CHENNAI MATHEMATICAL INSTITUTE

Postgraduate Programme in Mathematics
MSc/PhD Entrance Examination
18 May 2015

Instructions:

- Enter your *Registration Number* here: **CMI–PG–** | | | | | | | | |
- Enter the name of the city where you write this test: | | | | | | | | | | | | |
- The allowed time is 3 hours.
- This examination has two parts. You may use the blank pages at the end for your rough-work.
- Part A is worth 40 marks and will be used for screening. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **Record your answers to Part A in the attached bubble-sheet.**
- Part B is worth 60 marks. You should answer six (6) questions in Part B. In order to qualify the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17*)–(20*). **Indicate the six questions to be marked in the boxes in the bubble-sheet. Write your solutions to Part B in the page assigned to each question.**
- **Please read the further instructions given before Part A and inside each part carefully.**

For office use only

Part B

No.	Marks	Remarks
11		
12		
13		
14		
15		
16		

Part B (ctd.)

No.	Marks	Remarks
17*		
18*		
19*		
20*		

Further remarks:

Part A	
Part B	
Total	

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- *Registration Number*: **CMI–PG–** | | | | | | | | |
- City where you write this test: | | | | | | | | | | | | |

Bubble Sheet

Part A

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ
3. Ⓐ Ⓑ Ⓒ Ⓓ
4. Ⓐ Ⓑ Ⓒ Ⓓ
5. Ⓐ Ⓑ Ⓒ Ⓓ
6. Ⓐ Ⓑ Ⓒ Ⓓ
7. Ⓐ Ⓑ Ⓒ Ⓓ
8. ____________
9. ____________
10. ____________

Part B

In the boxes below, clearly indicate the SIX (6) solutions that should be marked. In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions (17^*)–(20^*).

For office use only:

Number of correct answers in Part A:	
Marks in Part A:	

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Important: Questions in Part A will be used for **screening**. There will be a cut-off for Part A, which will not be more than twenty (20) marks (out of 40). Your solutions to the questions in Part B will be marked only if your score in Part A places you over the cut-off. (In particular, if your score in Part A is at least 20 then your solutions to the questions in Part B will be marked.) However, note that the scores in both the sections will be taken into account while making the final decision. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.**

Notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ stand, respectively, for the sets of the natural numbers, of the integers, of the rational numbers, of the real numbers, and of the complex numbers. For a prime number p, $\mathbb{F}_p$ is the field with p elements. For a field F, $M_{m\times n}(F)$ stands for the set of $m \times n$ matrices over F and $\mathrm{GL}_n(F)$ is the set of invertible $n \times n$ matrices over F. If $F = \mathbb{R}$ or $F = \mathbb{C}$, we treat these sets as metric spaces with the metric $d(A, B) = \sqrt{\sum_{i,j} |a_{ij} - b_{ij}|^2}$ where $A = (a_{ij})$ and $B = (b_{ij})$.

Part A

Instructions: Each of the questions 1–7 has one or more correct answers. Record your answers on the attached bubble-sheet by filling in the appropriate circles. Every question is worth four (4) marks. A solution receives credit if and only if all the correct answers are chosen, and no incorrect answer is chosen.

(1) Which of the following topological spaces is/are connected?
(A) $\mathrm{GL}_1(\mathbb{R})$
(B) $\mathrm{GL}_1(\mathbb{C})$
(C) $\mathrm{GL}_2(\mathbb{R})$
(D) $\left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} : x, y \in \mathbb{R}, x^2 + y^2 = 1 \right\}$

(2) Consider $f : \{z \in \mathbb{C} : |z| > 1\} \longrightarrow \mathbb{C}, f(z) = \frac{1}{z}$. Choose the correct statement(s):
(A) There are infinitely many entire functions g such that $g(z) = f(z)$ for every $z \in \mathbb{C}$ with $|z| > 1$.
(B) There does not exist an entire function g such that $g(z) = f(z)$ for every $z \in \mathbb{C}$ with $|z| > 1$.
(C) $g : \mathbb{C} \longrightarrow \mathbb{C}$ with
$$g(z) = \begin{cases} 1 - \frac{1}{2}z^2, & |z| \le 1 \\ \frac{1}{z}, & |z| > 1 \end{cases}$$
is an entire function such that $g(z) = f(z)$ for every $z \in \mathbb{C}$ with $|z| > 1$.

(3) Let
$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.$$
Which of the following are true?
(A) G/N is isomorphic to $\mathbb{R}$ under addition.
(B) G/N is isomorphic to $\{a \in \mathbb{R} : a > 0\}$ under multiplication.
(C) There is a proper normal subgroup N' of G which properly contains N.
(D) N is isomorphic to $\mathbb{R}$ under addition.

(4) Choose the correct statement(s):
(A) There is a continuous surjective function from $[0,1)$ to $\mathbb{R}$;
(B) $\mathbb{R}$ and $[0,1)$ are homeomorphic to each other;
(C) There is a bijective function from $[0,1)$ to $\mathbb{R}$;
(D) Bounded subspaces of $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}$.

(5) Which of the following complex numbers has/have a prime number as the degree of its minimal polynomial over $\mathbb{Q}$?
(A) ζ_7, a primitive 7th root of unity;
(B) $\sqrt{2}+\sqrt{3}$;
(C) $\sqrt{-1}$;
(D) $\sqrt[3]{2}$.

(6) Let R be an integral domain such that every non-zero prime ideal of $R[X]$ (where X is an indeterminate) is maximal. Choose the correct statement(s):
(A) R is a field;
(B) R contains $\mathbb{Z}$ as a subring;
(C) Every ideal in $R[X]$ is principal;
(D) R contains $\mathbb{F}_p$ as a subring for some prime number p.

(7) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $\int_{-\infty}^{\infty} |f(x)| \mathrm{d}x < \infty$. Define $F : \mathbb{R} \longrightarrow \mathbb{R}$ by $F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t$. Choose the correct statement(s):
(A) f is continuous;
(B) F is continuous;
(C) F is uniformly continuous;
(D) There exists a positive real number M such that $|f(x)| < M$ for all $x \in \mathbb{R}$.

Instructions: The answers to questions 8–10 are integers. You are required to write the answers in decimal form in the attached bubble-sheet. Every question is worth four (4) marks.

(8) Let $\omega \in \mathbb{C}$ be a primitive third root of unity. How many distinct possible images of ω are there under all the field homomorphisms $\mathbb{Q}(\omega) \longrightarrow \mathbb{C}$.

(9) Let $C := \{z \in \mathbb{C} : |z| = 5\}$. What is value of M such that

$$2\pi\imath M = \int_C \frac{1}{z^2 - 5z + 6} \mathrm{d}z?$$

(10) Consider the set $\mathbb{R}[X]$ of polynomials in X with real coefficients as a real vector space. Let T be the $\mathbb{R}$-linear operator on $\mathbb{R}[X]$ given by

$$T(f) = \frac{\mathrm{d}^2 f}{\mathrm{d}X^2} - \frac{\mathrm{d}f}{\mathrm{d}X} + f.$$

What is the nullity of f?

Part B

Instructions: Answer six (6) questions from below. Provide sufficient justification. Write your solutions on the page assigned to each question. Each of the questions is worth ten (10) marks. **In order to qualify for the PhD Mathematics interview, you must obtain at least fifteen (15) marks from among the starred questions** (17*)–(20*)**.** Clearly indicate which six questions you would like us to mark in the six boxes in the bubble sheet. If the boxes are unfilled, we will mark the first six solutions that appear in your answer-sheet. If you do not want a solution to be considered, clearly strike it out.

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(11) Let $f \in \mathbb{R}[x, y]$ be such that there exists a non-empty open set $U \subseteq \mathbb{R}^2$ such that $f(x, y) = 0$ for every $(x, y) \in U$. Show that $f = 0$.

(12) Let $A \in M_{n\times n}(\mathbb{C})$.
(a) Suppose that $A^2 = 0$. Show that λ is an eigenvalue of $(I_n + A)$ if and only if $\lambda = 1$. (I_n is the $n \times n$ identity matrix.)
(b) Suppose that $A^2 = -1$. Determine (with proof) whether A is diagonalizable.

(13) Let f be a non-constant entire function satisfying the following conditions:
(a) $f(0) = 0$;
(b) For every positive real number M, the set $\{z : |f(z) < M\}$ is connected.
Prove that $f(z) = cz^n$ for some constant c and positive integer n.

(14) Let $(a_{mn})_{m\geq 1, n\geq 1}$ be a double sequence of real numbers such that
(a) For every n, $b_n := \lim\limits_{m\to\infty} a_{mn}$ exists;
(b) For all strictly increasing sequences $(m_k)_{k\geq 1}$ and $(n_k)_{k\geq 1}$ of positive integers, $\lim\limits_{k\to\infty} a_{m_k n_k} = 1$.
Show that the sequence $(b_n)_{n\geq 1}$ converges to 1.

(15) Let $f \in \mathbb{C}[x, y]$ be such that $f(x, y) = f(y, x)$. Show that there is a $g \in \mathbb{C}[x, y]$ such that $f(x, y) = g(x + y, xy)$.

(16) Let X be a topological space and $f : X \longrightarrow [0, 1]$ be a closed continuous surjective map such that $f^{-1}(a)$ is compact for every $0 \leq a \leq 1$. Prove or disprove: X is compact. (A map is said to be *closed* if it takes closed sets to closed sets.)

(17*) Determine the cardinality of set of subrings of $\mathbb{Q}$. (Hint: For a set P of positive prime numbers, consider the smallest subring of $\mathbb{Q}$ that contains $\{\frac{1}{p} \mid p \in P\}$.)

(18*) Let

$$f(x) = \sum_{n\geq 1} \frac{\sin(\frac{x}{n})}{n}.$$

Show that f is continuous. Determine (with justification) whether f differentiable.

(19*) Let m and n be positive integers and p a prime number. Let $G \subseteq \mathrm{GL}_m(\mathbb{F}_p)$ be a subgroup of order p^n. Let $U \subseteq \mathrm{GL}_m(\mathbb{F}_p)$ be the subgroup that consists of all the matrices with 1's on the diagonal and 0's below the diagonal. Show that there exists $A \in \mathrm{GL}_m(\mathbb{F}_p)$ such that $AGA^{-1} \subseteq U$.

(20*) Let m and n be positive integers and $0 \leq k \leq \min\{m, n\}$ an integer. Prove or disprove: The subspace of $M_{m\times n}(\mathbb{C})$ consisting of all matrices of rank equal to k is connected. (You may use the following fact: For $t \geq 2$, $\mathrm{GL}_t(\mathbb{C})$ is connected.)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (11)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (12)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (13)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (14)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (15)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (16)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (17*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (18*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (19*)

Please indicate in the bubble-sheet the questions in Part B to be marked.

Solution to Question (20*)