CHENNAI MATHEMATICAL INSTITUTE

Graduate Programme in Mathematics - M.Sc./Ph.D.

Solutions to the Entrance Examination, 2012

Part A

1. True. Take $\delta = \epsilon$. Then $\|(x_1, x_2 \cdots x_n) - (y_1, y_2 \cdots y_n)\|_2 < \delta$ implies that $|x_i - y_i| < \epsilon$ for all $i = 1, 2 \cdots n$, and hence $Max\{|x_i|\} < \epsilon$. So f is uniformly continuous.

2. True. If not, there exists an $\epsilon_0 > 0$, such that for all k, there there exists an $n_k > k$ satisfying $|x_k - x| \geq \epsilon_0$. The subsequence x_{n_k} does not have any subsequence converging to x.

3. False. Consider $f(x) = \frac{1}{x}$. Then $\{\frac{1}{n}\}$ is Cauchy but $\{n\}$ is not Cauchy.

4. Trues. Since f_n converges to f uniformly, f is also continuous. So $f(x_n)$ converges to $f(x)$. Now given $\epsilon > 0$

$$|f_n(x_n) - f(x)| \leq \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| + |f(x_n) - f(x)|.$$

5. True. If K is unbounded then the function $f(x) = \|x\|$ for $x \in \mathbb{R}^n$ is continuous and unbounded. On the other hand if K is not closed, that is there exists a sequence $\{x_n\} \subseteq K$ such that $x_n \mapsto x$ and $x \in K$, then the function $f(x) = \frac{1}{\|x - x_n\|}$ is continuous and unbounded.

6. True. It suffices to show that $X := \mathbb{R}^2 \setminus A$ is path-connected. Let $x, y \in X$. There are uncountably many lines passing through either of the points x or y; only countably many can go though a point in A; so there are lines l_x though x and l_y through y such that $l_x \subseteq X$ and $l_y \subseteq X$ and $l_x \cap l_y \neq \emptyset$.

7. False, as for any $z \in \mathbb{C}$, there exists a w such that $z^2 + w^2 = 1$

8. True, as the zeros of $f - g$ are isolated.

9. True. 121 is a prime power: $121 = 11^2$.

10. True. The minimal polynomial A is $(t - \pi)(t - \frac{22}{7})$. It has distinct roots since $\pi \neq \frac{22}{7}$.

11. False: take the subgroup $5\mathbb{Z} \subset \mathbb{Z}$, which is of index 5. ($\mathbb{Z}$ being the group of integers).

12. True: any group can be embedded in S_n for some n (by Cayley's theorem) and we embed S_n in an alternating group as follows: $i : S_n \subset A_{n+2}$ defined as follows: — if $\sigma \in S_n$ is even define $i(\sigma) = \sigma$, — if σ is odd define $i(\sigma) = \sigma.(n+1, n+2)$ and hence in both case $i(\sigma)$ is even.

13. False: S_3 is a group of order 6 which is non-abelian.

14. False: there are two non-isomorphic groups of order 4 and these are abelian, one is cyclic and the other a direct product of Z_2 with itself.

15. False. For any $x \neq y$ the interval (x, y) is connected and hence its image under the continuous function f. If $f(x) \neq f(y)$ then the interval contains irrational points and contained in the image of f, since any connected subset of $\mathbb{R}$ is an interval,

Part B

Each question carries 10 marks. Try to answer 5 questions.

1. Let $f(t) = (f_1, f_2(t), f_3(t))$ then $f'(t) = (f_1', f_2'(t), f_3'(t))$. $\|f(t)\| = 1$ implies that
$$f_1(t)^2 + f_2(t)^2 + f_3(t)^2 = 1$$
for all $t \in \mathbb{R}$. Differentiating we get
$$2f_1(t)f_1'(t) + 2f_2(t)f_2'(t) + 2f_3(t)f_3'(t) = 0$$
and hence $\langle f(t), f'(t) \rangle = 0$.

2. Since addition is continuous, B is open implies $a + B$ is open for any fixed $a \in A$. Now
$$A + B = \bigcup_{a \in A} (a + B).$$
Since arbitrary union of open sets is open $A + B$ is open. Note that it is enough if either A or B is open.

For closed sets this is not true. Take $A = \{(a, 0) : a \in \mathbb{R}\}$ and $B = \{(b, \frac{1}{b}) : b \in \mathbb{R} \setminus \{0\}\}$. Then both $A, B \subseteq \mathbb{R}^2$ are closed. But $A + B = \{(a + b, \frac{1}{b}) : a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\}\}$. The sequence $\{(0, \frac{1}{n})\} = \{(n - n, \frac{1}{n}\} \subseteq A + B$ but the limit $(0, 0) \notin A + B$.

3. We assume X and Y to be metric spaces. Let $U \subseteq Z$ be open, then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)$ is open since $g \circ f$ is continuous. Since X is compact and f is continuous, f is an open map ((i.e) f takes open sets to open sets. This is because continuous functions takes compact subsets to compact subsets, so when the space is compact, it takes closed sets to closed sets.) Now since f is onto, we have $g^{-1}(U) = f(f^{-1}(g^{-1}(U))$ and is open. Hence g is also continuous.

4. It is clear that B is a symmetric matrix and thus B can be written as $B = Q\Lambda Q^t$ where Λ is a diagonal matrix and Q is an orthogonal matrix (with $Q^{-1} = Q^t$). Now it can be seen that and $B^j = Q\Lambda^j Q^t$, $C_k = QD_kQ^t$ where
$$D_k = I + \sum_{j=1}^{k} \Lambda^j.$$
Since Λ is a diagonal matrix, say
$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$$
it follows that
$$D_k = \text{diag}(\phi_k(\lambda_1), \phi_k(\lambda_2), \ldots \phi_k(\lambda_n))$$
where $\phi_k(x) = 1 + x + \ldots x^k$. Since $B = AA^t$, it follows that the eigenvalues λ_j of B are non negative and thus $\alpha = \max_i \lambda_i$.

Since $\phi_k(x)$ converges if and only if $|x| < 1$ it follows that D_k (and hence C_k) converges if and only if $\alpha < 1$.

5. Without loss of generality, assume that $z = 1$ is a pole. Take $z = r$ real and let $r \to 1^-$. Now $|f(z)| \leq \sum |a_n| r^n$ and since a_n can be made arbitrarily small for all large n, conclude that $f(r) = \circ(1-r)^{-1}$ as $r \to 1^-$. This contradicts that $z = 1$ is a pole. The statement is false if (a_n) is only bounded, take $a_n = 1$ for all n, for instance.

6. Apply Schwarz's lemma to conclude that any automorphism f of the unit ball is necessarily of the form $f(z) = c\frac{z-a}{1-\bar{a}z}$ where $|c| = 1$. Since origin is mapped to itself, conclude that the map is a rotation.

7. (i) Note that number of elements in G is $(p-1)^2 p(p+1)$. Hence any subgroup of G having exactly p elements is a Sylow-p subgroup of G.

   Now let $H = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} | x \in \mathbb{F}_p\}$. Since $|H| = p$, H is a Sylow-p subgroup of G.

   Let N be the group of all upper triangular matrices in G. Clearly H is a normal subgroup of G.

   If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N_G(H)$, the normalizer of H in G, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ for some } y \in \mathbb{F}_p.$$

   Comparing the (2,2) entry both sides, we get $c = 0$. So $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N$. This proves (i).

   (ii) Let S denote the set of all one-dimensional subspaces of $\mathbb{F}_p^2$. Then $|S| = p+1$. Consider the action of G on S given by left multiplication: $A \in G$, $\begin{pmatrix} a \\ b \end{pmatrix} \in S$, then $A \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ is defined to be the matrix multiplication $A\begin{pmatrix} a \\ b \end{pmatrix} \in S$.

   This action of G on S is transitive. Moreover, the stabilizer of the one-dimensional subspace spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is N. Indeed, clearly N stabilizes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$, then $a = y$ and $c = 0$. So $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N$.

   Hence we conclude that $|G/N| = |S| = p+1$.

8. Let $\alpha = \sqrt{2}e^{\frac{2\pi i}{3}}$ and $\beta = \frac{-\alpha^2}{2}$. Then $\beta = -e^{\frac{4\pi i}{3}} = -e^{\pi i}e^{\frac{\pi i}{3}} = e^{\frac{\pi i}{3}}$. So $\beta^3 = -1$ and β satisfies the polynomial $Y^3 + 1$. Hence the minimal polynomial of β is $Y^2 - Y + 1$.

   Thus α satisfies the equation $(\frac{-\alpha^2}{2})^2 + \frac{\alpha^2}{2} + 1 = 0$. So α is a root of the polynomial $\frac{W^4}{4} + \frac{W^2}{2} + 1$.

Note that the degree of the minimal polynomial of α can not be smaller than 4. Indeed, 2 divides the degree and α does not satisfy a quadratic equation.

Clearing the denominators we conclude that the minimal polynomial of α is X^4+2X^2+4.

9. Embed G inside a suitable symmetric group; represent the symmetric group as the set of permutation matrices.

10. Let $n = \dim_F R$ and $\alpha_i, 1 \leq i \leq n$ be an F-basis of R. Let $\beta \neq 0 \in R$. Consider the vectors $\alpha_i\beta, 1 \leq i \leq n$. They must be linearly independent; for, otherwise, there will be a nonzero element $\gamma \in R$ such that $\beta\gamma = 0$. Hence they form an F-basis of R. Therefore there exists $\gamma \in R$ such that $\beta\gamma = 1$.