CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
18 May 2016

Instructions:

- Enter your *Registration Number* here: **CMI–PG–** ☐☐☐☐☐☐☐☐
- Enter the name of the city where you write this test: ☐☐☐☐☐☐☐☐☐☐☐☐

- The allowed time is 3 hours.
- This examination has two parts. You may use the blank pages at the end for your rough-work.
- Part A is worth 40 marks and will be used for screening. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **Record your answers to Part A in the attached bubble-sheet.**
- Part B is worth 60 marks. You should answer six (6) questions in Part B. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*). **Indicate the six questions to be marked in the boxes in the bubble-sheet. Write your solutions to Part B in the page assigned to each question.**
- **Please read the further instructions given before Part A and inside each part carefully.**

For office use only

Part B

No.	Marks	Remarks
11		
12		
13		
14		
15		
16		

Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
19*		
20*		

Further remarks:

Part A	
Part B	
Total	

This page is intentionally left blank.

CHENNAI MATHEMATICAL INSTITUTE
Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
18 May 2016

- *Registration Number*: **CMI–PG–** ☐☐☐☐☐☐☐☐
- City where you write this test: ☐☐☐☐☐☐☐☐☐☐☐☐

Bubble Sheet

Part A

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ
3. Ⓐ Ⓑ Ⓒ Ⓓ
4. Ⓐ Ⓑ Ⓒ Ⓓ
5. Ⓐ Ⓑ Ⓒ Ⓓ
6. Ⓐ Ⓑ Ⓒ Ⓓ
7. Ⓐ Ⓑ Ⓒ Ⓓ
8. Ⓐ Ⓑ Ⓒ Ⓓ
9. ____________
10. ____________

Part B

In the boxes below, clearly indicate the SIX (6) solutions that should be marked. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17^*)–(20^*).

For office use only:

Number of correct answers in Part A:	
Marks in Part A:	

This page is intentionally left blank.

CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics

MSc/PhD Entrance Examination

18 May 2016

Important: Questions in Part A will be used for **screening**. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17^*)–(20^*).

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ stand, respectively, for the sets of natural numbers, of integers, of rational numbers, of real numbers, and of complex numbers. For a prime number p, $\mathbb{F}_p$ is the field with p elements. For a field F, $M_{m\times n}(F)$ stands for the set of $m \times n$ matrices over F and $\mathrm{GL}_n(F)$ is the set of invertible $n \times n$ matrices over F. The symbol $\imath$ denotes a square-root of -1.

Part A

Instructions: Each of the questions 1–8 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) We say that two subsets X and Y of $\mathbb{R}$ are *order-isomorphic* if there is a bijective map $\phi : X \longrightarrow Y$ such that for every $x_1 \leq x_2 \in X$, $\phi(x_1) \leq \phi(x_2)$, where '$\leq$' denotes the usual order on $\mathbb{R}$. Choose the correct statement(s) from below:
(A) $\mathbb{N}$ and $\mathbb{Z}$ are not order-isomorphic;
(B) $\mathbb{N}$ and $\mathbb{Q}$ are order-isomorphic;
(C) $\mathbb{Z}$ and $\mathbb{Q}$ are order-isomorphic;
(D) The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are pairwise non-order-isomorphic.

(2) Let $x_n = \left(1 - \frac{1}{n}\right) \sin \frac{n\pi}{3}$, $n \geq 1$. Write $l = \liminf x_n$ and $s = \limsup x_n$. Choose the correct statement(s) from below:
(A) $-\frac{\sqrt{3}}{2} \leq l < s \leq \frac{\sqrt{3}}{2}$;
(B) $-\frac{1}{2} \leq l < s \leq \frac{1}{2}$;
(C) $l = -1$ and $s = 1$;
(D) $l = s = 0$.

(3) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}), & \text{if } x \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Choose the correct statement(s) from below:
(A) f is continuous;
(B) f is discontinuous at 0;
(C) f is differentiable;
(D) f is continuously differentiable.

(4) Let $A \in M_{m\times n}(\mathbb{R})$ be of rank m. Choose the correct statement(s) from below:
  (A) The map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $v \mapsto Av$ is injective;
  (B) There exist matrices $B \in M_m(\mathbb{R})$ and $C \in M_n(\mathbb{R})$ such that $BAC = \begin{bmatrix} I_m \mid \mathbf{0}_{n-m} \end{bmatrix}$;
  (C) There exist matrices $B \in \mathrm{GL}_m(\mathbb{R})$ and $C \in \mathrm{GL}_n(\mathbb{R})$ such that $BAC = \begin{bmatrix} I_m \mid \mathbf{0}_{n-m} \end{bmatrix}$;
  (D) For every $(B, C) \in M_m(\mathbb{R}) \times M_n(\mathbb{R})$ such that $BAC = \begin{bmatrix} I_m \mid \mathbf{0}_{n-m} \end{bmatrix}$, C is uniquely determined by B.

(5) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function such that $f(z + 1) = f(z + \imath) = f(z)$ for every $z \in \mathbb{C}$. Choose the correct statement(s) from below:
  (A) f is constant;
  (B) $f(z) = 0$ for every $z \in \mathbb{C}$;
  (C) There exist complex numbers a, b such that for every $x, y \in \mathbb{R}$, $f(x+\imath y) = a \sin(x)+ \imath b \cos(y)$;
  (D) f is not necessarily constant but $|f(z)|$ is constant.

(6) What is the cardinality of the centre of $O_2(\mathbb{R})$? (Definition: The *centre* of a group G is $\{g \in G \mid gh = hg \text{ for every } h \in G\}$. Hint: Reflection matrices and permutation matrices are orthogonal.)
  (A) 1;
  (B) 2;
  (C) The cardinality of $\mathbb{N}$;
  (D) The cardinality of $\mathbb{R}$.

(7) Let $U \subseteq \mathbb{R}$ be a non-empty open subset. Choose the correct statement(s) from below:
  (A) U is uncountable;
  (B) U contains a closed interval as a proper subset;
  (C) U is a countable union of disjoint open intervals;
  (D) U contains a convergent sequence of real numbers.

(8) Let R be a commutative ring. The *characteristic* of R is the smallest positive integer n such that $a + a + \cdots + a$ (n times) is zero for every $a \in R$, if such an integer exists, and zero, otherwise. Choose the correct statement(s) from below:
  (A) For every $n \in \mathbb{N}$, there exists a commutative ring whose characteristic is n;
  (B) There exists a integral domain with characteristic 57;
  (C) The characteristic of a field is either 0 or a prime number;
  (D) For every prime number p, every commutative ring of characteristic p contains $\mathbb{F}_p$ as a subring.

Instructions: The answers to questions 9–10 are integers. You are required to write the answers in decimal form in the attached bubble-sheet. Every question is worth <u>four</u> (4) marks.

(9) Consider the $\mathbb{Q}$-vector-space

$$\{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is continuous and } \mathrm{Image}(f) \subseteq \mathbb{Q}\}.$$

What is its dimension?

(10) Let p be a prime number and F a field of p^{23} elements. Let $\phi : F \longrightarrow F$ be the field automorphism of F sending a to a^p. Let $K := \{a \in F \mid \phi(a) = a\}$. What is the value of $[K : \mathbb{F}_p]$?

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

---

(11) Let $U = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4\}$. Let $p, q \in U$. Show that there is a continuous map $\gamma : [0, 1] \longrightarrow U$ such that $\gamma(0) = p$ and $\gamma(1) = q$ and such that γ is differentiable on $(0, 1)$.

(12) If I, J are two maximal ideals in a PID that is not a field, then show that IJ is never a prime ideal.

(13) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function. Suppose that $f(z) \in \mathbb{R}$ if z is on the real axis or on the imaginary axis. Show that $f'(z) = 0$ at $z = 0$.

(14) Let $A \subseteq \mathbb{R}^n$ be a closed proper subset. For $x, y \in \mathbb{R}^n$, denote the usual (Euclidean) distance between them by $d(x, y)$. Let $x \in \mathbb{R}^n \smallsetminus A$; define $\delta := \inf\{d(x, y) \mid y \in A\}$. Show that there exists $y \in A$ such that $\delta = d(x, y)$.

(15) Let F be a field and V a finite-dimensional vector-space over F. Let $T : V \longrightarrow V$ be a linear transformation, such that for every $v \in V$, there exists $n \in \mathbb{N}$ such that $T^n(v) = v$.
(A) Show that if $F = \mathbb{C}$, then T is diagonalizable.
(B) Show that if $\text{char}(F) > 0$, then there exists a non-diagonalizable T satisfying the above hypothesis.

(16) Let $F = \mathbb{Q}(\omega, \sqrt[3]{2})$, where $\omega \in \mathbb{C}$ is a primitive cube-root of unity. Find a $\mathbb{Q}$-basis for F (with proof). Let $\mu : F \longrightarrow F$ be the $\mathbb{Q}$-linear map given by $\mu(a) = \omega^2 a$. Find the matrix of μ with respect to the basis obtained above.

(17*) Let G be a non-trivial subgroup of the group $(\mathbb{R}, +)$. Show that either G is dense in $\mathbb{R}$ or that $G = \mathbb{Z} \cdot l$ where $l = \inf\{x \in G \mid x > 0\}$.

(18*) Let G be a subgroup of the group of permutations on a finite set X. Let F be the $\mathbb{C}$-vector-space of all the functions from X to $\mathbb{C}$. G acts on F by $(g \cdot f) : x \mapsto f(g^{-1}(x))$. Show that there is an $\phi \in F$ such that $g \cdot \phi = \phi$ for every $g \in G$. Show that there is a subspace F' of F such that $F = F' \oplus \mathbb{C}\langle\phi\rangle$ and such that $g \cdot f \in F'$ for every $g \in G$ and $f \in F'$.

(19*) (A) Let A and B be $n \times n$ matrices with entries in $\mathbb{N}$. Show that if $B = A^{-1}$ then A and B are permutation matrices. (A *permutation matrix* is a matrix obtained by permuting the rows of the identity matrix.)
(B) Let A be an $n \times n$ complex matrix that is not a scalar multiple of I_n. Show that A is similar to a matrix B such that $B_{1,1}$ (i.e. the top left entry of B) is 0.

(20*) Let $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. Consider the map $\text{Sq} : S^1 \longrightarrow S^1$,

$$\text{Sq}(z) = z^2.$$

Show that there does not exist a continuous map $\text{Sqrt} : S^1 \longrightarrow S^1$ such that $\text{Sq} \circ \text{Sqrt} = Id_{S^1}$? (That is, $(\text{Sqrt}(w))^2 = w$.) (Hint: If such a map existed, show that there would be a bijective continuous map $S^1 \times \{1, -1\} \longrightarrow S^1$.)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (12)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (13)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (14)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (15)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (16)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (17*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (18*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (19*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (20*)