identity matrix; the field will be clear from context. When considered as topological spaces, $\mathbb{R}^n$ or $\mathbb{C}$. are taken with the euclidean topology. Similarly, as topological spaces, $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ are taken with the euclidean topology. $\mathrm{GL}(n, \mathbb{R})$ is considered as a topological subspace of $M_n(\mathbb{R})$.

Part A

(1) Which of the following **can not** be the class equation for a group of appropriate order?
 (A) $14 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 7$.
 (B) $18 = 1 + 1 + 1 + 1 + 2 + 3 + 9$.
 (C) $6 = 1 + 2 + 3$.
 (D) $31 = 1 + 3 + 6 + 6 + 7 + 8$.

(2) Consider the improper integral $\int_2^\infty \frac{1}{x(\log x)^2} dx$ and the infinite series $\sum_{k=2}^\infty \frac{1}{k(\log k)^2}$. Which of the following is/are true?
 (A) The integral converges but the series does not converge.
 (B) The integral does not converge but the series converges.
 (C) Both the integral and the series converge.
 (D) The integral and the series both fail to converge.

(3) Let $A \in M_2(\mathbb{R})$ be a nonzero matrix. Pick the correct statement(s) from below.
 (A) If $A^2 = 0$, then $(I_2 - A)^5 = 0$.
 (B) If $A^2 = 0$, then $(I_2 - A)$ is invertible.
 (C) If $A^3 = 0$, then $A^2 = 0$.
 (D) If $A^2 = A^3 \neq 0$, then A is invertible.

(4) Let $f : [0, 1] \longrightarrow [0, 1]$ be a continuous function. Which of the following is/are true?
 (A) For every continuous $g : [0, 1] \longrightarrow \mathbb{R}$ with $g(0) = 0$ and $g(1) = 1$ there exists $x \in [0, 1]$ with $f(x) = g(x)$.

(B) For every continuous $g : [0,1] \longrightarrow \mathbb{R}$ with $g(0) < 0$ and $g(1) > 1$ there exists $x \in [0,1]$ with $f(x) = g(x)$.
(C) For every continuous $g : [0,1] \longrightarrow \mathbb{R}$ with $0 < g(0) < 1$ and $0 < g(1) < 1$ there exists $x \in [0,1]$ with $f(x) = g(x)$.
(D) For every continuous $g : [0,1] \longrightarrow [0,1]$ there exists $x \in [0,1]$ with $f(x) = g(x)$.

(5) Let I, J be nonempty open intervals in $\mathbb{R}$. Let $f : I \longrightarrow J$ and $g : J \longrightarrow \mathbb{R}$ be functions. Let $h : I \longrightarrow \mathbb{R}$ be the composite function $g \circ f$. Pick the correct statement(s) from below.
(A) If f, g are continuous, then h is continuous.
(B) If f, g are uniformly continuous, then h is uniformly continuous.
(C) If h is continuous, then f is continuous.
(D) If h is continuous, then g is continuous.

(6) Let A, B be non-empty subsets of $\mathbb{R}^2$. Pick the correct statement(s) from below:
(A) If A is compact, B is open and $A \cup B$ is compact, then $A \cap B \neq \emptyset$.
(B) If A and B are path-connected and $A \cap B \neq \emptyset$ then $A \cup B$ is path-connected.
(C) If A and B are connected and open and $A \cap B \neq \emptyset$, then $A \cap B$ is connected.
(D) If A is countable with $|A| \geq 2$, then A is not connected.

(7) Pick the correct statement(s) from below.
(A) $X = \prod_{n=1}^{\infty} X_n$ where $X_n = \{1, 2, \ldots, 2^n\}$ for $n \geq 1$ is not compact in the product topology.
(B) $Y = \prod_{n=1}^{\infty} Y_n$ where $Y_n = [0, 2^n] \subseteq \mathbb{R}$ for $n \geq 1$ is path-connected in the product topology.
(C) $Z = \prod_{n=1}^{\infty} Z_n$ where $Z_n = (0, \frac{1}{n}) \subseteq \mathbb{R}$ for $n \geq 1$ is compact in the product topology.
(D) $P = \prod_{n=1}^{\infty} P_n$ where $P_n = \{0, 1\}$ for $n \geq 1$ (with product topology) is homeomorphic to $(0, 1)$.

(8) Let $f(z) = \frac{e^z - 1}{z(z-1)}$ be defined on the extended complex plane $\mathbb{C} \cup \{\infty\}$. Which of the following is/are true?
(A) $z = 0, z = 1, z = \infty$ are poles.
(B) $z = 1$ is a simple pole.
(C) $z = 0$ is a removable singularity.
(D) $z = \infty$ is an essential singularity

(9) For $A \in M_3(\mathbb{C})$, let $W_A = \{B \in M_3(\mathbb{C}) \mid AB = BA\}$. Which of the following is/are true?
(A) For all diagonal $A \in M_3(\mathbb{C})$, W_A is a linear subspace of $M_3(\mathbb{C})$ with $\dim_{\mathbb{C}} W_A \geq 3$.
(B) For all $A \in M_3(\mathbb{C})$, W_A is a linear subspace of $M_3(\mathbb{C})$ with $\dim_{\mathbb{C}} W_A > 3$.
(C) There exists $A \in M_3(\mathbb{C})$ such that W_A is a linear subspace of $M_3(\mathbb{C})$ with $\dim_{\mathbb{C}} W_A = 3$.
(D) If $A \in M_3(\mathbb{C})$ is diagonalizable, then every element of W_A is diagonalizable.

(10) Let K be a field of order 243 and let F be a subfield of K of order 3. Pick the correct statement(s) from below.
(A) There exists $\alpha \in K$ such that $K = F(\alpha)$.
(B) The polynomial $x^{242} = 1$ has exactly 242 solutions in K.
(C) The polynomial $x^{26} = 1$ has exactly 26 roots in K.
(D) Let $f(x) \in F[x]$ be an irreducible polynomial of degree 5. Then $f(x)$ has a root in K.

Part B

(11) Let G be a finite group and X the set of all abelian subgroups H of G such that H is a maximal subgroup of G (under inclusion) and is not normal in G. Let M and N be distinct elements of X. Show the following:
  (A) The subgroup of G generated by M and N is contained in the centralizer of $M \cap N$ in G.
  (B) $M \cap N$ is the centre of G.

(12) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a smooth function whose derivative at every point is non-singular. Suppose that $f(0) = 0$ and for all $v \in \mathbb{R}^2$ with $|v| = 1$, $|f(v)| \geq 1$. Let D denote the open unit ball $\{v : |v| < 1\}$. Show that $D \subset f(D)$. (Hint: Show that $f(D) \cap D$ is closed in D.)

(13) Let X be a topological space and $x_0 \in X$. Let $\mathcal{S} = \{B \subseteq X \mid x_0 \in B \text{ and } B \text{ is connected}\}$. Let

$$A = \bigcup_{B \in \mathcal{S}} B.$$

Show that A is closed.

(14) Let $f : [1, \infty) \longrightarrow \mathbb{R} \setminus \{0\}$ be uniformly continuous. Show that the series $\sum_{n \geq 1} 1/f(n)$ is divergent.

(15) Show that $\int_0^\infty x^{\sqrt{10}} e^{-x^{1/100}} dx < \infty$.

(16) Consider the following statement: Let F be a field and $R = F[X]$ the polynomial ring over F in one variable. Let I_1 and I_2 be maximal ideals of R such that the fields $R/I_1 \simeq R/I_2 \not\simeq F$. Then $I_1 = I_2$.
  Prove or find a counterexample to the following claims:
  (A) The above statement holds if F is a finite field.
  (B) The above statement holds if $F = \mathbb{R}$.

(17*) Let $\mathrm{O}(2, \mathbb{R})$ be the subgroup of $\mathrm{GL}(2, \mathbb{R})$ consisting of orthogonal matrices, i.e., matrices A satisfying $A^{\mathrm{tr}} A = I$. Let $\mathrm{B}_+(2, \mathbb{R})$ be the subgroup of $\mathrm{GL}(2, \mathbb{R})$ consisting of upper triangular matrices with positive entries on the diagonal.
  (A) Let $A \in \mathrm{GL}(2, \mathbb{R})$. Show that there exist $A_o \in \mathrm{O}(2, \mathbb{R})$ and $A_b \in \mathrm{B}_+(2, \mathbb{R})$ such that $A = A_o A_b$. (Hint: use appropriate elementary column operations.)
  (B) Show that the map

$$\phi : \mathrm{O}(2, \mathbb{R}) \times \mathrm{B}_+(2, \mathbb{R}) \longrightarrow \mathrm{GL}(2, \mathbb{R}) \qquad (A', A'') \mapsto A'A''$$

  is injective.
  (C) Show that $\mathrm{GL}(2, \mathbb{R})$ is homeomorphic to $\mathrm{O}(2, \mathbb{R}) \times \mathrm{B}_+(2, \mathbb{R})$. (Hint: first show that the map $A \mapsto A_b$ is continuous.)

(18*) Let F be a field of characteristic $p > 0$ and V a finite-dimensional F-vector-space. Let $\phi \in \mathrm{GL}(V)$ be an element of order p^3. Show that there exists a basis of V with respect to which ϕ is given by an upper-triangular matrix with 1's on the diagonal.

(19*) Let $\zeta_5 \in \mathbb{C}$ be a primitive 5th root of unity; let $\sqrt[5]{2}$ denote a real 5th root of 2, and let $\imath$ denote a square root of -1. Let $K = \mathbb{Q}(\zeta_5, \sqrt[5]{2})$.
  (A) Find the degree $[K : \mathbb{Q}]$ of the field K over $\mathbb{Q}$.
  (B) Determine if $\imath \in \mathbb{Q}(\zeta_5)$. (Hint: You may use, without proof, the following fact: if $\zeta_{20} \in \mathbb{C}$ is a primitive 20th root of unity, then $[\mathbb{Q}(\zeta_{20}) : \mathbb{Q}] > 4$.)
  (C) Determine if $\imath \in K$.

(20*) Let a_0 and a_1 be complex numbers and define $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 2$.
  (A) Show that there are polynomials $p(z), q(z) \in \mathbb{C}[z]$ such that $q(0) \neq 0$ and $\sum_{n \geq 0} a_n z^n$ is the Taylor series expansion (around 0) of $\frac{p(z)}{q(z)}$.
  (B) Let $a_0 = 1$ and $a_1 = 2$. Show that there exist complex numbers $\beta_1, \beta_2, \gamma_1, \gamma_2$ such that

$$a_n = \beta_1 \gamma_1^{n+1} + \beta_2 \gamma_2^{n+1}$$

  for all n.