

NATIONAL BOARD FOR HIGHER MATHEMATICS

DOCTORAL SCHOLARSHIP SCHEME 2022 WRITTEN TEST

SATURDAY 18 JUNE 2022, 10:30 A.M. TO 1:30 P.M.

- Application number:  Roll number:
- Name in full in BLOCK letters:
- This test consists of 34 questions distributed over two parts (Part A and Part B). Answer as many as you can.
- **This test booklet must have 6 pages** (this cover page with instructions and 5 pages of questions). **Make sure that you have all 6 pages and all 34 questions in your booklet.**
- TIME ALLOWED: 180 minutes (three hours).
- QUESTIONS in each part are arranged rather randomly. They are not sorted by topic.
- MODE OF ANSWERING: Enter only your final answer in the box provided for it. This “answer box” has the following appearance: . It is neither necessary nor is there provision of space to indicate the steps taken to reach the final answer.

**Only the final answer, written legibly and unambiguously in the answer box, will be marked.**
- MARKING: The maximum possible score is 150. The marking scheme for each part is described in more detail at the beginning of that part. **There is negative marking in part B**, but not in Part A.
- NOTATION AND TERMINOLOGY: The questions make free use of standard notation and terminology. You too are allowed the use of standard notation. For example, answers of the form  $e + \sqrt{2}$  and  $2\pi/19$  are acceptable; both  $3/4$  and  $0.75$  are acceptable.
- DEVICES: Use of plain pencils, pens, and erasers is allowed. Mobile phones and calculators are not allowed inside the exam hall. More generally, any device (e.g. a smart watch) that can be used for communication or calculation or storage is prohibited. Invigilators have the right to impound any device that arouses their suspicion (for the duration of the test).
- ROUGH WORK: For rough work, you may use the sheets separately provided, in addition to the blank pages in the test booklet. You must:
  - Write your name and roll number on each such sheet (or set of sheets if stapled).
  - Return all these sheets to the invigilator along with this test booklet at the end of the test.
- Do **not** seek clarification from the invigilator or anyone else about any question. In the unlikely event that there is a mistake in any question, appropriate allowance will be made while marking.

**Part A (QUESTIONS 1 TO 25) Short Answer TYPE**

There are 25 questions in this part. Each question carries 4 marks and demands a short answer. The answers must be written only in the boxes provided for them. There is no negative marking in this part: in other words, there is no penalty for incorrect answers. There is no possibility of partial credit: either you get all 4 marks allotted to a question or none at all.

- (1) Let  $n$  be the smallest positive integer that has 2022 as a factor and has exactly 2022 factors (counting both 1 and  $n$  as factors of  $n$ ). Let  $d$  be the largest positive integer such that  $2^d$  divides  $n$ .

What is  $d$ ?

**‡ 336**

- (2) A box contains  $b$  blue balls and  $y$  yellow balls. One ball is drawn at random from the box, but is then put back into the box along with  $a$  additional balls of the same colour. Another ball is now drawn at random from the box. What is the probability that the first ball drawn was blue, given that the second ball drawn is yellow (in terms of  $b$ ,  $y$ , and  $a$ )?

**‡  $b/(b+y+a)$**

- (3) The inner surface of a radially symmetric coffee mug is described as follows: with the  $x$ -,  $y$ -, and  $z$ -axes being oriented as usual, consider the portion of the curve  $z = x^4$  in the  $x$ - $z$  plane that is below the horizontal line  $z = 1$ ; then rotate it about the  $z$ -axis to obtain the surface.

Find the volume of the coffee mug.

**‡  $2\pi/3$**

- (4) Consider the surface  $S$  in  $\mathbb{R}^3$  defined by the equation  $z^2 - xy = 0$ . What is the number of connected components of its complement (in  $\mathbb{R}^3$ )?

**‡ 3**

- (5) The complex numbers  $a_n$ ,  $n \geq 0$ , are defined by the condition that

$$\sum_{n \geq 0} a_n (z - 2i)^n = 1 + 2z + 3z^2 + \cdots = \sum_{n \geq 0} (n+1)z^n$$

holds for each  $z$  in  $\mathbb{C}$  such that  $|z| < 1$  and  $\Im(z) > 0$  (here  $\Im(z)$  denotes the imaginary part of  $z$ .)

What is  $\limsup \sqrt[n]{|a_n|}$ ?

**‡  $1/\sqrt{5}$**

- (6) Find the largest positive real number  $r$  such that  $|\Im \sin(z)| \leq 1$  for all  $z \in \mathbb{C}$  with  $|z| \leq r$ . (Here  $\Im \sin(z)$  denotes the imaginary part of the function  $\sin(z)$ .) If there is no such largest  $r$ , then write “infinity” in the answer box.

**‡  $\log(1 + \sqrt{2})$**

**‡  $\ln(1 + \sqrt{2})$**

- (7) Given two subsets  $C$  and  $S$  of a vector space  $V$ , their sum  $C + S$  is the subset of  $V$  defined as follows:  $C + S := \{c + s \mid c \in C, s \in S\}$ . Now take  $V$  to be the vector space  $\mathbb{R}^3$ . Let  $C$  be the cube and  $S$  the sphere defined as follows:

$$C := \{(x, y, z) \in \mathbb{R}^3 \mid |x| \leq 1, |y| \leq 1, |z| \leq 1\} \quad \text{and} \quad S := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}.$$

Find the volume of their sum  $C + S$ .

**‡  $32 + \frac{22\pi}{3}$**

- (8) Compute  $\iint_A (3x^2y - y^3) dx dy$  where  $A$  is the region  $\{(x, y) \mid x^2 + (y - 1)^2 \leq 1\}$ .

**‡  $-\pi$**

- (9) An element of the alternating group  $A_5$  is picked at random (each element is equally likely to be picked). What is the expected value of its order?  $\dagger$  **211/60**
- (10) On the set  $\mathcal{M}$  of all  $m \times n$  real matrices, introduce the following equivalence relation: two such matrices  $A$  and  $B$  are equivalent if there exist a real invertible  $m \times m$  matrix  $P$  and a real invertible  $n \times n$  matrix  $Q$  such that  $A = PBQ$ . Into how many disjoint equivalence classes is  $\mathcal{M}$  partitioned by this equivalence relation?  $\dagger$   **$\min(m, n) + 1$**
- (11) What is the area of the region  $\{z + \frac{z^2}{2} \mid z \in \mathbb{C}, |z| \leq 1\}$ ?  $\dagger$   **$3\pi/2$**
- (12) Let  $\Gamma := C[-1, 1]$  be the normed linear space of continuous real valued functions on the interval  $[-1, 1]$  with the supremum norm. Let  $K$  be the kernel of the linear functional  $I : f \mapsto \int_{-1}^1 f(x) dx$  on  $\Gamma$ . Consider the function  $f(x) = x^2$  in  $\Gamma$ . What is its distance from the subspace  $K$ ?  $\dagger$  **1/3**
- (13) Let  $X$  denote the set  $\{1, 2, \dots, n\}$  of the first  $n$  positive integers (we assume  $n \geq 3$ ). Let  $\mathcal{S}$  be the group of bijections of  $X$ , with multiplication being the composition. The group  $\mathcal{S}$  acts on  $X$  in the standard manner:  $\mathcal{S} \times X \rightarrow X$  being given by  $(\varphi, i) \mapsto \varphi \cdot i := \varphi(i)$ , where  $\varphi(i)$  denotes the image of  $i$  under  $\varphi$ . Consider the induced action of  $\mathcal{S}$  on  $X \times X \times X$  by  $\varphi \cdot (i, j, k) := (\varphi(i), \varphi(j), \varphi(k))$ . How many orbits does this induced action of  $\mathcal{S}$  on  $X \times X \times X$  have?  $\dagger$  **5**
- (14) Let  $k$  be a field with 8 elements and  $K$  a field with 64 elements. How many ring homomorphisms are there from  $k$  to  $K$ ?  $\dagger$  **3**
- (15) Let  $\mathcal{C}$  be the subset of real valued continuous functions  $f(x)$  on  $[-1, 1]$  such that  $f(x) \geq 0$  (for all  $x$  in  $[-1, 1]$ ) and  $\limsup_{n \rightarrow \infty} \int_{-1}^1 f(x)^n dx < \infty$ . What is the image of  $\mathcal{C}$  in  $\mathbb{R}$  under the map that takes  $f(x)$  in  $\mathcal{C}$  to  $\int_{-1}^1 f(x) dx$ ?  $\dagger$  **[0, 2]**
- (16) Let  $\mathcal{M}$  be the real vector space consisting of all  $n \times n$  real matrices with the inner product  $(A, B) := \text{trace}(AB^t)$ , where  $B^t$  denotes the transpose of  $B$ , and  $AB^t$  the usual matrix product. Let  $\mathcal{S}$  be the subspace of  $\mathcal{M}$  consisting of all symmetric  $n \times n$  matrices, and  $P : \mathcal{M} \rightarrow \mathcal{S}$  the orthogonal projection to  $\mathcal{S}$  with respect to the above inner product. For  $X$  an arbitrary element of  $\mathcal{M}$ , write a formula (in terms of  $X$ ) for the image  $P(X)$  of  $X$  under  $P$ .  $\dagger$   **$(X + X^t)/2$**
- (17) Let  $\mathcal{H} := \{a + bi \in \mathbb{C} \mid b > 0\}$  be the (open) upper half plane. Let  $\varphi$  be an analytic function on  $\mathcal{H}$  such that
- $$\varphi(\mathcal{H}) \subseteq \mathcal{H}, \quad \varphi(i) = i, \quad \text{and} \quad \varphi(2i) = i/2$$
- What is  $|\varphi(1 + i)|$ ? If the given information is not enough to determine the required answer, then write "indeterminable" in the answer box.  $\dagger$   **$1/\sqrt{2}$**
- (18) Let  $A$  be the abelian group  $\frac{\mathbb{Z}}{7\mathbb{Z}} \oplus \frac{\mathbb{Z}}{49\mathbb{Z}}$ , where  $\frac{\mathbb{Z}}{n\mathbb{Z}}$  denotes the cyclic group of integers modulo  $n$ . How many subgroups of order 7 does  $A$  have?  $\dagger$  **8**

- (19) Let  $\mathcal{M}$  be the algebra consisting of all  $3 \times 3$  complex matrices (with the usual operations of matrix addition, scalar multiplication, and matrix multiplication). For  $A$  in  $\mathcal{M}$ , let  $\Lambda_A : \mathcal{M} \rightarrow \mathcal{M}$  be the complex linear transformation on the complex vector space  $\mathcal{M}$  defined by  $X \mapsto AX - XA$ . What are the possible values of the rank of  $\Lambda_A$ , as  $A$  varies over all of  $\mathcal{M}$ ? **‡ 0,4,6**

- (20) Consider the topology on the set  $\mathbb{C}$  of complex numbers given by the following prescription: a subset  $Z$  of  $\mathbb{C}$  is closed if and only if there exists a polynomial  $f(x)$  with rational coefficients such that  $Z$  is precisely the set of zeroes of  $f(x)$ , that is,  $Z = \{z \in \mathbb{C} \mid f(z) = 0\}$ . (The whole set  $\mathbb{C}$  can be considered as the set of zeroes of the zero polynomial.) In the three boxes below, write respectively the cardinalities of the closures of the following three singleton sets:  $\{\sqrt{2} + \sqrt{3}\}$ ,  $\{e^{2\pi i/15}\}$ ,  $\{\pi\}$ . (You may write “ $\infty$ ” for infinity.) **‡ 4** **‡ 8** **‡  $\infty$  or infinity**

- (21) Subgroups  $H$  and  $H'$  of a group  $G$  are said to be conjugate if there exists an element  $g$  in  $G$  that conjugates  $H$  to  $H'$ , that is,  $gHg^{-1} = H'$ . The relation thus defined is an equivalence relation and the equivalence classes are called conjugacy classes (of subgroups). What is the number of conjugacy classes of cyclic subgroups of order 15 in the symmetric group  $S_{15}$ ? **‡ 5**

- (22) Let  $f(x)$  be the function on  $\mathbb{R}$  defined by  $f(x) := \sin(\pi x/2)$ . For  $y$  in  $\mathbb{R}$ , consider the sequence  $\{x_n(y)\}_{n \geq 0}$  defined by

$$x_0(y) := y \quad \text{and} \quad x_{n+1}(y) = f(x_n(y)) \quad \text{for all } n \geq 1,$$

and let  $g(y) := \lim_{n \rightarrow \infty} x_n(y)$ . Find  $\int_0^3 g(y) dy$ . **‡ 1**

- (23) Let  $t \mapsto y(t)$  be a real valued smooth function defined for all  $t$  in an open interval of the real line containing  $[0, 1]$ . Suppose that  $y(t)$  satisfies the following differential equation:

$$y''(t) + w(t)y(t) = \lambda y(t)$$

for some real constant  $\lambda$  and some function  $w(t)$  of  $t$ , and further suppose that  $y(t) > 0$  for all  $t \in [0, 1]$  and  $y'(0) = y'(1) = 0$ .

Given that  $\int_0^1 (\frac{y'}{y})^2 dt = 10$  and  $\int_0^1 w(t) dt = 20$ , what is  $\lambda$ ? **‡ 30**

- (24) Let  $u(x)$  be a real valued continuous function on  $[0, 1]$  such that  $u(\frac{1}{2}) = 1$  and:

$$u(x) \geq 0 \quad \text{and} \quad u(x) \leq \int_0^x u(t) dt \quad \text{for all } x \in [0, 1]$$

What is  $u(1)$ ? If the given information is insufficient to determine  $u(1)$ , write “indeterminable” in the answer box. If there is no function  $u(x)$  satisfying the given hypothesis, then write “no such  $u$ ” in the answer box. **‡ no such  $u$**

- (25) A smooth real valued function  $y = f(x)$  is defined for all  $x > 0$  in the real line and is positive (that is,  $f(x) > 0$  for all  $x > 0$ ). The graph  $C$  of  $f(x)$  has the following property: for any point  $(x_0, y_0)$  on  $C$ , if  $L(x_0, y_0)$  is the segment of the tangent line to  $C$  at  $(x_0, y_0)$  that lies in the first quadrant, then  $(x_0, y_0)$  is the mid-point of  $L(x_0, y_0)$ . Given that  $f(1) = 2$ , what is  $f(2022)$ ? **‡ 1/1011**

**Part B (QUESTIONS 26–34) True or False TYPE**

This part features 25 assertions grouped into 9 questions. For each assertion, you are required to determine its truth value and write either “True” or “False” in the corresponding answer box, as the case may be. Each correct response will fetch you 2 marks. But there is negative marking: **each incorrect response carries a penalty of 1 mark.** You do not gain or lose any marks on items left unanswered.

- (26) Suppose  $A$  is a  $n \times n$  complex matrix (for some  $n \geq 2$ ) with the property that every matrix that commutes with  $A$  is a polynomial in  $A$ .

(a) Such a matrix  $A$  is necessarily diagonalizable.

‡ False

(b) Such a matrix  $A$  cannot be nilpotent.

‡ False

(c) The characteristic and minimal polynomials coincide for every such  $A$ .

‡ True

- (27) Let  $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$  be a set with 7 elements. A simple graph with vertex set  $V$  is just a subset  $E$  of (some) subsets of cardinality 2 of  $V$ . The elements of  $E$  are called edges. An element  $v_i$  of  $V$  is said to be incident on an edge  $e$  if  $v_i$  belongs to  $e$ ; the valency  $p_i$  of  $v_i$  is defined to be the number of edges on which  $v_i$  is incident. We call  $(p_1, p_2, p_3, p_4, p_5, p_6, p_7)$  the valency tuple of the graph.

Which of the following tuples can occur as valency tuples of a simple graph structure on  $V$ ? Write “True” in the corresponding box if the tuple can occur and “False” if it cannot.

(a)  $(2, 3, 4, 4, 4, 3, 3)$

‡ False

(b)  $(2, 2, 4, 6, 4, 2, 2)$

‡ True

(c)  $(2, 6, 4, 6, 2, 6, 4)$

‡ False

- (28) Let  $f(z)$  be an analytic function on an open set of the complex plane containing the closed unit disk  $D := \{z \in \mathbb{C} \mid |z| \leq 1\}$ . Let  $m$  be the minimum of  $\{|f(z)| \mid z \in D\}$ , and  $M$  be the minimum of  $\{|f(z)| \mid z \in C\}$ , where  $C$  is the boundary  $\{z \in \mathbb{C} \mid |z| = 1\}$  of  $D$ . Assume that  $m < M$ . Then:

(a)  $f(z)$  admits a zero on  $D$ .

‡ True

(b)  $f(z)$  attains on  $D$  every complex value  $w$  such that  $|w| < M$ .

‡ True

- (29) Let  $S$  be the set  $\mathbb{R}^2 \setminus \{(0, y) \mid y \neq 0\}$  obtained by removing from  $\mathbb{R}^2$  the  $y$ -axis except for the origin. Let  $\tau_1$  denote the subspace topology on  $S$  induced from the usual topology of  $\mathbb{R}^2$ .

Now, consider the surjective map  $\pi : \mathbb{R}^2 \rightarrow S$  defined by

$$(x, y) \mapsto (x, y) \text{ for } x \neq 0, \text{ and } (0, y) \mapsto (0, 0) \text{ for all } y.$$

Let  $\tau_2$  be the resulting quotient topology on  $S$ . (In other words,  $\tau_2$  is the finest topology on  $S$  with respect to which  $\pi$  is continuous.)

Then:

(a) The topologies  $\tau_1$  and  $\tau_2$  are not comparable.

‡ False

(b)  $\tau_2$  is regular (that is, given a closed set and a point not in it, there exist disjoint open sets containing the closed set and the point respectively).

‡ True

(c)  $\tau_2$  admits a countable basis.

‡ False

(d)  $\tau_2$  is metrizable.

‡ False

(e) Any subset of  $S$  that is compact with respect to the topology  $\tau_2$  is also compact with respect to the topology  $\tau_1$ .

‡ False

- (30) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth, bounded function such that the limits  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow \infty} f(-x)$  exist. Then:
- (a) The function  $f$  is uniformly continuous.  $\dagger$  True
- (b) The derivative  $f'$  is uniformly continuous.  $\dagger$  False
- (31) Let  $A$  be an arbitrary special orthogonal  $3 \times 3$  real matrix. (In other words,  $A$  is a real  $3 \times 3$  matrix such that  $AA^t = A^tA = I$ , where  $A^t$  is the transpose of  $A$  and  $I$  is the  $3 \times 3$  identity matrix, and  $\det A = 1$ .) Then:
- (a)  $A$  admits 1 as an eigenvalue.  $\dagger$  True
- (b) If all the eigenvalues of  $A$  are real, then  $A$  is the identity  $3 \times 3$  matrix.  $\dagger$  False
- (c) There is a complex invertible matrix  $P$  such that  $PAP^{-1}$  is diagonal.  $\dagger$  True
- (32) Consider a sequence of independent and identically distributed random variables  $\{X_n\}_{n \geq 1}$ , taking values in  $\{1, -1\}$ , and satisfying  $\lim_{n \rightarrow \infty} X_n = X$  almost surely, for some random variable  $X$ . Then:
- (a)  $X$  is a constant, almost surely.  $\dagger$  True
- (b)  $X_1$  is a constant, almost surely.  $\dagger$  True
- (c)  $X$  is distributed as  $X_1$ .  $\dagger$  True
- (33) Let  $\mathbb{R}'$  be the set of real numbers with the topology for which  $\{[a, \infty) \mid a \in \mathbb{R}\}$  forms a basis. Consider the product topology on  $\mathbb{R}' \times \mathbb{R}'$ . Now let  $S$  be the subset  $\{(x, \sin(x)) \mid x \in \mathbb{R}\}$  of  $\mathbb{R}' \times \mathbb{R}'$  with the subspace topology. Consider the map  $\varphi : \mathbb{R} \rightarrow S$  defined by  $\varphi(x) := (x, \sin x)$ . Then:
- (a) The image of the interval  $(\pi/2, \pi)$  under  $\varphi$  is closed in  $S$ .  $\dagger$  False
- (b) The image of the interval  $(0, \pi/2)$  under  $\varphi$  is closed in  $S$ .  $\dagger$  False
- (34) Let  $d, e$  be positive integers and  $T : \mathbb{R}^d \rightarrow \mathbb{R}^e$  a linear transformation. Then:
- (a) The linear transformation  $T$  is an open map (that is, it maps open sets of  $\mathbb{R}^d$  to open sets of  $\mathbb{R}^e$ ) if and only if  $T$  is surjective.  $\dagger$  True
- (b)  $T$  is a closed map (that is, it maps closed sets of  $\mathbb{R}^d$  to closed sets of  $\mathbb{R}^e$ ) if and only if either  $T = 0$  or  $T$  is injective.  $\dagger$  True